TRACE FUNCTIONS IN THE RING OF FRACTIONS OF POLYCYCLIC GROUP RINGS

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Dedicated to the memory of I. N. Herstein

Abstract. Let $KG$ be the group ring of a polycyclic-by-finite group $G$ over a field $K$ of characteristic zero, $R$ be the Goldie ring of fractions of $KG$, $S$ be an arbitrary subring of $R_{n \times n}$. We prove that the intersection of the commutator subring $[S, S]$ with the center $Z(S)$ is nilpotent. This implies the existence of a nontrivial trace function in $R_{n \times n}$.

Let $G$ be a polycyclic-by-finite group, $K$ be a commutative field of characteristic zero. (Throughout this paper the term "field" is used in the sense of "skew field.") It is well known that the group ring $KG$ is semiprime Noetherian and hence has a Goldie ring of fractions which we denote by $R$. Let $S$ be a subring of the matrix ring $R_{n \times n}$, $Z(S)$ be its center and $[S, S]$ be the $K$-subalgebra of $R_{n \times n}$ generated by all the commutators $[x, y] = xy - yx$, $x, y \in S$. Our first main result is the following theorem which is motivated by R. Snider's article [1].

The intersection

\[ [S, S] \cap Z(S) \]

is a nilpotent ring (see Theorem 3). (It is known that (1.1) is a subring; the proof of this fact is easy.)

We obtain immediately from Theorem 3 an affirmative answer to the question, posed by R. Snider in [1]: Let $G$ be a poly-Z-group, $K$ be a commutative field of characteristic zero, $D$ be the field of fractions of $KG$. Does

\[ [D, D] \neq D? \]

In particular, does

\[ 1 \notin [D, D]. \]

We see thus that the relations (1.2) and (1.3) do hold in $D$. Furthermore, this result implies that there exists a nontrivial trace function $t: D \to D/[D, D]$, defined by

\[ t(d) = d + [D, D] \]
and this function can be extended to a function $T: D_{n \times n} \to D/[D, D]$ by

$$t(d_{ij}) = \sum_i t(d_{ii}),$$

where $(d_{ij})$ is an arbitrary matrix from $D_{n \times n}$ (see [1-3]). Snider proved in [1] the relation (1.3) and hence the existence of nontrivial trace functions in the case when $G$ is abelian-by-{infinite cyclic}.

The proof of Theorem 3 will be based on the following result (see Theorem 2):

Let $K$ be an arbitrary commutative field and $R$ be the ring of fractions of $KG$ and

$$(1.4) \quad x_j \quad (j = 1, 2, \ldots, m)$$

be given nonzero elements of $KG$. Then there exists an ideal $C \subseteq KG$ such that the quotient ring $(KG)/C$ is a finite-dimensional $K$-algebra $K[\tilde{G}]$, generated by a finite group $\tilde{G}$ which is the image of $G$ in $(KG)/C$. The homomorphism $\alpha: KG \to K[\tilde{G}]$ is extended to a specialization $\theta: R \to K[\tilde{G}]$, whose domain $R_0$ contains the elements (1.4). Furthermore the elements $\tilde{x}_j = \theta(x_j) \quad (j = 1, 2, \ldots, m)$ are nonzero elements of $K[\tilde{G}]$.

We will obtain one more result on specializations from $R$ to algebras finite-dimensional over their central subfields; this is Theorem 1 and its corollary. Let $H$ be a torsion-free normal subgroup of finite index in $G$ such that $H/H_1$ is free abelian, where $H_1$ is the Fitting radical of $H$. Then Theorem 1 essentially states that there exists a $G$-invariant ideal $A \subseteq KH_1$ and an ideal $B = (A)(KG)$ such that the quotient algebra $(KG)/B \simeq K[H]$, where the group $\overline{H}$ is abelian-by-finite; the images $\overline{x}_j \quad (j = 1, 2, \ldots, m)$ of the elements (1.4) are nonzero in $K[\overline{G}]$ and a given element $x_j$ is regular in $R$ iff its image $\overline{x}_j$ is regular in $K[\overline{G}]$. Roseblade's Theorem 11.2.9 in [4] implies that the ideal $B$ is localizable in $KG$.

It is worth remarking that Theorems 1 and 2 provide a method for construction of specializations from $R$ into finite-dimensional algebras over the same field $K$; they should be compared with the Reduction Theorem (see [5, Theorem 4.1], [6], or [7, 4.2.1]) which gives specializations into algebras over fields of finite characteristic (see a discussion on this in the book [7, p. 137]).

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Throughout this section let $D$ be a field, generated by a polycyclic-by-finite group $G$. Thus, $D$ is the field of fractions of its subring generated by the group $G$; we denote this subring by $T$. Thus, $T = Z[G]$ or $T = Z_p[G]$, depending on the characteristic of $D$.

**Lemma 1.** Let (1.4) be given nonzero elements of $T$. Then there exists an ideal $A \subseteq T$ such that the quotient ring $T/A \simeq \prod_{x \in F}$, where $\prod$ is a finite field and the images of the elements (1.4) are invertible in $T/A$.

