STABLE SPLITTINGS OF THE DUAL SPECTRUM OF THE CLASSIFYING SPACE OF A COMPACT LIE GROUP

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ABSTRACT. For a compact Lie group G, there is a map from the G-equivariant fixed point spectrum of the zero sphere to the dual spectrum of the classifying space of G, DBG_+. When G is finite, the affirmative solution to Segal's conjecture states that this map is an equivalence upon appropriate completion of the source. In the case of a compact Lie group, we obtain splitting results of DBG_+ via this map upon taking p-adic completions.

0. INTRODUCTION

Let G be a finite group, $A(G)$ be the Burnside ring of virtual finite G-sets, and $I(G)$ be the augmentation ideal of $A(G)$. Recall EG is a contractible G-CW complex acting freely by G whose quotient EG/G is the classifying space of G, denoted by BG. Let $(\ )_+$ denote the disjoint union with a base point. G. Segal in 1970 proposed the following

Conjecture. *The map of G-equivariant stable cohomotopy groups induced by the projection* $EG_+ \to S^0$,

$$\pi_G^*(S^0)^{\widehat{}}_{I(G)} \to \pi_G^*(EG_+) \cong \pi^*(BG_+),$$

is an isomorphism, where $(\ )^{\widehat{}}_{I(G)}$ *denotes the* $I(G)$*-adic completion.*

In 1982, G. Carlsson [4] completed the proof of Segal's conjecture for all finite groups which had been studied previously with the works of Lin, Adams, Gunawardena, Miller, and May, among others. For more historical background information, see [4].

As for compact Lie groups, Segal's conjecture can also be formulated as above with an appropriate modification of the Burnside ring. However, it is known that in this case the conjecture does not hold in general. Since $\pi_G^*(S^0)$ is actually computable, we may interpret the Segal conjecture for finite groups as a statement about $\pi^*(BG_+)$. From this point of view, we could ask to what extent information about $\pi^*(BG_+)$ is captured by $\pi_G^*(S^0)$ when G is a compact Lie group.

We shall state our results in terms of spectra. For general references, see Adams [1] and Lewis-May-Steinberger [9]. Now suppose G is a compact Lie

group. The projection $EG_+ \to S^0$ induces a map of spectra

$$\Phi : (S^0)^G = F(S^0, S^0)^G \to F(EG_+, S^0)^G \simeq F(GB_+, S^0) = DBG_+,$$

where $(S^0)^G$ denotes the equivariant fixed point spectrum of the sphere G-spectrum and $F(\ , \ )$ is the function space spectrum. By the results of tom Dieck [5, Theorem 2] and Lewis-May-Steinberger [9, Theorem 11.1], we have the following description:

$$(S^0)^G \simeq \bigvee_{(H)} BW_H{}^{\mathrm{Ad}\ W_H},$$

where W_H is NH/H, $BW_H{}^{\mathrm{Ad}\ W_H}$ is the Thom spectrum of the adjoint representation of W_H, and (H) ranges over all conjugacy classes of closed subgroups of G. Notice that the summand corresponding to the trivial group is $BG^{\mathrm{Ad}\ G}$. When G is finite, $BG^{\mathrm{Ad}\ G} = BG_+$ and the affirmative solution to the Segal conjecture implies that $BG_{+\hat{p}}$ is a summand of $DBG_{+\hat{p}}$ given by the map Φ where we completed the spectra at a prime p. When G is a compact Lie group, following evidence provided by Nishida [13], Ravenel [14], and Miller and Wilkerson [12], we prove the following:

Theorem 1.1. $BG^{\mathrm{Ad}\ G}{}_{\hat{p}}$ *is a summand of* $DBG_{+\hat{p}}$ *under* Φ.

In fact, we can extend this theorem to other conjugacy classes of finite p-subgroups of G. Specifically, we have

Theorem 3.1. *Let G be a compact Lie group and p be a fixed prime. Suppose H is a finite p-subgroup of G, then $BW_H{}^{\mathrm{Ad}\ W_H}{}_{\hat{p}}$ is a summand of $DBG_{+\hat{p}}$ under Φ. Furthermore, the finite wedge sum of any such spectra splits from $DBG_{+\hat{p}}$.*

There are technical difficulties in trying to extend Theorem 3.1 to conjugacy classes of closed infinite subgroups of G. In particular, the homotopy inverse limit of split cofiber sequences does not necessarily split. Nevertheless, we find the following weaker result to be true. Recall that a Lie group is said to be p-toral if it is an extension of a torus by a p-group.

Theorem 3.12. *Let G be a compact Lie group and p be a fixed prime. Then the following map is an injection:*

$$\pi_*(\Phi): \bigoplus_{\substack{(H) \subset G \\ H\ p\text{-toral}}} \pi_*(BW_H{}^{\mathrm{Ad}\ W_H}{}_{\hat{p}}) \to \pi_*(DBG_{+\hat{p}}).$$

Segal's conjecture for a general compact Lie group was first considered by M. Feshbach in [7]. There he analyzed $\pi^0(BG_+)$ and proved the conjecture via a different completion on $\pi^0_G(S^0)$ together with some additional assumptions at the prime 2. Consequently, only those closed subgroups having finite index in their normalizers are being considered. What we accomplish is to give splitting results corresponding to finite subgroups which do not necessarily have finite index in their normalizers. Furthermore the corresponding injection result of Feshbach also holds for higher cohomotopy groups.

As an illustration of our techniques, combined with splitting results of Mitchell and Priddy [11], we obtain the following splitting for $G = SO(3)$ at the prime 2.

Theorem 4.13.

$$DBSO(3)_{+}{}_{\widehat{2}} \simeq S^0{}_{\widehat{2}} \vee BSO(3)^{\mathrm{Ad}\ SO(3)}{}_{\widehat{2}}$$

$$\vee \prod_{1}^{\infty} BO(2)^{\mathrm{Ad}\ O(2)}{}_{\widehat{2}} \vee \prod_{1}^{\infty} BZ_{2+}{}_{\widehat{2}}.$$

The organization of this paper is as follows. In the first two sections, we shall concentrate on the proof of Theorem 1.1 in order to make the paper more readable. We first reduce to the case when G is an extension of a torus by a p-group in §1. Using Segal's conjecture for p-groups, we express $DBG_{+}{}_{\widehat{p}}$ as a certain homotopy inverse limit. In §2, we analyze this homotopy inverse limit. The proof of Theorem 1.1 follows after a homotopy inverse limit of transfer maps is identified as a Thom spectrum. In the first part of §3, we show the method of the preceding two sections can be easily generalized to give Theorem 3.1. However, the proof of Theorem 3.12 is much more technical in nature, although the essential ideas are already contained in the first two sections. The new ingredient is the construction of a finite filtration of a system of spectra. The splitting result of $G = SO(3)$ at the prime 2 is the focus of §4. There we utilize the fact that we have an explicit description of all conjugacy classes of closed subgroups of $SO(3)$ together with the splitting results of [11]. Combined with a theorem of Miller and Wilkerson [12], this gives the splitting of DBS^3_{+} at the prime 2.

Throughout this paper, we shall use X^G to denote the G-equivariant fixed point spectrum of a G-spectrum X. Homotopy inverse limits will be denoted by $\varprojlim$. (See Chapter XI of Bousfield-Kan [2] for details.) For a given nonequivariant spectrum X, the p-adic completion of X, denoted by $X_{\widehat{p}}$, is $\varprojlim_i X \wedge M(Z_{p^i})$, where $M(Z_{p^i})$ is the Moore spectrum of Z_{p^i}. Consult [14] for some of its basic properties.

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1. Preliminary reductions

Let G be a compact Lie group. Denote by G^0 the connected component of the identity. When we say a maximal torus T of G, we shall mean a maximal torus of G^0 in the usual sense. Consider the normalizer N of T in G. It is easy to see that N/T is finite since $N/N \cap G^0 \hookrightarrow G/G^0 = \Gamma$ is finite. Furthermore, we have $G^0/N \cap G^0 \xrightarrow{\cong} G/N$. For if $\bar{g} \in G/N$, then gTg^{-1}, being a maximal torus of G^0, would be a conjugate of T in G^0. Therefore, $\bar{g} \in G^0/N \cap G^0$. In particular, $\chi(G/N) = 1$ since it is well known that $\chi(G^0/N \cap G^0) = 1$.

Recall we want to prove

Theorem 1.1. $GB^{\mathrm{Ad}\ G}_{\widehat{p}}$ *is a summand of* $DBG_{+}{}_{\widehat{p}}$ *under* Φ.

As a preliminary reduction, we have

Lemma 1.2. *Let* N *be the normalizer of a maximal torus* T *of* G. *If* $BN^{\mathrm{Ad}\ N}_{\widehat{p}}$ *is a summand of* $DBN_{+}{}_{\widehat{p}}$ *under* Φ, *then Theorem* 1.1, *holds for* G.

Proof. We have the following diagram:

$$
\begin{array}{ccccc}
BN^{\mathrm{Ad}\ N} & \longrightarrow & (S^0)^N & \xrightarrow{\ \Phi\ } & DBN_+ \\
{\scriptstyle \mathrm{tr}}\uparrow & & \uparrow & & \uparrow{\scriptstyle D[B\pi]} \\
BG^{\mathrm{Ad}\ G} & \longrightarrow & (S^0)^G & \xrightarrow{\ \Phi\ } & DBG_+
\end{array}
$$

where π is the inclusion $N \hookrightarrow G$ and tr is the associated bundle transfer (cf. Mann-Miller-Miller [10] and Nishida [13]). The diagram commutes by [13, Lemma 2.3] and the naturality of Φ.

Consider the composite

$$
BG^{\mathrm{Ad}\ G} \xrightarrow{\ \mathrm{tr}\ } BN^{\mathrm{Ad}\ N} \xrightarrow{\ j\ } BN^{\mathrm{Ad}\ G} \xrightarrow{\ B\pi\ } BG^{\mathrm{Ad}\ G},
$$

where J is the inclusion $\mathrm{Ad}\ N \hookrightarrow \mathrm{Ad}\ G$.

Via Thom isomorphism, $\mathrm{tr}^*(\chi(\alpha) \cdot x) = \mathrm{tr}_0^*(x)$, where $\alpha = \mathrm{Ad}\ G/\mathrm{Ad}\ N$, $\chi(\alpha)$ is the Euler class of α, $x \in H^*(BN, Z_p)$, and tr_o is the ordinary Becker-Gottlieb transfer (cf. [13, Proposition 2.2]). Since j^* is multiplication by $\chi(\alpha)$ and $\chi(G/N) = 1$, it follows that the above composite induces an isomorphism on cohomology with Z_p coefficient. Therefore $BG^{\mathrm{Ad}\ G}\hat{}_p \xrightarrow{\ \mathrm{tr}\ } BN^{\mathrm{Ad}\ N}\hat{}_p$ has a retraction. Hence by completing the above commutative diagram at a prime p, we have the desired result. $\square$

We are reduced to the case of the normalizer of a maximal torus of G. Suppose we have the extension

$$
(*) \qquad\qquad 1 \to T \to N \xrightarrow{\ \psi\ } W \to 1,
$$

where T is a torus and W is a finite group.

Let W_p be a p-Sylow subgroup of W and N_p the preimage of W_p in N. We have

Lemma 1.3. *If $BN_p^{\mathrm{Ad}\ N_p}\hat{}_p$ is a summand of $DBN_{p+}\hat{}_p$, then $BN^{\mathrm{Ad}\ N}\hat{}_p$ is a summand of $DBN_+\hat{}_p$.*

Proof. The proof is identical to that of Lemma 2.1 for $\chi(N/N_p) \not\equiv 0 \mod p$. $\square$

Consequently, we may assume W in $(*)$ to be a p-group. The following result by Feshbach is useful in analyzing DBN_+.

Proposition 1.4 [7, Theorem 1.1]. *Let $|W| = n$, and $\tilde{T}$ be the subgroup of T generated by all elements of order n. Then there exists a subgroup $\tilde{F}$ of N such that $\tilde{F} \cap T = \tilde{T}$ and $\tilde{F}/\tilde{T} \cong W$. If T' is any invariant subgroup of T containing $\tilde{T}$, let F' denote the group generated by T' and $\tilde{F}$. Then $F' \cap T = T'$ and $F'/T' \cong W$.*

If W is a p-group and T_{Q_p} consists of all elements of order a power of p in T, then we have

Corollary 1.5 [7, Corollary 1.2]. *There exists a nested sequence $\{F_i\}$ of finite subgroups of N such that $F_i/F_i \cap T \cong W$ with $\bigcup F_i \cap T = T_{Q_p}$. We denote $\bigcup F_i$ by N_{Q_p}. Furthermore, we can assume $F_i \cap T = \{x \in T \,|\, x^{p^i|W|} = 1\}$.*

Thus we have the following approximation result for $DBN_+\hat{}_p$.

Proposition 1.6. $DBN_{+\hat{p}} = \varprojlim_i DBF_{i+\hat{p}}$.

Proof. It is easy to see that $H_*(BT_{Q_p}, Z_{p^n}) \xrightarrow{i_*} H_*(BT, Z_{p^n})$ is an isomorphism for $n \geq 1$, where $i: T_{Q_p} \to T$ is the inclusion. Consider the following map of extensions:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & T_{Q_p} & \longrightarrow & N_{Q_p} & \longrightarrow & W & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & T & \longrightarrow & N & \longrightarrow & W & \longrightarrow & 1
\end{array}
$$

The homology with a Z_{p^n} coefficient spectral sequence associated with the diagram induces an isomorphism of E_2 terms. Hence, $H_*(BN_{Q_p}, Z_{p^n}) \xrightarrow{\cong} H_*(BN, Z_{p^n})$. Taking the direct limit of n yields $H_*(BN_{Q_p}, Z_{p^\infty}) \xrightarrow{\cong} H_*(BN, Z_{p^\infty})$. Now, since $Z_{p^\infty} = Z[1/p]/Z$ and $\widetilde{H}_*(BN_{Q_p}, Z[1/p]) = 0$, we have $\widetilde{H}_*(BN_{Q_p}, Z) \cong \widetilde{H}_{*+1}(BN, Z_{p^\infty})$ or $BN_{Q_p+} \to BN_+ \wedge \Sigma^{-1} M(Z_{p^\infty})$ is an equivalence, where $M(Z_{p^\infty})$ denotes the Moore spectrum of Z_{p^∞}.

Hence, $DBN_{Q_p+} \xleftarrow{\simeq} D(BN_+ \wedge \Sigma^{-1} M(Z_{p^\infty})) \simeq DBN_{+\hat{p}}$ and therefore we have $\varprojlim_i DBF_{i+\hat{p}} = DBN_{Q_p+\hat{p}} \simeq DBN_{+\hat{p}}$ as the desired result. □

Since the F_i are p-groups, the affirmative solution to Segal's conjecture for finite p-groups yields [4]:

$$\pi: (S^0)^{F_i}_{\hat{p}} \xrightarrow{\simeq} DBF_{i+\hat{p}}.$$

Furthermore, we have the commutative diagram:

$$
\begin{array}{ccc}
BF_{i+\hat{p}} & \xrightarrow{\Phi} & DBF_{i+\hat{p}} \\
{\scriptstyle tr}\downarrow & & \downarrow{\scriptstyle D[Bj]} \\
BF_{i-1+\hat{p}} & \xrightarrow{\Phi} & DBF_{i-1+\hat{p}}
\end{array}
$$

Here j is the inclusion $F_{i-1} \hookrightarrow F_i$ and tr is the transfer associated to j. Even though $BF_{i+\hat{p}}$ is a summand of $DBF_{i+\hat{p}}$ under Φ, it is not true that the splitting respects the maps in the inverse system $\{DBF_{i+\hat{p}}\}$. The method then is to find appropriate summands in $DBF_{i+\hat{p}}$ that will respect these maps in the inverse system. We begin by recalling the following

Definition. A *family* $\mathscr{F}$ in G is a set of subgroups G which is closed under conjugation and passage to subgroups.

Fix a closed subgroup T' of T. Consider the set

$$\mathscr{F}_{T'} = \{H \subset N | H \cap T \text{ is a conjugate of } T'\}.$$

We have the following finiteness property of $\mathscr{F}_{T'}$.

