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3-MANIFOLD GROUPS WITH THE FINITELY GENERATED INTERSECTION PROPERTY

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ABSTRACT. In this paper, first we consider whether the fundamental groups of certain geometric 3-manifolds have FGIP or not. Next we give the sufficient conditions that FGIP for 3-manifold groups is preserved under torus sums or annulus sums and connect this result with a conjecture by Hempel [4].

A group G is said to have the *finitely generated intersection property* (for short FGIP) if, for each pair of finitely generated subgroups $H, K \subset G$, $H \cap K$ is finitely generated. Greenberg [2] proved that the fundamental groups of surfaces have FGIP. For given 3-manifolds M, we would like to know if their fundamental groups $\pi_1(M)$ have FGIP or not. In the case where $\pi_1(M)$ does not have FGIP, certain structures on $H \cap K$ for finitely generated subgroups H, K of $\pi_1(M)$ are studied by Kakimizu [6]. In [5, Chapter V], Jaco proved that, for every surface bundle M over S^1 with fiber F of negative Euler number, $\pi_1(M)$ does not have FGIP, hence in particular, the group $(\mathbf{Z} * \mathbf{Z}) \times \mathbf{Z}$ does not have FGIP. This result implies that, if the following Conjecture 1 proposed by Thurston [12] is true, then Conjecture 2 is also true (see Hempel [4]).

Conjecture 1. Every hyperbolic 3-manifold of finite volume is finitely covered by a surface bundle over the circle.

Conjecture 2. The fundamental group of every hyperbolic 3-manifold of finite volume does not have FGIP.

In [4], Hempel proved that every geometrically finite Kleinian group Γ of the second kind has FGIP. Here Γ of the *second kind* means that the limit set of Γ is not equal to the sphere S^2_∞ at infinity. By using this result, it is not hard to prove that the fundamental group of every hyperbolic 3-manifold of infinite volume has FGIP, see Proposition 1 in §1. We also consider the fundamental groups of 3-manifolds with the geometric structures other than the hyperbolic structure. For every 3-manifold M with $\mathbf{S}^3$, $\mathbf{S}^2 \times \mathbf{E}^1$, $\mathbf{E}^3$, Nil or Sol structure, $\pi_1(M)$ has FGIP (Proposition 2), and for every 3-manifold M with $\mathbf{H}^2 \times \mathbf{E}^1$ or $\widetilde{\mathrm{SL}_2(\mathbf{R})}$ structure of finite volume, $\pi_1(M)$ does not have FGIP (Proposition 3).

According to Baumslag [1], the free product $A * B$ of two groups A and B with FGIP has also FGIP. This result implies that, if two 3-manifolds have

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fundamental groups with FGIP, then that of their connected sum also has FGIP.

The following question is the torus sum version of this result.

Question. Let M be a 3-manifold and let T be an embedded, two-sided incompressible torus in M. If, for each component N of $M - T$, $\pi_1(N)$ has FGIP, does $\pi_1(M)$ have FGIP?

Let N_i $(i = 1, 2, \ldots, n)$ be 3-manifolds whose boundaries ∂N_i contain incompressible torus components and such that all $\pi_1(N_i)$ have FGIP, and let M be a 3-manifold obtained from $\{N_i\}$, for some pairs (T, T') of torus boundary components, by identifying T with T'. The following theorem gives a sufficient condition for $\pi_1(M)$ to have FGIP.

Theorem 1. *With the notation as above, we suppose that, for each* i, $\mathrm{int}\, N_i$ *is homeomorphic to* $\mathbf{H}^3/\Gamma$, *where* Γ *is a geometrically finite Kleinian group of the second kind. Then* $\pi_1(M)$ *has FGIP.*

To prove Theorem 1, in §2, we will define a geometric model M_g for M and piecewise geodesic loops in M_g.

This theorem asserts that, under torus sums for certain 3-manifolds, FGIP (for the fundamental groups) is preserved. The following corollary implies that, if Conjecture 2 is true, then FGIP is preserved under torus sums for 3-manifolds.

Corollary. *Let* T *be a union of mutually disjoint, two-sided incompressible tori in a connected* 3*-manifold* M *(possibly noncompact, nonorientable or reducible). If Conjecture* 2 *is true and* $\pi_1(N)$ *has FGIP for every component* N *of* $M - T$, *then* $\pi_1(M)$ *has FGIP.*

Under annulus sums for 3-manifolds, FGIP is not preserved. In §3, we will give a simple counterexample.

