MULTIPLIERS OF FAMILIES OF CAUCHY-STIELTJES TRANSFORMS

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This paper is dedicated to Glenn Schober

Abstract. For $\alpha > 0$ let $\mathcal{F}_\alpha$ denote the class of functions defined for $|z| < 1$ by integrating $1/(1-xz)^\alpha$ against a complex measure on $|x| = 1$. A function $g$ holomorphic in $|z| < 1$ is a multiplier of $\mathcal{F}_\alpha$ if $f \in \mathcal{F}_\alpha$ implies $gf \in \mathcal{F}_\alpha$. The class of all such multipliers is denoted by $\mathcal{M}_\alpha$. Various properties of $\mathcal{M}_\alpha$ are studied in this paper. For example, it is proven that $\alpha < \beta$ implies $\mathcal{M}_\alpha \subset \mathcal{M}_\beta$, and also that $\mathcal{M}_\alpha \subset H^\infty$. Examples are given of bounded functions which are not multipliers. A new proof is given of a theorem of Vinogradov which asserts that if $f'$ is in the Hardy class $H^1$, then $f \in \mathcal{M}_1$. Also the theorem is improved to $f' \in H^1$ implies $f \in \mathcal{M}_\alpha$, for all $\alpha > 0$. Finally, let $\alpha > 0$ and let $f$ be holomorphic in $|z| < 1$. It is known that $f$ is bounded if and only if its Cesàro sums are uniformly bounded in $|z| \leq 1$. This result is generalized using suitable polynomials defined for $\alpha > 0$.

Let $\Delta = \{z: |z| < 1\}$ and $\Gamma = \{z: |z| = 1\}$, and let $\mathcal{M}$ denote the set of complex-valued Borel measures on $\Gamma$. For $\alpha > 0$, let $\mathcal{F}_\alpha$ denote the family of functions $f$ for which there exists $\mu \in \mathcal{M}$ such that

$$f(z) = \int_{\Gamma} \frac{1}{(1-xz)^\alpha} d\mu(x), \quad |z| < 1.$$ (1)

Here we choose the branch of $1/(1-z)^\alpha$ which equals 1 when $z = 0$.

This class of functions has been studied extensively in the case $\alpha = 1$ [1, 7, 8, 10, 15, 16]. More recently, the families $\mathcal{F}_\alpha (\alpha \neq 1)$ were introduced in [13]. Closure properties of the families $\mathcal{F}_\alpha$ were studied by the present authors in [9].

The following two results were proven in [13], and will be useful here.

Theorem A. For $\alpha > 0$, $f \in \mathcal{F}_\alpha$ if and only if $f' \in \mathcal{F}_{\alpha+1}$.

Theorem B. If $f \in \mathcal{F}_\alpha$ and $g \in \mathcal{F}_\beta$, then $fg \in \mathcal{F}_{\alpha+\beta}$.

For $f \in \mathcal{F}_\alpha$, let

$$\|f\|_{\mathcal{F}_\alpha} = \inf \{\|\mu\|: \mu \in \mathcal{M} \text{ such that (1) holds}\}.$$ (2)

With this norm, $\mathcal{F}_\alpha$ is a Banach space. As an example, suppose that $f \in \mathcal{F}_\alpha$, $\mu$ is a positive measure, and (1) holds. Then $\|f\|_{\mathcal{F}_\alpha} = \|\mu\|$. In the case $\alpha = 1$,
this was first observed by P. Bourdon and J. A. Cima, who showed in [1] that if \( \nu \in \mathcal{M} \) is any other representing measure for \( f \), then

\[ \| \mu \| = \mu(\Gamma) = f(0) = \int_{\Gamma} 1 \, d\nu(x) \leq \| \nu \|. \]

We note that by an easy argument, the infimum in (2) is actually attained.

Let \( \{ f_n : n = 1, 2, \ldots \} \) be a sequence of functions in \( \mathcal{F}_\alpha \) and suppose that \( f_n \to f \) in the norm (2). It is easy to show that this implies that \( f_n \to f \) uniformly on compact sets. To see that the converse is false in the case \( \alpha = 1 \), let \( f_n(z) = z^n \) for \( |z| < 1 \). Then \( f_n \) converges uniformly on compact sets to the function \( f(z) = 0 \). On the other hand, suppose that \( \mu_n \in \mathcal{M} \) is any measure representing \( f_n \). Then since

\[ z^n = \int_{\Gamma} \frac{1}{1-xz} \, d\mu_n(x), \]

it follows that

\[ 1 = \int_{\Gamma} x^n \, d\mu_n(x) \leq \int_{\Gamma} 1 \, d|\mu_n|(x) = \| \mu_n \|. \]

This shows that for each \( n \), \( \| f_n \|_{\mathcal{F}_1} \geq 1 \), so that the sequence \( f_n \) does not converge to \( f \) in norm. In the case \( \alpha \neq 1 \), a similar example can be constructed.

**Definition.** Suppose that \( f \) is holomorphic in \( \Delta \). Then \( f \) is called a multiplier of \( \mathcal{F}_\alpha \) if \( g \in \mathcal{F}_\alpha \Rightarrow fg \in \mathcal{F}_\alpha \).

The family of all such multipliers is denoted by \( \mathcal{M}_\alpha \).

Suppose that \( f \in \mathcal{M}_\alpha \) for some \( \alpha > 0 \). An application of the Closed Graph Theorem shows that the map \( \Lambda : \mathcal{F}_\alpha \to \mathcal{F}_\alpha \) defined by \( \Lambda(g) = fg \) is continuous. Equivalently, \( \Lambda \) is a bounded operator on \( \mathcal{F}_\alpha \), so that

\[ \sup \{ \| fg \|_{\mathcal{F}_\alpha} : g \in \mathcal{F}_\alpha, \| g \|_{\mathcal{F}_\alpha} \leq 1 \} < \infty. \]

This last quantity will be denoted by \( \| f \|_{\mathcal{M}_\alpha} \), and with this norm \( \mathcal{M}_\alpha \) is itself a Banach space.

This paper is concerned with the multiplier families \( \mathcal{M}_\alpha \). The family \( \mathcal{M}_1 \) has been studied in [10], [15], and [16], and various properties of \( \mathcal{M}_1 \) which were developed there will be generalized to \( \mathcal{M}_\alpha \) for \( \alpha \neq 1 \). For example, S. A. Vinogradov [16] has shown that if \( f' \) is in the Hardy space \( H^1 \), then \( f \in \mathcal{M}_1 \).

We give a new proof of this result, and show that if \( f' \in H^1 \), then \( f \in \mathcal{M}_\alpha \), for every \( \alpha > 0 \). Also we show that if \( f \in \mathcal{M}_\alpha \), then \( f \) is bounded, and that \( f \) has a number of other properties. Examples are given of bounded functions which are not in any \( \mathcal{M}_\alpha \) for \( \alpha > 0 \).

Finally, suppose that \( f' \) is holomorphic in \( \Delta \), and let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \). Let

\[ \sigma_n(z) = \sum_{j=0}^{n} \frac{n-j+1}{n+1} a_j z^j. \]

It is a classical result that \( f \) is bounded if and only if the Cesàro sums \( \sigma_n(z) \) are uniformly bounded for \( |z| \leq 1 \), and that in this case \( \| \sigma_n \|_{H^\infty} \leq \| f \|_{H^\infty} \),
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This result is generalized here where $\sigma_n$ is replaced by suitable polynomials depending on $\alpha > 0$.

In this section various properties of the families $\mathcal{M}_\alpha$ are studied. The following lemma will be useful.

