AN UPPER BOUND FOR THE LEAST DILATATION

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ABSTRACT. We given an upper bound for the least dilatation arising from a pseudo-Anosov map of a closed surface of genus greater or equal to three.

1. INTRODUCTION AND BACKGROUND

Introduction. Throughout the paper, $F = F_g$ will be a closed surface of genus $g$ with negative Euler characteristic. Suppose that $(\mathcal{F}, \nu)$ is a measured foliation (see [FLP]) and $\phi$ is an orientation preserving homeomorphism of $F$, then we define $\phi(\mathcal{F})$ to be the foliation whose leaves are the images of the leaves of $\mathcal{F}$. Furthermore, $\phi_*(\nu)$ is a measure on $\phi(\mathcal{F})$ that is defined as the push forward of the measure $\nu$ under $\phi$. To be more explicit, if $\alpha$ is an arc transverse to the foliation $\phi(\mathcal{F})$, then $\phi_*(\nu)(\alpha) = \nu(\phi^{-1}(\alpha))$. We define $\phi(\mathcal{F}, \nu) = (\phi(\mathcal{F}), \phi_*(\nu))$.

An orientation preserving homeomorphism $\phi$ of $F$ is pseudo-Anosov (or p.A.) if there is a pair of transverse arational (i.e. no closed leaves) measured foliations $(\mathcal{F}, \nu)$ and $(\mathcal{F}^\perp, \nu^\perp)$ in $F$, such that $\phi(\mathcal{F}, \nu) = (\mathcal{F}, \lambda \nu)$ and $\phi(\mathcal{F}^\perp, \nu^\perp) = (\mathcal{F}^\perp, (1/\lambda)\nu^\perp)$, for some $\lambda > 1$. $\lambda$ is called the dilatation of $\phi$, and we define the 'spectrum' of $F$ as

$$\text{Spec}(F) = \{\log \lambda : \lambda \text{ is the dilatation of a p.A. self-map of } F\} \subset \mathbb{R}.$$ 

Spec$(F)$ has a geometric interpretation as the collection of Teichmüller distances between Riemann surfaces of the same topological type as $F$ (see [Ab]). Furthermore, a pseudo-Anosov map $\phi$ realizes the smallest topological entropy in its homotopy class and the topological entropy of $\phi$ is given by the logarithm of its dilatation (see [FLP]). It is known that Spec$(F)$ is discrete (see [AY]).

The main goal of this note is to show that the smallest element $S_g$ of Spec$(F_g)$ allows the upper bound

$$\delta_g \leq \frac{\log 6}{g}, \quad \text{for all } g \geq 3.$$ 

This is an improvement of the upper bound given in [P2].

Background. We begin by establishing some notation. The group of $l \times l$ matrices over $\mathbb{Z}$ is denoted by Mat$_l(\mathbb{Z})$, and the spectrum of $A \in \text{Mat}_l(\mathbb{Z})$ is the set of eigenvalues of $A$ listed with multiplicity. We say that an eigenvalue $\lambda$ in
the spectrum of $A$ has maximum modulus if the modulus of $\lambda$ strictly exceeds the modulus of any other element of the spectrum of $A$. Suppose next that $A$ is an $n \times m$ matrix with $n, m \geq 1$, then by an expression of the form $A > 0$, $A \geq 0$, etc. we mean that the relevant relation holds for each component of $A$. A matrix $A \in \text{Mat}_1(\mathbb{Z})$ is said to be Perron-Frobenius or P.F. if $A \geq 0$ and for some $n \geq 1$ we have $A^n > 0$.

Perron-Frobenius matrices have the following well known spectral properties (see [Ga]).

**Theorem (Perron-Frobenius).** The spectrum of a P.F. matrix $A$ contains an element $\lambda$ of maximum modulus that is positive real with corresponding eigenvector $x^*$ strictly positive. $x^*$ is the unique positive eigenvector and $\lambda$ is a simple root of the characteristic polynomial of $A$.