Let $N = N_1 \cup \cdots \cup N_n$ be a disjoint union of n connected 3-manifolds, and let $A = A_1^+ \cup A_1^- \cup \cdots \cup A_m^+ \cup A_m^-$ be a disjoint union of $2m$ annuli in ∂N which are incompressible in N.

Suppose M is the 3-manifold obtained from N by identifying A_s^+ and A_s^- for all $s = 1, \ldots, m$ by some homeomorphisms $A_s^+ \to A_s^-$. For each pair i, j (possibly $i = j$), let A_{ij} be the union of components of A such that $A_{ij} \supset A_s^+$ (resp. A_s^-) if and only if $A \cap \partial N_i \supset A_s^+$ (resp. A_s^-) and $A \cap \partial N_j \supset A_s^-$ (resp. A_s^+). We note that $A_{ij} \subset A \cap \partial N_i$. When $i \neq j$, this A_{ij} nonempty means that N_i is adjacent to N_j in M.

Theorem 2. *With the notation as above, if the following two conditions are satisfied, then* $\pi_1(M)$ *has FGIP.*

(i) *For each* N_j, $\pi_1(N_j)$ *has FGIP.*

(ii) *For each pair* N_i, N_j *(possibly* $i = j$*) with* $A_{ij} \neq \varnothing$, *at least one of* $(N_i, A \cap N_i)$ *and* $(N_j, A \cap N_j)$ *contains no properly embedded essential annuli or Möbius bands.*

The proof of Theorem 2 is similar to that of the Corollary, but in this case, we do not need the assumption that Conjecture 2 is true.

1. Proofs of Propositions

We refer to Hempel [3] and Jaco [5] for the notation on the 3-dimensional topology and to Scott [10] and Thurston [13] for the notation on hyperbolic 3-manifolds and other 3-dimensional geometric structures.

The following lemma is an elementary exercise.

Lemma 1. *Let A, B, C be subgroups of a group G such that A and B are finitely generated and C is of finite index in G. Then $A \cap B$ is finitely generated if and only if $A \cap B \cap C$ is finitely generated. In particular, G has FGIP if and only if C has FGIP.* □

We say that a 3-manifold M is *atoroidal* if, for every incompressible torus T in M, at least one of the components of $M - T$ is homotopic to the torus. According to Thurston [13, Proposition 5.4.4], every complete hyperbolic 3-manifold is atoroidal.

Proposition 1. *The fundamental group of every hyperbolic* 3*-manifold M of infinite volume has FGIP.*

Proof. We may assume that M is orientable and $\pi_1(M)$ is nonabelian and finitely generated. Furthermore, by Baumslag [1], we may also assume that $\pi_1(M)$ is indecomposable. Note that even after these reductions, we may assume that M still has infinite volume since any covering space of M also has infinite volume. By Scott [8], M contains a compact submanifold N such that the inclusion $N \subset M$ is homotopy equivalent and ∂N is incompressible in M. Since M is irreducible and atoroidal and since ∂N is incompressible in M, N is also atoroidal and irreducible. Since the volume of M is infinite, $\partial N \neq \varnothing$. If the euler number $\chi(\partial N) = 0$, then ∂N would consist of a finite number of tori. Since M is atoroidal, $M - \operatorname{int} N$ would consist of parabolic cusps of M. This contradicts that M has infinite volume. Therefore the Euler number $\chi(\partial N)$ is negative and hence by Hempel [4, Theorem 1.3], $\pi_1(M)$ ($\cong \pi_1(N)$) has FGIP. □

Lemma 2. *Let M be an orientable torus bundle over S^1. Every subgroup A of $\pi_1(M)$ is either of finite index in $\pi_1(M)$ or A contains a free abelian subgroup with rank at most* 2 *of finite index. Hence, in particular, A is finitely generated.*

Proof. Let $p: \widetilde{M} \to M$ be the covering associated to A. The covering space $\widetilde{M}$ has the surface bundle structure $\mathscr{S}$ induced from the torus bundle structure on M. A fiber F in $\mathscr{S}$ is either a torus or an open annulus or an open disk, and the base space is either S^1 or $\mathbf{R}$. If the base space is $\mathbf{R}$, then $\pi_1(F) \cong \pi_1(\widetilde{M}) \cong A$ is free abelian with rank at most 2. So we may assume that the base space is S^1. If F is a torus, then $\widetilde{M}$ is a closed 3-manifold and hence $\pi_1(\widetilde{M})$ is of finite index in $\pi_1(M)$. If F is an open disk (resp. an open annulus), then $\pi_1(\widetilde{M})$ is isomorphic to Z (resp. to the fundamental group of either a torus or a Klein bottle). □