**Lemma 2.1.** Let $f$ be holomorphic in $\Delta$, and let $\alpha > 0$. Then $f \in \mathcal{M}_\alpha$ if and only if $f(z)/(1-xz)^\alpha \in \mathcal{T}_\alpha$ for every $x$ with $|x| = 1$ and there exists a constant $M$ such that $\|f(z)/(1-xz)^\alpha\|_{\mathcal{S}_\alpha} \leq M$ for $|x| = 1$.

**Proof.** First suppose that $f \in \mathcal{M}_\alpha$. Then multiplication by $f$ is a bounded operator on $\mathcal{T}_\alpha$, and there is a constant $M$ such that

$$\|fg\|_{\mathcal{S}_\alpha} \leq M\|g\|_{\mathcal{S}_\alpha}$$

for all $g \in \mathcal{T}_\alpha$. In particular, (3) holds for all functions of the form $g(z) = 1/(1-xz)^\alpha$, where $|x| = 1$. Since $\|1/(1-xz)^\alpha\|_{\mathcal{S}_\alpha} = 1$, this implies that $\|f(z)/(1-xz)^\alpha\|_{\mathcal{S}_\alpha} \leq M$ for all $|x| = 1$.

For the converse, let $g \in \mathcal{T}_\alpha$. Then for some $\mu \in \mathcal{M}$,

$$g(z) = \int_{\Gamma} \frac{1}{(1-xz)^\alpha} d\mu(x).$$

To show that $fg \in \mathcal{T}_\alpha$, it is enough to consider the case in which $\mu$ is a probability measure. Then $g$ is the limit in the topology of uniform convergence on compact subsets of $\Delta$ of functions of the form

$$h(z) = \sum_{k=1}^{n} \mu_k \frac{1}{(1-x_k z)^\alpha}$$

where $\mu_k \geq 0$, $\sum_{k=1}^{n} \mu_k = 1$, $|x_k| = 1$, and $n$ is a natural number.

For such a function $h$,

$$f(z)h(z) = \sum_{k=1}^{n} \mu_k \frac{f(z)}{(1-x_k z)^\alpha}. \hspace{1cm} (4)$$

By the assumption, there is a measure $\nu_k \in \mathcal{M}$ with $\|\nu_k\| \leq M$ such that

$$\frac{f(z)}{(1-x_k z)^\alpha} = \int_{\Gamma} \frac{1}{(1-x z)^\alpha} d\nu_k(x).$$

Letting $\lambda = \sum_{k=1}^{n} \mu_k \nu_k$, (4) can be written as

$$f(z)h(z) = \int_{\Gamma} \frac{1}{(1-x z)^\alpha} d\lambda(x),$$

where $\lambda \in \mathcal{M}$ and $\|\lambda\| \leq \sum_{k=1}^{n} \mu_k \|\nu_k\| \leq M \sum_{k=1}^{n} \mu_k = M$.

Since $\{\lambda \in \mathcal{M} : \|\lambda\| \leq M\}$ is compact, an argument using subsequences now yields a measure $\sigma \in \mathcal{M}$ with $\|\sigma\| \leq M$ and $f(z)g(z) = \int_{\Gamma} 1/(1-xz)^\alpha d\sigma(x)$. Therefore $fg \in \mathcal{T}_\alpha$, and $f \in \mathcal{M}_\alpha$. 

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Theorem 2.2. If $0 < \alpha < \beta$, then $M_\alpha \subset M_\beta$.

Proof. Let $f \in M_\alpha$. By 2.1, it is enough to show that $f(z)/(1 - xz)^\beta \in S_\beta$ for every $x$ with $|x| = 1$, and to show that there is a constant $N$ such that $\|f(z)/(1 - xz)^\beta\| S_\beta \leq N$, for $|x| = 1$.

Since $f \in M_\alpha$, the lemma implies that there is a constant $M$ with

$$\|f(z)/(1 - xz)^\alpha\| S_\beta \leq M,$$

for $|x| = 1$.

Equivalently, for any $x$ with $|x| = 1$, there is a measure $\mu_x \in M$ such that

$$\frac{f(z)}{(1 - xz)^\beta} = \int_\Gamma \frac{1}{1 - yz} d\mu_x(y) \leq M.$$

and $\|\mu_x\| \leq M$.

Since

$$\frac{f(z)}{(1 - xz)^\beta} = \frac{f(z)}{(1 - xz)^\alpha} \frac{1}{(1 - xz)^\beta - \alpha},$$

(5) yields that

$$\frac{f(z)}{(1 - xz)^\beta} = \left\{ \int_\Gamma \frac{1}{1 - yz} d\mu_x(y) \right\} \frac{1}{(1 - xz)^\beta - \alpha} = \int_\Gamma \frac{1}{1 - yz} \frac{1}{(1 - xz)^\beta - \alpha} d\mu_x(y).$$

For every $x$ and $y$ with $|x| = |y| = 1$, there is a probability measure $\nu_{x,y}$ such that

$$\frac{1}{(1 - yz)^\alpha} \frac{1}{(1 - xz)^\beta - \alpha} = \int_\Gamma \frac{1}{1 - wz} d\nu_{x,y}(w).$$

Therefore,

$$\frac{f(z)}{(1 - xz)^\beta} = \int_\Gamma \int_\Gamma \frac{1}{1 - wz} d\nu_{x,y}(w) d\mu_x(y).$$

Because $\|\nu_{x,y}\| \leq 1$ and $\|\mu_x\| \leq M$, an argument as in the proof of Lemma 2.1 shows that there is a measure $\lambda \in M$ with $\|\lambda\| \leq M$ and such that

$$\frac{f(z)}{(1 - xz)^\beta} = \int_\Gamma \frac{1}{1 - sz} d\lambda(s).$$

This shows that $f(z)/(1 - xz)^\beta \in S_\beta$, and that $\|f(z)/(1 - xz)^\beta\| S_\beta \leq M$.

Next we obtain several properties of functions in $M_\alpha$. First it is shown that such functions are bounded.

Theorem 2.3. Let $\alpha > 0$ and let $f \in M_\alpha$. Then $f \in H^\infty$, and $\|f\|_{H^\infty} \leq \|f\|_{M_\alpha}$.

Proof. Let $M$ be a constant with $\|f\|_{M_\alpha} < M$. Let $z_0 = re^{i\theta}$ ($0 \leq r < 1$) and let $x = e^{-i\theta}$.

Since $f \in M_\alpha$, there is a measure $\mu_x \in M$ with $\|\mu_x\| < M$ and such that

$$\frac{f(z)}{(1 - xz)^\alpha} = \int_\Gamma \frac{1}{1 - yz} d\mu_x(y).$$

It follows that

$$f(z) = \int_\Gamma \left( \frac{1 - xz}{1 - yz} \right)^\alpha d\mu_x(y).$$
Letting $z = z_0$ in (6) yields

$$
|f(re^{i\theta})| = \left| \int_\Gamma \left( \frac{1 - r}{1 - rxy} \right)^\alpha \, d\mu_x(y) \right| \leq \int_\Gamma |d\mu_x(y)| < M.
$$

Since (7) holds for all $r$ and $\theta$, it follows that $f \in H^\infty$ and $\|f\|_{H^\infty} < M$, for every $M$ with $M > \|f\|_{\mathcal{A}_\alpha}$. Therefore, $\|f\|_{H^\infty} \leq \|f\|_{\mathcal{A}_\alpha}$.

**Theorem 2.4.** Let $\alpha > 0$, and let $f \in \mathcal{M}_\alpha$. Then $f \in \mathcal{F}_\alpha$, and $\|f\|_{\mathcal{F}_\alpha} \leq \|f\|_{\mathcal{A}_\alpha}$.