The main tool in the present investigation is the theory of train tracks as introduced by Thurston [Th]. The material that is needed for the present investigation is collected in the section on train tracks in [Ba]. We refer the reader who wishes more information on surface homeomorphisms and train tracks to [CB, FLP, Th, Pa, PP] and the monograph on train tracks [HP]. We remark in particular that if $\tau$ is a train track embedded in the surface $F$, and if $V(\tau)$ denotes the set of measures on $\tau$, then there is an embedding $\mathcal{J}$ of $V(\tau)$ into the space of (equivalence classes of) measured foliations $\mathcal{MF}(F)$. Moreover, if $\tau$ is invariant under a homeomorphism $\phi$ of $F$, then there are induced maps $\hat{\phi} : V(\tau) \to V(\tau)$ and $\check{\phi} : \mathcal{MF}(F) \to \mathcal{MF}(F)$ such that $\mathcal{J} \circ \hat{\phi} = \check{\phi} \circ \mathcal{J}$.

2. AN UPPER BOUND BY EXAMPLE

**A class of pseudo-Anosov maps.** We describe in this section the map that yields the upper bound mentioned in the introduction.

If $k$ is a simple closed curve embedded in $F$, then we denote the (right handed) Dehn twist along $k$ by $\tau_k$. (For a definition of Dehn twist see for example [Ba].) For $g \geq 3$, we take the surface $F_g$ as a sphere with $g$ handles attached so that $F_g$ is invariant under the rotation $\rho = \rho_g$ by $2\pi/g$ about the axis $L$ through the north and south pole of the sphere. Figure 1 shows the case $g = 3$. We then define

$$\psi = \psi_g = \rho \circ \tau_c^{-1} \circ \tau_b \circ \tau_a.$$  

Whenever the dependency of an object on the genus $g$ is clear we will simplify notation by suppressing this dependency. For example the maps $\rho, \tau_c^{-1}, \tau_b$ and $\tau_a$ depend on $g$.

Our first goal is to show that $\psi_g$ is (isotopic to) a pseudo-Anosov homeomorphism using the following pseudo-Anosov recognition theorem (see [CB]).

**Theorem (Casson).** A homeomorphism $\phi$ of $F$ is isotopic to a p.A. map if there is a $\phi$ invariant train track $\tau$ that fills $F$, such that no proper subtrack $\tau'$ of $\tau$ is invariant under $\phi$, and such that if $\tau$ itself is a subtrack of a $\phi$ invariant train track $\tau''$ (not necessarily proper) then the induced map $\phi'' : V(\tau'') \to V(\tau'')$ has no nonzero fixed point.

The train track that will allow us to apply the previous theorem is $\tau(g)$ as shown in Figure 2 in case $g = 3$. It is clear how $\tau(g)$ is defined (up to isotopy) for $g \geq 3$. We isotope $\tau(g)$ so that $\tau(g)$ is invariant under $\rho$. Note that
\( \tau(g) \) has as complementary regions \( 2g \) tri-gons and \( 2g \) g-gons, hence \( \tau(g) \) fills the surface. The number of branches of \( \tau(g) \) is \( 7g \).

We need to be more precise as to how \( \psi_g \) acts on \( F_g \) and to that end we use the notation for branches of \( \tau(g) \) as illustrated in Figure 2 in case \( g = 3 \). Clearly we can isotope \( \tau_b \circ \tau_a \) so that it fixes the switches of \( \tau(g) \) and all of the branches except for \( b_1, b_2, \) and \( b_3 \). We also isotope \( \tau_c^{-1} \) such that it fixes the switches of \( \tau(g) \) and only moves the branches \( b_4 \) and \( b_5 \).

By choosing the central ties of a standard tie neighborhood of \( \tau(g) \) carefully, we may assume that \( \tau_b \circ \tau_a(b_1) \) intersects the central ties corresponding to the branches of \( \{\rho^{g-1}(b_5), \rho^{g-1}(a_2), b_3, a_1, b_4\} \); also \( \tau_b \circ \tau_a(b_2) \) intersects the central tie corresponding to the branch \( b_1 \), and \( \tau_b \circ \tau_a(b_3) \) intersects the central ties corresponding to the branches of \( \{\rho^{g-1}(b_4), b_2, b_5\} \). Similarly, we may assume that \( \tau_c^{-1}(b_4) \) intersects the central ties corresponding to the branches of \( \{\rho(b_3), a_2, b_3, b_4\} \), and \( \tau_c^{-1}(b_5) \) intersects the ones corresponding to \( \{b_3, a_1, \rho(b_3), b_2, b_5\} \).