Proposition 2. *For every* 3*-manifold M with* $\mathbf{S}^3$, $\mathbf{S}^2 \times \mathbf{E}^1$, $\mathbf{E}^3$, Nil *or* Sol *structure, $\pi_1(M)$ has FGIP.*

Proof. If M has $\mathbf{S}^3$, $\mathbf{S}^2 \times \mathbf{E}^1$ or $\mathbf{E}^3$ structure, then $\pi_1(M)$ has an abelian group of finite index. Hence $\pi_1(M)$ has FGIP. If M has Nil or Sol structure, then M is finitely covered by a torus bundle over S^1. Hence, by Lemmas 1 and 2, $\pi_1(M)$ has FGIP. □

Proposition 3. *For every* 3*-manifold M with* $\mathbf{H}^2 \times \mathbf{E}^1$ *or* $\widetilde{\mathrm{SL}_2(\mathbf{R})}$ *structure of finite volume, $\pi_1(M)$ does not have FGIP.*

Proof. There exists an S^1-bundle $\widetilde{M}$ over a surface F with $\chi(F) < 0$ which finitely covers M. Let $p: \widetilde{M} \to F$ be the fibration. The base surface F contains mutually disjoint, noncontractible, simple loops l_1, l_2 which are nonparallel in F. Let α be a simple arc in F connecting l_1 and l_2 and with $\operatorname{int}\alpha \cap (l_1 \cup l_2) = \varnothing$. We set $C = p^{-1}(l_1 \cup \alpha \cup l_2)$. Since $\pi_1(C)$ is isomorphic to $(\mathbf{Z} * \mathbf{Z}) \times \mathbf{Z}$ and since the homomorphism $\pi_1(C) \to \pi_1(\widetilde{M})$ induced by the inclusion is injective, $\pi_1(\widetilde{M})$ and hence $\pi_1(M)$ do not have FGIP. □

2. Proof of Theorem 1

Let N_i be compact, orientable 3-manifolds whose interiors admit complete hyperbolic structures, and let M be a 3-manifold obtained from $\{N_i\}$, for some pairs (T, T') of torus boundary components, by identifying T with T' by some diffeomorphisms.

Let A be a finitely generated subgroup of $\pi_1(M)$ and let $g \in \pi_1(M)$. By Bass-Serre Theory, $\pi_1(M)$ is the fundamental group of a graph of groups (see [11, §5]), and hence so is A. Since A is finitely generated and the edge-groups are finitely generated (subgroups of $\mathbf{Z} \times \mathbf{Z}$), it is an exercise to show that the vertex groups $B = A \cap g\pi_1(N_i)g^{-1}$ are finitely generated. Thus we have the following:

Lemma 3. *For every finitely generated subgroup A of $\pi_1(M)$ and $g \in \pi_1(M)$, $A \cap g\pi_1(N_i)g^{-1}$ is finitely generated.* □

We will define the geometric model M_g for the 3-manifold M given as above and the piecewise geodesic loops in M_g. From now on, we identify $\operatorname{int} N_i$ with $\mathbf{H}^3/\Gamma_i$ for some finitely generated Kleinian group Γ_i. Let $H_i^{(k)}$ be mutually disjoint neighborhoods of the parabolic cusps of N_i, which are covered by horoballs in $\mathbf{H}^3$. We set $\overline{N}_i = N_i - \bigcup_k \operatorname{int} H_i^{(k)}$. We can construct a 3-manifold M_g from $\{\overline{N}_i\}$, for some pairs $\{T, T'\}$ of boundary components, by identifying T and T' so that $\operatorname{int} M_g$ is homeomorphic to $\operatorname{int} M$. The set $C = M_g - \bigcup_i \operatorname{int} \overline{N}_i$ consists of incompressible tori and open annuli in M_g. We will equip each component C_j of C with a complete euclidean structure. Even in the case where $C_j \subset \partial \overline{N}_i$, the structure on C_j may not be that induced from $\overline{N}_i$. This is because, in general, the structures on C_j induced from the 3-manifolds on the right and left sides of C_j are distinct. The 3-manifold M_g with the hyperbolic structures on $\{\overline{N}_i\}$ and with the euclidean structures on $\{C_j\}$ is called a *geometric model* for M.