Lemma 1.7. *There are a finite number of conjugacy classes in $\mathscr{F}_{T'}$.*

Proof. Recall we have the extension

$$1 \to T \to N \to W \to 1.$$

Since W is finite, it is sufficient to consider the subgroups in $\mathcal{F}_{T'}$ whose image in W is some fixed J. Suppose H is such a subgroup. Let N' be the preimage of J in N. Since T' is normal in N', we have the extension

$$(1.7.1) \qquad 1 \to \frac{T}{T'} \to \frac{N'}{T'} \to J \to 1 .$$

Now H corresponds to a splitting of (1.7.1). Two such splittings differ from a crossed homomorphism $J \to T/T'$. Hence, it suffices to check $H^1(J, T/T')$ is finite. It is now clear that this reduces to showing $H^1(J, (S^1)^n)$ is finite, where J acts on $(S^1)^n$ by automorphisms. Of course automorphisms of $(S^1)^n$ are induced from automorphisms of R^n, where $(S^1)^n = R^n/Z^n$. Thus we have an exact sequence of $Z[J]$-modules:

$$0 \to Z^n \to R^n \to (S^1)^n \to 0 .$$

Taking the corresponding long exact sequence yields

$$0 \to H^0(J, Z^n) \to H^0(J, R^n) \to H^0(J, (S^1)^n) \to H^1(J, Z^n)$$
$$\to H^1(J, R^n) \to H^1(J, (S^1)^n) \to H^2(J, Z^n) \to H^2(J, R^n) \to \cdots .$$

Since J is finite and R^n is divisible with no elements of finite order, we have $H^m(J, R^n) = 0$ for $m \geq 1$. However, $H^2(J, Z^n)$ is a finitely generated Z-module with J finite. Therefore, $H^2(J, Z^n)$ is finite and so is $H^1(J, (S^1)^n)$. □

Let $I_m = \{x \in T | x^{p^m} = 1\}$. Consider the family $\mathcal{F}_m = \{H \subset N | H \cap T \subset I_m\}$. The following two results from [7] show that for each $H \in \mathcal{F}_m$, there exists a subgroup $H' \in \mathcal{F}_m$ with $H' \subset N_{Q_p}$ and $(H') = (H)$.

Proposition 1.8 [7, Theorem 1.4]. *Suppose W is a p-group. Let H be a subgroup of N such that $H \cap T = T_{Q_p}$ and $H/T_{Q_p} \cong W$. Then H is conjugate to N_{Q_p} in N.*

Corollary 1.9 [7, Corollary 1.5]. *Suppose H is a subgroup of N such that $H \cap T \subset T_{Q_p}$. Then H is conjugate to a subgroup of N_{Q_p}.*

Now given a family $\mathcal{M}$ in G, there exists a universal G-space $E\mathcal{M}$ such that the H-fixed point of $E\mathcal{M}$ is empty if $H \notin \mathcal{M}$ and contractible otherwise (cf. tom Dieck [6, Proposition 7.2.1]).

Recall we have the family $\mathcal{F}_n$. $E\mathcal{F}_{n+}$ is an N-space. Upon restriction to F_i, $E\mathcal{F}_{n+}$ becomes an F_i-space.

Proposition 1.10. $\varprojlim_i (E\mathcal{F}_{n+})^{F_i}\widehat{}_p$ *is a summand of* $DBN_{+}\widehat{}_p$.

Proof. Recall $DBN_{+}\widehat{}_p = \varprojlim_i DBF_{i+}\widehat{}_p$. By Segal's conjecture for finite p-groups, $DBF_{i+}\widehat{}_p = (S^0)^{F_i}\widehat{}_p$. Consider the map $(E\mathcal{F}_{n+})^{F_i} \xrightarrow{j} (S^0)^{F_i}$ induced from $E\mathcal{F}_{n+} \to S^0$. By the naturality of the following splitting due to tom Dieck [5, Theorem 2],

$$\pi_*^G(X) \cong \bigoplus_{(H)} \pi_*^{W_H}(EW_{H+} \wedge X^H) ,$$

we see that

$$j: (E\mathcal{F}_{n+})^{F_i} \simeq \bigvee_{\substack{(H) \\ H \subset F_i \\ H \cap T \subset I_n}} BW_{H+}^i \to \bigvee_{\substack{(H) \\ H \subset F_i}} BW_{H+}^i \simeq (S^0)^{F_i}$$

is the obvious map of inclusion, where $W_H^i = N_i H/H$, $N_i H$ is the normalizer of H in F_i.

Suppose we identify each summand above by its corresponding conjugacy class (H). Clearly j has a splitting map which collapses all the summands whose corresponding conjugacy class (H) are such that $H \cap T \not\subset I_n$ to the base point. We claim that this splitting map is compatible with $(S^0)^{F_i} \xrightarrow{\gamma} (S^0)^{F_{i-1}}$ for all sufficiently large i, where γ is induced from $F_{i-1} \hookrightarrow F_i$. The proposition then follows upon taking homotopy inverse limit.

We will show below that under γ, (H) will map to summands of the form $\{(H^\alpha \cap F_{i-1})\}$, where $\alpha \in F_i$ and $H^\alpha = \alpha H \alpha^{-1}$. Taking this for granted and using the fact that I_n is normal, it is now easy to check that for all sufficiently large i, $H \cap T \not\subset I_n$ implies the same for $H^\alpha \cap F_{i-1}$. $\quad\square$

We verify the claim made during the proof of Proposition 1.10.

Proposition 1.11. *Let $K_1 \subset K_2$ be the inclusion of finite groups. Suppose we identify each summand of $(S^0)^{K_1}$ (resp. $(S^0)^{K_2}$) by its corresponding conjugacy class (H). Then under the map $(S^0)^{K_2} \xrightarrow{\gamma} (S^0)^{K_1}$, (H) maps to $\{(H^{g_i} \cap K_1)\}$, where $\{g_i\}$ is a set of representatives for the double coset space $K_1 \backslash {}^{K_2}/N_{K_2}H$. Furthermore, the map is given by*

$$BW_{H+} \xrightarrow{C_{g_i}} B(W_H)^{g_i}{}_+ \xrightarrow{\text{tr}} B\frac{(NH)^{g_i} \cap K_1}{H^{g_i} \cap K_1}{}_+ \xrightarrow{B_i} B\frac{N_1(H^{g_i} \cap K_1)}{H^{g_i} \cap K_1}{}_+,$$

where C_{g_i} is induced from conjugation and $N_1(H^{g_i} \cap K_1)$ denotes the normalizer of $H^{g_i} \cap K_1$ in K_1.

Proof. We first recall that tom Dieck's splitting theorem, mentioned in the proof of Proposition 1.10, is given by (when G is finite)

$$\pi_*^{W_H}(EW_{H+}) \xrightarrow{\alpha} \pi_*(BW_{H+}) \xrightarrow{j} \pi_*^G(BW_{H+})$$
$$\xrightarrow{\tau} \pi_*^G(G_+ \wedge_{NH} EW_{H+}) \xrightarrow{p} \pi_*^G(S^0),$$

where the first left arrow is Adam's isomorphism, j is the induced map of $G \to \{e\}$, τ is the G-equivariant transfer associated to $G_+ \wedge_{NH} EW_{H+} \to BW_{H+}$, and p is the evident projection map (cf. Lewis-May-McClure [8, Theorem 1]).

We have the following commutative diagram:

$$
\begin{array}{ccc}
\pi_*(BW_{H+}) & = & \pi_*(BW_{H+}) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\
\pi_*^{K_2}(BW_{H+}) & \longrightarrow & \pi_*^{K_1}(BW_{H+}) \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\
\pi_*^{K_2}(K_{2+} \wedge_{NH} EW_{H+}) & \longrightarrow & \pi_*^{K_1}(K_{2+} \wedge_{NH} EW_{H+}) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
\pi_*^{K_2}(S^0) & \longrightarrow & \pi_*^{K_1}(S^0)
\end{array}
$$

where the horizontal maps are induced from $K_1 \hookrightarrow K_2$.

Now we recall the construction of the G-equivariant transfer τ. First there exists a G-map $\tilde{\tau}: S^0 \to G/H_+$, called the pretransfer. Then taking the smash product with EW_{H+} and the W_H orbit gives τ.

In the above diagram, we are interested in understanding the K_2-equivariant transfer as a K_1 map. Using the additivity property of transfers, we want to decompose K_2/H as a K_1 space. The point is we have such a decomposition derived from that of K_2/NH. So suppose $\{g_i\}$ is a set of representatives of the double coset space $K_1 \backslash {}^{K_2}/NH$. Of course, by definition we have

$$K_2/NH = \bigvee_{\{g_i\}} K_1/((NH)^{g_i} \cap K_1).$$

However, it is easy to see that as K_1 spaces, we also have

$$K_2/H = \bigvee_{\{g_i\}} (K_1/H^{g_i} \cap K_1) \times_{((NH)^{g_i}\cap K_1)/(H^{g_i}\cap K_1)} (W_H)^{g_i}.$$

This observation gives rise to the following commutative diagram:

$$
\begin{array}{ccc}
\pi_*^{K_1}(BW_{H+}) & =\!=\!=\!=\!= & \pi_*^{K_1}(BW_{H+}) \\
\Big\downarrow{\tau} & & \Big\downarrow{\oplus\{\theta_{g_i}\}} \\
\pi_*^{K_1}(K_{2+} \wedge_{NH} EW_{H+}) & \xrightarrow{\ \cong\ } & \bigoplus_{\{g_i\}} \pi_*^{K_1}(K_{1+} \wedge_{(NH)^{g_i}\cap K_1} E(W_H)^{g_i}{}_+)
\end{array}
$$

where θ_{g_i} is induced from appropriate equivariant transfers.

By using the commutativity property of transfers, we have the following diagram describing θ_{g_i}:

$$
\begin{array}{ccc}
\pi_*^{K_1}(BW_{H+}) & \xrightarrow{\ \theta_{g_i}\ } & \pi_*^{K_1}(K_{1+} \wedge_{(NH)^{g_i}\cap K_1} E(W_H)^{g_i}{}_+) \\
{\scriptstyle \tau' \circ C_{g_i}}\Big\downarrow & & \tau'\Big\downarrow{\scriptstyle \cong} \\
\pi_*^{K_1}\!\left(B\frac{(NH)^{g_i}\cap K_1}{H^{g_i}\cap K_1}{}_+ \right) & \xrightarrow{\ \tau\ } & \pi_*^{K_1}\!\left(K_{1+} \wedge_{(NH)^{g_i}\cap K_1} E\frac{(NH)^{g_i}\cap K_1}{H^{g_i}\cap K_1}{}_+ \right)
\end{array}
$$

Here τ' is the evident transfer.

Combining the above commutative diagrams gives the following:

$$
\begin{array}{ccc}
\pi_*(BW_{H+}) & \xrightarrow{\ \oplus\{\tau' \circ C_{g_i}\}\ } & \bigoplus_{\{g_i\}} \pi_*\!\left(B\left(\frac{(NH)^{g_i}\cap K_1}{H^{g_i}\cap K_1}\right)_+ \right) \\
j\Big\downarrow & & \Big\downarrow{\oplus j} \\
\pi_*^{K_2}(BW_{H+}) & & \bigoplus_{\{g_i\}} \pi_*^{K_1}\!\left(B\left(\frac{(NH)^{g_i}\cap K_1}{H^{g_i}\cap K_1}\right)_+ \right) \\
\tau\Big\downarrow & & \Big\downarrow{\oplus \tau} \\
\pi_*^{K_2}(K_{2+} \wedge_{NH} EW_{H+}) & & \bigoplus_{\{g_i\}} \pi_*^{K_1}\!\left(K_{1+} \wedge_{(NH)^{g_i}\cap K_1} E\frac{(NH)^{g_i}\cap K_1}{H^{g_i}\cap K_1}{}_+ \right) \\
p\Big\downarrow & & \Big\downarrow{p} \\
\pi_*^{K_2}(S^0) & \xrightarrow{\qquad\qquad} & \pi_*^{K_1}(S^0)
\end{array}
$$

Now

$$\begin{array}{ccc}
K_{1+} \wedge_{(NH)^{g_i} \cap K_1} E\frac{(NH)^{g_i} \cap K_1}{H^{g_i} \cap K_1}{}_+ & \longrightarrow & K_{1+} \wedge_{N_1(H^{g_i} \cap K_1)} E\frac{N_1(H^{g_i} \cap K_1)}{H^{g_i} \cap K_1}{}_+ \\
\downarrow & & \downarrow \\
B\frac{(NH)^{g_i} \cap K_1}{H^{g_i} \cap K_1}{}_+ & \longrightarrow & B\frac{N_1(H^{g_i} \cap K_1)}{H^{g_i} \cap K_1}{}_+
\end{array}$$

is a pullback diagram. So

$$\begin{array}{ccc}
\pi_*^{K_1}\left(B\frac{N_1(H^{g_i} \cap K_1)}{H^{g_i} \cap K_1}{}_+\right) & \xrightarrow{\tau} & \pi_*^{K_1}\left(K_{1+} \wedge_{N_1(H^{g_i} \cap K_1)} E\frac{N_1(H^{g_i} \cap K_1)}{H^{g_i} \cap K_1}{}_+\right) \\
\uparrow & & \uparrow \\
\pi_*^{K_1}\left(B\frac{(NH)^{g_i} \cap K_1}{H^{g_i} \cap K_1}{}_+\right) & \xrightarrow{\tau} & \pi_*^{K_1}\left(K_{1+} \wedge_{(NH)^{g_i} \cap K_1} E\frac{(NH)^{g_i} \cap K_1}{H^{g_i} \cap K_1}{}_+\right)
\end{array}$$

commutes. It is clear that we also have

$$\begin{array}{ccc}
\pi_*^{K_1}(K_{1+} \wedge_{(NH)^{g_i} \cap K_1} E(W_H)^{g_i}{}_+) & \longleftarrow & \pi_*^{K_1}\left(K_{1+} \wedge_{(NH)^{g_i} \cap K_1} E\frac{(NH)^{g_i} \cap K_1}{H^{g_i} \cap K_1}{}_+\right) \\
& & \downarrow \\
p\downarrow & & \pi_*^{K_1}\left(K_{1+} \wedge_{(NH)^{g_i} \cap K_1} E\frac{N_1(H^{g_i} \cap K_1)}{H^{g_i} \cap K_1}{}_+\right) \\
& & \downarrow \\
\pi_*^{K_1}(S^0) & \xleftarrow{p} & \pi_*^{K_1}\left(K_{1+} \wedge_{N_1(H^{g_i} \cap K_1)} E\frac{N_1(H^{g_i} \cap K_1)}{H^{g_i} \cap K_1}{}_+\right)
\end{array}$$

where all unmarked arrows are induced from evident inclusions.

Therefore, by combining these commutative diagrams, we see that the summand corresponding to (H) goes to summands corresponding to $\{(H^{g_i} \cap K_1)\}$ with the prescribed maps as asserted. $\square$

For a general compact Lie group, tom Dieck's splitting theorem for $\pi_*^G(S^0)$ is given as follows:

$$\pi_*^{W_H}(EW_{H+}) \xrightarrow{\cong} \pi_*(BW_H{}^{\mathrm{Ad}\ W_H}) \to \pi_*^G(BW_H{}^{\mathrm{Ad}\ W_H})$$
$$\xrightarrow{\tilde{\tau}} \pi_*^G(G_+ \wedge_{NH} BW_H{}^{\mathrm{Ad}\ W_H}) \xrightarrow{1 \wedge \tau} \pi_*^G(G_+ \wedge_{NH} EW_{H+}) \xrightarrow{p} \pi_*^G(S^0),$$

where $\tilde{\tau}$ is the pretransfer associated with $G/NH \to 1$ and τ is the dimension-shifting transfer associated with $W_H \to 1$ (cf. [9, Chapter IV]). The analogue of the previous proposition is the following

Proposition 1.12. *Let* K_1, $H \subset K_2$ *be inclusions of compact Lie groups. Consider the double coset space* $K_1 \backslash {}^{K_2}/N_{K_2}H$. *Let* $\{M_\alpha\}$ *be the set of all orbit type manifold components of the double coset space. Choose* $g_\alpha \in N$ *for each* α *which projects into* M_α. *Suppose we identify each summand of* $(S^0)^{K_1}$ *(resp.* $(S^0)^{K_2}$*) by its corresponding conjugacy class* (H). *Then under the map* $(S^0)^{K_2} \xrightarrow{\gamma} (S^0)^{K_1}$, (H) *maps to* $\{(H^{g_\alpha} \cap K_1)\}$. *Furthermore, each map is given by* $\chi^{\#}(M_\alpha)$ *times*

$$BW_H{}^{\mathrm{Ad}\ W_H} \xrightarrow{C_{g_\alpha}} BW_{H^{g_\alpha}}{}^{\mathrm{Ad}\ W_{H^{g_\alpha}}} \xrightarrow{\mathrm{tr}} BW_{H_1}{}^{\mathrm{Ad}\ W_{H_1}} \xrightarrow{Bi} BW_{H_2}{}^{\mathrm{Ad}\ W_{H_2}},$$

where $\chi^{\#}(M_\alpha) = \chi(\overline{M}_\alpha) - \chi(\overline{M}_\alpha - M_\alpha)$ *is the internal Euler characteristic,* C_{g_α} *is induced from conjugation,* $W_{H_1} = ((NH)^{g_\alpha} \cap K_1)/(H^{g_\alpha} \cap K_1)$, $W_{H_2} = (N_1(H^{g_\alpha} \cap K_1))/(H^{g_\alpha} \cap K_1)$, *and* $N_1(H^{g_\alpha} \cap K_1)$ *denotes the normalizer of* $H^{g_\alpha} \cap K_1$ *in* K_1.