**Proof.** Let $I(z) = 1$ for $|z| < 1$. Since

$$
I(z) = \int_\Gamma \frac{1}{(1 - xz)^\alpha} \, dm(x),
$$

where $m$ denotes normalized Lebesgue measure, $I \in \mathcal{F}_\alpha$. Also, since $m$ is a positive measure, the remark in §1 shows that

$$
\|I\|_{\mathcal{F}_\alpha} = \|m\| = 1.
$$

Since $f \in \mathcal{M}_\alpha$ and $I \in \mathcal{F}_\alpha$, it follows that $f = fI \in \mathcal{F}_\alpha$. Also, since

$$
\|f\|_{\mathcal{F}_\alpha} = \|fI\|_{\mathcal{F}_\alpha} \leq \|f\|_{\mathcal{A}_\alpha} \|I\|_{\mathcal{F}_\alpha},
$$

(8) implies that

$$
\|f\|_{\mathcal{F}_\alpha} \leq \|f\|_{\mathcal{A}_\alpha}.
$$

We note that the inequality (9) is sharp, because $I \in \mathcal{M}_\alpha$ and $\|I\|_{\mathcal{F}_\alpha} = 1$.

As an application of Theorem 2.4, let

$$
\frac{1}{(1 - z)^\alpha} = \sum_{n=0}^\infty A_n(\alpha) z^n \quad (|z| < 1),
$$

and suppose that $f \in \mathcal{M}_\alpha$ where $f(z) = \sum_{n=0}^\infty a_n z^n$ ( $|z| < 1$). The theorem asserts that for some $\mu \in \mathcal{M}$,

$$
f(z) = \int_\Gamma \frac{1}{(1 - xz)^\alpha} \, d\mu(x).
$$

Equations (10) and (11) imply that

$$
a_n = A_n(\alpha) \int_\Gamma x^n \, d\mu(x).
$$

Since $A_n(\alpha) = O(n^{\alpha-1})$, and since $|\int_\Gamma x^n \, d\mu(x)| \leq \|\mu\|$, this shows that the coefficients of $f$ obey $|a_n| = O(n^{\alpha-1})$.

In the case $0 < \alpha < 1$, this coefficient estimate provides additional information on functions in $\mathcal{M}_\alpha$. Suppose that $f$ is holomorphic in $\Delta$, and that $f(z) = \sum_{n=0}^\infty a_n z^n$. In [16] it was shown that if $\sum_{n=0}^\infty |a_n| \log(n + 2) < \infty$, then $f \in \mathcal{M}_1$. In particular, the function $f(z) = \sum_{n=0}^\infty (1/n^3) z^{2n}$ is in $\mathcal{M}_1$, but for $m = 2^n$, $a_m \neq O(m^{\alpha-1})$, for each $\alpha$ ($0 < \alpha < 1$). This shows that $f \notin \mathcal{M}_\alpha$ for $\alpha < 1$. The first author and E. A. Nordgren have shown that $\mathcal{M}_1 \neq \mathcal{M}_2$, and also that for $0 < \alpha < \beta < 1$, $\mathcal{M}_\alpha \neq \mathcal{M}_\beta$. It is an open question to determine if $\mathcal{M}_\alpha \neq \mathcal{M}_\beta$ for all $\alpha \neq \beta$.

It was shown in [9] that $\mathcal{F}_\alpha$ is closed under composition with disk automorphisms $z \rightarrow (z + \xi)/(1 + \xi z)$, where $|\xi| < 1$. This will be used in the proof of the next theorem, which asserts the same result for $\mathcal{M}_\alpha$. 

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Theorem 2.5. Let \( \alpha > 0 \). If \( f \in \mathcal{M}_\alpha \), \( |\xi| < 1 \), and \( g(z) = f((z + \xi)/(1 + \overline{\xi}z)) \), then \( g \in \mathcal{M}_\alpha \).

Proof. Let \( h \in \mathcal{F}_\alpha \), and let \( k(z) = h((z - \xi)/(1 - \overline{\xi}z)) \). Since the map \( w = (z - \xi)/(1 - \overline{\xi}z) \) is an automorphism of \( \Delta \), the result in [9] quoted above shows that \( k \in \mathcal{F}_\alpha \). Since \( f \in \mathcal{M}_\alpha \), it follows that \( m = fk \in \mathcal{F}_\alpha \). A second application of the result in [9] implies that \( m((z + \xi)/(1 + \overline{\xi}z)) \in \mathcal{F}_\alpha \). Since

\[
m \left( \frac{z + \xi}{1 + \overline{\xi}z} \right) = f \left( \frac{z + \xi}{1 + \overline{\xi}z} \right) k \left( \frac{z + \xi}{1 + \overline{\xi}z} \right) = g(z)h(z),
\]

this shows that \( g \in \mathcal{M}_\alpha \).

The following theorem generalizes a result in [16], which showed that if \( f \in \mathcal{M}_1 \), then \( f \) has finite radial variation in every direction.

Theorem 2.6. For each \( \alpha > 0 \) there is a constant \( A_\alpha \) such that if \( f \in \mathcal{M}_\alpha \), then the radial variation of \( f \) in the direction \( \theta \) obeys \( V(f, \theta) \leq A_\alpha \|f\|_{\mathcal{M}_\alpha} \) for all \( \theta \).

Proof. Suppose that \( f \in \mathcal{M}_\alpha \) for some \( \alpha > 0 \). If \( |\xi| = 1 \) then there is a measure \( \mu_\xi \) such that

\[
f(z) = \frac{1}{(1 - \xi z)^\alpha} = \int \frac{1}{(1 - xz)^\alpha} d\mu_\xi(x).
\]

Also, if \( M = \|f\|_{\mathcal{M}_\alpha} \), and \( \epsilon > 0 \), then \( \|\mu_\xi\| \leq M + \epsilon \) for \( |\xi| = 1 \).

It follows from (12) that

\[
f'(z) = \alpha \int \frac{(1 - \xi z)^{\alpha-1}(x - \xi)}{(1 - xz)^\alpha} d\mu_\xi(x),
\]

and therefore

\[
\int_0^1 |f'(r\xi)| dr \leq \alpha \int \left[ \int_0^1 \frac{(1 - r)^{\alpha-1}|x - \xi|}{|1 - rx\xi|^{\alpha+1}} dr \right] d|\mu_\xi|(x).
\]

Let \( I \) denote the inner integral on the right-hand side of (13). Because

\[
|1 - rx\xi|^{\alpha+1} = \left(1 - r^2|\xi|^2\right)^{\alpha+1/2} = \left((1 - r)^2 + r|1 - x\xi|^2\right)^{\alpha+1/2} \geq \left((1 - r)^2 + r^2|1 - x\xi|^2\right)^{\alpha+1/2},
\]

it follows that

\[
I \leq \int_0^1 \frac{(1 - r)^{\alpha-1}b}{\left((1 - r)^2 + r^2|1 - x\xi|^2\right)^{\alpha+1/2}} dr \equiv J,
\]

where \( b = |1 - x\xi| \). The change of variables \( y = rb/(1 - r) \) shows that \( J = \int_0^\infty \frac{1}{(1 + y^2)^{\alpha+1/2}} dy \equiv B_\alpha \). This integral converges since \( \int_1^\infty 1/y^{\beta} dy \) converges for \( \beta > 1 \). Therefore (13) yields that

\[
\int_0^1 |f'(r\xi)| dr \leq \alpha \int \left[ \int_0^1 B_\alpha d|\mu_\xi|(x) \leq A_\alpha (M + \epsilon),
\]

where \( A_\alpha = \alpha B_\alpha \). Let \( \epsilon \to 0 \), the theorem is established. \( \square \)

Let \( f \in \mathcal{M}_\alpha \). As a consequence of Theorem 2.6, the radial limit \( \lim_{r \to 1} f(re^{i\theta}) \) exists for all \( \theta \). Also, note that the conclusion of the theorem implies that \( f \) is bounded.
As an application of Theorem 2.6, we next give a number of simple examples of bounded functions which are not in $M_\alpha$ for any $\alpha > 0$.