For future reference, we summarize

**Lemma 1.** For \( g \geq 3 \), \( \tau(g) \) is a \( \psi_g \) invariant train track that fills \( F_g \).

Note that if we define \( \overline{B} = \bigcup_{i=0}^{g-1} B_i \), where \( B_i = \{\rho^i(b_1), \ldots, \rho^i(b_5)\} \), for \( 0 \leq i \leq g - 1 \), then \( \tau_c^{-1} \circ \tau_b \circ \tau_a \) only affects the branches \( B_0 \). Whenever we need to assume that the branches of \( \tau(g) \) are ordered, we choose the following ordering

\[
(p_0^0(b_1), \ldots, p_0^0(b_5), \ldots, p^{g-1}_g(b_1), \ldots, p^{g-1}_g(b_5),
\]

\[
p^0_0(a_1), p^0_0(a_2), \ldots, p^{g-1}_g(a_1), p^{g-1}_g(a_2)\).

Note that this defines an induced ordering of the branches \( \overline{B} \).

If \( \phi \) is a composition of \( \rho \), \( \tau_c^{-1} \), and \( \tau_b \circ \tau_a \), then we define Inc(\( \phi \)) to be the incidence matrix of \( \phi \) with respect to the ordering of the branches of \( \tau \) specified above. We remark that Inc(\( \phi \)) is well defined by our convention of how \( \rho \), \( \tau_c^{-1} \), and \( \tau_b \circ \tau_a \) act on \( \tau(g) \). Note that the distributive property of the ‘hat’ operator (i.e. \( \hat{\phi_1} \hat{\phi_2} = \hat{\phi_1 \phi_2} \)) implies that for example Inc(\( \tau_c^{-1} \circ \tau_b \circ \tau_a \)) = Inc(\( \tau_c^{-1} \)) Inc(\( \tau_b \circ \tau_a \)).

The following lemma is crucial in showing that \( \psi_g \) is p.A.
Lemma 2. If \( w^{(0)} \in \mathbb{R}^{7g} \) represents a nonzero weight on \( \tau(g) \) (we do not require that the switch conditions hold), such that \( w^{(0)} \) is positive on at least one branch of \( B \), then there is an \( n_0 \) such that for \( n > n_0 \), \( \text{Inc}(\psi^n)w^{(0)} > 0 \).

Proof. Recall that \( \tau_c^{-1} \circ \tau_b \circ \tau_a \) only moves the branches \( B_0 \). This will be most important and used repeatedly without being mentioned.

Claim 1. Suppose that \( T \in \text{Mat}_5(\mathbb{Z}) \) consists of those rows and columns of the incidence matrix \( \text{Inc}(\tau_c^{-1} \circ \tau_b \circ \tau_a) \) that correspond to the branches \( B_0 \), then \( T^4 > 0 \).

We readily see that

\[
T = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
2 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix},
\]

and compute.

Claim 2. If \( w \in \mathbb{R}^{7g} \) represents a weight on \( \tau(g) \) that is positive on a set \( B_0' \subset B_0 \), then the weight \( \text{Inc}(\psi^{-1} \circ \rho)w \) is also positive on \( B_0' \).

This is immediate.
Claim 3. If for some \(0 < r < g - 1\), \(w \in \mathbb{R}_{\tau}^g\) is a weight that is positive on all branches of \(B_t\), then \(\text{Inc}(\psi^r)w\) is also positive on the branches \(B_t\).

Indeed, \(w' = \text{Inc}(\psi^{g-r})w\) is positive on the branches \(B_0\), and as each row of \(T\) contains at least one positive entry, \(w'' = \text{Inc}(\tau_c^{-1} \circ \tau_b \circ \tau_a)w'\) is still positive on the branches \(B_0\). But then \(\text{Inc}(\psi^{t-1})w''\) is positive on \(B_t\) as desired.

Claim 4. If \(w\) is a weight that is positive on the branches of \(B_0\), then \(w^* = M(\tau_c^{-1} \circ \tau_b \circ \tau_a)w\) is positive on \(B_0 \cup \{a_1, a_2\}\).