Remark. We will omit the proof since it proceeds similarly to that of Proposition 1.11. Note the internal Euler characteristic comes in when one decomposes a certain K_2-space as a K_1-space. See Theorem 2.11 of Chapter IV in [9] for more details.

2. PROOF OF THEOREM 1.1

Consider the map $(E\mathscr{I}_{n+})^N_{\hat{p}} \xrightarrow{\gamma} \varprojlim_i (E\mathscr{I}_{n+})^{F_i}_{\hat{p}}$ induced from inclusions $F_i \hookrightarrow N$. Notice that we have

$$(E\mathscr{I}_{n+1})^N_{\hat{p}} = \bigvee_{\substack{(H) \\ H \cap T \subset I_n}} BW_H{}^{\mathrm{Ad}\, W_H}{}_{\hat{p}}.$$

Since $BN^{\mathrm{Ad}\, N}_{\hat{p}} \xrightarrow{\Phi} DBN_{+\hat{p}}$ is compatible with γ, it therefore follows from Proposition 1.10 that it suffices to show γ is an equivalence. We might add that it would be enough to assume $n = 0$ for proving Theorem 1.1. However, for later purposes, we will prove γ is an equivalence for all n.

In order to analyze $\varprojlim_i (E\mathscr{I}_{n+})^{F_i}_{\hat{p}}$, we use Lemma 1.7. Let $\mathscr{F}_1 \subset \mathscr{F}_2$ be adjacent families of N contained in $\mathscr{I}_n$, i.e., $\mathscr{F}_2 - \mathscr{F}_1 = (H)$. We have the following map of cofibrations:

$$
\begin{array}{ccccc}
(E\mathscr{F}_{1+})^N & \longrightarrow & (E\mathscr{F}_{2+})^N & \longrightarrow & (E(\mathscr{F}_2, \mathscr{F}_1)_+)^N \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\
(E\mathscr{F}_{1+})^{F_i} & \longrightarrow & (E\mathscr{F}_{2+})^{F_i} & \longrightarrow & (E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_i}
\end{array}
$$

where the K-fixed point of $E(\mathscr{F}_2, \mathscr{F}_1)$ is $*$ if $(K) \neq (H)$ in N, or S^0 if $(K) = (H)$ in N. By completing the above map of cofibration sequences at the prime p and taking the homotopy inverse limit, we have the following map of cofibration sequences:

$$
\begin{array}{ccccc}
(E\mathscr{F}_{1+})^N_{\hat{p}} & \longrightarrow & (E\mathscr{F}_{2+})^N_{\hat{p}} & \longrightarrow & (E(\mathscr{F}_2, \mathscr{F}_1)_+)^N_{\hat{p}} \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\
\varprojlim_i (E\mathscr{F}_{1+})^{F_i}_{\hat{p}} & \longrightarrow & \varprojlim_i (E\mathscr{F}_{2+})^{F_i}_{\hat{p}} & \longrightarrow & \varprojlim_i (E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_i}_{\hat{p}}
\end{array}
$$

Hence, it suffices to prove

Proposition 2.1. *Let* $\mathscr{F}_1 \subset \mathscr{F}_2$ *be adjacent families of* $\mathscr{I}_n$. *Then we have the equivalence*

$$(E(\mathscr{F}_2, \mathscr{F}_1)_+)^N_{\hat{p}} \xrightarrow[\gamma]{\simeq} \varprojlim_i (E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_i}_{\hat{p}}.$$

We need some preliminary results. Fix $(H) = \mathscr{F}_2 - \mathscr{F}_1$. The map

$$(E(\mathscr{F}_2, \mathscr{F}_1)_+)^N_{\hat{p}} \xrightarrow{\gamma} (E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_i}_{\hat{p}}$$

is the same as

$$(2.1.1) \qquad BW_H{}^{\mathrm{Ad}\ W_H}\widehat{}_p \to \bigvee_{\substack{(H') \\ H' \subset F_i \\ (H')=(H) \text{ in } N}} BN_i(H')/H'_+\widehat{}_p,$$

where $N_i(H')$ denote the normalizer of H' in F_i. Notice the map in (2.1.1) is described by Proposition 1.12.

By Corollary 1.9, we may assume $H \subset F_{i_0}$ for some i_0. We give an explicit description of the (H')'s in (2.1.1).

Lemma 2.2. *The NH orbit of $(F_i\backslash N)^H$ with action on the right is in one-to-one correspondence with $\{(H')|H' \subset F_i, (H') = (H) \text{ in } N\}$. The assignment is given by $\beta\colon \{\overline{g}\} \to (H^g)$, where $H \subset F_i$. In particular, the set $\{\overline{g}\}$ is finite.*

Proof. Let $\overline{g} \in F_i\backslash N$ be fixed under H. Therefore we have $\overline{g}H = \overline{g}$ or $gHg^{-1} \subset F_i$. Furthermore, let $k \in NH$ and $z \in F_i$; then $H^{zgk} = zgkHk^{-1}g^{-1}z^{-1} = zgHg^{-1}z^{-1} \subset zF_iz^{-1} = F_i$. Hence, β is well defined since $(H^{zgk}) = (H^g)$ in F_i.

Suppose $\overline{g}_1, \overline{g}_2$ are two representatives such that $(H^{g_1}) = (H^{g_2})$ in F_i. This implies $g_2^{-1}hg_1 \in NH$ for some $h \in F_i$. Hence, $\overline{g}_1 = \overline{g}_2(g_2^{-1}hg_1)$ or $\overline{g}_1 \sim \overline{g}_2$ in the NH orbit of $(F_i\backslash N)^H$. β is injective.

Clearly if $(H') = (H)$ in N, then $H' = H^g$ for some $g \in N$. Thus, $\beta(\overline{g}) = (H^g)$ with $\overline{g}$ fixed under H. So β is surjective and we are done. □

Hence to each $i \geq i_0$, we have a set of representatives $\{\overline{g}_\alpha^{(i)}\}$ in the NH orbit of $(F_i\backslash N)^H$ corresponding to summands of $(E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_i}\widehat{}_p$. From Proposition 1.11, we get

Lemma 2.3. *Let $j\colon (F_i\backslash N)^H/NH \to (F_{i+1}\backslash N)^H/NH$ be the induced map of $F_i \hookrightarrow F_{i+1}$. Suppose*

$$j^{-1}(\overline{g}_\alpha^{(i+1)})) = \{\overline{g}_{\alpha_1}^{(i)}, \overline{g}_{\alpha_2}^{(i)}, \ldots, \overline{g}_{\alpha_m}^{(i)}\}.$$

Then the summand $(H^{g_\alpha^{(i+1)}})$ in $(E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_{i+1}}$ maps to summands $(H^{g_{\alpha_1}^{(i)}})$, $(H^{g_{\alpha_2}^{(i)}}), \ldots, (H^{g_{\alpha_m}^{(i)}})$ in $(E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_i}$ under the inclusion $F_i \hookrightarrow F_{i+1}$.

Proof. Without loss of generality, we may assume $g_\alpha^{(i+1)} = g_{\alpha_1}^{(i)} = g$. Let $\{z_\gamma\}$ be the set of representatives of the double coset:

$$F_i\backslash {}^{F_{i+1}}/(NH)^g \cap F_{i+1}.$$

Proposition 1.11 shows that (H^g) maps to $\{(H^{z_\gamma g} \cap F_i)\}$ in $(S^0)^{F_{i+1}} \to (S^0)^{F_i}$. Hence, we only have to consider those z_γ such that $H^{z_\gamma g} \subset F_i$. Now $j(\overline{z}_\gamma g) = \overline{g}$ and hence $\overline{z}_\gamma g \in \{\overline{g}_{\alpha_1}^{(i)}, \ldots, \overline{g}_{\alpha_m}^{(i)}\}$. □

Proposition 2.4.

$$\varprojlim_i (E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_i}\widehat{}_p \xrightarrow{\cong} \varprojlim_{\substack{\gamma \\ \mathrm{tr}}} \left(\frac{NH \cap F_i}{H}\right)\widehat{}_{+p}.$$

Proof. From the previous lemma, it is obvious that only the summand corresponding to (H) in $(E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_{i+1}}$ will map to the summand corresponding to (H) in $(E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_i}$. By Proposition 1.11, $B((NH \cap F_{i+1})/H)_+$ maps

to $B((NH \cap F_i)/H)_+$ by transfer. Hence we have the following map of cofiber sequences:

$$
\begin{array}{ccccc}
\mathscr{L}_{i+1} & \longrightarrow & (E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_{i+1}}\widehat{_p} & \longrightarrow & B\left(\frac{NH \cap F_{i+1}}{H}\right)\widehat{_{+p}} \\
\downarrow & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{tr}} \\
\mathscr{L}_i & \longrightarrow & (E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_i}\widehat{_p} & \longrightarrow & B\left(\frac{NH \cap F_i}{H}\right)\widehat{_{+p}}
\end{array}
$$

where $\mathscr{L}_{i+1}$ and $\mathscr{L}_i$ are the fibers of right horizontal maps.

Now Lemma 2.3 gives us a description of γ in terms of the corresponding representatives $\{\overline{g}_\alpha^{(i)}\}$. If we can show for a fixed i that there exists $d(i)$ so that we can choose the $g_\alpha^{(i)}$'s belonging to $F_{d(i)}$ for all α, then we would be done. This is because $\mathscr{L}_{d(i)} \to \mathscr{L}_i$ would then be the trivial map. Upon taking the homotopy inverse limit this yields

$$
\varprojlim_{\mathrm{tr}} B\left(\frac{NH \cap F_i}{H}\right)\widehat{_{+p}} \xrightarrow{\simeq} \varprojlim_i (E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_i}\widehat{_p}.
$$

The claim follows from the following lemma. $\quad\square$

Lemma 2.5. *Let $\{\overline{g}_\alpha^{(i)}\}$ be representatives of $(F_i \backslash N)^H / NH$. There exists $d(i)$ such that $\{g_\alpha^{(i)}\}$ can be chosen so that $g_\alpha^{(i)} \in F_{d(i)}$ for all α.*

Proof. Since $\widetilde{F} \hookrightarrow F_i$, we can assume each $g_\alpha^{(i)} \in T$. For each $x \in H$, pick $\tilde{x} \in \widetilde{F}$ so that $\tilde{x}^{-1}x \in T$. Therefore $g_\alpha^{(i)} x g_\alpha^{(i)^{-1}} = \tilde{x}(\tilde{x}^{-1}x)(\tilde{x}^{-1}g_\alpha^{(i)}\tilde{x})g_\alpha^{(i)^{-1}} \in F_i$ since $\tilde{x}^{-1}g_\alpha^{(i)}\tilde{x}$, $\tilde{x}^{-1}x \in T$. But elements of $F_i \cap T$ have order dividing p^m for some fixed integer m. Therefore we have $(\tilde{x}^{-1}g_\alpha^{(i)}\tilde{x})^{p^m}(g_\alpha^{(i)})^{-p^m} = 1$. Or equivalently, $x^{-1}g_\alpha^{(i)p^m}x g_\alpha^{(i)-p^m} = 1$. Hence, $g_\alpha^{(i)p^m} \in NH \cap T$.

Next, we show $\{g_\alpha^{(i)}\}$ can be chosen to have finite order. Let $z = g_\alpha^{(i)}$. Then z generates an abelian group L. Since the closure $\overline{L}$ is a compact abelian Lie group with a cyclic generator, it therefore has the form of $J \times T^k$ with J cyclic. Similarly, z^{p^m} generates L' with $\overline{L}' = J' \times T^{k'}$. Since $y \in J \times T^k \Rightarrow y^{p^m} \in J' \times T^{k'}$, it is clear that $\overline{L}' = J' \times T^k$ with J' a cyclic subgroup of J.

The above analysis shows that there exists $z' \in \overline{L}'$ such that $(z')^{-1}z$ is of finite order. Furthermore, $z^{p^m} \in NH \Rightarrow \overline{L}' \subset NH$ and so $z' \in NH$. Therefore we may assume $g_\alpha^{(i)}$ to be of finite order.

So we suppose $(g_\alpha^{(i)})^M = 1$. Write $g_\alpha^{(i)} = (g_\alpha^{(i)})^{1-p^l}(g_\alpha^{(i)})^{p^l}$. Also $(g_\alpha^{(i)})^{p^l} \in NH$ if $l \geq m$. We can pick $l \geq m$ so that $(g_\alpha^{(i)})^{1-p^l}$ is of order a power of p. We have proved the lemma. $\quad\square$

By Proposition 2.4, it suffices to know the map between

$$
BW_H{}^{\mathrm{Ad}\ W_H}\widehat{_p} = (E(\mathscr{F}_2, \mathscr{F}_1)_+)^N\widehat{_p} \quad \text{and} \quad B\left(\frac{NH \cap F_i}{H}\right)\widehat{_{+p}}.
$$

From Proposition 1.12, this amounts to evaluating the internal Euler characteristics of the orbit type manifold component corresponding to 1 in $F_i \backslash^N / NH$.

Proposition 2.6. *Let $H \subset K_1 \subset K_2$ be an inclusion of compact Lie groups. Denote by NH the normalizer of H in K_2. Let $M_\circ$ be the orbit type manifold*

component corresponding to 1 *in* $K_1 \setminus {}^{K_2}/NH$. *Then* $M_\circ$ *is a point. In particular,* $\chi^\#(M_\circ) = 1$.

Proof. Suppose z has the same orbit type as 1. This would mean that $K_1^{z^{-1}} \cap NH$ is a conjugate of $K_1 \cap NH$ in NH. So $K_1^{\gamma z^{-1}} \cap NH = K_1 \cap NH$ for some $\gamma \in NH$. However, since $H \subseteq K_1 \cap NH$, this implies $H \subseteq K_1^{z^{-1}}$ or $H^z \subseteq K_1$. In other words, z actually belongs to the NH orbit of $(K_1 \setminus K_2)^H$. But it was shown in Bredon [3] that $(K_1 \setminus K_2)^H/NH$ is finite. Therefore we can conclude that z is not in the same orbit type manifold component as 1 unless $z = 1$. Therefore $M_\circ$ is a point. $\square$

The above discussion together with Proposition 1.12 implies that

$$(E(\mathscr{F}_2, \mathscr{F}_1)_+)^N \hat{}_p \to \varprojlim_i (E(\mathscr{F}_2, \mathscr{F}_1)_+)^{F_i} \hat{}_p$$

is equivalent to

$$BW_H{}^{\mathrm{Ad}\ W_H} \hat{}_p \xrightarrow{\ \mathrm{tr}\ } \varprojlim_{\mathrm{tr}} B\left(\frac{NH \cap F_i}{H}\right) \hat{}_{+p}.$$

In order to show tr is an equivalence, we need to understand how far off $\bigcup_i (NH \cap F_i)$ is from NH.

Lemma 2.7. $NH = (\bigcup_i (NH \cap F_i)) \cdot (NH \cap T)$ $(H \subset F_{i_\circ})$.

Proof. Let $z \in NH$. We can find $\tilde{z} \in \widetilde{F}$ such that $\tilde{z}^{-1} z \in T$. Since $zHz^{-1} = H$ and $\widetilde{F}, H \subset F_{i_\circ}$, an easy calculation shows that $(\tilde{z}^{-1} z)^{p^m} \in NH \cap T$ for some m depending only on $i_\circ$. Therefore, as in Lemma 2.5, we can write $\tilde{z}^{-1} z$ as $y_1 y_2$, where $y_1, y_2 \in T$, y_1 has order a power of p, and $y_2 \in NH \cap T$. So $z = \tilde{z}(\tilde{z}^{-1} z) = \tilde{z} y_1 y_2$, $z, y_2 \in NH \Rightarrow \tilde{z} y_1 \in NH$. But notice that $\tilde{z} y_1 \in F_k$ for some large k. We are done. $\square$

We now have the following map of extensions:

$$(2.8.1)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \dfrac{NH \cap T}{H \cap T} & \longrightarrow & \dfrac{NH}{H} & \longrightarrow & J & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \dfrac{NH \cap T \cap F_i}{H \cap T \cap F_i} & \longrightarrow & \dfrac{NH \cap F_i}{H \cap F_i} & \longrightarrow & J_i & \longrightarrow & 1 \end{array}$$

Of course J is finite since $NH/NH \cap T$ is finite.

Corollary 2.8. *For sufficiently large* i, $J_i \to J$ *is an isomorphism.*

Proof. For injectivity, suppose $x \in NH \cap F_i$ such that $xh = z \in NH \cap T$ for some $h \in H$. If $H \subset F_i$, then $z \in F_i$ and thus $z \in NH \cap T \cap F_i$.

For surjectivity, since J is finite, it suffices to show for each $z \in J$ that there exists $z' \in J_i$ which maps to z for some i. Let $\overline{y} \in NH/H$ map to z. By Lemma 2.7, we can write $y = y'h$, where $y' \in NH \cap F_i$ and for some i $h \in NH \cap T$. Then $\overline{y}' \in NH \cap F_i/H$ has the same image as $\overline{y} \in NH/H$ in J. $\square$

We are ready to prove

Proposition 2.9.

$$BW_H{}^{\text{Ad } W_H}\widehat{}_p \xrightarrow{\text{ tr }} \varprojlim_i B\left(\frac{NH \cap F_i}{H}\right)\widehat{}_{+p}$$

is an equivalence.