**Proof.** Wehrfritz proved (see [8] or [7, 4.3.12]) that if $R$ is a finitely generated subring of $D$, then there exists an ideal $C$ of $R$ of finite index with $\bigcap_{n=1}^{\infty} C^n = 0$; furthermore, every quotient ring $R/C^n \quad (n = 1, 2, \ldots)$ is finite. We apply this theorem to the subring $S$ of $D$, generated by the elements $x_j, x_j^{-1} \quad (j = $
1, 2, ..., m) and find an ideal $B \subseteq S$ such that the ring $\overline{S} = S/B$ is finite. Since the images of the elements $x_j \in T$ ($j = 1, 2, \ldots, m$) are invertible in the finite ring $\overline{S}$ they must be invertible in the subring $\overline{T}/(T \cap B)$. We see now that an arbitrary maximal ideal $A \supseteq (T \cap B)$ satisfies the conclusions of the assertion.

**Remark.** The current proof of Lemma 1 is somewhat shorter than the proof given in the original version of the paper, where Lemma 1 was obtained as one of the corollaries of the Reduction Theorem [5].

Now let $\Pi[G]$ be a domain, generated by a polycyclic-by-finite group $G$ over a finite field $\Pi$. We see that $\Pi[G] \approx \mathbb{Z}_p[G_1]$, where $G_1$ is the subgroup of units of $\Pi[G]$, generated by $G$ and the multiplicative group of $\Pi$. We see thus that Lemma 1 is true for this case, when $T = \Pi[G]$. We will use it in this form in the proof of Proposition 1 below.

**Proposition 1.** Let $K$ be an arbitrary commutative field, $G$ be a torsion-free polycyclic group and let (1.4) be given nonzero elements of $KG$. Then there exists a maximal ideal $A \subseteq KG$ such that the quotient algebra $(KG)/A$ is generated over $K$ by a finite group $G$, the image of $G$ under the natural homomorphism $(KG) \rightarrow (KG)/A$, and the images of the elements (1.4) in the ring $K[G]$ are invertible.

**Proof.** We reduce first the proof to the case when the field $K$ is finitely generated. Indeed, assume that the theorem is proved for this special case. Let $K_1$ be the finitely generated subfield of $K$, such that $K_1G$ contains all the elements (1.4) and $A_1 \subseteq K_1G$ be the ideal, which satisfies all the conclusions of the theorem. Since

$$(KG)/(KA_1) \approx K \otimes ((K_1G)/A_1),$$

we obtain an ideal $KA_1 \subseteq KG$ such that the algebra $(KG)/(KA_1)$ is generated by a finite group and the images of the elements (1.4) are invertible in it. Since images of the elements (1.4) are invertible in the algebra $(KG)/(KA_1)$ they are invertible in every simple homomorphic image of it; this implies easily that an arbitrary maximal ideal $A \subseteq KG$, which contains $KA_1$, satisfies the conclusion of the theorem.

We can assume therefore that the field $K$ is finitely generated. Let $K_0 \subseteq K$ be a finitely generated subring such that $K$ is the field of fractions of $K_0$. We have the following representations for the elements (1.4)

$$(2.1) \quad x_j = \sum_i c_{ij} g_i \quad (c_{ij} \in K; \ j = 1, 2, \ldots, m).$$

An arbitrary coefficient $c_{ij}$ in (2.1) has a representation

$$(2.2) \quad c_{ij} = a_{ij} b_{ij}^{-1} \quad (a_{ij}, b_{ij} \in K_0).$$

We can find a maximal ideal $\mathcal{P} \subseteq K_0$ which defines a $p$-adic valuation in $K_0$ and contains no one of the elements $a_{ij}, b_{ij}$ in (2.2). If $K_\mathcal{P}$ is the ring of fractions of $K_0$ with respect to $\mathcal{P}$ then all the coefficients $c_{ij}$ in (2.1) belong to $K_\mathcal{P}$ and hence

$$(2.3) \quad x_j \in K_\mathcal{P}G \quad (j = 1, 2, \ldots, m).$$

Now consider the natural homomorphism

$$\varphi : K_\mathcal{P}G \rightarrow (K_\mathcal{P}G)/(\mathcal{P}),$$
where \( (\mathcal{P}) \) is the ideal of \( K_{(\mathcal{P})} G \), generated by the ideal \( \mathcal{P} \subseteq K_{(\mathcal{P})} G \). We observe that the ring \( (K_{(\mathcal{P})} G)/(\mathcal{P}) \) is isomorphic to the group ring \( \Pi G \), where \( \Pi \simeq (K_{(\mathcal{P})})/(\mathcal{P}) \) is a finite field and the elements \( \varphi(x_j) \) \( (j = 1, 2, \ldots, m) \) are nonzero. Lemma 1 implies that there exists an ideal \( B \subseteq \Pi G \) such that \( (\Pi G)/B \) is a simple finite ring and the images \( \overline{x}_j \) of the elements \( \varphi(x_j) \) \( (j = 1, 2, \ldots, m) \) are invertible in the ring \( (\Pi G)/B \). This together with the homomorphism (2.3) implies that there exists a homomorphism
\[
\psi : K_{(\mathcal{P})} G \rightarrow (\Pi G)/B
\]
such that the elements
\[
\overline{x}_j = \psi(x_j) \quad (j = 1, 2, \ldots, m)
\]
are invertible in the ring \( (\Pi G)/B \); clearly, \( (\Pi G)/B \) is generated over \( \Pi \) by the finite group \( \overline{G} = \psi(G) \), i.e.,
\[
(\Pi G)/B \simeq \Pi[\overline{G}].
\]