As a first example, let $f(z) = (1 - z)^{-1}$, using the principal branch of the logarithm. Then $f$ is holomorphic in $\Delta$, and since $|f(z)| = e^{-\text{Arg}(1-z)}$, it follows that $|f(z)| < e^{\pi/2}$ for $|z| < 1$. It is easy to verify that $f$ maps the interval $[0, 1)$ onto the circle $\Gamma$ covered infinitely often and hence the curve $w = f(r)$, $0 \leq r < 1$, is not rectifiable. It follows by Theorem 2.6 that $f \notin M_\alpha$ for any $\alpha > 0$.

In [9], it was shown that if $f$ is holomorphic in $\Delta$, then $f \in M_\alpha$ for all $\alpha > 0$. In particular, this implies that a finite Blaschke product belongs to $M_\alpha$ for $\alpha > 0$. Theorem 2.5 provides a second proof of this fact, as follows. Let $I(z) = z$ for $|z| < 1$. It is clear that $I \in M_\alpha$ for $\alpha > 0$. If $|\zeta| < 1$, then Theorem 2.5 implies that

$$I \left( \frac{z + \zeta}{1 + \zeta \bar{z}} \right) = \frac{z + \zeta}{1 + \zeta \bar{z}} \in M_\alpha, \quad \text{for } \alpha > 0.$$ 

Since the finite product of functions in $M_\alpha$ is itself in $M_\alpha$, this proves the assertion.

We next show that there are infinite Blaschke products which are not in $M_\alpha$ for any $\alpha > 0$. Let $f(z) = \prod_{n=1}^{\infty} (a_n - z)/(1 - a_n z)$ where $a_n = 1 - 1/2^n$, $n = 1, 2, \ldots$. In [6] it was shown that there is a constant $A > 0$ such that if $\rho_n = 1/2(a_n + a_{n+1})$ then $|f(\rho_n)| \geq A$ for $n = 1, 2, \ldots$. It follows that $\int_0^1 |f'(r)| \, dr = \infty$, so that by Theorem 2.6, $f \notin M_\alpha$ for $\alpha > 0$.

We note that in [10], it was proved that an inner function belongs to $M_1$ if and only if it is a Blaschke product with the sequence of zeros satisfying the Frostman condition.

The next example shows that a function holomorphic in $\Delta$ and continuous in $\overline{\Delta}$ need not be in $M_\alpha$ for any $\alpha > 0$. In [17], L. Zalcman described a bounded region $D$ such that $\partial D$ is a Jordan curve, $z = 1 \in \partial D$, and $z = 1$ is not rectifiably accessible from the interior of $D$. Since $\partial D$ is a Jordan curve, any conformal mapping of $\Delta$ onto $D$ extends continuously to $\overline{\Delta}$. Let $f$ be such a map with $f(1) = 1$. Then $f \notin M_\alpha$, since the curve $w = f(r)$, $0 \leq r \leq 1$, is not rectifiable. The argument in [17] even shows that the power series for $f$ is uniformly convergent on $\partial \Delta$. Hence even with this additional condition we can still have $f \notin M_\alpha$ for all $\alpha > 0$.

The examples above give bounded functions for which the radial variation in one direction is infinite. A stronger result is presented in [14], where examples are given of infinite Blaschke products $B(z)$ for which the radial variation $V(B, \theta) = \infty$ for almost all $\theta$. Also, [14] includes the construction of a function $f$ holomorphic in $\Delta$ and continuous in $\overline{\Delta}$ for which $V(f, \theta) = \infty$ for almost all $\theta$.

In this section a condition is shown to be sufficient for membership in $M_\alpha$ for every $\alpha > 0$. Let $H^1$ denote the Hardy space of functions $f$ that are holomorphic in $\Delta$ and such that

$$\sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta < \infty.$$
In [16, p. 20] it was proved by Vinogradov that if \( f' \in H^1 \) then \( f \in \mathcal{M}_1 \). This result is generalized to \( f' \in H^1 \) implies \( f \in \mathcal{M}_\alpha \) for every \( \alpha > 0 \). This strengthens the result in [9] which asserts that if \( f \) is holomorphic in \( \Delta \) then \( f \in \mathcal{M}_\alpha \) for every \( \alpha > 0 \).

We begin by giving a new proof of Vinogradov's theorem. It may have independent interest especially since it shows that this result is related to the class of functions of bounded mean oscillation [5, p. 222]. Let \( \mathcal{B} \) denote the set of functions \( f \) holomorphic in \( \Delta \) which can be expressed as \( f = g + h \), where \( g \) and \( h \) are holomorphic in \( \Delta \), \( \Re g \) is bounded in \( \Delta \), and \( \Im h \) is bounded in \( \Delta \). If \( f \in \mathcal{B} \) then \( \|f\|_\mathcal{B} \) is defined by \( \inf(\|\Re g\|_\infty + \|\Im h\|_\infty) \) where \( g \) and \( h \) vary over all pairs as above. Here \( \|u\|_\infty = \sup_{|z|<1} |u(z)| \) for any function \( u \) defined in \( \Delta \).

**Lemma 3.1.** Let \( f \) be holomorphic in \( \Delta \) and suppose that there is a holomorphic function \( g \) and a constant \( M > 0 \) such that

\[
|f(z) + g(\overline{z})| \leq M \quad \text{for } |z| < 1.
\]

Then \( f \in \mathcal{B} \) and \( \|f\|_\mathcal{B} \leq M \).

**Proof.** Let \( s = \Re f \), \( t = \Im f \), \( u = \Re g \), and \( v = \Im g \). The function \( G \) defined by \( G(z) = \frac{1}{2}[f(z) + g(\overline{z})] \) is holomorphic in \( \Delta \) and \( \Re G(z) = \frac{1}{2}[s(z) + u(\overline{z})] \). Hence (14) implies that \( |\Re G(z)| \leq \frac{1}{2}M \) for \( |z| < 1 \). The function \( H \) defined by \( H(z) = \frac{1}{2}[f(z) - g(\overline{z})] \) is holomorphic in \( \Delta \) and \( |\Im H(z)| \leq \frac{1}{2}M \) for \( |z| < 1 \). Since \( f = G + H \) this yields \( f \in \mathcal{B} \). Moreover \( \|f\|_\mathcal{B} \leq \|\Re G\|_\infty + \|\Im H\|_\infty \leq M \).

**Lemma 3.2.** Let \( f \in H^\infty \) and let \( g \) be defined by

\[
g(z) = \frac{1}{z} \int_0^z \frac{f(w)}{1-w} \, dw
\]

for \( |z| < 1 \). Then \( |g'(z)| \leq B\|f\|_{H^\infty}/|1-z| \) for \( |z| < 1 \), where \( B \) is an absolute constant.