Remark. This will prove Proposition 2.1and hence Theorem 1.1.

Proof. Following Nishida [13], it is sufficient to check that

$$\varinjlim_{\text{tr}^*} \overline{H}^*\left(B\left(\frac{NH \cap F_i}{H}\right)_+, Z_p\right) \to \overline{H}^*(BW_H{}^{\text{Ad } W_H}, Z_p)$$

is an isomorphism.

Suppose we are in the following situation. Let

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

be an extension of groups. Consider the space $EG_3 \times_{G_3} (EG_2)/G_1$, where all actions are from the left and $G_3 \cong G_2/G_1$ acts on $(EG_2)/G_1$ via the action of G_2 on EG_2. Since EG_2 is G_2 free, $(EG_2)/G_1$ is G_2/G_1 free. This implies that $EG_3 \times_{G_3} (EG_2)/G_1$ is equivalent to $((EG_2)/G_1)/G_3 = (EG_2)/G_2$.

Suppose further that we have the following map of extensions:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 1 \\
& & i_1 \big\uparrow & & i_2 \big\uparrow & & \cong \big\uparrow & & \\
1 & \longrightarrow & G_1' & \longrightarrow & G_2' & \longrightarrow & G_3' & \longrightarrow & 1
\end{array}
$$

where i_1, i_2 are inclusions. We want to understand the transfer $BG_{2+} \xrightarrow{\tau} BG_{2+}'$. Recall we have a pretransfer $\tilde{\tau}\colon S^0 \to (G_2/G_2')_+$, where $\tilde{\tau}$ is a G_2 map. Taking the smash product with EG_{2+} and the G_2 orbit yields:

$$\tau\colon EG_2/G_{2+} \xrightarrow{1 \wedge_{G_2} \tilde{\tau}} EG_{2+} \wedge_{G_2} G_2/G_{2+}' \cong EG_2/G_{2+}'.$$

We claim that the following diagram commutes:

$$
\begin{array}{ccc}
EG_{3+} \wedge_{G_3} EG_2/G_{1+} & \xrightarrow{1 \wedge_{G_3} \tau'} & EG_{3+} \wedge_{G_3} EG_2/G_{1+}' \\
\text{proj} \Big\downarrow \cong & & \cong \Big\downarrow \text{proj} \\
EG_2/G_{2+} & \xrightarrow{\quad \tau \quad} & EG_2/G_{2+}'
\end{array}
$$

where τ' is the G_3 equivariant transfer associated to $G_1' \hookrightarrow G_1$.

Observe that $G_2' \cap G_1 = G_1'$ and hence $G_1/G_1' \cong G_2/G_2'$ as G_1 spaces. If $\tilde{\tau}'$ is the pretransfer associated to τ', then this observation implies that $\tilde{\tau}' = \tilde{\tau}$. Hence the diagram commutes.

Now, using $1 \wedge_{G_3} \tau'$, we can filter EG_{3+} by skeleta to get a map of spectral sequences with E_2 terms:

$$\overline{H}^*(BG_3, \overline{H}^*(BG_1', Z_p)) \to \overline{H}^*(BG_3, \overline{H}^*(BG_1, Z_p))$$

converging to

$$\overline{H}^*(BG_2', Z_p) \xrightarrow{\tau^*} \overline{H}^*(BG_2, Z_p).$$

Now, similarly, we have the following maps of spectral sequences when we use dimension shifting pretransfers

$$S^0 \to \Sigma^{-A} F_+ ,$$

where $F = G_2/G_2'$ and A is the adjoint representation of F derived from $\{e\}$.

Now rewrite (2.8.1) as

(2.9.1)
$$
\begin{array}{ccccccccc}
1 & \longrightarrow & T_H & \longrightarrow & W_H & \longrightarrow & J & \longrightarrow & 1 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & T_{H^{(i)}} & \longrightarrow & W_{H^{(i)}} & \longrightarrow & J_i & \longrightarrow & 1
\end{array}
$$

Applying the previous discussion to (2.9.1) yields a map of spectral sequences with E_2 terms:

$$\overline{H}^*(J_i, \overline{H}^*(T_{H^{(i)}}, Z_p)) \xrightarrow{\tau^*} \overline{H}^*(J, \overline{H}^*(B(T_H)^{\mathrm{Ad}(T_H)}, Z_p))$$

converging to

$$\overline{H}^*(W_{H^{(i)}}, Z_p) \xrightarrow{\tau^*} \overline{H}^*(BW_H^{\mathrm{Ad}\,W_H}, Z_p),$$

where $i \geq M$ for some large M so that $J \cong J_i$. Since the maps in the spectral sequence are compatible as we vary i, it would be sufficient to show that

$$\varinjlim_{\tau^*} \overline{H}^*(J_i, \overline{H}^*(T_{H^{(i)}}, Z_p)) \xrightarrow{\tau^*} \overline{H}^*(J, \overline{H}^*(B(T_H)^{\mathrm{Ad}(T_H)}, Z_p))$$

is an isomorphism. But

$$\varinjlim_{\tau^*} \overline{H}^*(J_i, \overline{H}^*(T_{H^{(i)}}, Z_p)) = \overline{H}^* \left(J_i, \varinjlim_{\tau^*} \overline{H}^*(T_{H^{(i)}}, Z_p) \right) .$$

So we are reduced to proving that

$$\varinjlim_{\tau^*} \overline{H}^*(T_{H^{(i)}}, Z_p) \xrightarrow{\tau^*} (B(T_H)^{\mathrm{Ad}(T_H)}, Z_p))$$

is an isomorphism. Notice that T_H is a compact abelian group. Hence, $T_H = C \times T^k$ for some finite abelian group C and a torus of rank k, T^k. But $T_{H^{(i)}} = C^{(p)} \times (T_{Q_p}^k \cap I_{n(i)})$ for sufficiently large i, where $C^{(p)}$ is the unique p-Sylow subgroup of C and $n(i)$ is an increasing function of i. Since we have cohomology in the Z_p coefficient, it suffices to verify that

$$\varinjlim_{\tau^*} \overline{H}^*(T_{Q_p}^k \cap I_{n(i)}, Z_p) \xrightarrow{\tau^*} \overline{H}^*(BT^{k\,\mathrm{Ad}\,T^k}, Z_p)$$

is an isomorphism. Since it is well known that (cf. [13, Proposition 3.2])

$$\varinjlim_{r} \overline{H}^*(BZ_{p^r}, Z_p) \xrightarrow{\tau^*} \overline{H}^*(\Sigma BS^1, Z_p)$$

is an isomorphism, we are done. $\square$

3. Some generalizations

In this section, we begin by proving an easy splitting result generalizing Theorem 1.1 to other conjugacy classes of p-subgroups of G. Afterward, we shall consider those conjugacy classes of closed infinite subgroups and verify the injection result of Theorem 3.12.

Our first aim is to prove

Theorem 3.1. *Let G be a compact Lie group and p be a fixed prime. Suppose H is a finite p-subgroup of G; then $BW_H^{\mathrm{Ad}\, W_H}{}_{\hat{p}}$ is a summand of $DBG_{+\hat{p}}$ under Φ. Furthermore, the finite wedge sum of any such spectra splits from $DBG_{+\hat{p}}$.*

We first give some preliminary results. As before, we continue to assume N is an extension of a torus T by a p-group unless stated otherwise.

Let $\mathscr{R}_i$ denote the wedge of those summands of $(S^0)^{F_i}{}_{\hat{p}}$ whose corresponding subgroup (H) is such that $H \cap T \not\subset F_{i-1} \cap T$. Denote by $\mathscr{S}_i$ the wedge of those summands in $(S^0)^{F_i}{}_{\hat{p}}$ that does not belong to $\mathscr{R}_i$.

Lemma 3.2. *There is a cofiber sequence*

$$\varprojlim_i \mathscr{R}_i \to \varprojlim_i (S^0)^{F_i}{}_{\hat{p}} \to \varprojlim_i \mathscr{S}_i.$$

Proof. Recall we have $I_n = \{x \in T \,|\, x^{p^n} = 1\}$. Since I_n is normal in N, summands of $\mathscr{R}_i$ will map to summands of $\mathscr{R}_{i-1}$ under the map $(S^0)^{F_i} \to (S^0)^{F_{i-1}}$. This gives rise to the following diagram:

$$
\begin{array}{ccccc}
\mathscr{R}_i & \longrightarrow & (S^0)^{F_i}{}_{\hat{p}} & \longrightarrow & \mathscr{S}_i \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{R}_{i-1} & \longrightarrow & (S^0)^{F_{i-1}}{}_{\hat{p}} & \longrightarrow & \mathscr{S}_{i-1}
\end{array}
$$

Here the right vertical map will collapse to the base point of $\mathscr{S}_{i-1}$ those summands of $\mathscr{S}_i$ whose corresponding subgroup (H) is such that $H \cap T \not\subset F_{i-2} \cap T$. Taking the homotopy inverse limit of the diagram will give us the desired cofiber sequence. $\square$

Corollary 3.3.

$$\varprojlim_i \mathscr{S}_i = \prod_{\substack{(H) \subset N \\ H \text{ a finite } p\text{-group}}} BW_H^{\mathrm{Ad}\, W_H}{}_{\hat{p}}.$$

Remark. $\prod$ denotes the product of spectra (cf. [1, Part III]).

Proof. We have the family $\mathscr{I}_n = \{H \,|\, H \cap T \subset I_n\}$. Notice that given $(H) \in \mathscr{I}_n - \mathscr{I}_{n-1}$, then $(H^\alpha \cap F_m) \in \mathscr{I}_n - \mathscr{I}_{n-1}$ for all m sufficiently large. It follows from Lemma 3.2 that

$$\varprojlim_i \mathscr{S}_i = \prod_{n=0}^{\infty} \varprojlim_i \left(E(\mathscr{I}_n, \mathscr{I}_{n-1})_+\right)^{F_i}{}_{\hat{p}}.$$

This, combined with Proposition 2.1, gives the desired result. $\square$

Corollary 3.4. *Let $\mathscr{C}$ be a finite collection of conjugacy classes of p-subgroups in N. Then the following map splits:*

$$\Phi: \bigvee_{(H) \in \mathscr{C}} BW_H{}^{\text{Ad } W_H}\hat{}_p \to \varprojlim_i (S^0)^{F_i}\hat{}_p \cong DBN_+\hat{}_p.$$

Proof. Let $\mathscr{F}$ be the family of all conjugacy classes of finite p-subgroups of N. Consider the map

$$(E\mathscr{F}_+)^N\hat{}_p \xrightarrow{\gamma} \varprojlim_i (S^0)^{F_i}\hat{}_p \to \varprojlim_i \mathscr{S}_i.$$

Corollary 3.3 says this is the inclusion of wedges of spectra into products of spectra. Since $\mathscr{C}$ is finite, we therefore have the desired result. □

Proof of Theorem 3.1. Let W_p be a p-Sylow subgroup of the Weyl group $W = N/T$ and let N_p be its preimage in N. Suppose H is any closed subgroup of G with $\pi_0(H)$ a p-group. From Bredon [3], we have $\chi((N_p\backslash G)^H) \equiv \chi(N_p\backslash G) \bmod p$. Since $\chi(N_p\backslash G) \not\equiv 0 \bmod p$, this implies that $(N_p\backslash G)^H$ is not empty. In particular, H is conjugate to a subgroup of N_p. Furthermore, he showed that the NH orbit of $(N_p\backslash G)^H$ is finite. Let $\{g_i\}$ be a set of representatives. We then have

$$(N_p\backslash G)^H = \coprod_i NH \cap N_p^{g_i^{-1}}\backslash NH.$$

Since $\chi((N_p\backslash G)) \not\equiv 0 \bmod p$, this says $\chi(NH \cap N_p^{g_i^{-1}}\backslash NH) \not\equiv 0 \bmod p$ for some g_i. In other words, $\chi(NH^{g_i} \cap N_p\backslash NH^{g_i}) \not\equiv 0 \bmod p$.

Consider the following commutative diagram:

$$
\begin{array}{ccc}
(S^0)^G\hat{}_p & \xrightarrow{\ \Phi\ } & DBG_+\hat{}_p \\
\downarrow{\scriptstyle\gamma} & & \downarrow \\
(S^0)^{N_p}\hat{}_p & \xrightarrow{\ \Phi\ } & DBN_{p+}\hat{}_p
\end{array}
$$

Let K be a p-subgroup of N_p. Recall that, under γ, the summand corresponding to (K) maps to a finite number of summands corresponding to $\{(K^{g_\alpha^{(i)}}\cap N_p)\}$ which are also p-subgroups of N_p. By Corollary 3.4, it suffices to check γ is a splitting map.

Now $K^{g_\alpha^{(i)}} \cap N_p$ is not a conjugate of K in N_p unless $\bar{g}_\alpha^{(i)} \sim \bar{1} \in L = N_p\backslash G/NK$. By Propositions 1.12 and 2.6, the map between the summands (K) of $(S^0)^G\hat{}_p$ and (K) of $(S^0)^{N_p}\hat{}_p$ is the bundle transfer:

$$BW_K{}^{\text{Ad } W_K}\hat{}_p \xrightarrow{\text{tr}} BW_{K^N}{}^{\text{Ad } W_{K^N}}\hat{}_p,$$

where $W_{K^N} = (NK\cap N_p)/K$. Therefore we see that $BW_K{}^{\text{Ad } W_K}\hat{}_p$ is a summand of $DBG_+\hat{}_p$ if tr has a retraction.

From Lemma 1.2 and the previous calculation of the Euler characteristics, this holds in particular for $K = H^{g_i}$.

Now let $\mathscr{C}$ be any finite collection of conjugacy classes of p-subgroups in G. Without loss of generality, we may assume each representation H of its conjugacy class is chosen so that

$$H \subset N_p \quad \text{and} \quad \chi(NH \cap N_p\backslash NH) \not\equiv 0 \bmod p.$$

Let $\mathscr{C}'$ be a collection of such representatives.

We want to show that

$$\bigvee_{H \in \mathscr{C}'} BW_H{}^{\mathrm{Ad}\ W_H}\hat{}_p \xrightarrow{\gamma} \bigvee_{\substack{(K) \subset N_p \\ K \text{ a } p\text{-group}}} BW_{K^N}{}^{\mathrm{Ad}\ W_{K^N}}\hat{}_p$$

is a splitting map. This would prove the second part of the proposition. For convenience, we may suppose that if $(K) = (H)$ in N_p for $H \in \mathscr{C}'$, then in fact $K = H$.

For each $H \in \mathscr{C}'$, consider the retraction of tr given specifically in Lemma 1.2, denoted by ret_H. Let Θ be the map

$$\bigvee_{\substack{(K) \subset N_p \\ K \text{ a } p\text{-group}}} BW_{K^N}{}^{\mathrm{Ad}\ W_{K^N}}\hat{}_p \to \bigvee_{(H) \in \mathscr{C}'} BW_H{}^{\mathrm{Ad}\ W_H}\hat{}_p$$

given by ret_K if $K \in \mathscr{C}'$ and trivial otherwise. We claim that

$$(3.1.1) \qquad \bigvee_{H \in \mathscr{C}'} BW_H{}^{\mathrm{Ad}\ W_H}\hat{}_p \xrightarrow{\gamma} \bigvee_{\substack{(K) \subset N_p \\ K \text{ a } p\text{-group}}} BW_{K^N}{}^{\mathrm{Ad}\ W_{K^N}}\hat{}_p \xrightarrow{\Theta} \bigvee_{H \in \mathscr{C}'} BW_H{}^{\mathrm{Ad}\ W_H}\hat{}_p$$

is an equivalence. This would then complete the proof.