Now take a minimal left ideal \( V \) in the matrix ring \( \Pi[\overline{G}] \); this ideal affords a representation \( \rho \) of the group \( \overline{G} \) and \( \rho(\Pi G) \simeq \Pi[\overline{G}] \). Let \( \widetilde{K}_0 \) be the \( p \)-adic completion of \( K_0 \), \( (\pi) \) be the maximal ideal of \( \widetilde{K}_0 \). Since \( G \) is polycyclic, the group \( \overline{G} \) is solvable and Fong-Swan’s Theorem implies that there exists a \( \widetilde{K}_0 \overline{G} \)-module \( \widetilde{V} \), free over \( \widetilde{K}_0 \), such that \( \widetilde{V}/(\pi) \widetilde{V} \simeq V \). (In fact, this theorem is proven in [9, 22.1] for the case when the group is \( p \)-solvable and \( \widetilde{K}_0 \) contains a primitive root of degree \( (G:1) \) from 1 but the last condition is unnecessary (see [10]); this can be shown also by a standard argument based on the Galois theory.) If \( \lambda \) is the representation afforded by \( \widetilde{V} \) and \( \lambda(\widetilde{K}_0 G) \simeq \Pi[\overline{G}] \) it is important that the ideal \( \pi R \) is quasiregular in \( R \).

There exists therefore a system of homomorphisms
\[
(2.5) \quad \widetilde{K}_0 G \xrightarrow{\lambda_1} \widetilde{K}_0 \overline{G} \xrightarrow{\lambda_2} R \xrightarrow{\lambda_3} \Pi[\overline{G}]
\]
where \( \lambda_1 \) and \( \lambda_2 \) are homomorphisms of \( \widetilde{K}_0 \)-algebras.

The homomorphism
\[
(2.6) \quad \lambda_2 \lambda_1 : \widetilde{K}_0 G \rightarrow \Pi[\overline{G}]
\]
maps the elements (1.4) into invertible elements \( \overline{x}_j \) \( (j = 1, 2, \ldots, m) \). Since the kernel of \( \lambda_2 \) is a quasiregular ideal we conclude easily that the images of the elements (1.4) under the homomorphism
\[
(2.7) \quad \lambda \lambda_1 : \widetilde{K}_0 G \rightarrow R
\]
are invertible elements of \( R \). Since the field of fractions of \( K_0 \) coincides with \( K \) we see that the field of fractions of \( \widetilde{K}_0 \) is isomorphic to the \( p \)-adic completion \( \widetilde{K} \) of \( K \); homomorphism (2.7) is extended to a homomorphism of \( \widetilde{K} \)-algebras
\[
(2.8) \quad \mu : \widetilde{K} G \rightarrow \widetilde{K} R.
\]

Since the algebra \( \widetilde{K} R \) is generated over \( \widetilde{K} \) by the finite group \( \overline{G} \), we see that the \( K \)-algebra \( \mu(KG) \) is also generated over \( K \) by the group \( \overline{G} \), i.e.,
\[
(2.9) \quad \mu(KG) \simeq K[\overline{G}].
\]
The homomorphism (2.8) carries out the elements (1.4) into invertible elements of $\bar{K}R$; we obtain therefore that the images of these elements under the homomorphism (2.9) are invertible elements of $K[\bar{G}]$. We found thus a homomorphism

$$KG \to (KG)/A \cong K[\bar{G}]$$

which maps the elements (1.4) into invertible elements of $K[\bar{G}]$. We can assume, of course, that $K[\bar{G}]$ is simple, i.e. the ideal $A$ is maximal. The proof is complete.

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Let $G$ be a polycyclic-by-finite group, $\rho(G)$ be the Fitting radical of $G$. It is not difficult to verify that $G$ contains a torsion-free normal subgroup $H$ of finite index such that the quotient group $H/\rho(H)$ is free abelian; it is more convenient to denote the subgroup $\rho(H)$ by $H_1$. We observe that if $A$ is an arbitrary $G$-invariant ideal of $KH_1$ then $B = A(KG)$ is an ideal of $KG$ and $(KG)/B \cong K[\bar{G}]$, where the group $\bar{G}$ is an extension of the normal subgroup $H_1$ by the group $\bar{G}/H_1 \cong G/H_1$. Thus, the algebra $K[\bar{G}]$ is isomorphic to an appropriate cross product of the algebra $K[\bar{H}_1]$ and the group $G/H_1$ and $K[\bar{H}] \cong K[\bar{H}_1]* (H/H_1)$.

**Theorem 1.** Let $K$ be an arbitrary commutative field, $\text{char } K = p \geq 0$, and assume that nonzero elements (1.4) of $KG$ are given. Then there exists a $G$-invariant ideal $A \subseteq KH_1$ and an ideal $B = (A)KG$ such that

(i) The image $\bar{H}_1$ of the group $H_1$ under the natural homomorphism

$$\varphi : KG \to (KG)/B \cong K[\bar{G}]$$

is a finite $p'$-group and hence the group $\bar{H}$ is finite-by-free abelian. Furthermore, there exists a free abelian normal subgroup $N \subseteq \bar{G}$ of finite index, which is contained in $\bar{H}$ and central in it, and whose elements are linearly independent over $K$; hence $K[N]$ is isomorphic to the group ring $KN$.