Let $* \in M_g - C$ be the base point of M and let l be a noncontractible loop in M_g containing $*$. We will define the piecewise geodesic loop in M_g homotopic to l fixing $*$. Modifying l by a homotopy fixing $*$, we may assume that l meets C transversely and the number of the points of $l \cap C$ is least among all loops in M_g homotopic to l fixing $*$. Let $\alpha_1, \dots, \alpha_n$ be the closures of the components of $l - (l \cap C) \cup \{*\}$ such that $\alpha_1 \cap \alpha_n = \{*\}$ and, for each i, α_i and α_{i+1} are adjacent in l. We suppose that α_j is contained in $\overline{N}_i$. If $1 < j < n$, then α_j connects two neighborhoods $H_i^{(p)}$ and $H_i^{(q)}$ (possibly $p = q$). Then $(\alpha_j, \partial\alpha_j)$ is homotopic to a unique geodesic arc $(\beta_j, \partial\beta_j)$ in $(N_i, \partial H_i^{(p)} \cup \partial H_i^{(q)})$ such that β_j meets $\partial H_i^{(p)} \cup \partial H_i^{(q)}$ orthogonally. Note that, in general, β_j is not contained in $\overline{N}_i$. Let γ_j be the arc in $\overline{N}_i$ homotopic fixing

$\partial\gamma_j$ to β_j in N_i and hence to α_j such that $\gamma_j \cap \operatorname{int}\overline{N}_i = \beta_j \cap \operatorname{int}\overline{N}_i$ and, for each arc component $\beta_j^{(s)}$ of $\beta_j - \beta_j \cap (\operatorname{int}\overline{N}_i)$, γ_j has the geodesic arc $\gamma_j^{(s)}$ in $\partial\overline{N}_i$ connecting the two points of $\partial\beta_j^{(s)}$ and homotopic to $\beta_j^{(s)}$ fixing $\partial\gamma_j^{(s)}$ in N_i. When $j = 1$ or n, the arc γ_j in $\overline{N}_i$ connecting $*$ with $\partial H_i^{(p)}$ can be defined similarly. When $\alpha_j \subset \overline{N}_i$ and $\alpha_{j+1} \subset \overline{N}_i$, (possibly $i = i'$), let C_j be the component of $\partial\overline{N}_i \cap \partial\overline{N}_{i'}$ containing the point $p = \partial\alpha_j \cap \partial\alpha_{j+1}$. A proper homotopy from α_j to γ_j traces an arc s_j in C_j connecting p with $\partial\gamma_j \cap C_j$. Similarly an arc s_j' in C_j connecting p with $\partial\gamma_{j+1} \cap C_j$ is defined. Let t_j be the geodesic arc in C_j homotopic to $s_j \cup s_j'$ fixing ∂t_j. We say that $l_g = \gamma_1 \cup t_1 \cup \cdots \cup \gamma_{n-1} \cup t_{n-1} \cup \gamma_n$ is a *piecewise geodesic loop* (for short p.g. loop) in M_g homotopic to l fixing $*$.

The following lemma is straightforward from the definition of p.g. loops.

Lemma 4. *If l_g and l_g' are p.g. loops homotopic fixing $*$ to the same loop, then $l_g = l_g'$.* □

The proof of Theorem 1 is based on the argument in Hempel [4].