Define $\mathscr{C}_i' = \{H \in \mathscr{C}' \mid |H| \leq i\}$. Observe that since $|H^{g_\alpha^{(i)}} \cap N_p| \leq |H|$ for all $g_\alpha^{(i)} \in L$, we can filter (3.1.1) so that it suffices to prove the equivalence

$$(3.1.2) \qquad \bigvee_{H \in \mathscr{C}_i' - \mathscr{C}_{i-1}'} BW_H{}^{\mathrm{Ad}\ W_H}\hat{}_p \xrightarrow{\gamma} \bigvee_{\substack{(K) \subset N_p \\ K \text{ a } p\text{-group} \\ |K| = i}} BW_{K^N}{}^{\mathrm{Ad}\ W_{K^N}}\hat{}_p$$

$$\xrightarrow{\Theta} \bigvee_{H \in \mathscr{C}_i' - \mathscr{C}_{i-1}'} BW_H{}^{\mathrm{Ad}\ W_H}\hat{}_p .$$

Since $|H^{g_\alpha^{(i)}} \cap N_p| = |H|$ implies $H^{g_\alpha^{(i)}} \cap N_p$ is a conjugate of H in G, the map Θ will map trivially on $(H^{g_\alpha^{(i)}} \cap N_p)$ unless $\bar{g}_\alpha^{(i)} \sim \bar{1} \in L$. Therefore (3.1.2) is a wedge sum of equivalences. $\square$

Corollary 3.5. *The following map is an injection*:

$$\pi_*(\Phi): \bigoplus_{\substack{(H) \\ H \subset G \text{ a } p\text{-group}}} \pi_*(BW_H{}^{\mathrm{Ad}\ W_H}\hat{}_p) \to \pi_*(DBG_+\hat{}_p) .$$

We want to extend Corollary 3.5 to conjugacy classes of closed infinite subgroups of G. Let $\mathscr{B}_2$ be the family of all conjugacy classes of closed subgroups of N and $\mathscr{B}_1$ be the family of those conjugacy classes in N which are finite. Observe $E\mathscr{B}_{2+}$ is N-equivariantly homotopic equivalent to S^0.

Lemma 3.6. *There is a map of cofiber sequences*:

$$
\begin{array}{ccccc}
(E(\mathscr{B}_2, \mathscr{B}_1)_+)^N\hat{}_p & \longrightarrow & (E\mathscr{B}_{2+})^N\hat{}_p & \longrightarrow & (E\mathscr{B}_{1+})^N\hat{}_p \\
\downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma_1} \\
\varprojlim_i \mathscr{R}_i & \longrightarrow & \varprojlim_i (S^0)^{F_i}\hat{}_p & \longrightarrow & \varprojlim_i \mathscr{S}_i
\end{array}
$$

where γ_1 and γ_2 are induced from γ.

Proof. We note that given any closed infinite subgroup H of N, $H^\alpha \cap T = (H \cap T)^\alpha$ has an infinite number of elements. By Corollary 3.3 and Proposition 1.12, this implies that the summand in $(E\mathscr{B}_{2+})^N{}_{\hat{p}}$ corresponding to (H) will project trivially into $\varprojlim_i \mathscr{S}_i$ under γ. $\square$

Of course, it is obvious that γ_1 does not induce an injection on π_* due to the fact that there are non-p-groups in $\mathscr{B}_1$. To correct this, let $\mathscr{P}$ be the set $\{(H) \subset N | H \; p\text{-toral}\}$, and let $\mathscr{P}_1 = \mathscr{P} \cap \mathscr{B}_1$ and $\mathscr{P}_2 = \mathscr{P} \cap (\mathscr{B}_2 - \mathscr{B}_1)$. We then have the following map of cofiber sequences:

(3.6.1)

$$
\begin{array}{ccccc}
\bigvee_{(H)\in\mathscr{P}_2} BW_H^{\mathrm{Ad}\; W_H}{}_{\hat{p}} & \longrightarrow & \bigvee_{(H)\in\mathscr{P}} BW_H^{\mathrm{Ad}\; W_H}{}_{\hat{p}} & \longrightarrow & \bigvee_{(H)\in\mathscr{P}_1} BW_H^{\mathrm{Ad}\; W_H}{}_{\hat{p}} \\
\downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_1} \\
(E(\mathscr{B}_2,\mathscr{B}_1)_+)^N{}_{\hat{p}} & \longrightarrow & (E\mathscr{B}_{2+})^N{}_{\hat{p}} & \longrightarrow & (E\mathscr{B}_{1+})^N{}_{\hat{p}}
\end{array}
$$

where the i's are obvious inclusions. Notice that the map $\gamma_1 \circ i_1$ is an inclusion of wedges of spectra into products of spectra.

Suppose we are given a system of p-completed spectra $\{\mathscr{Z}_{i\hat{p}}\}$. Recall from [2] that we have the following exact sequence of groups:

$$
1 \to \varprojlim{}^1 \pi_{i+1}(\mathscr{Z}_{i\hat{p}}) \to \pi_i(\varprojlim \mathscr{Z}_{i\hat{p}}) \to \varprojlim \pi_i(\mathscr{Z}_{i\hat{p}}) \to 1.
$$

If each $\mathscr{Z}_i$ is of finite type, then $\varprojlim{}^1 \pi_{i+1}(\mathscr{Z}_{i\hat{p}})$ would be trivial. Applying this to our situation, it follows that we have the diagram:

$$
\begin{array}{ccccccccc}
1\to & \bigoplus_{(H)\in\mathscr{P}_2} \pi_*(BW_H^{\mathrm{Ad}\; W_H}{}_{\hat{p}}) & \to & \bigoplus_{(H)\in\mathscr{P}} \pi_*(BW_H^{\mathrm{Ad}\; W_H}{}_{\hat{p}}) & \to & \bigoplus_{(H)\in\mathscr{P}_1} \pi_*(BW_H^{\mathrm{Ad}\; W_H}{}_{\hat{p}}) & \to 1 \\
& \downarrow{\scriptstyle \gamma_2 \circ i_2} & & \downarrow{\scriptstyle \gamma \circ i} & & \downarrow{\scriptstyle \gamma_i \circ i_1} & \\
1\to & \varprojlim_i \pi_*(\mathscr{R}_i) & \to & \varprojlim_i \pi_*((S^0)^{F_i}{}_{\hat{p}}) & \to & \varprojlim_i \pi_*(\mathscr{S}_i) & \to 1
\end{array}
$$

since $\varprojlim_i{}^1 \pi_*(\mathscr{R}_i)$ is trivial. Therefore in order to show $\pi_*(\gamma \circ i)$ is injective, it suffices to show $\pi_*(\gamma_2 \circ i_2)$ is injective by Corollary 3.5.

To accomplish that, we need to break up $\varprojlim_i \mathscr{R}_i$ into pieces more amenable to calculations. This motivates the following

Definition. Recall we have a nested sequence $\{F_i\}$ of finite subgroups of N constructed in Corollary 1.5. Let H be a subgroup of F_i. We define the *type* of H to be $|H/H \cap T|$ and the *form* of H to be $|H \cap T/H \cap F_{i-1} \cap T|$ which are denoted $t_i(H)$ and $f_i(H)$, respectively. Note that the definition of type and form is well defined on each conjugacy class in F_i. For a general $H \subset N$, we can also define its type and form by $t(H) = |H/H \cap T|$ and $f(H) = \varinjlim_i |H \cap F_i \cap T/H \cap F_{i-1} \cap T|$.

Remark. We observe that given $H \subset N$, $t_{i-1}(H \cap F_{i-1}) \le t_i(H \cap F_i)$. Similarly, $f_i(H \cap F_i) \le f_{i-1}(H \cap F_{i-1})$ via the power map $x \to x^p$.

Lemma 3.7. *There exists a finite number of spectra $\{\mathscr{U}_j\}$ and a sequence of maps $\{\lambda_j\}$,*

$$
* \xrightarrow{\lambda_k} \mathscr{U}_k \xrightarrow{\lambda_{k-1}} \mathscr{U}_{k-1} \xrightarrow{\lambda_{k-2}} \cdots \xrightarrow{\lambda_0} \mathscr{U}_0 = \varprojlim_i \mathscr{R}_i,
$$

such that the cofiber of each λ_j is a homotopy inverse limit of a system consisting of all the summands having the same type and form.

Proof. It is clear that there are only a finite number of distinct pairs (type of H, form of H). By the above remark and the fact that $H \subset F_i$ will go to $\{(H^\alpha \cap F_{i-1})\}$ between consecutive maps in the inverse system, we can filter the system $\{\mathscr{R}_i\}$ so that each successive quotient will be a system having all summands of the same type and form. □

Corollary 3.8. *Recall $\mathscr{P}$ is the set $\{(H) \subset N \,|\, \pi_0(H)$ is a p-group$\}$. Let $\mathscr{P}_{v,\nu}^\infty = \{(H) \in \mathscr{P} \,|\, t(H) = v,\, f(H) = \nu\}$. Then we can filter $\bigvee_{(H) \in \mathscr{P}_{v,\nu}^\infty} BW_H^{\mathrm{Ad}\ W_H}\hat{}_p$ in a compatible way with the $\mathscr{U}_k$'s of Lemma 3.7 so that the map between corresponding cofibers is given by*

$$\bigvee_{(H) \in \mathscr{P}_{v,\nu}^\infty} BW_H^{\mathrm{Ad}\ W_H}\hat{}_p \xrightarrow{\gamma_{v,\nu}} \varprojlim_i \bigvee_{\substack{(H),\, H \subseteq F_i \\ t_i(H) = v \\ f_i(H) = \nu}} BW_H^{(i)}\hat{}_p = \varprojlim_i \mathscr{D}_{v,\nu}^{(i)}.$$

We are reduced to studying the map $\gamma_{v,\nu}$. Let Δ be the inverse system $\{\mathscr{D}_{v,\nu}^{(i)}\}$. We first want to study the structure of Δ. Let $\mathscr{G}_i$ be a set of representatives of conjugacy classes of subgroups of F_i with type v and form ν such that given any $H' \subset F_i$ with same type and form, there exist $(H) \in \mathscr{G}_i$ and $\alpha \in W$ so that $H' \cap T = H^\alpha \cap T$. Similarly, we define $\mathscr{G}$ as a set of representatives of conjugacy classes of closed subgroups in N satisfying the same condition. Since T is normal in N, $\mathscr{G}_i$ and $\mathscr{G}$ are well defined.

Rewrite $\mathscr{D}_{v,\nu}^{(i)}$ as $\bigvee_{(H) \in \mathscr{G}_i} \mathscr{D}_H^{(i)}$, where $\mathscr{D}_H^{(i)}$ denotes the wedge of all summands in $\mathscr{D}_{v,\nu}^{(i)}$ whose corresponding representative in $\mathscr{G}_i$ is (H). It follows from Proposition 1.12 that $\mathscr{D}_H^{(i)}$ will either map to $\mathscr{D}_{H^\alpha \cap F_{i-1}}^{(i-1)}$ in Δ for some α or a map trivially otherwise. Geometrically then, the structure of Δ is a directed tree $\mathscr{T}$ with each node being represented by some $\mathscr{D}_H^{(i)}$.

Lemma 3.9. *Let (H) be a conjugacy class in $\mathscr{P}_{v,\nu}^\infty$. Then under $\gamma_{v,\nu}$, the summand corresponding to (H) will map to a nontrivial directed path in $\mathscr{T}$. Furthermore, distinct conjugacy classes (H) and (H') with the same type and form will map into the same directed path in $\mathscr{T}$ under $\gamma_{v,\nu}$ if and only if $H \cap T$ and $H' \cap T$ are conjugates in N.*

Proof. Since T is normal in N and the summand corresponding to (H) will map to summands corresponding to $\{(H^\alpha \cap F_i)\}$ under $\gamma_{v,\nu}$, we only need to show that given a conjugacy class (H) in $\mathscr{P}_{v,\nu}^\infty$, we can pick a representative $H_\circ$ so that

$$(3.9.1) \qquad t(H_\circ) = t_i(H_\circ \cap F_i) \quad \text{and} \quad f(H_\circ) = f_i(H_\circ \cap F_i)$$

for all large i. Clearly, by definition, any representative will suffice for the form of H. As for the type of H, following a proof similar to the proof of Proposition 1.4, we can find a subgroup H_{Q_p} of H such that $H_{Q_p} \cap T$ consists of all elements in $H \cap T$ of order a power of p and $H_{Q_p}/H_{Q_p} \cap T \cong H/H \cap T$. By Corollary 1.9, H_{Q_p} is conjugate to a subgroup of N_{Q_p}. Let $\beta \in W$ be such that $H_{Q_p}^\beta \subset N_{Q_p}$. Since H is the closure of H_{Q_p} in N, this implies that $t(H^\beta) = t_i(H^\beta \cap F_i)$ for all large i.

It remains to prove that given $(H), (H') \in \mathscr{P}_{v,\nu}^{\infty}$ such that $H \cap F_j \cap T$ and $H' \cap F_j \cap T$ are conjugates for all large j, then $H \cap T$ and $H' \cap T$ are in fact conjugates. We may assume H and H' satisfy (3.9.1). Since N/T is finite, there exists $\sigma \in W = N/T$ such that $H \cap F_j \cap T = H'^{\sigma} \cap F_j \cap T$ for infinitely many j's. Consider the unions

$$\bigcup H \cap F_j \cap T, \qquad \bigcup H'^{\sigma} \cap F_j \cap T.$$

Since $\pi_0(H)$ and $\pi_0(H')$ are p-groups, the closure of these unions will give $H \cap T$ and $H'^{\sigma} \cap T$, respectively. Hence we have the equality $H \cap T = H'^{\sigma} \cap T$. $\square$

Remark. From now on, we will assume the representative of each conjugacy class in $\mathscr{P}_{v,\nu}^{\infty}$ is chosen to satisfy (3.9.1).

Lemma 3.10. *For each* $(H) \in \mathscr{P}_{v,\nu}^{\infty}$, *let* J_H *be the induced directed path in* $\mathscr{T}$ *given by the representative* $(H \cap F_i) \in \mathscr{G}_i$ *at each level* i. *Then the canonical map*

$$\bigvee_{\substack{(H) \in \mathscr{G} \\ J_H}} \varprojlim_{J_H} \mathscr{D}_{H \cap F_i}^{(i)} \to \varprojlim_{i} \mathscr{D}_{v,\nu}^{(i)}$$

induces an injection on π_*.

Proof. Given $(H) \in \mathscr{G}$, we claim that there exists an integer r such that $H' \cap F_i \cap T$ is not a conjugate of $H \cap F_i \cap T$ for all $i \geq r$ if $(H') \neq (H) \in \mathscr{G}$. If not, then it is immediate that we may assume $H \cap F_i \cap T = H' \cap F_i \cap T$. Let $(H \cap T)_\circ$ and $(H' \cap T)_\circ$ be the components of 0 in $H \cap T$ and $H' \cap T$, respectively. Since form $= \nu$, we can write $H \cap T = C_\circ \times T^\nu$ and $H' \cap T = C'_\circ \times (T')^\nu$, where the C's are cyclic groups.

Suppose $C_\circ \subset I_n$. We may choose n so that $|C_\circ| \leq p^n$. Pick r such that $F_r \cap T \supset I_{n+1}$. This r will do the job. We have two cases: $T^\nu = T'^\nu$ or $T^\nu \neq T'^\nu$. In the first case, the equation $|H \cap F_i \cap T| = |H' \cap F_i \cap T|$ implies that $C'_\circ \cap F_i$ contains only elements of order less than p^n. Hence, $C'_\circ \subset I_n$ and $H \cap T = H' \cap T$. In the second case, since $T^\nu \neq T'^\nu$, $|C'_\circ \cap F_i| \geq p^m$, where $I_m = F_i \cap T$. Hence,

$$|H' \cap F_i \cap T| \geq (p^{m\nu})(p^m) \geq (p^{m\nu})(p^{n+1}) \gneq (p^{m\nu})|C_\circ| = |H \cap F_i \cap T|.$$

This implies the directed path associated to each $(H') \neq (H) \in \mathscr{G}$ will not intersect J_H from the level r onward. Since $\mathscr{G}$ is countable, we can order $\mathscr{G}$ as $\{(H_1), (H_2), \ldots\}$. Choose integer r_i for each (H_i) as above. Without loss of generality, we may assume $r_i \geq r_j$ if $i \geq j$. For each i, define $\mathscr{H}_i = \{(H) \subset F_i \mid H \cap F_{r_j} \sim H_j \cap F_{r_j} \in \mathscr{G}_{r_j}, \text{ for some } j \text{ such that } i \geq r_j\}$. Let

$$\mathscr{D}_{v,\nu}^{(i)'} = \begin{cases} \bigvee_{(H) \in \mathscr{H}_i} BW_{H+\widehat{p}} & \text{if } \mathscr{H}_i \text{ is nonempty}, \\ * & \text{otherwise}. \end{cases}$$

Notice that $\mathscr{D}_{v,\nu}^{(i)'}$ consists of those summands in $\mathscr{T}$ which go to $\mathscr{D}_{H_j \cap F_{r_j}}^{(r_j)}$ for some j such that $i \geq r_j$. Form the inverse system $\{\mathscr{D}_{v,\nu}^{(i)'}\}$ where the maps are induced from those of Δ. This gives a canonical map of systems $\{\mathscr{D}_{v,\nu}^{(i)}\} \to \{\mathscr{D}_{v,\nu}^{(i)'}\}$. It is clear that $\varprojlim_{i} \mathscr{D}_{v,\nu}^{(i)'}$ is a product of homotopy inverse limits,

and the composite

$$\bigvee_{\substack{(H)\in\mathscr{G} \\ J_H}} \varprojlim_{J_H} \mathscr{D}_{H\cap F_i}^{(i)} \to \varprojlim_{i} \mathscr{D}_{v,\nu}^{(i)} \to \varprojlim_{i} \mathscr{D}_{v,\nu}^{(i)'}$$

maps distinct wedge summands of the source into distinct product factors of the target. However, upon taking π_*, each wedge summand maps injectively into its target since an inverse limit of inclusions of groups is an inclusion. Therefore this gives the desired result. $\square$

Proposition 3.11. *Fix a* $(H) \in \mathscr{P}_{v,\nu}^{\infty}$. *The following map induces an injection on* π_*:

$$\gamma_{v,\nu}^{(H)}: \bigvee_{\substack{(H') \\ (H')\sim(H)\in\mathscr{G}}} BW_{H'}^{\mathrm{Ad}\ W_{H'}}\widehat{_p} \to \varprojlim_{J_H} \mathscr{D}_{H\cap F_i}^{(i)}.$$

Here $\gamma_{v,\nu}^{(H)}$ *is the restriction of* $\gamma_{v,\nu}$.