Indeed, we already remarked in the proof of Claim 3 that \(w^*\) is positive on branches of \(B_0\), and the fact that \(w^*\) is positive on the branches \(a_1\) and \(a_2\) can be seen directly from our convention of how \(\tau_c^{-1}\) and \(\tau_b \circ \tau_a\) acts on \(\tau(g)\).

We now take \(w(0)\) as in the statement of the lemma, hence there exists \(t \in \{0, \ldots, g - 1\}\), such that \(w(0)\) is positive on a branch of \(B_t\).

We first prove that \(\text{Inc}(\psi^n)w(0)\), for \(n\) large enough, is positive on the branches of \(\overline{B}\). The proof is by induction on the number of \(i\) such that \(\text{Inc}(\psi^n)w(0)\), for \(n\) large enough, is positive on the branches of \(B_i\). We take the subscripts of \(B_i\) as elements of the cyclic group on \(g\) elements.

For the basis step, we remark that \(w(1) = \text{Inc}(\psi^{g-1})w(0)\) is positive on a branch of \(B_0\). It follows from Claim 1 and Claim 2 that \(w(2) = \text{Inc}(\psi^{4g})w(1)\) is positive on all branches of \(B_0\).

For the induction step we assume that for some \(n\), \(w(3) = \text{Inc}(\psi^n)w(0)\) is positive on the branches of \(B_0, \ldots, B_j\), for some \(j \in \{0, \ldots, g - 1\}\). We first note that \(w(4) = \text{Inc}(\psi^{g-1})w(3)\) is positive on the branches of \(B_0\). But then \(w(5) = \text{Inc}(\tau_c^{-1} \circ \tau_b \circ \tau_a)w(4)\) is positive on the branches \(B_0 \cup \{\rho(b_3)\}\) (as follows from our convention of how \(\psi_g\) acts on \(\tau(g)\)), hence \(w(6) = \text{Inc}(\psi^{8-2})w(5)\) is positive on the branches \(B_{g-1} \cup \{b_3\}\). Using the fact that \(w(6)\) is positive on a branch of \(B_0\), we see as above, using Claim 1 and Claim 2, that \(w(7) = \text{Inc}(\psi^{8})w(6)\) is positive on all branches of \(B_0\). We conclude that \(w(8) = \text{Inc}(\psi^{j+1})w(7)\) is positive on the branches of \(B_{j+1}\). To complete the induction step, note first that \(w(8) = \text{Inc}(\psi^{6})w(3)\). As \(w(3)\) is positive on the branches of \(B_0, \ldots, B_j\), so is \(w(8)\) as follows from Claim 3.

We showed that for some \(m \geq 1\) we have that \(\text{Inc}(\psi^m)w(0)\) is positive on the branches of \(\overline{B}\). To finish the proof of the lemma, we use Claim 4 to see that \(\text{Inc}(\psi^{m+g})w(0)\) is positive on all branches of \(\tau\) and the claim follows by taking \(n_0 = m + g\). Q.E.D.

In order to show that \(\psi_g\) is pseudo-Anosov, we need to analyze the incidence matrix \(\text{Inc}(\psi_g)\). We claim

**Lemma 3.** (a)

\[
\text{Inc}(\psi_g) = \begin{pmatrix} S_g & 0 \\ \overline{C} & P \end{pmatrix} \in \text{Mat}_{7g}(\mathbb{Z}),
\]

where \(\overline{C}\) is the \(5g \times 2g\) zero matrix and \(P \in \text{Mat}_{2g}(\mathbb{Z})\) is a permutation matrix. (Of course \(C\) is a \(2g \times 5g\) matrix.) Moreover,

(b) the ‘small’ incidence matrix \(S = S_g \in \text{Mat}_{5g}(\mathbb{Z})\) of \(\psi\) with respect to the branches \(\overline{B}\) of \(\tau(g)\) is P.F.