Proof of Theorem 1. With the notation as above, we suppose furthermore that each Γ_i with $\mathbf{H}^3/\Gamma_i = \operatorname{int} N_i$ is geometrically finite and of the second kind. Let A_1 and A_2 be two finitely generated subgroups of $\pi_1(M) = \pi_1(M_g)$ and, for $j = 1, 2$, let $p_j\colon \widetilde{M}_j \to M_g$ be the covering associated to A_j. Let G_j be a finite 1-graph in $\widetilde{M}_j$ with the base point of $\widetilde{M}_j$ as a unique vertex and such that $i_*(\pi_1(G_j)) = \pi_1(\widetilde{M}_j)$, where $i\colon G_j \to M_j$ is the inclusion. Let R_j be the finite union of the closures $S_j^{(k)}$ of those components of $\widetilde{M}_j - p_j^{-1}(C)$ that meet G_j nontrivially. We will construct a certain compact core of R_j. Here a *core* of R_j is a connected subset of R_j such that the inclusion is homotopy equivalent. By Lemma 3, the Kleinian group $\Gamma_j^{(k)}$ associated to $S_j^{(k)}$ is finitely generated, hence it is geometrically finite, see [7, Proposition 7.1]. Hence $C_j^{(k)} \cap S_j^{(k)}$ is compact, where $C_j^{(k)}$ is the smallest closed convex core of $\mathbf{H}^3/\Gamma_j^{(k)}$. If $S_j^{(k)}$ is the closure of the component containing the base point $\tilde{*}$, we may assume that $C_j^{(k)} \ni \tilde{*}$. let $\Lambda_j^{(k)} \subset S_\infty^2$ be the limit set of $\Gamma_j^{(k)}$ and let $\Omega_j^{(k)} = S_\infty^2 - \Lambda_j^{(k)}$. Here, we define that, if $\Gamma_j^{(k)} = \{1\}$, then $\Lambda_j^{(k)} = \varnothing$, and if $\Gamma_j^{(k)}$ is abelian, then $\Lambda_j^{(k)}$ is the set of the fixed points for $\Gamma_j^{(k)}$. The *Kleinian manifold* $O_j^{(k)}$ is defined by $(\mathbf{H}^3 \cup \Omega_j^{(k)})/\Gamma_j^{(k)}$, see [13, Definition 8.3.5]. Let $q\colon \mathbf{H}^3 \to \mathbf{H}^3/\Gamma_j^{(k)}$ be the universal covering, and let $\{B_s\}$ be the set of horoballs in $\mathbf{H}^3$ such that $q^{-1}(S_j^{(k)}) = \mathbf{H}^3 - \bigcup_s \operatorname{int} B_s$. We say that the fixed point in S_∞^2 of any parabolic transformation fixing a horoball B is the *base point* of B. Let $x_1, \ldots, x_r$ be the finite points in $\partial O_j^{(k)}$ corresponding to the base points of horoballs B_s connected to another B_t by an arc in $q^{-1}(G_j \cap S_j^{(k)})$. Let $CH_j^{(k)}$ be the convex hull of $\Lambda_j^{(k)} \cup q^{-1}(\{x_1, \ldots, x_r\})$ and let $\widehat{C}_j^{(k)} = CH_j^{(k)}/\Gamma_j^{(k)}$. Let $H_1, \ldots, H_n$ be those components of $\mathbf{H}^3/\Gamma_j^{(k)} - \operatorname{int} S_j^{(k)}$ corresponding to parabolic cusps of $\mathbf{H}^3/\Gamma_j^{(k)}$ and let $P_j^{(k)} = O_j^{(k)} - \bigcup_i \operatorname{int} H_i$. Since each component of $P_j^{(k)} - \operatorname{int} C_j^{(k)} \cap P_j^{(k)}$ is homeomorphic to (a compact surface) $\times$ [0, 1], $P_j^{(k)}$ is