Proof. We first prove the proposition for the summand corresponding to (H). For each i, define

$$\mathscr{Z}_i = \{(K) \subset F_i | (K) \sim (H \cap F_i) \in \mathscr{G}_i \text{ and } (K)$$
$$\text{is a conjugate of } (H^t \cap F_i) \text{ for some } t \in T\}.$$

Consider the cofiber sequence

$$\bigvee_{(K)\in\mathscr{Z}_i} BW_{K+}^{(i)}\widehat{_p} \to \mathscr{D}_{H\cap F_i}^{(i)} \to \mathscr{F}_i,$$

where $\mathscr{F}_i$ is the cofiber of the left arrow. Notice that under $F_i \hookrightarrow F_{i+1}$, the $\mathscr{Z}_i$'s are compatible by Proposition 1.11. Since $\gamma_{v,\nu}^{(H)}$ restricted to $BW_H^{\mathrm{Ad}\ W_H}\widehat{_p}$ actually factors through the homotopy inverse limit $\varprojlim_i \bigvee_{(K)\in\mathscr{Z}_i} BW_{K+}^{(i)}\widehat{_p}$, it suffices to check that

$$\gamma_{v,\nu}^{(H)}: BW_H^{\mathrm{Ad}\ W_H}\widehat{_p} \to \varprojlim_i \bigvee_{(K)\in\mathscr{Z}_i} BW_{K+}^{(i)}\widehat{_p}$$

induces an injection on π_*.

Consider the double coset $F_i\backslash^N/NH$. Let $L_i = \{g \in F_i\backslash^N/NH | t_i(H^g \cap F_i) = v, \ f_i(H^g \cap F_i) = \nu\}$. We define an equivalence relation on L_i by $g_1 \sim g_2$ if $H^{g_1} \cap F_i$ is a conjugate of $H^{g_2} \cap F_i$ in F_i. It is easy to check that this equivalence relation is well defined. By definition, each $(K) \in \mathscr{Z}_i$ gives rise to an equivalence class in L_i.

We claim that for all sufficiently large i, the equivalence relation induces a well-defined map $j: L_i/\sim \to L_{i+1}/\sim$ coming from $F_i \hookrightarrow F_{i+1}$. That is if $g_1 \sim g_2 \in L_i$, then $j(g_1) \sim j(g_2) \in L_{i+1}$. We may assume $g_1, g_2 \in T$. Since g_1, g_2 are chosen so that $t(H) = t_i(H^{g_1} \cap F_i) = t_i(H^{g_2} \cap F_i)$, we have

$$H^{g_k} = H^{g_k} \cap F_i \cdot H^{g_k} \cap T = H^{g_k} \cap F_i \cdot H \cap T, \qquad k = 1, 2.$$

Since $H \cap F_{Q_p} \cap T$ is dense in $H \cap T$ and N/T is finite, we can find some sufficiently large $m_\circ$ such that if $\sigma \in N$ normalizes $H \cap F_{m_\circ} \cap T$, then σ also normalizes $H \cap T$. So pick $i \geq m_\circ$. If $(H^{g_1} \cap F_i)^\sigma = H^{g_2} \cap F_i$ for some $\sigma \in F_i$, then this implies that $(H^{g_1} \cap F_i)^\sigma \cap T = H^{g_2} \cap F_i \cap T$. But $g_1, g_2 \in T$ and thus

σ normalizes $H \cap T$. Hence, we have $H^{\sigma g_1} = H^{g_2}$. Consequently, $H^{g_1} \cap F_{i+1}$ is a conjugate of $H^{g_2} \cap F_{i+1}$ in F_{i+1}.

Now the analogue in Lemma 2.3 shows that the summand corresponding to $(H^g \cap F_{i+1}) \in \mathscr{Z}_{i+1}$ will map to summands corresponding to $\{(H^{(g_k)} \cap F_i) \in \mathscr{Z}_i\}$ for all sufficiently large i, where $j^{-1}(g) = \{g_1, g_2, \dots\}$. Similarly, using the fact that the types of these groups are all the same, we have the analogue of Lemma 2.5. In summary, we see that it suffices to verify that

$$BW_H{}^{\mathrm{Ad}\ W_H}\widehat{{}_p} \to \varprojlim_i B\left(\frac{N(H \cap F_i) \cap F_i}{H \cap F_i}\right)^{\widehat{}}_{+p}$$

is an equivalence.

A priori, by Proposition 1.12, the maps in the homotopy inverse limit above are compositions of transfers and natural inclusions, i.e.,

$$B\frac{N(H \cap F_{i+1}) \cap F_{i+1}}{H \cap F_{i+1}}{}^{\widehat{}}_{+p} \xrightarrow{\mathrm{tr}} B\frac{N(H \cap F_{i+1}) \cap F_i}{H \cap F_i}{}^{\widehat{}}_{+p}$$

$$\xrightarrow{Bi} B\frac{N(H \cap F_i) \cap F_i}{H \cap F_i}{}^{\widehat{}}_{+p}.$$

However, we want to show that for all sufficiently large i,

$$NH \cap F_i = N(H \cap F_i) \cap F_i.$$

If this is indeed the case, then $N(H \cap F_{i+1}) \cap F_i = NH \cap F_{i+1} \cap F_i = N(H \cap F_i) \cap F_i$ and thus

$$\varprojlim_i B\left(\frac{N(H \cap F_i) \cap F_i}{H \cap F_i}\right)^{\widehat{}}_{+p} = \varprojlim_{i,\mathrm{tr}} B\left(\frac{NH \cap F_i}{H \cap F_i}\right)^{\widehat{}}_{+p}.$$

Clearly, $NH \cap F_i \subset N(H \cap F_i) \cap F_i$ for all i. As before, we can find some sufficiently large m_0 so that is $\sigma \in N/T$ normalizes $N \cap F_i \cap T$ and $i \geq m_0$, then σ also normalizes $H \cap T$. Since $t(H) = t_i(H \cap F_i)$ for all large i, the equation $H = H \cap F_i \cdot H \cap T$, which holds for such i's, implies $N(H \cap F_i) \cap F_i \subset NH \cap F_i$.

Henceforth we want to check the following equivalence:

$$(3.11.1) \qquad BW_H{}^{\mathrm{Ad}\ W_H}\widehat{{}_p} \xrightarrow{\mathrm{tr}} \varprojlim_{i,\mathrm{tr}} B\left(\frac{NH \cap F_i}{H \cap F_i}\right)^{\widehat{}}_{+p}.$$

Consider the following map of extensions:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \dfrac{NH \cap T}{H \cap T} & \longrightarrow & \dfrac{NH}{H} & \longrightarrow & D & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \dfrac{NH \cap T \cap F_i}{H \cap T \cap F_i} & \longrightarrow & \dfrac{NH \cap F_i}{H \cap F_i} & \longrightarrow & D_i & \longrightarrow & 1
\end{array}
$$

We claim $D_i \to D$ is an isomorphism for sufficiently large i. The argument used in Proposition 2.8 would then prove (3.11.1).

It is easy to see $D_i \hookrightarrow D$. So suppose $z \in NH$. Write $z = \tilde{z} \cdot t$, where $t \in T$ and $\tilde{z} \in \tilde{F}$. Pick i so that $t_i(H \cap F_i) = t(H)$. This together with $H^z \cap F_i = H \cap F_i$ implies that $t^{p^k} \in NH$ for some integer k. As in the proof

of Lemma 2.5, we could write $t = t_1 \cdot t_2$, where $t_2 \in NH$ and t_1 is of order a power of p. Therefore we have shown surjectivity.

Hence we have proved the injection result for the single summand corresponding to (H). As for other summands, we note that if $(H') \neq (H)$ and $H' \cap T = H \cap T$, then $H' \cap F_i$ is not a conjugate of $H^t \cap F_i$ in F_i for all sufficiently large i and all $t \in T$. The proof of this fact is just the argument used for showing $j: L_i/ \sim \rightarrow L_{i+1}/ \sim$ is well defined. Now apply the above analysis of $\varprojlim_i \bigvee_{(K)\in\mathscr{Z}_i} BW_K^{(i)} {}_+ \widehat{}_p$ to $\varprojlim_i \mathscr{F}_i$ and combine with Lemma 1.7 to complete the proof. □

The generalization of Corollary 3.5 is

Theorem 3.12. *The following map is an injection*:

$$\pi_*(\Phi): \bigoplus_{\substack{(H)\subset G \\ \pi_0(H) \text{ a } p\text{-group}}} \pi_*(BW_H^{\text{Ad } W_H}\widehat{}_p) \to \pi_*(DBG_+\widehat{}_p).$$

Proof. It suffices to show

$$(3.12.1) \qquad \bigoplus_{\substack{(H)\subset G \\ \pi_0(H) \text{ a } p\text{-group}}} \pi_*(BW_H^{\text{Ad } W_H}\widehat{}_p) \xrightarrow{\gamma} \pi_*^{N_p}(S^0\widehat{}_p) \xrightarrow{\pi_*(\Phi_{N_p})} \pi_*(DBN_{p+}\widehat{}_p)$$

is an injection.

By combining Lemmas 3.8 and 3.9 and Proposition 3.11, we notice that the theorem holds for N_p. Recall that, under γ, the summand corresponding to (H) of G maps to a finite number of summands corresponding to $\{(H^{g_\alpha^{(i)}} \cap N_p)\}$ of N_p. As in the proof of Theorem 3.1, we may assume each representative H belongs to N_p whenever $(H) \in \mathscr{P}$. Furthermore, a slight modification of the proof given there shows that the map

$$\bigoplus_{\substack{(H)\subset G \\ \pi_0(H) \text{ a } p\text{-group}}} \pi_*(BW_H^{\text{Ad } W_H}\widehat{}_p) \xrightarrow{\gamma} \pi_*^{N_p}(S^0\widehat{}_p)$$

is an injection. Of course, this does not immediately give us what we want.

Let z be a nonzero element in $\pi_*^{N_p}(S^0\widehat{}_p)$. We can rewrite z uniquely as a sum of x and y, where $x \neq 0$ comes from summands whose corresponding conjugacy class (H) is such that the dimension of H is maximal possible. Notice the analysis of $\gamma \circ i$ in (3.6.1) could equally well be applied to γ itself even though we will not have injections in general. Therefore we observe that the argument presented in the proof of Lemma 3.10 together with Corollary 3.8 actually shows $\pi_*(\Phi_{N_p})(z) = 0$ implies $\pi_*(\Phi_{N_p})(x) = 0$.

Clearly for $(H) \subset G$ with $\pi_0(H)$ a p-group, the dimension of H is greater than or equal to the dimension of $H^{g_\alpha^{(i)}} \cap N_p$ and equality holds only if $\pi_0(H^{g_\alpha^{(i)}} \cap N_p)$ is a p-group. The above discussion then proves (3.12.1) is an injection. □

In our discussion up to now we have not mentioned those conjugacy classes of subgroups of G which are not subconjugates of N. The following proposition explains the reason.

Proposition 3.13. *Let* (H) *be a conjugacy class of* N. *Then under the map*

$$(S^0)^N \xrightarrow{D\tau} (S^0)^G,$$

the summand corresponding to (H) *of* $(S^0)^N$ *goes to the corresponding* (H) *of* $(S^0)^G$ *given by a map induced by the inclusion* $N_N H \hookrightarrow N_G H$, *where* τ *is the G-transfer* $S^0 \to G/N_+$.

Proof. From tom Dieck's splitting, for a given $f \in \pi_*(BW_H{}^{\mathrm{Ad}\ W_H})$, the image of f in $\pi_*^G(S^0)$ is the following composite:

$$f_1 \colon S^0 \xrightarrow{f} BW_H{}^{\mathrm{Ad}\ W_H} \xrightarrow{\tau} N_+ \wedge_{N_N H} BW_H{}^{\mathrm{Ad}\ W_H}$$

$$\xrightarrow{1 \wedge \tau} N_+ \wedge_{N_N H} EW_{H+} \xrightarrow{\mathrm{pr}} S^0,$$

where pr is the obvious projection map. If we identify $\{G/N_+, S^0\}_*^G$ with $\{S^0, S^0\}_*^N$, the above composite becomes:

$$f_2 \colon G/N_+ \wedge S^0 \xrightarrow{1 \wedge f_1} G/N_+ \wedge S^0 \xrightarrow{\mathrm{pr}} S^0.$$

Now, under the G-transfer $S^0 \xrightarrow{\tau} G/N_+$, f_2 sends to $f_3 = f_2 \circ \tau$. Consider the following diagram:

$$
\begin{array}{ccc}
BW_H{}^{\mathrm{Ad}\ W_H} & \xrightarrow{\quad j \quad} & BW_H'{}^{\mathrm{Ad}\ W_H'} \\
{\scriptstyle \tau \wedge 1}\Big\downarrow & & \Big\downarrow{\scriptstyle \tau \wedge 1} \\
G \wedge_{N_G H} BW_H{}^{\mathrm{Ad}\ W_H} & \xrightarrow{\quad j \quad} & G \wedge_{N_G H} BW_H'{}^{\mathrm{Ad}\ W_H'} \\
{\scriptstyle 1 \wedge \tau \wedge 1}\Big\downarrow & & \Big\downarrow{\scriptstyle 1 \wedge \tau' \wedge 1} \\
G \wedge_{N_G H} N_G H \wedge_{N_N H} BW_H{}^{\mathrm{Ad}\ W_H} & \xrightarrow{\ j\ } & G \wedge_{N_G H} N_G H \wedge_{N_N H} BW_H'{}^{\mathrm{Ad}\ W_H} \\
{\scriptstyle 1 \wedge \tau}\Big\downarrow & & \Big\downarrow{\scriptstyle 1 \wedge \tau'} \\
G \wedge_{N_G H} N_G H \wedge_{N_N H} W_H \wedge_{W_H} EW_{H+} & \xrightarrow{\ j\ } & G \wedge_{N_G H} EW_{H+}'
\end{array}
$$

where $W_H' = N_G H / H$, τ (τ') are appropriate transfers (dimension-shifting transfers), and j is induced from inclusions $\mathrm{Ad}\ W_H \hookrightarrow \mathrm{Ad}\ W_H'$ and $W_H \hookrightarrow W_H'$. Since the diagram commutes, we have the desired result. $\square$

Notice the same argument applies if we replace N by N_p in the proposition. Consider the following commutative diagram:

$$
\begin{array}{ccc}
(S^0)^G\widehat{}_p & \xrightarrow{\ \Phi\ } & DBG_+\widehat{}_p \\
{\scriptstyle i}\Big\downarrow & & \Big\downarrow \\
(S^0)^{N_p}\widehat{}_p & \xrightarrow{\ \Phi\ } & DBN_{p+}\widehat{}_p \\
{\scriptstyle j}\Big\downarrow & & \Big\downarrow \\
(S^0)^G\widehat{}_p & \xrightarrow{\ \Phi\ } & DBG_+\widehat{}_p
\end{array}
$$

where i is induced from inclusion and j is induced from $\tau \colon S^0 \to G/N_{p+}$. Observe that the vertical composite of the right side is an equivalence. By the

preceding proposition and Proposition 1.12, the summand corresponding to (H) will map to summands of the form $H^\alpha \cap N_p$ under the composite $j \circ i$. If $(H) \not\subset (N_p)$, then $(H^\alpha \cap N_p) \neq (H)$ for any α. Therefore we conclude that the results of this section are not far from being optimal.

4. $G = SO(3)$

In this section we want to consider the case when $G = SO(3)$. Of course, the previous sections gave a partial description of $DBSO(3)_+$ at all primes. However, we can actually do better and obtain an explicit splitting of $DBSO(3)_+$ at the prime 2.

We note that the proofs of Theorems 1.1 and 3.1 can basically be reduced to the case when G is an extension of a torus by a p-group. Therefore we begin by deriving a crude structural theorem of $DBG_{+\hat{p}}$ in that case.

Let $\eta\colon J_1 \to J_2$ be a surjection of finite p-groups with kernel K. We want to understand the induced map $DBJ_{2+} \xrightarrow{DB\eta} DBJ_{1+}$. By Segal's conjecture for p-groups, this is equivalent to studying $(S^0)^{J_2} \to (S^0)^{J_1}$.