(ii) The images

$$\bar{x}_j \quad (j = 1, 2, \ldots, m)$$

of the elements (1.4) are nonzero elements of $K[\bar{G}]$. Furthermore, a given element $x_j$ in (1.4) is regular in $KG$ if and only if its image $\bar{x}_j$ is regular in $K[\bar{G}]$.

(iii) The ideal $B$ is localizable in $KG$.

**Proof.** Let $g_1, g_2, \ldots, g_n$ be a transversal for $H$ in $G$. The group ring $KH$ contains no zero divisors of $KG$ and we can form the ring $R$ of fractions of $KG$ with respect to the set $(KH)\setminus 0$. If $D$ is the field of fractions of $KH$ then $R \cong D \otimes_{KH} KG$ and the transversal $g_1 = 1, g_2, \ldots, g_n$ gives a basis of the left vector space $R$ over $D$.

We can assume without loss of generality that the set (1.4) contains regular elements and these are the first $m_1$ elements

$$x_1, x_2, \ldots, x_{m_1}.$$

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These elements must be invertible in $R$; this implies easily that there exist nonzero elements $x'_j$ ($j = 1, 2, \ldots, m_1$) in $KG$ such that

\begin{align}
    y_j &= x'_j x_j \in (KH) \setminus 0 \quad (j = 1, 2, \ldots, m_1), \\
    x'_j x_j &= 0 \quad (j = m_1 + 1, \ldots, m).
\end{align}

(3.2)

Now let

\begin{align}
    x_j &= \sum_{\alpha=1}^{n} c_{\alpha j} g_\alpha, \\
    x'_j &= \sum_{\alpha=1}^{n} c'_{\alpha j} g_\alpha
\end{align}

(3.3)

\[(c_{\alpha j}, c'_{\alpha j} \in KH; \ \alpha = 1, 2, \ldots, n; \ j = 1, 2, \ldots, m)\]

be the representations of the elements $x_j$, $x'_j$ ($j = 1, 2, \ldots, m$). Let

\begin{align}
    c_1, c_2, \ldots, c_r
\end{align}

(3.4)

be all the nonzero coefficients $c_{ij}$ in (3.3). Let $h_i$ ($i \in I$) be a transversal for $H_1$ in $H$ and

\begin{align}
    c_\beta &= \sum_i \lambda_{i\beta} h_i \quad (\lambda_{i\beta} \in KH_1; \ \beta = 1, 2, \ldots, r).
\end{align}

(3.5)

Similarly, we have for the elements $y_j$ in (3.2)

\begin{align}
    y_j &= \sum_i \mu_{ij} h_i \quad (\mu_{ij} \in KH_1; \ j = 1, 2, \ldots, m_1).
\end{align}

(3.6)

Apply now Proposition 1 and find a maximal ideal $A \subseteq KH$ such that $(KH)/A \simeq K[\tilde{H}]$, where $\tilde{H}$ is a finite group and for all the elements $\lambda_{i\beta}$, $\mu_{ij}$ from (3.5) and (3.6) the images of the elements

\begin{align}
    g_\alpha^{-1} \lambda_{i\beta} g_\alpha, \ g_\alpha^{-1} \mu_{ij} g_\alpha \quad (\alpha = 1, 2, \ldots, n)
\end{align}

(3.7)

are invertible in $K[\tilde{H}]$. Let

\begin{align}
    A_1 &= \bigcap_{\alpha=1}^{n} g_\alpha^{-1} A g_\alpha, \\
    A_2 &= A_1 \cap KH_1, \\
    B &= (A_2)KG.
\end{align}

(3.8)

Clearly, $A_1$ is a $G$-invariant ideal of $KH$ and as a result of this $A_2$ is a $G$-invariant ideal of $KH_1$. Hence $B$ is an ideal in $KG$. We have already pointed out that the quotient ring $(KH)/B \simeq K[\tilde{G}]$, where the group $\tilde{G}$ is an extension of the group $\tilde{H}_1$ by the group $\tilde{G}/\tilde{H}_1 \simeq G/H_1$; the group $G/H_1$ is an extension of the free abelian group $H/H_1$ by the finite group $G/H$. On the other hand, we obtain from (3.8),

\begin{align}
    (KH_1)/(KH_1 \cap B) \simeq (KH_1)/A_2 \simeq (KH_1)/(KH_1 \cap A_1).
\end{align}

(3.9)