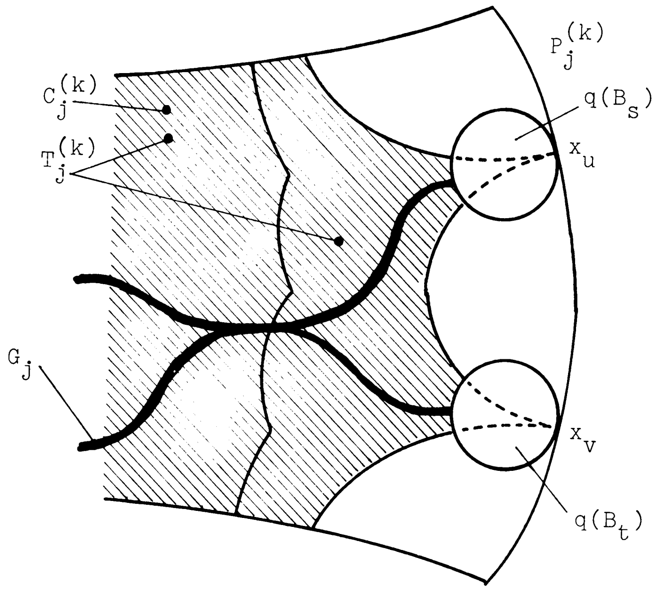

FIGURE 1

compact. We set $T_j^{(k)} = \widehat{C}_j^{(k)} \cap S_j^{(k)}$ and $U_j^{(k)} = P_j^{(k)} \cap \widehat{C}_j^{(k)} - \bigcup_s \operatorname{int} q(B_s)$, see Figure 1. Since $U_j^{(k)}$ is compact and since $T_j^{(k)}$ is the complement of the set $\{x_1, \dots, x_r\}$ of isolated points in $U_j^{(k)}$, $T_j^{(k)}$ is also compact and hence the number of the components of $\partial S_j^{(k)}$ meeting $T_j^{(k)}$ nontrivially is finite. Let $\widetilde{C}_u$ be any component of $p_j^{-1}(C)$ meeting some $T_j^{(k)}$ nontrivially. If $\pi_1(\widetilde{C}_u) = \{1\}$ (resp. $\cong \mathbf{Z}$), there exists a closed convex disk (resp. closed annulus with geodesic boundary) D_u in $\widetilde{C}_u$ such that $\widetilde{C}_u \cap T_j^{(k)} \subset \operatorname{int} D_u$. In the case where $\widetilde{C}_u$ meets two $T_j^{(k)}$ and $T_j^{(l)}$, we choose D_u so that $\widetilde{C}_u \cap (T_j^{(k)} \cup T_j^{(l)}) \subset \operatorname{int} D_u$. Then $K_j = (\bigcup_k T_j^{(k)}) \cup (\bigcup_u D_u)$ is the compact set in R_j such that $(e_j)_*(\pi_1(K_j)) = \pi_1(R_j) \cong A_j$, where $e_j: K_j \subset R_j$, see Figure 2.

Let $f: (K, *) \to (M_g, *)$ be the pull-back of the two maps $p_j \circ e_j: (K_j, *) \to (M_g, *)$, where $j = 1, 2$. Since K_1 and K_2 are compact, K is also compact, hence in particular, $\pi_1(K)$ is finitely generated. By Lemma 4, every element of $A_1 \cap A_2$ is represented by the unique p.g. loop l_g in M_g. Let l_j be the p.g. loop in $\widetilde{M}_j$ passing through the base point and covering l_g.

Now we show that l_j is contained in K_j. For $i = 2, \dots, n-1$, let $\gamma_i \subset S_j^{(k)}$ be the part of l_j obtained from the geodesic arc β_i in $\mathbf{H}^3/\Gamma_j^{(k)}$ meeting $\partial S_j^{(k)}$ orthogonally at $\partial \beta_i$ by replacing each component of $\beta_i - \beta_i \cap S_j^{(k)}$ by a certain geodesic arc in $p_j^{-1}(C)$. Let $\tilde{\beta}_i \subset \mathbf{H}^3$ be a lift of β_i. Let B_s, B_t be the horoballs connected each other by $\tilde{\beta}_i$ and let x_s, x_t be the base points of B_s, B_t. Since $\tilde{\beta}_i$ meets $\partial B_s \cup \partial B_t$ orthogonally, $\tilde{\beta}_i$ can be extended to the geodesic line $\hat{\beta}_i$ in $\mathbf{H}^3$ connecting x_s with x_t. Since $\hat{\beta}_i$ is contained in the convex hull $CH_j^{(k)}$, β_i is contained in $\widehat{C}_j^{(k)}$. Since every D_u is convex in

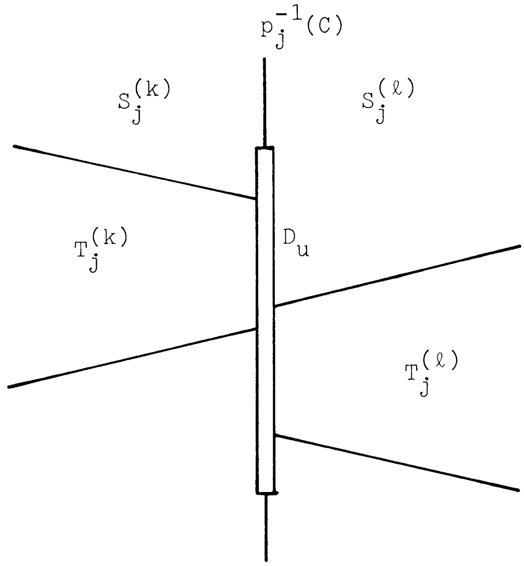

FIGURE 2

$p_j^{-1}(C)$, γ_i is contained in $T_j^{(k)} \cup (\bigcup_u D_u) \subset K_j$. Similarly the both parts γ_1, γ_n of l_j containing $\tilde{*}$ are contained in K_j. Again by using the convexity of D_u, it is proved easily that each component t_i of $l_j - \bigcup_i \gamma_i$ is contained in $\bigcup_u D_u$. Therefore we have $l_j \subset K_j$.