As before, we have a commutative diagram:

$$
\begin{array}{ccc}
\pi_*(BW_{\overline{H}+}) & ===== & \pi_*(BW_{\overline{H}+}) \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\
\pi_*^{J_2}(BW_{\overline{H}+}) & \longrightarrow & \pi_*^{J_1}(BW_{\overline{H}+}) \\
{\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau} \\
\pi_*^{J_2}(J_{2+} \wedge_{N\overline{H}} EW_{\overline{H}+}) & \longrightarrow & \pi_*^{J_1}(J_{2+} \wedge_{N\overline{H}} EW_{\overline{H}+}) \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\
\pi_*^{J_2}(S^0) & \longrightarrow & \pi_*^{J_1}(S^0)
\end{array}
$$

where the horizontal maps are induced from $J_1 \xrightarrow{\eta} J_2$ and $\overline{H}$ is a subgroup of J_2. Let H be the preimage of $\overline{H}$ in J_1.

Lemma 4.1. $J_{2+} \wedge_{N\overline{H}} EW_{\overline{H}+} = J_{1+} \wedge_{NH} EW_{H+}$ *as* J_1-*spaces.*

Proof. Observe that $N(\overline{H}) = \overline{NH}$. Hence, $J_2/N(\overline{H}) = J_1/K \times \overline{NH} = J_1/NH$ as J_1-spaces. Since $N(\overline{H})/\overline{H} = \overline{NH}/\overline{H} \cong NH/H$ and clearly the action of NH on $EW_{\overline{H}}$ is the same as that of NH on EW_H, we have the lemma. $\square$

Corollary 4.2. *The summand of* $(S^0)^{J_2}$ *corresponding to* $(\overline{H})$ *maps under* η *to the summand* (H) *of* $(S^0)^{J_1}$ *via an equivalence induced from* η^{-1}.

We can apply this simple result to the following special situation. Suppose that

$$1 \to T \to N \to W \to 1$$

is a split extension with W a p-group. That is, N is a semidirect product of W and T, i.e., $N = W \ltimes T$. Recall $I_m = \{x \in T \mid x^{p^m} = 1\}$ and let $N_m = W \ltimes I_m$. We have a surjection $\eta\colon N_{m+1} \to N_m$ induced from the power

map $I_{m+1} \xrightarrow{\ (\ )^p\ } I_m$. Consider the following commutative diagram:

$$
(4.3.1) \qquad
\begin{array}{ccccc}
DBN_{m+} & \xrightarrow{D[B\eta]} & DBN_{m+1+} & \longrightarrow & \mathscr{A}_{m+1} \\
{\scriptstyle D[Bi]}\downarrow & & \downarrow{\scriptstyle D[Bi]} & & \downarrow \\
DBN_{m-1+} & \xrightarrow[D[B\eta]]{} & DBN_{m+} & \longrightarrow & \mathscr{A}_m
\end{array}
$$

where i is the inclusion $N_m \hookrightarrow N_{m+1}$ and $\mathscr{A}_m$ is the cofiber of $D[B\eta]$. As a matter of fact, from Corollary 4.2 the map $D[B\eta]$ splits and $\mathscr{A}_m$ is a wedge of those summands of $(S^0)^{N_m}_p{}^\frown$ corresponding to (H) such that $I_1 \not\subset H$. It is clear then under $D[Bi]$, $\mathscr{A}_{m+1}$ maps into $\mathscr{A}_m$ only. Therefore, the diagram (4.3.1) splits naturally. Taking the homotopy inverse limit with respect to m yields

$$ DBN_{+\hat{p}} \xrightarrow{D[B\overline{\eta}]} DBN_{+\hat{p}} \longrightarrow \varprojlim_m \mathscr{A}_m. $$

Here $\overline{\eta}$ is induced from $T \xrightarrow{\ (\ )^p\ } T$ and of course the sequence splits. By iterating $D[B\overline{\eta}]$, we get:

$$
\begin{array}{ccccc}
DBN_{+\hat{p}} & \xrightarrow{D[B\overline{\eta}^{n+1}]} & DBN_{+\hat{p}} & \longrightarrow & \bigvee_1^{n+1}(\varprojlim_m \mathscr{A}_m) \\
{\scriptstyle D[B\overline{\eta}]}\downarrow & & \| & & \downarrow \\
DBN_{+\hat{p}} & \xrightarrow{D[B\overline{\eta}^n]} & DBN_{+\hat{p}} & \longrightarrow & \bigvee_1^{n}(\varprojlim_m \mathscr{A}_m)
\end{array}
$$

where the right vertical map collapses the $(n+1)$th wedge summand to the base point. Taking the homotopy inverse limit with respect to n yields the split cofiber sequence

$$ \varprojlim_{D[B\overline{\eta}]} DBN_{+\hat{p}} \longrightarrow DBN_{+\hat{p}} \longrightarrow \prod_1^\infty \left(\varprojlim_m \mathscr{A}_m \right). $$

Now

$$ \varprojlim_{D[B\overline{\eta}]} DBN_{+\hat{p}} = D\left(\varprojlim_{B\overline{\eta}} \widetilde{BN}_+ \right), $$

where $\sim$ denotes the cocompletion at p or equivalently, the smash product with the Moore spectrum $\Sigma^{-1}M(Z_{p^\infty})$. But

$$ \varinjlim_{B\overline{\eta}} \widetilde{BN}_+ = \varinjlim_{B\overline{\eta}} \left(\varinjlim_{Bi} \widetilde{BN}_{m+} \right) = \varinjlim_{Bi} \left(\varinjlim_{B\overline{\eta}} \widetilde{BN}_{m+} \right) = \varinjlim_{Bi} \widetilde{BW}_+ = \widetilde{BW}_+. $$

Therefore

$$ DBN_{+\hat{p}} = DBW_{+\hat{p}} \vee \prod_1^\infty \left(\varinjlim_m \mathscr{A}_m \right). $$

Proposition 4.3. *Let* $1 \to T \to N \to W \to 1$ *be an arbitrary extension with* W *a p-group, p a prime. Then* $DBN_{+}\widehat{}_p$ *can be written in the form*

$$DBW_{+}\widehat{}_p \vee \mathscr{A}' \vee \prod_1^\infty \mathscr{A}$$

for some p-completed spectra $\mathscr{A}$ *and* $\mathscr{A}'$.

Proof. From Proposition 1.4, there exists a subgroup $\widetilde{F}$ of N such that $\widetilde{F} \cap T$ equals the subgroup $\widetilde{T}$ of T generated by all elements of order $n = |W|$. Corollary 1.5 says we can find a nested sequence $\{F_i\}$ of finite subgroups of N such that $F_i/F_i \cap T \cong W$ with $\bigcup F_i \cap T = T_{Q_p}$. Consider the following diagram:

$$
\begin{array}{ccccc}
DB(F_i/\widetilde{T})_+ & \xrightarrow{D[Br]} & DBF_{i+} & \longrightarrow & \mathscr{B}_i \\
\downarrow & & \downarrow & & \downarrow \\
DB(F_{i-1}/\widetilde{T})_+ & \xrightarrow{D[Br]} & DBF_{i-1+} & \longrightarrow & \mathscr{B}_{i-1}
\end{array}
$$

where $r\colon F_i \to F_i/\widetilde{T}$ is the quotient map and the vertical maps are induced from $F_{i-1} \hookrightarrow F_i$. By taking the homotopy inverse limit and observing the naturality of the splittings, it is immediate that we have

$$DBN_{+}\widehat{}_p = DB(N/\widetilde{T})_{+}\widehat{}_p \vee \varprojlim_i \mathscr{B}_i.$$

However, the extension

$$1 \to \frac{T}{\widetilde{T}} \to \frac{N}{\widetilde{T}} \to W \to 1$$

splits since $\widetilde{F}/\widetilde{T} \cong W$. Therefore, applying the previous discussion to this situation results in the assertion. $\square$

We get the following result of Ravenel [14, Theorem 1.11].

Corollary 4.4.

$$DBS^1{}_{+}\widehat{}_p = S^0{}_{p}\widehat{} \vee \prod_1^\infty BS^{1\,\mathrm{Ad}\,S^1}{}_p\widehat{}.$$

As we shall see later, Proposition 4.3 also recovers a result of Miller and Wilkerson [12].

We are now ready to describe $DBO(2)_{+}\widehat{}_p$. Recall that the dihedral group D_n of order $2n$ is given by

$$\mathrm{gp}\langle s, t \,|\, s^n = 1 = t^2,\ t^{-1}st = s^{-1} \rangle, \qquad n \geq 1.$$

We have the following maps:

$$i\colon D_{2^n} \to D_{2^{n+1}}, \qquad i \text{ the inclusion map given by } s \to s^2,\ t \to t,$$
$$r\colon D_{2^n} \to D_{2^{n-1}}, \qquad r \text{ the reduction map given by } s \to s,\ t \to t.$$

Lemma 4.5. *The cofiber* $\mathscr{C}_n$ *of* $D[Br]$ *is* $BD_{2^n+2}\widehat{} \vee BZ_{2+2}\widehat{} \wedge BZ_{2+2}\widehat{}$.

Proof. The conjugacy classes of subgroups of D_{2^n} consist of

$$T_{2^j} = \mathrm{gp}\langle s^{2^{n-j}}\rangle, \qquad j = 0, 1, \ldots, n,$$

$$D_{2^j} = \mathrm{gp}\langle s^{2^{n-j}}, t\rangle, \qquad j = 0, 1, \ldots, n,$$

$$D'_{2^j} = \mathrm{gp}\langle s^{2^{n-j}}, st\rangle, \qquad j = 0, 1, \ldots, n-1.$$

By Corollary 4.2, $\mathscr{C}_n$, the cofiber of $D[Br]$, is

$$BW_{T_1+\widehat{2}} \vee BW_{D_1+\widehat{2}} \vee BW_{D'_1+\widehat{2}}.$$

Now

$$NT_{2^j} = D_{2^n},$$
$$ND_{2^j} = D_{2^{j+1}} \quad \text{if } j \neq n,$$
$$ND'_{2^j} = D'_{2^{j+1}} \quad \text{if } j \neq n-1,$$
$$ND'_{2^{n-1}} = D_{2^n}.$$

Hence, $\mathscr{C}_n = BD_{2^n+\widehat{2}} \vee BZ_{2+\widehat{2}} \vee BZ_{2+\widehat{2}}.$ $\quad\square$

Proposition 4.6.

$$DBO(2)_{+}\widehat{2} = S^0\widehat{2} \vee BZ_{2+\widehat{2}} \vee \prod_1^\infty BO(2)^{\mathrm{Ad}\ O(2)}\widehat{2} \vee \prod_1^\infty BZ_{2+\widehat{2}}.$$

Proof. Notice that

$$\varprojlim_{D[Bi]} \mathscr{C}_n = \varprojlim_n (E\mathscr{F}_+)^{D_{2^n}}\widehat{2},$$

where $\mathscr{F} = \{H \subset O(2) | H \cap T = 1\}$. Therefore, by Proposition 2.1, $\varprojlim_{D[Bi]} \mathscr{C}_n$ $= (E\mathscr{F}_+)^{O(2)}\widehat{2}$. But then $\mathscr{F}$ only consists of two conjugacy classes, namely (1) and (D_1). Hence, $\varprojlim_{D[Bi]} \mathscr{C}_n = BO(2)^{\mathrm{Ad}\ O(2)}\widehat{2} \vee BZ_{2+\widehat{2}}$. The rest follows from the discussion after Corollary 4.2. $\quad\square$

We record the description of the map $(E\mathscr{F}_+)^{O(2)}\widehat{2} \xrightarrow{\gamma} (E\mathscr{F}_+)^{D_2}\widehat{2}$ to be needed later. For convenience, we denote by $BD_{1+\widehat{2}}$ the summand corresponding to (D_1) in $DBD_{2+\widehat{2}}$ and by $BD'_{1+\widehat{2}}$ the summand corresponding to (D'_1).

Lemma 4.7. *In* $(E\mathscr{F}_+)^{O(2)}\widehat{2} \xrightarrow{\gamma} (E\mathscr{F}_+)^{D_2}\widehat{2}$, *we have*

(1) $BO(2)^{\mathrm{Ad}\ O(2)}\widehat{2}$ *maps to* $BD_{2+\widehat{2}}$ *by the bundle transfer and is trivial otherwise;*

(2) $BZ_{2+\widehat{2}}$ *maps to* $BD_{1+\widehat{2}}$ *and* $BD'_{1+\widehat{2}}$ *by the identity;*

(3) $BZ_{2+\widehat{2}}$ *maps to* $BD_{2+\widehat{2}}$ *by* $(-1) \cdot Bi$, *where* $i: Z_2 \hookrightarrow D_2$.

Proof. The proof follows easily from Proposition 1.12. $\quad\square$

Given a compact Lie group G with normalizer N of a maximal torus T, an element $x \in \pi^*(N)$ is *stable* with respect to G if

$$(4.8.1) \qquad \rho^*(N \cap N^g, N^g)C_{g^{-1}}(x) = \rho^*(N \cap N^g, N)(x) \quad \text{for all } g \in G.$$

Here $\rho^*(H_1, H_2)$ is the map $\pi^*(BH_2) \to \pi^*(BH_1)$ and $C_{g^{-1}}$ is the conjugation isomorphism $\pi^*(BN) \to \pi^*(BN^g)$. Let $(\pi^*(BN))^s$ denote these stable elements of $\pi^*(BN)$. From Feshbach [7, Theorem 2.2], we have

$$(4.8.2) \qquad\qquad \pi^*(BG) \xrightarrow[\cong]{\rho(N,G)} (\pi^*(BN))^s.$$

Applying (4.8.2) to the situation where $G = SO(3)$ and $N = O(2)$, we have to understand those $g \in G$ which give nontrivial stability conditions (4.8.1). Recall we have the following list of conjugacy classes of proper closed subgroups of $SO(3)$ and $O(2)$ from [6]:

$$O(2): \quad (T_n) \ N_{O(2)}T_n = O(2)$$
$$(D_n) \ N_{O(2)}D_n = D_{2n}, \ n \geq 1$$
$$(S^1) \ N_{O(2)}S^1 = O(2)$$

$$SO(3): \quad (T_n) \ N_{SO(3)}T_n = O(2)$$
$$(D_n) \ N_{SO(3)}D_n = D_{2n}, \ n \geq 3$$
$$(D_2) \ N_{SO(3)}D_2 = S_4$$
$$(A_4) \ N_{SO(3)}A_4 = S_4$$
$$(S_4) \ N_{SO(3)}S_4 = S_4$$
$$(A_5) \ N_{SO(3)}A_5 = A_5$$
$$(S^1) \ N_{SO(3)}S^1 = O(2)$$
$$(O(2)) \ N_{SO(3)}O(2) = O(2)$$

Notice that closed subgroups in $O(2)$ of order greater than two which are conjugate in $SO(3)$ are in fact conjugate in $O(2)$. Now (4.8.1) is equivalent to

$$(4.8.3) \qquad C_{g^{-1}}\rho^*(N \cap N^g, N)(x) = \rho^*(N \cap N^{g^{-1}}, N)(x).$$

Clearly, $O(2) \cap O(2)^g$ is conjugate to $O(2) \cap O(2)^{g^{-1}}$ in $SO(3)$. Using the fact that $SO(3) = SU(2)/\pm I$, it is easy to check that $O(2) \cap O(2)^g$ is not a conjugate of T_2 in $O(2)$ for all $g \in SO(3)$. Therefore, $O(2) \cap O(2)^g$ and $O(2) \cap O(2)^{g^{-1}}$ are indeed conjugate in $O(2)$. Let $(O(2) \cap O(2)^g)^\alpha = O(2) \cap O(2)^{g^{-1}}$ for some $\alpha \in O(2)$. Of course, we have

$$
\begin{array}{ccc}
\pi^*(BO(2)) & \xrightarrow{C_\alpha = \mathrm{id}} & \pi^*(BO(2)) \\
\rho^* \downarrow & & \downarrow \rho^* \\
\pi^*(BO(2) \cap O(2)^{g^{-1}}) & \xrightarrow{C_\alpha} & \pi^*(BO(2) \cap O(2)^g)
\end{array}
$$

Combining this with (4.8.3), it suffices to consider the elements $x \in \pi^*(BO(2))$ such that

$$(4.8.4) \qquad \rho^*(O(2) \cap O(2)^g, O(2))(x) = C_{g'}\rho^*(O(2) \cap O(2)^g, O(2))(x),$$

where g' normalizes $O(2) \cap O(2)^g$ with $g \in SO(3)$. Notice that we only have to consider one representative from each conjugacy class. Clearly (4.8.4) will hold if $g' \in O(2)$ or if $|O(2) \cap O(2)^g| \leq 2$. Hence we want to consider those closed subgroups of $O(2)$ with order at least three which has normalizers not contained in $O(2)$. From the given list, we see that the only possibility is D_2.