**Proof.** For the proof of part (a) we remark that \(\rho\) and hence \(\psi\) induces a permutation of the branches \(A = \{\rho^i(a_j) : 0 \leq i \leq g - 1\text{ and } j = 1, 2\}\). To be more specific, if \(w\) is a weight on \(\tau(g)\) that is positive on a single branch
For the proof of part (b) we define $e_i \in \mathbb{R}^{5g}$ to be the $i$th unit vector in $\mathbb{R}^{5g}$. To show that $S$ is P.F. it is of course enough to show that there exists an $n \geq 1$ such that for $1 \leq i \leq 5g$ we have $S^n e_i > 0$. If for $1 \leq i \leq 5g$, we define $\tilde{e}_i \in \mathbb{R}^{7g}$ to be the $i$th unit vector, then, using part (a), we readily see that it is enough to show that there exists an $n$ such that for $1 \leq i \leq 5g$ we have that $\text{Inc}(\psi^n)\tilde{e}_i$ is positive on branches of $\overline{B}$. This, however, follows from Lemma 2. Q.E.D.

We can now prove

**Proposition 4.** For $g \geq 3$, $\psi_g$ is pseudo-Anosov.

**Proof.** We need to check the conditions of Casson’s theorem.

The first two conditions, namely that $\tau(g)$ fills $F_g$ and is invariant under $\psi_g$ follow from Lemma 1.

To show the third condition, suppose that $\tau' = \tau'(g)$ is a subtrack of $\tau = \tau(g)$ that is invariant under $\psi_g$. We will show that $\tau'$ cannot be a proper subtrack of $\tau(g)$.

We choose a nonzero measure $\mu'$ on $\tau'$ and extend $\mu'$ to a nonzero measure $\mu \in V(\tau)$ by defining $\mu(b) = \mu'(b)$, if $b$ is a branch of $\tau'$ and $\mu(b) = 0$, otherwise. It is easy to see that any nonzero measure on $\tau$ is positive on some branch of $\overline{B}$, hence by Lemma 2, we can find $n > 0$ such that $\phi^n(\mu)$ is positive on all branches of $\tau$. (\(\phi\) being the self-map of $V(\tau)$ that is induced by $\psi$.)

As we assumed that $\tau'$ is invariant under $\psi$, and as $\mu$ is zero on branches of $\tau \setminus \tau'$, the measure $\phi^n(\mu)$ must also be zero on these branches. It follows that $\tau' = \tau$ as desired.

We are left to check the last condition in Casson’s theorem, namely if $\tau$ is a subtrack of a $\psi$ invariant train track $\tau'' = \tau''(g)$, then the induced map $\phi'' : V(\tau'') \to V(\tau'')$ has no nonzero fixed point. An Euler characteristic argument shows that for $\tau''$ to satisfy the fourth condition in the definition of train track, the branches of $\tau'' \setminus \tau$ must be contained in the closure of the two complementary $g$-gons of $\tau$. But as $\psi$ induces a rotation of these complementary $g$-gons, we readily see that $\tau''$ can only be $\psi$ invariant and satisfy the fourth condition in the definition of train track if $\tau'' = \tau$.

Assume now to derive a contradiction that $\mu \in V(\tau)$ is a nonzero fixed point of $\phi : V(\tau) \to V(\tau)$, hence we have a nonzero fixed point $x \in \mathbb{R}^{7g}$ of $\text{Inc}(\psi)$. As remarked above, any nonzero measure on $\tau$ is positive on some branch of $\overline{B}$, hence Lemma 2 applies to show that $x > 0$. It follows from Lemma 3 that $x$ restricts to a nonzero fixed point $x$ of $B$. But as $S \in \text{Mat}_{5g} (\mathbb{Z})$ is P.F. and integral, $S^n \geq 1$, for $n$ large enough, and as $5g > 1$, we see that $S$ cannot have a nonzero fixed point. This contradiction shows that our assumption of $\psi$ having a nonzero fixed point was absurd.

We showed that all the conditions in Casson’s theorem are satisfied and conclude that $\psi_g$ is indeed p.A. Q.E.D.

**The upper bound.** We will show

**Proposition 5.** For $g \geq 3$, the dilatation $\lambda_g$ of $\psi_g$ satisfies

$$\log \lambda_g \leq \frac{\log 6}{g}.$$
As a corollary we get our main result.

**Theorem 6.** The smallest element $\delta_g$ in $\text{Spec}(F_g)$ satisfies

$$\delta_g \leq \frac{\log 6}{g}, \text{ for } g \geq 3.$$  

Before we prove the proposition some remarks are in order.