Thus $f_*(\pi_1(K)) = A_1 \cap A_2$ and hence $A_1 \cap A_2$ is finitely generated. □

3. PROOFS OF COROLLARY AND THEOREM 2

Proof of Corollary. Let M be a connected 3-manifold and let T be a union of two-sided incompressible tori in M satisfying the assumptions of Corollary. By a combination of Scott's Theorem [9], Baumslag's Theorem [1] and Lemma 1, we may assume that M is compact, orientable and irreducible. We separate M into the *simple pieces* $S_1, \dots, S_n$ (that is, every incompressible torus in S_j is parallel to a torus component of ∂S_j) by the union T_* of incompressible tori in $\operatorname{int} M$ with $T_* \supset T$. By Thurston's Uniformization Theorem (see [7]), for each j, either S_j is Seifert fibered or $\operatorname{int} S_j$ is homeomorphic to $\mathbf{H}^3/\Gamma_j$, where Γ_j is a geometrically finite Kleinian group. Since $\pi_1(S_j)$ is isomorphic to a subgroup of $\pi_1(N)$ for some component N of $M - T$, it has FGIP. If S_j is Seifert-fibered, then, by Proposition 3, it is homeomorphic to either $T^2 \times [0, 1]$ or the twisted I-bundle over a Klein bottle. If necessary, replacing M by its certain double covering, we may assume that M contains no π_1-injectively embedded Klein bottles, in particular that every Seifert piece S_j is homeomorphic to $T^2 \times [0, 1]$. If $\operatorname{int} S_j$ is hyperbolic and if Conjecture 2 is true, then Γ_j is of the second kind. Therefore, by Theorem 1, $\pi_1(M)$ has FGIP. □

The following simple example implies that FGIP for 3-manifold groups is not closed under annulus sums for 3-manifolds.

Example. Let M_1, M_2 be 3-manifolds homeomorphic to $T^2 \times [0, 1]$. For $i = 1, 2$, let A_i be a noncontractible annulus in ∂M_i. Let M be the 3-manifold obtained from M_1 and M_2 by identifying A_1 and A_2 by some homeomorphism $A_1 \to A_2$. Then $\pi_1(M_i)$ is isomorphic to $\mathbf{Z} \times \mathbf{Z}$, hence in particular, it has FGIP. On the other hand, since $\pi_1(M) \cong (\mathbf{Z} * \mathbf{Z}) \times \mathbf{Z}$, it does not have FGIP.

Let $f: (A, \partial A) \to (M, B)$ be a proper embedding (resp. 2-fold covering of a Möbius band embedded in M) from an annulus to a 3-manifold, where B is a subsurface in ∂M. We say that the annulus (resp. Möbius band) $f(A)$ is *essential* in (M, B) if $f_*: \pi_1(A) \to \pi_1(M)$ is injective and if a simple arc α in $f(A)$ connecting the two components of ∂A is not homotopic fixing $\partial\alpha$ to an arc in B.

Note that, in the above example, a component of $\partial M_i - \operatorname{int} A_i$ is an essential annulus in (M_i, A_i).

Proof of Theorem 2. Let $q: N_1 \cup \cdots \cup N_n \to M$ be the natural quotient map. We set $q(A) = A'$ and $q(A_{ij}) = q(A_{ji}) = A'_{ij}$. As in the proof of Corollary, we may assume that M is compact, orientable and irreducible and that M contains no π_1-injectively embedded Klein bottles. For any N_j, if $\partial N_j - \operatorname{int}(A \cap N_j)$ contains an annulus component which is inessential in $(N_j, A \cap N_j)$, then $\pi_1(M)$ is isomorphic to $\pi_1(M - q(N_j))$. So we may assume that

(3.1) each annulus component of $\partial N_j - \operatorname{int}(A \cap N_j)$ is essential in $(N_j, A \cap N_j)$.

We will separate N_j into simple factors $S_1^j, \dots, S_{n_j}^j$. Let $S_k^j \subset N_j$ and $S_l^u \subset N_u$ (possibly $j = u$ or $k = l$) be simple pieces such that $A'_{ju} \cap q(S_k^j) \cap q(S_l^u)$ is nonempty. Now we show the following (3.2).