Since D_2 is abelian, $S_4/D_2 = S_3$ acts on D_2 by conjugation. It is easy to check that in fact S_3 gives all the automorphisms of D_2. It remains to check $O(2) \cap O(2)^g = D_2$ for $g \in S_4 - D_4$ since the normalizers of D_2 in $O(2)$ is D_4. So suppose $D_2 \subset K \subset O(2) \cap O(2)^g$. This implies that $K^{g^{-1}} \subset O(2)$. As before,

this implies $K^{g^{-1}}$ is conjugate to K in $O(2)$. This means the normalizer of K in $SO(3)$ is not contained in $O(2)$. There is only one such conjugacy class, namely (D_2). In summary, we have

Proposition 4.8.

$$\pi^*(BSO(3)) = \{x \in \pi^*(BO(2))| \, \rho^*(D_2, O(2))(x) \text{ is invariant}$$
$$\text{under the action of } \mathrm{Aut}(D_2)\}.$$

We can interpret the proposition in terms of spectra. That is, $DBSO(3)_{+\hat{2}}$ is the summand of $DBO(2)_{+\hat{2}}$ whose image in $DBD_{2+\hat{2}}$ is invariant under the action of $\mathrm{Aut}(D_2)$. An obvious approach to calculate $DBSO(3)_{+\hat{2}}$ is to split $BD_{2+\hat{2}}$ and $BO(2)^{\mathrm{Ad}\ O(2)}_{\hat{2}}$ in a useful manner that will allow us to see the action of $\mathrm{Aut}(D_2)$. Recall the following result of Mitchell and Priddy.

Proposition 4.9 [11, Theorem C]. $BO(2)_{+\hat{2}} = BSO(3)_{+\hat{2}} \vee F$, where $BZ_{2+\hat{2}}$ is a summand of F.

Using this, we are able to prove

Proposition 4.10. $BO(2)^{\mathrm{Ad}\ O(2)}_{\hat{2}} = BSO(3)^{\mathrm{Ad}\ SO(3)}_{\hat{2}} \vee F$, where $F \to BO(2)_{+\hat{2}}$ $\xrightarrow{j} BO(2)^{\mathrm{Ad}\ O(2)}_{\hat{2}}$ and $BSO(3)^{\mathrm{Ad}\ SO(3)}_{\hat{2}} \xrightarrow{\mathrm{tr}} BO(2)^{\mathrm{Ad}\ O(2)}_{\hat{2}}$ are the maps that give the equivalence. Note j is induced from $\{e\} \hookrightarrow \mathrm{Ad}\ O(2)$.
Proof. It suffices to prove isomorphism in $\mod 2$ cohomology. Recall we have

$$H^*(BSO(3), Z_2) = Z_2[\overline{\omega}_2, \overline{\omega}_3], \qquad H^*(BO(2), Z_2) = Z_2[\omega_1, \omega_2],$$

where $\pi^*(\overline{\omega}_2) = \omega_2 + \omega_1^2$, $\pi^*(\overline{\omega}_3) = \omega_1\omega_2$, and $\pi: BO(2) \to BSO(3)$. Via the Thom isomorphism,

$$H^*(BO(2)^{\mathrm{Ad}\ O(2)}, Z_2) \xrightarrow{j^*} H^*(BO(2), Z_2)$$

can be identified with

$$\overline{H}^{*-1}(BO(2), Z_2) \xrightarrow{\times \omega_1} H^*(BO(2), Z_2)$$

since the Euler class of $\mathrm{Ad}\ O(2)$ is ω_1.
We want to show that

$$\overline{H}^*(BO(2)^{\mathrm{Ad}\ O(2)}, Z_2) \to \overline{H}^*(BSO(3)^{\mathrm{Ad}\ SO(3)}, Z_2) \oplus \overline{H}^*(F, Z_2)$$

is an isomorphism. Since $BO(2)_{+\hat{2}} = BSO(3)_{+\hat{2}} \vee F$, the kernel of

$$H^*(BO(2), Z_2) \to H^*(F, Z_2)$$

can be identified via $H^*(BSO(3), Z_2)$ as

$$\sum a_{ij}(\omega_2\omega_1)^i(\omega_2 + \omega_1^2)^j.$$

Hence the kernel of

$$H^*(BO(2)^{\mathrm{Ad}\ O(2)}, Z_2) \xrightarrow{j^*} H^*(BO(2), Z_2) \to H^*(F, Z_2)$$

can be written as

$$K = \sum a'_{ij}\omega_2(\omega_2\omega_1)^i(\omega_2 + \omega_1^2)^j.$$

We claim $\mathrm{tr}^*(K) \neq 0$ of $K \neq 0$, where $\mathrm{tr}: BSO(3)^{\mathrm{Ad}\,BO(3)}\hat{}_2 \to BO(2)^{\mathrm{Ad}\,O(2)}\hat{}_2$. By Thom isomorphism, tr^* corresponds to

$$t_!: H^*(BO(2), Z_2) \to H^{*-2}(BSO(3), Z_2).$$

Since $SO(3)/O(2) = RP^2$ and $i^*(\omega_1^2) = [F]$, the fundamental class of RP^2 where $RP^2 \xrightarrow{i} BO(2)$, we have $t_!(\omega_1^2) = 1$ (cf. [13, Proposition 2.2]). By naturality, we have $t_!(\omega_1^2 + \omega_2) = t_!(\pi^*(\overline{\omega}_2)) = t_!(1)\overline{\omega}_2 = 0$. Therefore $t_!(\omega_2) = 1$. Hence,

$$
\begin{aligned}
t_!(K) &= t_! \left(\sum a'_{ij} \omega_2 (\omega_2 \omega_1)^i (\omega_2 + \omega_1^2)^j \right) \\
&= t_! \left(\sum a'_{ij} \omega_2 \pi^*(\overline{\omega}_3)^i \pi^*(\overline{\omega}_2)^j \right) \\
&= \sum a'_{ij} \overline{\omega}_3^i \overline{\omega}_2^j \\
&\neq 0 \quad \text{unless } a'_{ij} = 0 \text{ for all } i, j.
\end{aligned}
$$

To show surjectivity, we just check the Poincaré series of the target and the source are equal. Therefore we are done. □

Theorem 4.11 [11, Theorem A]. $BD_{2+}\hat{}_2 = BA_{4+}\hat{}_2 \vee F \vee F$, where the first summand F can be mapped into the second summand F by the automorphism $\tau\sigma$ in $\mathrm{Aut}(D_2) = GL_2(Z_2)$; here $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Remark. F coincides with the summand F in the splitting of $BO(2)_{+}\hat{}_2$. We discuss some details of the splitting so that we can understand the action of $\mathrm{Aut}(D_2)$ with respect to this decomposition. First of all, the summands $F_1 \vee F_2$ (the subscripts are for distinction) come from splitting using idempotents in $Z_2[GL_2(Z_2)]$. Specifically, let $\pi_2 = \sigma + \sigma^2$, $\rho_1 = (1 + \tau\sigma)(1 + \tau)$, and $\rho_2 = (1 + \tau)(1 + \tau\sigma)$. We have $\pi_2 = \rho_1 + \rho_2$, $\rho_1\rho_2 = \rho_2\rho_1 = 0$. Then $H^*(F_i, Z_2) = \mathrm{Im}\,\rho_i^* \subset H^*(BD_2, Z_2)$ for $i = 1, 2$. We remark that elements are acting from the right on cohomology.

We claim that $\mathrm{Aut}(D_2)$ acts as automorphism $GL_2(Z_2)$ on $F_1 \vee F_2$. To see the reason, we already know ρ_1, ρ_2 are projection onto the first or second coordinate while $\tau\sigma$ switch the coordinates. Consider τ. We have $\rho_1\tau = \rho_1$. This implies that the action of τ on F_1 maps it to itself. $\rho_2\tau\rho_1 = \rho_2(\sigma\tau)$, and $\rho_2\tau\rho_2 = \rho_2$ implies that τ maps F_2 to $F_1 \vee F_2$ by diagonal. Similarly, we get all the other elements of $GL_2(Z_2)$.

As for $BA_{4+}\hat{}_2$, this comes from the transfer associated to $D_2 \hookrightarrow A_4$. Therefore it is obvious that the action of $\mathrm{Aut}(D_2)$ on $BA_{4+}\hat{}_2$ is trivial.

Lemma 4.12. *Let K be a compact Lie group and $\sigma \in \mathrm{Aut}(K)$. Then the action of σ on $(S^0)^K$ is given by $BW_H^{\mathrm{Ad}\,W_H} \to BW_{\sigma H}^{\mathrm{Ad}\,W_{\sigma H}}$.*

Proof. To avoid confusion, write $\sigma: K_1 \to K_2$. As a K_1-space, $K_2/NH = K_1/\sigma^{-1}(NH) = K_1/N(\sigma^{-1}H)$. By the naturality of tom Dieck's splitting with respect to the conjugation isomorphism, we have the result. □

By the lemma, it is easy to see that $\mathrm{Aut}(D_2) = S_3$ acts on DBD_{2+} by conjugation on the summand BD_{2+}, permuting $BZ_{2+} \vee BZ_{2+} \vee BZ_{2+}$, and leaves S^0 alone.

Theorem 4.13.

$$DBSO(3)_{+\hat{2}} \simeq S^0{}_{\hat{2}} \vee BSO(3)^{\mathrm{Ad}\ SO(3)}{}_{\hat{2}}$$

$$\vee \prod_1^\infty BO(2)^{\mathrm{Ad}\ O(2)}{}_{\hat{2}} \vee \prod_1^{\cdot\infty} BZ_{2+\hat{2}}\,.$$

Proof. Consider the split cofiber sequence

(4.13.1) $DBO(2)_{+\hat{2}} \xrightarrow{D[Br]} BBO(2)_{+\hat{2}} \to BO(2)^{\mathrm{Ad}\ O(2)}{}_{\hat{2}} \vee BZ_{2+\hat{2}}\,.$

We may rewrite (4.13.1) as

$$DBW_{+\hat{2}} \vee \mathscr{F} \xrightarrow{D[Br]} DBO(2)_{+\hat{2}} \to BO(2)^{\mathrm{Ad}\ O(2)}{}_{\hat{2}} \vee BZ_{2+\hat{2}}\,,$$

where $W = Z_2$ is the Weyl group of $O(2)$ and $\mathscr{F} = \prod_1^\infty BO(2)^{\mathrm{Ad}\ O(2)}{}_{\hat{2}} \vee \prod_1^\infty BZ_{2+\hat{2}}$. Let Θ be the composite

$$\mathscr{F} \hookrightarrow BDW_{+\hat{2}} \vee \mathscr{F} \xrightarrow{D[Br]} DBO(2)_{+\hat{2}} \xrightarrow{D[Bj]} DBD_{2+\hat{2}}\,.$$

By the commutative diagram

$$
\begin{array}{ccc}
DBO(2)_{+\hat{2}} & \xrightarrow{\ D[Br]\ } & DBO(2)_{+\hat{2}} \\[4pt]
{\scriptstyle D[Bj]}\big\downarrow & & \big\downarrow{\scriptstyle D[Bj]} \\[4pt]
DBD_{1+\hat{2}} & \xrightarrow{\ D[Br]\ } & DBD_{2+\hat{2}}
\end{array}
$$

we see that Θ is a map between $\mathscr{F}$ and $BZ_{2+\hat{2}}$ if we identify $DBD_{1+\hat{2}}$ as $S^0{}_{\hat{2}} \vee BZ_{2+\hat{2}}$. Consider the map

$$\mathscr{F} \xrightarrow{\mathrm{id}\vee\Theta} \mathscr{F} \vee BW_{+\hat{2}} \hookrightarrow \mathscr{F} \vee DBW_{+\hat{2}} \xrightarrow{D[Br]} DBO(2)_{+\hat{2}} \xrightarrow{D[Bj]} DBD_{2+\hat{2}}\,.$$

By construction, this is the zero map. Thus $\mathscr{F}$ is a summand of $DBO(2)_{+\hat{2}}$, via $\mathrm{id}\vee\Theta$, whose image in $DBD_{2+\hat{2}}$ is invariant under the action of $\mathrm{Aut}(D_2)$.

Now the summand $BZ_{2+\hat{2}}$ maps to $DBD_{2+\hat{2}}$ under $D[Bj]$ according to the description in Lemma 4.7, and $BW_{+\hat{2}}$ maps to the summand in $DBD_{2+\hat{2}}$ corresponding to (T_2) by the identity. From the comment made after Lemma 4.12, it is clear that we have to analyze the map $BZ_{2+\hat{2}} \xrightarrow{Bi} BD_{2+\hat{2}}$ and understand how it relates to the bundle transfer tr: $BO(2)^{\mathrm{Ad}\ O(2)}{}_{\hat{2}} \to BD_{2+\hat{2}}$. We use the splittings of $BO(2)^{\mathrm{Ad}\ O(2)}{}_{\hat{2}}$ and $BD_{2+\hat{2}}$ from Propositions 4.10 and 4.11 to analyze the bundle transfer map tr.

In [11, Theorem C], it was shown that the summand F of $BO(2)_{+\hat{2}}$ factors through $BD_{2+\hat{2}} \xrightarrow{Bi} BO(2)_{+\hat{2}}$, where F maps to the summand F_1. The composite

$$BO(2)_{+\hat{p}} \to BO(2)^{\mathrm{Ad}\ O(2)}{}_{\hat{2}} \xrightarrow{\mathrm{tr}} BD_{2+\hat{2}}$$

is equal to the ordinary transfer associated to $D_2 \hookrightarrow O(2)$. Therefore we have the following diagram:

$$
\begin{array}{ccccc}
F & \longrightarrow & BO(2)_{+\hat{2}} & \xrightarrow{\ j\ } & BO(2)^{\mathrm{Ad}\ O(2)}{}_{\hat{2}} \\[4pt]
\big\downarrow & & & & \big\downarrow{\scriptstyle \mathrm{tr}} \\[4pt]
BD_{2+\hat{2}} & \xrightarrow[Bi]{} & BO(2)_{+\hat{2}} & \xrightarrow[\mathrm{tr}]{} & BD_{2+\hat{2}}
\end{array}
$$

By the double coset theorem, $BD_{2+\widehat{2}} \xrightarrow[Bi]{} BO(2)_{+\widehat{2}} \xrightarrow[\text{tr}]{} BD_{2+\widehat{2}}$ is the same as $1 + C_{\tau\sigma}$. Therefore F of $BO(2)^{\text{Ad } O(2)}\widehat{_2}$ maps diagonally into $F_1 \vee F_2$ of $BD_{2+\widehat{2}}$. Clearly, the summand $BSO(3)^{\text{Ad } SO(3)}\widehat{_2}$ of $BO(2)^{\text{Ad } O(2)}\widehat{_2}$ maps to the summand $BA_{4+\widehat{2}}$ of $BD_{2+\widehat{2}}$ by bundle transfer.

Since we remarked that $BZ_{2+\widehat{2}}$ is a summand of F, it is clear then from the discussion following Proposition 4.11 that the summand $F \vee BZ_{2+\widehat{2}} \vee BW_{+\widehat{2}}$ of $DBO(2)_{+\widehat{2}}$ cannot contribute to a stable summand of $DBSO(3)_{+\widehat{2}}$. Therefore we have

$$DBSO(3)_{+\widehat{2}} \simeq S^0\widehat{_2} \vee \mathscr{F} \vee BSO(3)^{\text{Ad } SO(3)}\widehat{_2}. \quad \square$$

Recall Miller and Wilkerson proved

Theorem 4.14 [12, Theorem B]. $DBS^3_{+\widehat{2}} \simeq BS^{3\,\text{Ad } BS^3}\widehat{_2} \vee DBSO(3)_{+\widehat{2}}$.

Theorem 4.15 [12, Theorem B]. $DBN_{2+\widehat{2}} \simeq BN_2^{\text{Ad } BN_2}\widehat{_2} \vee DBO(2)_{+\widehat{2}}$, where N_2 denotes the normalizer of the maximal torus in S^3.

Remark. We can recover Theorem 4.15 from Proposition 4.3 for there is no nontrivial 2-group in N_2 whose intersection with the normal subgroup $\{\pm I \in S^3\}$ is trivial.

From Theorem 4.13 and Proposition 4.6, we get

Corollary 4.16.

$$DBS^3_{+\widehat{2}} \simeq S^0\widehat{_2} \vee BS^{3\,\text{Ad } BS^3}\widehat{_2} \vee BSO(3)^{\text{Ad } SO(3)}\widehat{_2}$$
$$\vee \prod_1^\infty BO(2)^{\text{Ad } O(2)}\widehat{_2} \vee \prod_1^\infty BZ_{2+\widehat{2}}.$$

Corollary 4.17.

$$DBN_{2+\widehat{2}} = S^0\widehat{_2} \vee BZ_{2+\widehat{2}} \vee BN_2^{\text{Ad } BN_2}\widehat{_2}$$
$$\vee \prod_1^\infty BO(2)^{\text{Ad } O(2)}\widehat{_2} \vee \prod_1^\infty BZ_{2+\widehat{2}}.$$

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