Remark. (a) One can define $F_g$ and $\phi_g$, for $g = 2$, but a computation shows that the dilatation $\lambda_2$ of $\psi_2$ satisfies $\lambda_2^2 \approx 6.018 > 6$.

(b) [P2] shows, using [P1], that $\rho \circ \tau_c \circ \tau_{b}^{-1} \circ \tau_a$ is p.A. with dilatation $l_g$ that satisfies $l_g \leq (\log 11)/g$. The fact that $\psi_g$ is p.A. does not follow from [P1] as we perform right-handed Dehn twists along two curves that intersect. The method in this note can be extended to produce a large class of p.A. maps. (See [Ba].)

**Proof of Proposition 5.**

**Step 1.** We choose $g \geq 3$ and show that we can find the dilatation of $\psi = \psi_g$ by spectrally analyzing the incidence matrix $\text{Inc}(\psi)$. This is a standard technique. We then proceed to demonstrate that the dilatation of $\psi$ is in fact given by the spectral radius of the smaller P.F. matrix $S = S_g$ (as in Lemma 3).

If we identify $V(\tau)$ with a closed cone in $\mathbb{R}^{7g}$, then $(V(\tau) \setminus \{0\})/\mathbb{R}_+$, where $\mathbb{R}_+$ denotes the positive real numbers, can be represented by a closed cell. $\text{Inc}(\psi)$ induces a continuous self-map of this cell and we conclude from the Brouwer fixed point theorem that there exist $\sigma = \sigma_g > 0$ and $x \in \mathbb{R}^{7g}$ such that $\text{Inc}(\psi)x = \sigma x$. Moreover, $x$ corresponds to a nonzero measure on $\tau$.

We show next that $\sigma > 1$ and that $\sigma$ is the spectral radius of $S$. Indeed, we conclude from Lemma 3 that if we define $\bar{x} \in \mathbb{R}^{5g}$ to be the restriction of $x$ onto its first $5g$ coordinates, then $S\bar{x} = \sigma \bar{x}$. Moreover, $\bar{x} > 0$, as follows from Lemma 2, and hence $\bar{x} > 0$. The uniqueness statement of the Perron-Frobenius theorem shows that $\sigma$ is the spectral radius of $S$. As $S$ is P.F. and integral, we conclude that $\sigma > 1$.

We claim that $\sigma$ equals the dilatation $\lambda = \lambda_g$ of $\psi_g$, and to that end we define $\mu \in V(\tau)$ to be the measure that corresponds to $x \in \mathbb{R}^{7g}$. We further define $(\mathcal{F},\nu) = \mathcal{F}(\mu)$. We showed above that $\text{Inc}(\psi)x = \sigma x$, hence $\psi_\mu = \sigma \mu$. As $\mathcal{F} \circ \psi = \hat{\psi} \circ \mathcal{F}$, we conclude that $\psi(\mathcal{F},\nu) = (\mathcal{F},\sigma \nu)$. But there is only one such foliation (class) for a p.A. map with $\sigma > 1$, hence $\sigma = \lambda$ as desired. The claim follows.

**Step 2.** We need to bound above the spectral radius $\lambda$ of $S$. To that end we think of the matrix $S$ and its powers as having $g$ columns, the $g$ elements of each column being $5 \times 5$ matrices. We find

$$S = \begin{pmatrix}
A_1 & 0 & 0 & 0 & \cdots & 0 & 0 & I \\
A_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
A_3 & I & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & I & 0 & 0 & \cdots & 0 & 0 & 0 \\
& & & & & & & \\
& & & & & & & \\
0 & 0 & 0 & 0 & \cdots & 0 & I & 0
\end{pmatrix} \in \text{Mat}_{5g}(\mathbb{Z}),$$
where

\[ A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

\( I \in \text{Mat}_5(\mathbb{Z}) \) is the identity matrix, and \( \bar{0} \in \text{Mat}_5(\mathbb{Z}) \) is the zero matrix.