**Proof.** We first show that if \( |z| < 1 \) and \( \alpha \) is the line segment from \( w = 0 \) to \( w = z \) then

\[
\int_{\alpha} \frac{1}{|1-w|^2} \, |dw| \leq \frac{\pi}{2} \frac{|z|}{|1-z|}.
\]

This is clear if \( z = 0 \). Also if \( z \) is real and \( z \neq 0 \) then we have

\[
\int_{\alpha} \frac{1}{|1-w|^2} \, |dw| = |z| \int_0^1 \frac{1}{(1-tz)^2} \, dt = \frac{|z|}{1-z},
\]

and hence (16) follows. Henceforth assume that \( |z| < 1 \) and \( z \) is not real. Then

\[
\int_{\alpha} \frac{1}{|1-w|^2} \, |dw| = |z| \int_0^1 \frac{1}{(1-tz)(1-t\overline{z})} \, dt
\]

\[
= \frac{|z|}{z-\overline{z}} \left\{ \log \frac{1}{1-z} - \log \frac{1}{1-\overline{z}} \right\}
\]

\[
= \frac{|z|}{z-\overline{z}} \int_{\beta} \frac{1}{1-w} \, dw
\]
where $\beta$ is the arc on the circle that is centered at $w = 1$ and goes from $z$ to $\overline{z}$. Let $\theta$ denote the angle subtended by the arc $\beta$ and let $L$ denote the length of $\beta$. Then $|z - \overline{z}| = 2|1 - z|\sin(\theta/2)$ and $L = |1 - z|\theta$. Therefore

$$\int_0^1 \frac{1}{|1 - w|^2} \, dw \leq \frac{|z|}{|1 - z|} \frac{1}{|z - \overline{z}|} L = \frac{\theta/2}{\sin(\theta/2)} \frac{|z|}{|1 - z|} \leq \frac{\pi}{2} \frac{|z|}{|1 - z|},$$

since $0 < \theta/2 \leq \pi/2$. This proves (16).

From (15) we obtain $z g'(z) + g(z) = f(z)/(1 - z)$. Hence an integration by parts yields

$$z^2 g'(z) = \frac{zf(z)}{1 - z} - zg(z) = \frac{zf(z)}{1 - z} - \int_0^z \frac{f(w)}{1 - w} \, dw$$

$$= \frac{zf(z)}{1 - z} - h(z) + \int_0^z \frac{h(w)}{(1 - w)^2} \, dw,$$

where

$$h(z) = \int_0^z f(w) \, dw$$

for $|z| < 1$. Clearly (17) implies $\|h\|_{H^\infty} \leq \|f\|_{H^\infty} = M$. It follows that

$$|z^2 g'(z)| \leq \frac{M}{|1 - z|} + \frac{M}{|1 - z|} + M \int_0^1 \frac{1}{|1 - w|^2} |dw|.$$ 

Therefore (16) implies that

$$|z^2 g'(z)| \leq \left(2 + \frac{\pi}{2}\right) M \frac{1}{|1 - z|} \quad \text{for } |z| < 1.$$

The function $G$ defined by $G(z) = (1 - z)z^2 g'(z)$ is analytic in $\Delta$, has at least a second order zero at $z = 0$ and satisfies $|G(z)| \leq BM$ for $|z| < 1$ where $B = 2 + \pi/2$. Hence $|G(z)| \leq BM|z|^2$ for $|z| < 1$ and therefore $|g'(z)| \leq BM/|1 - z|$.

**Lemma 3.3.** Suppose that $f \in H^\infty$ and $g$ is defined by

$$g(z) = \frac{1}{z} \int_0^z \frac{f(w)}{1 - w} \, dw$$

for $|z| < 1$. Then $g \in \mathcal{B}$ and $\|g\|_{\mathcal{B}} \leq A\|f\|_{H^\infty}$ where $A$ is an absolute constant.

**Proof.** By equation (18) and Lemma 3.2, there is an absolute constant $B$ such that

$$|g'(z)| \leq \frac{B\|f\|_{H^\infty}}{|1 - z|} \quad \text{for } |z| < 1.$$

Let $|z| < 1$ and let $\gamma$ denote the circle centered at 1 which passes through $z$ and has radius $r = |1 - z|$. Let $\delta$ denote the subarc of $\gamma$ from $\overline{z}$ to $z$. Then

$$g(z) - g(\overline{z}) = \int_\delta g'(w) \, dw$$

and hence (19) implies that

$$|g(z) - g(\overline{z})| \leq \frac{B\|f\|_{H^\infty}}{r} (\text{length of } \delta) \leq \frac{\pi}{2} B\|f\|_{H^\infty}.$$
An application of Lemma 3.1 in the special case where the functions there are related by $g = -f$ implies that $g \in \mathcal{B}$ and $\|g\|_{\mathcal{B}} \leq A\|f\|_{H^\infty}$ where $A = \pi B/2$. \hfill $\square$

**Lemma 3.4.** Suppose that $f$ and $g$ are functions holomorphic in $\overline{\Delta}$ and let $F$ and $G$ be defined by

(20) \[ F(z) = \frac{1}{1 - z} \int_z^1 f(w) \, dw \]

and

(21) \[ G(z) = \frac{1}{z} \int_0^z \frac{1}{1 - w} g(w) \, dw. \]

Then

(22) \[ \int_0^{2\pi} f(e^{i\theta}) G(e^{-i\theta}) \, d\theta = \int_0^{2\pi} F(e^{i\theta}) G(e^{-i\theta}) \, d\theta. \]

**Proof.** There is a number $R > 1$ such that $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$ for $|z| < R$. Then $F$ also is holomorphic in $\{z: |z| < R\}$ and $G$ is holomorphic in $\Delta$ except possibly for a logarithmic singularity at $z = 1$. In particular, $G \in H^1$ (in fact, $G \in H^p$ for all $p > 0$). For $|z| < R$ we have

\[
F(z) = \frac{1}{1 - z} \int_z^1 \left( \sum_{n=0}^\infty a_n w^n \right) \, dw = \frac{1}{1 - z} \sum_{n=0}^\infty a_n \left( 1 - z^{n+1} \right) 
\]

\[
= \sum_{n=0}^\infty \left\{ \frac{a_n}{n+1} \sum_{k=0}^n z^k \right\} = \sum_{n=0}^\infty \left\{ \sum_{k=n}^\infty \frac{a_k}{k+1} \right\} z^n.
\]

Therefore

(23) \[ \int_0^{2\pi} F(e^{i\theta}) G(e^{-i\theta}) \, d\theta = 2\pi \sum_{n=0}^\infty \left( \sum_{k=n}^\infty \frac{a_k}{k+1} \right) b_n = 2\pi \sum_{n=0}^\infty \left\{ \frac{a_n}{n+1} \sum_{k=0}^n b_k \right\}. \]

For $|z| < 1$, we have

\[
G(z) = \frac{1}{z} \int_0^z \left( \sum_{n=0}^\infty b_n w^n \right) \left( \sum_{n=0}^\infty a_n w^n \right) \, dw 
\]

\[
= \frac{1}{z} \int_0^z \left( \sum_{n=0}^\infty \left( \sum_{k=0}^n b_k \right) w^n \right) \, dw 
\]

\[
= \sum_{n=0}^\infty \left( \frac{1}{n+1} \sum_{k=0}^n b_k \right) z^n.
\]

If $0 < r < 1$ then

\[ \int_0^{2\pi} f(e^{i\theta}) G(re^{-i\theta}) \, d\theta = 2\pi \sum_{n=0}^\infty \left( \frac{a_n}{n+1} \sum_{k=0}^n b_k \right) r^n \equiv H(r). \]

Since the series defining $H$ converges at $r = 1$, Abel's theorem gives

(24) \[ \lim_{r \to 1^-} \int_0^{2\pi} f(e^{i\theta}) G(re^{-i\theta}) \, d\theta = \lim_{r \to 1^-} H(r) = 2\pi \sum_{n=0}^\infty \left( \frac{a_n}{n+1} \sum_{k=0}^n b_k \right). \]
Also, because $f(e^{i\theta})$ is bounded and $G \in H^1$ it follows that

$$\lim_{r \to 1-} \int_0^{2\pi} f(e^{i\theta})G(re^{-i\theta})d\theta = \int_0^{2\pi} f(e^{i\theta})G(e^{-i\theta})d\theta.$$  

Therefore by (23), (24), and (25),

$$\int_0^{2\pi} F(e^{i\theta})g(e^{-i\theta})d\theta = 2\pi \sum_{n=0}^{\infty} \left( \frac{a_n}{n+1} \sum_{k=0}^{n} b_k \right) = \int_0^{2\pi} f(e^{i\theta})G(e^{-i\theta})d\theta. \quad \square$$

We thank D. J. Hallenbeck for pointing out and rectifying an error in our initial proof of Lemma 3.4.