The first relation in (3.8) shows that the image of $KH$ under the natural homomorphism $(KH) \to (KH)/A_1$ is a subdirect sum of the rings $(KH)/(g_\alpha^{-1} A g_\alpha)$ ($\alpha = 1, 2, \ldots, n$) which are isomorphic to the simple artinian ring $(KH)/A \simeq K[\tilde{H}]$; a routine argument (see [5, Lemma 2.9]) implies that in fact $(KH)/A_1$ is a direct sum of rings isomorphic to $K[\tilde{H}]$. This, together with the relation (3.9), implies first of all that the group $\tilde{H}_1$ which is the image of $H_1$ under the homomorphism $KG \to (KG)/B$, is finite. Furthermore, the images of the
elements (3.7) under the homomorphism \( KH \to (KH)/A \) are invertible. This implies that the elements

\[(3.10) \quad \lambda_{i\beta}, \quad \mu_{ij}\]

become invertible modulo the ideals \( g_{\alpha}^{-1}Ag_{\alpha} \) \((\alpha = 1, 2, \ldots, n)\) and hence they are invertible modulo the ideal \( A_1 = \bigcap_{\alpha=1}^{n} g_{\alpha}^{-1}Ag_{\alpha} \). Since the elements (3.10) belong to \( KH \), the second and the third relations in (3.8) imply that they are invertible modulo the ideal \( B \). We have already observed that the image of \( KH \) in \( (KG)/B \) is isomorphic to

\[(3.11) \quad K[\overline{H}] \simeq K[\overline{H_1}] * (H/H_1).\]

Since the group \( H/H_1 \) is free abelian and all the elements

\[(i = 1, 2, \ldots, n)\]

are invertible in \( K[\overline{H_1}] \) we conclude easily that the elements

\[(3.5') \quad \overline{\gamma}_\beta = \sum_i \overline{\lambda}_{i\beta}h_i \quad (\beta = 1, 2, \ldots, r)\]

and

\[(3.6') \quad \overline{\nu}_j = \sum_i \overline{\mu}_{ij}h_i \quad (j = 1, 2, \ldots, n_1)\]

are regular in \( K[\overline{H}] \). Since \( K[\overline{G}] \) is a free \( K[\overline{H}] \)-module a routine argument shows that these elements are also regular in \( K[\overline{G}] \). We obtain from (3.3)

\[(3.3') \quad \overline{x}_j = \sum_{\alpha=1}^{n} \overline{c}_{\alpha j}g_{\alpha}, \quad \overline{x}'_j = \sum_{\alpha=1}^{n} \overline{c}'_{\alpha j}g_{\alpha}\]

\[(\overline{c}_{\alpha j}, \overline{c}'_{\alpha j} \in K[\overline{H}], \alpha = 1, 2, \ldots, n; j = 1, 2, \ldots, m).\]

Since the elements (3.5') are nonzero we obtain from (3.3') that \( \overline{x}_j \neq 0 \) \((j = 1, 2, \ldots, m)\). The relations

\[(3.2') \quad \overline{\nu}_j = \overline{x}_j^j \overline{x}_j \quad (j = 1, 2, \ldots, m_1)\]

imply, via the regularity of the elements (3.6'), that the elements \( \overline{x}_j \) \((j = 1, 2, \ldots, m_1)\) are regular in \( K[\overline{G}] \). Similarly, the relations \( \overline{x}_j\overline{x}'_j = 0 \) \((j = m_1 + 1, \ldots, m)\) imply that the elements \( \overline{x}_j \) \((j = m_1 + 1, \ldots, m)\) are zero divisors. We completed thus the proof of statement (ii).

To prove statement (iii) we observe that the ideal \( B = (A_2)KG \), where \( A_2 \) is an ideal in the group ring of the nilpotent group \( H_1 \). Since \( G \) is polycyclic-by-finite Roseblade's Theorem 11.2.9 in [5] implies that \( B \) is localizable and (iii) is proved.

We have already shown that \( (KH)/A_1 \) is a direct sum of rings isomorphic to \( (KH)/A_1 \simeq K[\overline{H}] \), where \( \overline{H} \) is a finite group and \( A \) is a maximal ideal of \( KH \). Hence the ring \( (KH)/A_1 \) is semisimple artinian. Furthermore, we have a homomorphism

\[(3.12) \quad K[\overline{H}] \to (K[\overline{H}])/A_1 \simeq K[H]/A_1\]

and the second relation (3.9) implies that

\[(3.13) \quad \overline{A}_1 \cap K[\overline{H}_1] = \overline{0}.\]
We have already shown that the group $\overline{H}_1$ is finite. Assume now that
$\text{char } K = p$ and prove that $p \mid (\overline{H}_1 : 1)$. Indeed, we observe first of all that the
group $\overline{H}_1$ is nilpotent since $H_1$ is. Assume now that $p \mid (\overline{H}_1 : 1)$, let $P$ be the
Sylow $p$-subgroup of $\overline{H}_1$ and let $\overline{H}_1 = P \times Q$. The elements $h - 1$ ($h \in P$)
generate a nonzero nilpotent ideal in $K[\overline{H}_1]$ because $P$ is a normal subgroup
of $\overline{H}_1$. Since $K[H]/A_1$ is semisimple we obtain from (3.12) that $(h - 1) \in A_1$
($h \in P$) which contradicts (3.13). We proved thus that $\overline{H}_1$ is a finite $p'$-group.

To complete the proof we need the following assertion which is part of
Lemma 3.2 in [5].

Lemma 2. Let $K$ be an arbitrary commutative field and $K[U]$ be a ring, generated
by a group $U$, which is an extension of a finite group $V$ be a polycyclic-by-
finite group $U/V$. Assume also that $K[U] \simeq K[V]^\ast(U/V)$. Then there exists a
characteristic poly{infinite cyclic} subgroup $F \subseteq U$ of finite index such that the
elements of $F$ are linearly independent over $K$ and, hence, $K[F] \simeq KF$.