(3.2) At least one of $(S_k^j, A \cap S_k^j)$ and $(S_l^u, A \cap S_l^u)$ contains no essential annuli.

If both $(S_k^j, A \cap S_k^j)$, $(S_l^u, A \cap S_l^u)$ contained essential annuli, then for the original M before the reductions and for the original N_s's and A_{st}'s, we would have N_s and N_t such that $A_{st} \neq \varnothing$ and both $(N_i, A \cap N_i)$ $(i = s, t)$ contain nondegenerate, immersed annuli. If N_i is orientable, then by the Annulus Theorem (see [5, VIII.13]) $(N_i, A \cap N_i)$ contains an essential annulus. When N_i is nonorientable, let $p: \widetilde{N}_i \to N_i$ be the orientable double covering. Again by the Annulus Theorem, $(\widetilde{N}_i, p^{-1}(A \cap N_i))$ contains an essential annulus $\widetilde{A}$. By the elementary cut and paste argument, we may assume that $\widetilde{A}$ is equivariant under the covering transformation. Then $p(\widetilde{A})$ is either an essential annulus or an essential Möbius band in $(N_i, A \cap N_i)$. This contradicts the assumption (ii) and hence (3.2) holds.

Now we return to the reduced case. For the union T_0 of the tori used for the torus decompositions of all N_j, we set $T'_0 = q(T_0) \subset M$. By (3.2), for any component U of $M - T'_0$, any essential torus in $\operatorname{int} U$ is ambient isotopic to a torus disjoint from $A' \cap U$. So we have the disjoint union T_U of essential tori in $\operatorname{int} U$ defining a torus decomposition of U with $A' \cap T_U = \varnothing$. The union T_* of T'_0 and T_U's for all components U of $M - T'_0$ separates M into simple pieces $U_1, \dots, U_m$. If $A' \cap U_r = \varnothing$, then $\pi_1(U_r)$ has FGIP, and hence either U_r is homeomorphic to $T^2 \times [0, 1]$ or $\operatorname{int} U_r$ is complete hyperbolic. We may assume that all these U_r are in the latter case. If $A' \cap U_r \neq \varnothing$, then by

(3.1) and (3.2) $\chi(\partial U_r) < 0$. By Thurston's Uniformization Theorem, $\operatorname{int} U_r$ is homeomorphic to $\mathbf{H}^3/\Gamma_r$, where Γ_r is a geometrically finite Kleinian group of the second kind. Therefore the geometric model M_g for M is defined. Let B_1 and B_2 be finitely generated subgroups of $\pi_1(M_g)$ and let $f_j: \widetilde{M}_j \to M_g$ be the covering associated to B_j. As in the proof of Theorem 1, there exists a finite union R_j if the closures V_j^k of components of $\widetilde{M}_j - f_j^{-1}(T_*)$ such that $i_*(\pi_1(R_j)) = \pi_1(\widetilde{M}_j)$. Let K_j be the submanifold of R_j obtained by replacing all the V_j^k such that $f_j(V_j^k) \cap A' \neq \varnothing$ by compact convex cores T_j^k defined as in Theorem 1. Let X_j be the union of these T_j^k and let Y_j be the closure of $K_j - X_j$. We set $g_j = f_j | K_j$ and denote by $g: (K, *) \to (M_g, *)$ the pull back of g_1 and g_2. Note that K is a closed set contained in $K_1 \times K_2 = (X_1 \times X_2) \cup (X_1 \times Y_2) \cup (Y_1 \times X_2) \cup (Y_1 \times Y_2)$. Since $K \cap (X_1 \times Y_2)$ and $K \cap (Y_1 \times X_2)$ are contained in $(Y_1 \times Y_2)$, $K = (K \cap (X_1 \times X_2)) \cup (K \cap (Y_1 \times Y_2))$. Since $g_1(Y_1)$ and $g_2(Y_2)$ are contained in $M_g - A$, by the assumption (i), for each component N of $K \cap (Y_1 \times Y_2)$, $\pi_1(N)$ is finitely generated. Since $K \cap (X_1 \times X_2)$ is compact, $\pi_1(K)$ is finitely generated. As in Theorem 1, $\pi_1(K)$ is isomorphic to $B_1 \cap B_2$. This completes the proof. □

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