**Theorem C (Vinogradov).** If $f' \in H^1$, then $f \in \mathcal{M}_1$.

**Proof.** Suppose that $f' \in H^1$ and $|\xi| = 1$. We first note that

$$\frac{f(z)}{\xi - z} = \frac{1}{\xi} \frac{f(z)}{1 - \xi z}. \quad \text{Therefore by Lemma 2.1, it is enough to show that } f(z)/(\xi - z) \in \mathcal{F}_1, \text{ and that there is a constant } M > 0 \text{ such that } \|f(z)/(\xi - z)\|_{\mathcal{F}_1} \leq M \text{ for all } |\xi| = 1. \quad \text{Also note that}$$

$$\frac{f(z)}{\xi - z} = \frac{1}{\xi} \int_0^z f'(w)dw + \frac{f(0)}{\xi - z}. \quad \text{Since } f(0)/(\xi - z) \in \mathcal{F}_1 \text{ and since } \|f(0)/(\xi - z)\|_{\mathcal{F}_1} = |f(0)|, \text{ it suffices to show that the function } (\xi - z)^{-1} \int_0^z f'(w)dw \text{ belongs to } \mathcal{F}_1 \text{ and that for some } M > 0, \| (\xi - z)^{-1} \int_0^z f'(w)dw \|_{\mathcal{F}_1} \leq M \text{ for all } |\xi| = 1. \text{ The argument is carried out with } \xi = 1 \text{ and a similar argument serves for all } \xi \text{ providing the same bound on the norm.}$$

In our formulation we replace $f'$ by $f$. In other words, assume that $f \in H^1$ and let

$$g(z) = \frac{1}{1 - z} \int_0^z f(w)dw \quad \text{for } |z| < 1.$$  

Then $g(z) = b/(1 - z) - (1 - z)^{-1} \int_1^z f(w)dw$, where $b = \int_0^1 f(w)dw$.

First note that

$$|b| \leq \int_0^1 |f(w)||dw| \leq \int_1^1 |f(w)||dw| \leq \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|d\theta = \pi\|f\|_{H^1}. \quad [4, \text{ p. 46}].$$

It follows that

$$\left\| \frac{b}{1 - z} \right\|_{\mathcal{F}_1} \leq \pi\|f\|_{H^1} \quad (27)$$

Next let $k(z) = (1 - z)^{-1} \int_1^z f(w)dw$. Let $A$ denote the space of functions holomorphic in $\Delta$ and continuous in $\bar{\Delta}$. To show that $k \in \mathcal{F}_1$ it suffices to prove that there is a constant $A > 0$ such that

$$\int_0^{2\pi} k(re^{i\theta})h(e^{-i\theta})d\theta \leq A\|h\|_{H^\infty} \quad (28)$$
for $0 < r < 1$ and for all $h \in A$. This inequality will be obtained where $A = B\|f\|_{H^1}$ and $B$ is an absolute constant. This will imply that

$$\|k\|_{X_1} \leq B\|f\|_{H^1}$$

and it then follows from (26), (27), and (29) that $\|g\|_{X_1} \leq (\pi + B)\|f\|_{H^1}$.

By first making the change of variables $z \rightarrow \rho z$ where $0 < \rho < 1$ and then letting $\rho \rightarrow 1$, we may assume that $f$ and $h$ are holomorphic in $\Delta$. Then $k$ is holomorphic in $\Delta$. We now show that it suffices to prove that

$$\int_0^{2\pi} k(e^{i\theta})h(e^{-i\theta}) d\theta \leq C\|f\|_{H^1}\|h\|_{H^\infty},$$

where $C$ is an absolute constant. For $0 \leq r \leq 1$ let $F(r) = \int_0^{2\pi} k(re^{i\theta})h(e^{i\theta}) d\theta$. Assuming (30) we get $|F(1)| \leq C\|f\|_{H^1}\|h\|_{H^\infty}$. Since $F$ is continuous in $[0, 1]$, there exists $r_0$ ($0 < r_0 < 1$) such that $|F(r)| \leq 2|F(1)|$ for $r_0 \leq r \leq 1$. Therefore

$$|F(r)| \leq 2C\|f\|_{H^1}\|h\|_{H^\infty},$$

for $r_0 \leq r < 1$.

Suppose now that $0 \leq r \leq r_0$. Then

$$|F(r)| \leq \int_0^{2\pi} |k(re^{i\theta})||h(e^{i\theta})| d\theta \leq \|h\|_{H^\infty} \int_0^{2\pi} |k(re^{i\theta})| d\theta.$$

Without loss of generality we may assume that $f \neq 0$. Then $k \neq 0$, $\|f\|_{H^1} > 0$, and $\int_0^{2\pi} |k(re^{i\theta})| d\theta > 0$. Therefore for some $D > 0$, $\int_0^{2\pi} |k(re^{i\theta})| d\theta = D\|f\|_{H^1}$. It follows that

$$|F(r)| \leq D\|f\|_{H^1}\|h\|_{H^\infty},$$

for $0 \leq r \leq r_0$.

Letting $B = \max(2C, D)$, relations (31) and (32) imply that

$$|F(r)| \leq B\|f\|_{H^1}\|h\|_{H^\infty},$$

for $0 \leq r \leq 1$.

This proves (28).

It remains to prove the assertion (30). Let $m(z) = z^{-1} \int_0^z (1 - w)^{-1} h(w) dw$. Lemma 3.3 implies that $m \in \mathcal{B}$ and $\|m\|_{\mathcal{B}} \leq C\|h\|_{H^\infty}$ for an absolute constant $C$. We have $m = p + q$ where $p$ and $q$ are holomorphic in $\Delta$ and $u = \text{Re } p$ and $v = \text{Im } q$ are bounded and $\|u\|_{\infty} + \|v\|_{\infty} \leq C\|h\|_{H^\infty}$. Now

$$\int_0^{2\pi} f(e^{i\theta})m(e^{-i\theta}) d\theta = \int_0^{2\pi} f(e^{i\theta})p(e^{-i\theta}) d\theta + \int_0^{2\pi} f(e^{i\theta})q(e^{-i\theta}) d\theta.$$

Using power series and the orthonormal relations for the trigonometric functions, this equals

$$\int_0^{2\pi} f(e^{i\theta})u(e^{-i\theta}) d\theta + i \int_0^{2\pi} f(e^{i\theta})v(e^{-i\theta}) d\theta.$$

Hence

$$\int_0^{2\pi} f(e^{i\theta})m(e^{-i\theta}) d\theta \leq \|u\|_{\infty}\|f\|_{H^1} + \|v\|_{\infty}\|f\|_{H^1}$$

$$= (\|u\|_{\infty} + \|v\|_{\infty})\|f\|_{H^1}$$

$$\leq C\|f\|_{H^1}\|h\|_{H^\infty}.$$
Because of Lemma 3.4, this yields
\[
\left| \int_0^{2\pi} k(e^{i\theta})h(e^{-i\theta}) \, d\theta \right| \leq C\|f\|_{H^1}\|h\|_{H^\infty},
\]
which is the required inequality. □

The argument used to prove Theorem C does not depend on the duality theorem about \( H^1 \) and BMO proved by C. Fefferman [5, p. 245]. It is interesting to note that the function \( g \) defined in Lemma 3.3 can be shown to have bounded mean oscillation by a fairly direct argument.