Proof. Let $F$ be a poly-infinite cyclic characteristic subgroup of finite index in
$U$. Then $F \cap V = 1$ and it is not difficult to verify that the elements of $F$ are
linearly independent over $K[V]$ and hence over $K$.

We complete now the proof of Theorem 1. Since $\overline{H}_1$ is finite, $\overline{H}/\overline{H}_1$ is free
abelian, and $H$ is finitely generated we conclude that $\overline{H}/Z$ is finite, where $Z$
is the center of $\overline{H}$. The relation (3.11) implies, via Lemma 2, the existence of
a characteristic subgroup $F \subseteq \overline{H}$ of finite index such that $K[F] \simeq KF$. Take
now $N = F \cap Z$ and statement (iii) follows. The proof is completed.

Let $R$ and $\overline{R}$ be the ring of fractions of $KG$ and $K[\overline{G}]$ correspondingly.
The ring $\overline{R}$ is isomorphic to the ring of fractions of $K[\overline{G}]$ with respect to the
subring $KN$; since $(\overline{G}:N)$ is finite we conclude easily that $\overline{R}$ has a finite left
dimension over the subfield $T = (KN)(KN)^{-1}$ and as a result of it is finite-
dimensional over a central subfield $Z \subseteq T$. Furthermore, $\overline{R}$ is a homomorphic
image of a suitable cross product $T \ast \overline{G}/N$; when char $K = 0$ this cross product
is semisimple artinian and so is $\overline{R}$.

If now nonzero elements (1.4) in $R$ are given then

\begin{equation}
(3.14) \quad x_j = a_j b_j^{-1} \quad (a_j \in KG; b_j \in (KG)\setminus 0; j = 1, 2, \ldots, m).
\end{equation}

We apply Theorem 1 to the set of elements $a_j, b_j \in KG$ ($j = 1, 2, \ldots, n$) and
obtain via well-known facts of the localization theory the following corollary.

Corollary. Let nonzero elements (3.14) in $R$ be given. Then there exists a
localizable ideal $B \subseteq KG$ such that the elements (3.14) belong to the subring
$S \subseteq R$, obtained by the localization of the ideal $B$, and $S/BS \simeq \overline{R}$, where $\overline{R}$ is
the ring of fractions of $K[\overline{G}]$; the ring $\overline{R}$ has a finite dimension over its central
subfield $Z$. Clearly, the ideal $BS$ of $S$ is quasiregular,

Let $Q$ be an arbitrary ring. We recall (see Cohn [11] and Passman [12])
that a specialization from $Q$ on ring $\overline{Q}$ is a homomorphism $\alpha: Q_0 \to \overline{Q}$ such
that $\ker \alpha$ is a quasiregular ideal of $Q_0$; $Q_0$ is the domain of $\alpha$. Theorem
1 thus gives a method for constructing specializations from the $K$-algebra $R$
to algebras finite-dimensional over their central subfields. Another system of
specializations to algebras finite-dimensional over $K$ is obtained from the fol-
lowing theorem.
Theorem 2. Let $R$ be the ring of fractions of $KG$ and (3.14) be given nonzero elements of $R$. Then there exists an ideal $C \subseteq KG$ such that the quotient ring $(KG)/C$ is a finite-dimensional $K$-algebra, generated by a finite group $\tilde{G}$, which is the image of $G$ in $(KG)/C$. The homomorphism $\alpha: KG \to K[\tilde{G}]$ is extended to a specialization $\theta: R \to K[\tilde{G}]$, whose domain $R_0$ contains the elements (3.14). Furthermore, $\tilde{x}_j = \theta(x_j)$ $(j = 1, 2, \ldots, m)$ are nonzero elements of $K[\tilde{G}]$.

Proof. Apply first Theorem 1 and its Corollary and obtain a homomorphism

$$\beta: KG \to (KG)/B \simeq K[\tilde{G}]$$

such that the elements (3.14) belong to the subring $S \subseteq R$, the domain of the specialization $\pi: R \to \tilde{R}$ which extend $\beta$, and

$$\bar{x}_j = \pi(x_j) \neq 0 \quad (j = 1, 2, \ldots, m). \quad (3.16)$$

We recall that $\tilde{G}$ contains a free abelian normal subgroup $N$ of finite index such that $K[N] \simeq K\tilde{N}$. Let $T$ be the field of fractions of $KN$ and $g_1, g_2, \ldots, g_r$ be a system of elements of $\tilde{G}$ which form a basis of the left vector space $\tilde{R}$ over $T$. Let

$$x_j = \sum_{i=1}^r a_{ij} g_i \quad (a_{ij} \in T; j = 1, 2, \ldots, m). \quad (3.17)$$

Let $a_1, a_2, \ldots, a_s$ be all the elements of $KN$ which occur in the numerators and denominators of the nonzero elements $a_{ij}$ in (3.17); clearly, every element $a_k$ $(k = 1, 2, \ldots, s)$ has a finite number of $\tilde{G}$-conjugates. Then apply Proposition 1 and find an ideal $A \subseteq KN$ such that the quotient algebra $(KN)/A \simeq K[\tilde{N}]$ where $\tilde{N}$ is a finite group and

$$g^{-1} a_k g \notin A \quad (k = 1, 2, \ldots, s; g \in \tilde{G}).$$

Let $A_1 = \bigcap_{g \in \tilde{G}} A$ and $\overline{C} = A_1(K\tilde{G})$. The same argument as in the proof of Theorem 1 shows that $\overline{C}$ contains no one of the elements (3.17) and $K[\overline{G}]/\overline{C} \simeq K[\tilde{G}]$, where $\tilde{G}$ is a finite group.