Note that \( A_2^2 = \bar{0} \). We define \( A_4 = A_3A_1 + A_2 \) and \( A_5 = A_1A_3 + A_4 \), and find that in case \( g = 3 \),

\[ S^3 = \begin{pmatrix} A_5 & A_1 & A_3 \\ A_2A_3 & A_2 & A_2A_1 \\ A_3 + A_2A_1 & A_3 & A_4 \end{pmatrix}. \]

Furthermore, for \( g \geq 4 \), we claim that

\[ S^g = \begin{pmatrix} A_5 & A_1 & A_3 \\ A_2A_3 & A_2 & A_2A_1 \\ A_3 & A_4 & A_2A_1 \\ A_3 + A_2A_1 & A_3 & A_4 \\ A_2A_1 & A_3 & A_4 \\ A_2A_1 & A_3 & A_4 \\ 0 & 0 & A_3 \\ 0 & 0 & A_4 \\ 0 & 0 & A_2A_1 \\ 0 & 0 & A_2A_1 \\ 0 & 0 & A_2A_1 \\ 0 & 0 & A_2A_1 \\ 0 & 0 & A_2A_1 \end{pmatrix}. \]

To show that, we choose \( g \geq 4 \), and define \( c(i, n) \) as the \( i \)th column of \( S^n \), for \( 1 \leq i \leq g \) and \( 1 \leq n \leq g \). We will only demonstrate that \( c(i, g) \) is as asserted for \( i \in \{3, \ldots, g-1\} \). The remaining cases are similar.

Note first that \( c(i, 1) \) has \( I \) in its \((i-1)\)th position and zero matrices \( \bar{0} \) in the other positions. We further compute

\[ c(i, g-i) = (\bar{0}, \ldots, \bar{0}, I), \]
\[ c(i, g-i+1) = (I, \bar{0}, \ldots, \bar{0}), \]
\[ c(i, g-i+2) = (A_1, A_2, A_3, \bar{0}, \ldots, \bar{0}), \]
\[ c(i, g-i+3) = (\bar{0}, A_2A_1, A_4, A_3, \bar{0}, \ldots, \bar{0}), \]
and

\[ c(i, g) = (\bar{0}, \ldots, \bar{0}, A_2A_1, A_4, A_3, \bar{0}, \ldots, \bar{0}), \]

where in the last equation \( A_2A_1 \) is in the \((i-1)\)th position as desired.

We now choose \( g \in \{3, 4, \ldots\} \), and note that as \( \psi = \psi_\xi \) is p.A. with dilatation \( \lambda = \zeta_\xi \), then \( \psi^g \) is p.A. with dilatation \( \zeta^g \). We showed in Step 1 that \( \lambda \) is the spectral radius of the P.F. matrix \( \psi \), hence \( \zeta^g \) is the spectral radius of the P.F. matrix \( \psi^g \). The spectral radius \( \zeta^g \) of the P.F. matrix \( \psi^g \) satisfies (see [Ga])

\[ \zeta^g = \min \left\{ \max_{1 \leq i \leq g} \frac{(\psi^g x)_i}{x_i} : x \in \mathbb{R}^g, \ x > 0 \right\}, \]
and hence
\[ \lambda^g \leq \max_{1 \leq i \leq 3g} \frac{\langle S^g x \rangle_i}{x_i}, \]
for any \( x \in \mathbb{R}^{5g}, x > 0 \).

If we let
\[
y_1 = (.04, .22, 1.2, .82, .28), \quad y_2 = (.11, .65, 1.47, .48, .8), \\
y_3 = (.08, .47, 2.2, .48, .52), \quad \text{and} \quad x = (y_1, y_2, y_3, \ldots, y_3) \in \mathbb{R}^{5g},
\]
then we compute that
\[
\frac{\langle S^g x \rangle_i}{x_i} \leq 6, \quad \text{for } i = 1, \ldots, 5g.
\]
This completes the proof of the proposition. Q.E.D.

**Remark.** The estimate can be improved by better choice of \( x \) in the proof of Proposition 5. Some values for \( (\lambda_g)^g \) (as provided by Matlab) and

<table>
<thead>
<tr>
<th>( g )</th>
<th>( (\lambda_g)^g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( \approx 5.50 )</td>
</tr>
<tr>
<td>4</td>
<td>( \approx 5.35 )</td>
</tr>
<tr>
<td>5</td>
<td>( \approx 5.28 )</td>
</tr>
<tr>
<td>6</td>
<td>( \approx 5.25 )</td>
</tr>
<tr>
<td>9</td>
<td>( \approx 5.21 )</td>
</tr>
</tbody>
</table>

**References**


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