The essential ideas for proving Theorem C as developed above are due to Boris Korenblum [12]. The authors would like to thank Korenblum for several helpful conversations about multipliers.

**Theorem 3.5.** If \( f' \in H^1 \), then \( f \in \mathcal{M}_\alpha \) for all \( \alpha > 0 \).

**Proof.** Let \( f' \in H^1 \). By Theorem C, \( f \in \mathcal{M}_1 \), and by Theorem 2.2 it follows that \( f \in \mathcal{M}_\alpha \) for every \( \alpha > 1 \).

In the case \( 0 < \alpha < 1 \), let \( g \in \mathcal{F}_\alpha \), and let \( h = fg \). By Theorem A, it suffices to show that \( h' \in \mathcal{F}_{\alpha+1} \).

Since \( g \in \mathcal{F}_\alpha \), Theorem A implies that \( g' \in \mathcal{F}_{\alpha+1} \). By the previous part of the proof, \( f \in \mathcal{M}_{\alpha+1} \), and therefore
\[
(33) \quad fg' \in \mathcal{F}_{\alpha+1}.
\]
Because \( f' \in H^1 \), it follows that \( f' \in \mathcal{F}_1 \) [4, p. 34]. By assumption, \( g \in \mathcal{F}_\alpha \) and so Theorem B implies that
\[
(34) \quad f'g \in \mathcal{F}_{\alpha+1}.
\]
Since \( h' = fg' + f'g \), (33) and (34) show that \( h' \in \mathcal{F}_{\alpha+1} \), or equivalently, \( h \in \mathcal{F}_\alpha \). This proves that \( f \in \mathcal{M}_\alpha \) for \( 0 < \alpha < 1 \). □

Theorem 3.5 is sharp, since there are functions \( f \) such that \( f' \in H^p \) \((0 < p < 1)\) and \( f \) is not bounded. By Theorem 2.3, such functions are not multipliers.

One example where Theorem 3.5 applies concerns bounded convex maps. Suppose that \( f \) is holomorphic in \( \Delta \) and that \( f \) maps \( \Delta \) one-to-one onto a bounded convex region. Since the boundary \( C \) of such a region is rectifiable and since \( C \) is a Jordan curve, it follows that \( f' \in H^1 \) [4, p. 44]. Therefore, \( f \in \mathcal{M}_\alpha \), for \( \alpha > 0 \).

4

Suppose that \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is holomorphic in \( \Delta \). Let
\[
s_n(z) = \sum_{j=0}^{n} a_j z^j
\]
and
\[
\sigma_n(z) = \frac{1}{n+1} \sum_{j=0}^{n} s_j(z).
\]
By a classical result [3, p. 439], the function \( f \) is bounded if and only if the sequence \( \sigma_n(z) \) is uniformly bounded for \( n = 0, 1, \ldots \) and for \( |z| \leq 1 \), and
in this case, \( \|f\|_{H^\infty} = \sup\{\|\sigma_n\|_{H^\infty}: n = 0, 1, \ldots \} \). This result is generalized in this section, in terms of polynomials which are generated in the study of the multiplier problem.

**Definition.** For \( f(z) = \sum_{n=0}^{\infty} a_n z^n \ (|z| < 1) \), let
\[
P_n(z; \alpha) = \frac{1}{A_n(\alpha)} \{ A_n(\alpha)a_0 + A_{n-1}(\alpha)a_1 z + \ldots + A_1(\alpha)a_{n-1} z^{n-1} + A_0(\alpha)a_n z^n \}
\]
where \( \alpha > 0 \), \( n = 0, 1, \ldots \), and \( z \in \mathbb{C} \).

**Theorem 4.1.** If \( f \in \mathcal{M}_\alpha \), then \( \|P_n(z; \alpha)\|_{H^\infty} \leq \|f\|_{\mathcal{M}_\alpha} \) for \( n = 0, 1, \ldots \).

**Proof.** Let \( f \in \mathcal{M}_\alpha \) and suppose that \( M > \|f\|_{\mathcal{M}_\alpha} \). If \( |x| = 1 \) then we have
\[
\begin{align*}
f(z)/(1 - xz)^\alpha &\in \mathcal{F}_\alpha. \\
\end{align*}
\]
Also,
\[
\left\| f(z)/(1 - xz)^\alpha \right\|_{\mathcal{F}_\alpha} \leq M \text{ for all } |x| = 1.
\]
Therefore for each \( x \ (|x| = 1) \) there is a measure \( \mu_x \in \mathcal{M} \) such that
\[
f(z)/(1 - xz)^\alpha = \int_{(1 - yz)^\alpha} \frac{1}{(1 - yz)^\alpha} d\mu_x(y),
\]
and \( \|\mu_x\| \leq M \) for \( |x| = 1 \).

If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then \( f(z)/(1 - xz)^\alpha = \sum_{n=0}^{\infty} b_n z^n \) where
\[
b_n = A_0(\alpha)a_n + A_1(\alpha)a_{n-1}x + \ldots + A_{n-1}(\alpha)a_1 x^{n-1} + A_n(\alpha)a_0 x^n.
\]
If
\[
\begin{align*}
\int_{(1 - yz)^\alpha} \frac{1}{(1 - yz)^\alpha} d\mu_x(y) &= \sum_{n=0}^{\infty} c_n z^n, \\
\end{align*}
\]
then
\[
c_n = A_n(\alpha) \int_{(1 - yz)^\alpha} y^n d\mu_x(y).
\]
Because of (35), \( b_n = c_n \), or
\[
x^n P_n\left(\frac{1}{x}; \alpha\right) = \int_{(1 - yz)^\alpha} y^n d\mu_x(y).
\]

Since \( \|\mu_x\| \leq M \) for \( |x| = 1 \), (36) implies that \( |P_n(1/x; \alpha)| \leq M \) for \( |x| = 1 \) and \( n = 0, 1, \ldots \). Equivalently \( |P_n(z; \alpha)| \leq M \) for \( |z| = 1 \) and hence \( \|P_n(z; \alpha)\|_{H^\infty} \leq M \). Since this holds for every \( M > \|f\|_{\mathcal{M}_\alpha} \), this proves the theorem. \( \Box \)

The next results generalize the statement made previously concerning the Cesàro sums \( \sigma_n(z) \) for a function holomorphic in \( \Delta \). Note that \( \sigma_n(z) = P_n(z; 2) \) since the binomial coefficient \( A_n(2) = n + 1 \) for \( n = 0, 1, \ldots \).