The ideal $\overline{C}$ is localizable in $K[\tilde{G}]$; this can be verified in a straightforward way or obtained from Roseblade's theorem in [5]. We see therefore that the homomorphism $\gamma: K[\overline{G}] \to K[\tilde{G}]$ is extended to a specialization $\tau: \overline{R} \to K[\tilde{G}]$ and

$$\tilde{x}_j = \tau(\overline{x}_j) \neq 0. \quad (3.18)$$

Finally, let $C$ be the inverse image of the ideal $\overline{C}$ in $KG$. Clearly,

$$(KG)/C \simeq (K[\overline{G}])/\overline{C} \simeq K[	ilde{G}].$$

Furthermore, the natural homomorphism

$$\alpha: KG \to (KG)/C \simeq K[\tilde{G}]$$

is a composition of two homomorphisms $\beta$ and $\gamma$, which are extended to specializations $\pi$ and $\tau$ correspondingly. We obtain from this (see [8, Chapter 6] or [11]) that $\alpha$ can be extended to a specialization $\tau \pi = \theta: \overline{R} \to K[\tilde{G}]$,
whose domain contains the elements (3.14). The assertion follows now from (3.16) and (3.18).

4

We will need in the proof of Theorem 3 the following fact:

**Lemma 3.** Let $D$ be a field, $x$ be a given matrix from $D_{n \times n}$. Assume that $D$ has a system of subrings $T_i$ $(i \in I)$ such that $x \in (T_i)_{n \times n}$ for all $i \in I$ and
1. given any finite set of elements $M \subseteq D$ there is a $T_i$ with $M \subseteq T_i$,
2. each $T_i$ has an ideal $U_i \neq T_i$ such that the image of the matrix $x$ in the quotient ring
   $$(T_i)_{n \times n}/(U_i)_{n \times n} \simeq (T_i/U_i)_{n \times n}$$
is nilpotent. Then the matrix $x$ is nilpotent.

**Proof.** Assume that $x$ is not nilpotent and hence

$$(4.1) \quad x^n \neq 0.$$  

The powers of $x$ are linearly dependent over $D$; there exists therefore elements $0 \neq d_j \in D$ $(j = 1, 2, \ldots, r)$ such that

$$(4.2) \quad \sum_{j=1}^{r} d_j x^{n_j} = 0 \quad (1 \leq n_1 < n_2 < \cdots < n_r \leq n^2 + 1).$$

Find in the system of subrings $T_i$ $(i \in I)$ a subring $T$ and its ideal $U \subseteq T$ such that $T$ contains all the elements $d_j, d_j^{-1}$ $(j = 1, 2, \ldots, r)$, all the nonzero entries of the matrix $x$ (and $x^n$) and the inverses of these entries. Since all these elements are invertible in $T$ and $U \neq T$, their images in $T/U$ are nonzero. Let $\bar{X}$ denote the image of a subset $X \subseteq T_{n \times n}$ under the homomorphism $(T)_{n \times n} \to (T/U)_{n \times n}$. We see that the elements $\bar{d}_j$ are invertible in $(T/U)_{n \times n}$, the element $\bar{x}$ is nilpotent but

$$(4.3) \quad \bar{x}^n \neq \bar{0}$$

and

$$(4.2') \quad \sum_{j=1}^{r} \bar{d}_j \bar{x}^{n_j} = \bar{0}.$$  

Now let $k$ be the smallest natural number such that $\bar{x}^k = 0$. It follows from (4.3) that $k > n$. We multiply (4.2') on the right by $\bar{x}^{k-n_1-1}$ and obtain that $\bar{d}_1 \bar{x}^{k-1} = 0$. Since $\bar{d}_1$ is invertible we see that $\bar{x}^{k-1} = 0$ which contradicts (4.3). Thus assumption (4.1) leads to a contradiction, i.e., $x$ is nilpotent.

The following fact is known (see [13, Lemma II.5.4]).

**Lemma 4.** Let $U$ be a finite-dimensional algebra over a field $K$ of characteristic zero, $Z$ be its center. Then the intersection $[U, U] \cap Z$ is a nilpotent ring.

We can now prove our main result.

**Theorem 3.** Let $G$ be a polycyclic-by-finite group, $K$ be a field of characteristic zero and $R$ be the ring of fractions of $KG$. Let $S$ be a subring of the matrix ring $R_{n \times n}$, $Z$ be its center. Then the intersection $[S, S] \cap Z$ is a nilpotent ring.