**Theorem 4.2.** Suppose that \( f \) is holomorphic in \( \Delta \) and that \( |P_n(z; \alpha)| \leq M \) for \( |z| \leq 1 \) and \( n = 0, 1, \ldots \). Then \( f \in H^\infty \) and \( \|f\|_{H^\infty} \leq M \).
Proof. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) for \( |z| < 1 \). Assume that \( 0 < r < 1 \) and \( |x| = 1 \). Then

\[
\frac{1}{(1-r)^\alpha} f(rx) = \left\{ \sum_{n=0}^{\infty} A_n(\alpha) r^n \right\} \left\{ \sum_{n=0}^{\infty} a_n r^n x^n \right\}
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} A_{n-k}(\alpha) a_k x^k \right) r^n
= \sum_{n=0}^{\infty} A_n(\alpha) P_n(x; \alpha) r^n.
\]

Therefore

\[
\frac{1}{(1-r)^\alpha} |f(rx)| \leq \sum_{n=0}^{\infty} A_n(\alpha) |P_n(x; \alpha)| r^n
\leq M \sum_{n=0}^{\infty} A_n(\alpha) r^n = M \frac{1}{(1-r)^\alpha},
\]

and so \( |f(rx)| \leq M \). Since this holds for all \( r \) and \( x \), it follows that \( |f(z)| \leq M \) for \( |z| < 1 \). \( \square \)

The following lemma will be used to establish a partial converse to Theorem 4.2. The kernels \( T_n(\theta; \alpha) \) introduced in the lemma are well known, and are studied in [18].

Lemma 4.3. Let \( \mu_0 = \frac{1}{2} \) and for \( k = 1, 2, \ldots \) let \( \mu_k(\theta) = \cos k\theta \). Also let

\[
T_n(\theta; \alpha) = \frac{1}{A_n(\alpha)} \sum_{k=0}^{n} A_{n-k}(\alpha) \mu_k(\theta).
\]

(a) If \( \alpha \geq 2 \) then \( T_n(\theta; \alpha) \geq 0 \) for \( 0 \leq \theta \leq 2\pi \) and \( n = 0, 1, \ldots \).

(b) If \( 1 < \alpha < 2 \) there is a constant \( B(\alpha) \) such that

\[
\frac{1}{2\pi} \int_{0}^{2\pi} |T_n(\theta; \alpha)| d\theta \leq B(\alpha) \quad \text{for} \quad n = 0, 1, \ldots.
\]

Proof. First consider the case \( \alpha = 2 \). Then (a) is a known fact and the argument for it is as follows. Since \( A_n(2) = n + 1 \) for \( n = 0, 1, \ldots \),

\[
T_n(\theta; 2) = \frac{1}{n+1} \left\{ \frac{n+1}{2} + \sum_{k=1}^{n} (n-k+1) \cos k\theta \right\}
= \frac{1}{2} \sum_{k=-n}^{n} \left( 1 - \frac{|k|}{n+1} \right) e^{ik\theta} = \frac{1}{2} \frac{1}{n+1} \left\{ \frac{\sin \frac{n+1}{2} \theta}{\sin \frac{1}{2} \theta} \right\}^2 \geq 0.
\]

This proves (a) when \( \alpha = 2 \).
Suppose that \(\alpha > 0\) and \(\beta > 0\). Then
\[
\sum_{n=0}^{\infty} A_n(\alpha + \beta) z^n = \frac{1}{(1 - z)^{\alpha+\beta}} = \frac{1}{(1 - z)^{\alpha}} \frac{1}{(1 - z)^{\beta}}
\]
\[
= \sum_{n=0}^{\infty} A_n(\alpha) z^n \sum_{n=0}^{\infty} A_n(\beta) z^n
\]
\[
= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} A_{n-k}(\alpha) A_k(\beta) \right\} z^n.
\]
This shows that
\[
(37) \quad A_n(\alpha + \beta) = \sum_{k=0}^{n} A_{n-k}(\alpha) A_k(\beta).
\]

Now assume that \(\alpha > 2\). From (37), it follows that
\[
A_n(\alpha) T_n(\theta; \alpha) = \sum_{k=0}^{n} A_{n-k}(\alpha) \mu_k(\theta)
\]
\[
= \sum_{k=0}^{n} \left\{ \sum_{j=0}^{n-k} A_{n-k-j}(2) A_j(\alpha - 2) \right\} \mu_k(\theta)
\]
\[
= \sum_{j=0}^{n} \left\{ \sum_{k=0}^{n-j} A_{n-j-k}(2) \mu_k(\theta) \right\} A_j(\alpha - 2)
\]
\[
= \sum_{j=0}^{n} T_{n-j}(\theta; 2) A_{n-j}(2) A_j(\alpha - 2).
\]
Because \(A_{n-j}(2) > 0\), \(A_j(\alpha - 2) > 0\), and \(T_{n-j}(\theta; 2) \geq 0\), this implies that \(A_n(\alpha) T_n(\theta; \alpha) \geq 0\). This proves (a) for \(\alpha > 2\).

A proof of (b) is contained in [18, Vol. 1, p. 94], where it is shown that the kernel
\[
K_n^\beta(\theta) = \frac{1}{A_n(\beta + 1)} \sum_{k=0}^{n} A_{n-k}(\beta) D_k(\theta)
\]
is "quasipositive" for \(0 < \beta < 1\). Here \(D_k(\theta)\) denotes the Dirichlet kernel \(\frac{1}{2} \sum_{j=-k}^{k} e^{ij\theta} \). Note that \(K_n^{\alpha-1}(\theta) = T_n(\theta; \alpha)\), and since \(1 < \alpha < 2\) by assumption, this establishes (b). \(\Box\)

The authors would like to thank B. Muckenhoupt, who provided the proof of (a) for \(\alpha > 2\), and who pointed out that this fact is known.

**Theorem 4.4.** For each \(\alpha > 1\) there is a constant \(C(\alpha)\) such that if \(f \in H^\infty\), then
\[
(38) \quad \|P_n(z; \alpha)\|_{H^\infty} \leq C(\alpha) \|f\|_{H^\infty},
\]
for \(n = 0, 1, \ldots\). When \(\alpha \geq 2\), (38) holds with \(C(\alpha) = 1\).

**Proof.** The orthonormal relations for the trigonometric functions imply that
\[
(39) \quad \frac{1}{2\pi} \int_0^{2\pi} f(ze^{i\theta}) T_n(\theta; \alpha) d\theta = \frac{1}{2} P_n(z; \alpha)
\]
for \(|z| < 1\).
Suppose that $\alpha \geq 2$, $|z| < 1$, and $f \in H^\infty$. Then (39) and (a) in Lemma 4.3 imply that
\[
\frac{1}{2} |P_n(z ; \alpha)| \leq \frac{1}{2\pi} \int_0^{2\pi} \|f\|_{H^\infty} T_n(\theta ; \alpha) d\theta = \frac{1}{2\pi} \int_0^{2\pi} |T_n(\theta ; \alpha)| d\theta.
\]
This proves the theorem in the case $\alpha \geq 2$.

Now suppose that $1 < \alpha < 2$, $|z| < 1$, and $f \in H^\infty$. Then (39) and (b) in Lemma 4.3 imply that
\[
\frac{1}{2} |P_n(z ; \alpha)| \leq \|f\|_{H^\infty} \frac{1}{2\pi} \int_0^{2\pi} |T_n(\theta ; \alpha)| d\theta \leq B(\alpha) \|f\|_{H^\infty}.
\]
This proves the theorem where $C(\alpha) = 2B(\alpha)$. □

The assertion in Theorem 4.4 does not hold for $\alpha = 1$. This is because there are functions bounded and holomorphic in $\Delta$ such that the sequence of partial sums $s_n$ is not uniformly bounded in $\Delta$ [3, p. 444]. Also note that $P_n(z ; 1) = s_n(z)$.

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