**Proof.** In order to prove Theorem 3 it is enough to prove that the ring $[S, S] \cap Z$ is nil because a nil subring of a matrix ring over the artinian ring $R$ must be nilpotent.
Let thus \( z \in ([S, S] \cap Z) \), where \( S \subseteq R_{m \times m} \). There exist therefore elements \( u_i, v_i \in S \) \( (i = 1, 2, \ldots, r) \) such that

\[
\sum_{i=1}^{r} [u_i, v_i] = z.
\]

Pick in \( R \) an arbitrary finite subset which has a form

\[
x_1, x_2, \ldots, x_k; \quad x_1^{-1}, x_2^{-1}, \ldots, x_k^{-1}
\]

and contains all the nonzero entries of the matrices \( u_i, v_i \) \( (i = 1, 2, \ldots, r) \) (and of \( z \)). Apply Theorem 2 and find a subring \( T \subseteq R \) and an ideal \( U \subseteq T \) such that elements (4.5) belong to \( T \) and \( T/U \cong K[G] \), where \( K[G] \) is a finite-dimensional algebra over \( K \). Relation (4.4) implies the following relation in \( (T/U)_{m \times m} \) for the images of the elements \( u_i, v_i, z \) \( (i = 1, 2, \ldots, r) \):

\[
\sum_{i=1}^{r} [\overline{u}_i, \overline{v}_i] = \overline{z}.
\]

Since the element \( \overline{z} \) commutes with all the elements \( \overline{u}_i, \overline{v}_i \) \( (i = 1, 2, \ldots, r) \), we obtain from Lemma 4 that \( \overline{z} \) is nilpotent. Lemma 3 now implies that \( z \) is nilpotent which completes the proof of Theorem 3.

**Corollary 1.** Let the subring \( S \) in Theorem 3 be semiprime. Then \( [S, S] \cap Z = 0 \).

Now let \( G \) be a residually torsion-free nilpotent group, \( K \) be a commutative field. Let

\[
G = N_1 \supseteq N_2 \supseteq \cdots
\]

be a series of normal subgroups in \( G \) such that every quotient group \( G/N_i \) \( (i = 1, 2, \ldots) \) is torsion-free nilpotent and \( \bigcap_{i=1}^{\infty} N_i = 1 \). It is not difficult to define in \( G \) an order such that all the homomorphisms \( G \to G/N_i \) are homomorphisms of ordered groups (see [14]). Let \( K(G) \) be the appropriate Malcev-Neumann power series ring and \( \Delta \) be its subfield, generated by the group ring. We will give now a sketch of proof of the following result.

**Proposition 2.**

(i) If \( \text{char } K = 0 \) then the conclusion of Theorem 3 is valid for an arbitrary subring \( S \subseteq \Delta_{n \times n} \).

(ii) If \( K \) has an arbitrary characteristic then

\[
1 \not\in [\Delta, \Delta].
\]

**Proof.** Let \( \Delta_i \) be the field of fractions of the group ring \( K(G/N_i) \). The results of [14] imply that for every given \( i \) there exists a specialization \( \theta_i: \Delta \to \Delta_i \), extending the natural homomorphism \( G \to G/N_i \) and that for every given elements of \( D \),

\[
x_1, x_2, \ldots, x_k; \quad x_1^{-1}, x_2^{-1}, \ldots, x_k^{-1}
\]

an index \( i_0 \) can be found such that for every \( i \geq i_0 \) these elements belong to the domain \( T_i \) of the specialization \( \theta_i \). Since Theorem 3 holds for the subrings of \( (\Delta_i)_{n \times n} \) we obtain now easily from Lemma 3 the statement (i).
We prove now (ii). A routine argument reduces the proof to the case when
the group $G$ is finitely generated; we can assume also that the field $K$ is al-
gebraically closed. Assume that $1 \in [\Delta, \Delta]$, i.e. there exist nonzero elements
$u_j, v_j \in \Delta$ ($j = 1, 2, \ldots, s$) such that

\[(4.7) \quad 1 = \sum_{j=1}^{s} [u_j, v_j].\]

Apply Proposition 2.8 in [15] and find a specialization $\pi: \Delta \rightarrow K[\tilde{G}]$ such
that $K[\tilde{G}]$ is a simple algebra generated by a finite $q$-group $\tilde{G}$ where $q$ is an
arbitrary prime number unequal to char $K$ and the domain $T$ of $\pi$ contains
all the elements $u_j, v_j$ from (4.7). The relation (4.7) now yields the following
relation in $K[\tilde{G}]$,

\[(4.7') \quad \tilde{1} = \sum_{j=1}^{s} [\tilde{u}_j, \tilde{v}_j].\]

Since $K[\tilde{G}]$ is a simple algebra over an algebraically closed field $K$ and $q \neq$
char $K$ we obtain that $K[\tilde{G}]$ is isomorphic to a matrix algebra of degree $q^m$
over $K$. The relation (4.7') however is impossible in the algebra $K_{q^m \times q^m}$ since
the trace of the right side is zero whereas $Tr(\tilde{1}) = q^m \neq 0$. This completes the
proof.

Since free groups and free soluble groups are residually torsion-free nilpotent,
we obtain that Propositon 2 is valid for the universal field of fractions of free
group rings or for Ore fields of fractions of group rings of free soluble groups.

The truth of (4.6) for a ring of fractions $R$ of a ring $(KG)/P$, where $G$ is a
finitely generated nilpotent group, char $K = 0$ and $P$ is a prime ideal of $KG$,
was established by M. Lorenz in [16].

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