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A BOUNDED MOUNTAIN PASS LEMMA WITHOUT THE (PS) CONDITION AND APPLICATIONS

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ABSTRACT. We present a version of the mountain pass lemma which does not require the (PS) condition. We apply this version to problems where the (PS) condition is not satisfied.

1. INTRODUCTION

The mountain pass lemma has been a very interesting tool in solving variational problems (cf., e.g., [4, 6, 7, 8, 9, 10]). It concerns a real-valued C' functional $G(u)$ defined on a real Banach space X for which one desires to find a critical point, i.e., a point where $G'(u) = 0$. In the simplest version one finds two points e_1, e_2 which are separated by a set M such that for some number a

$$G(e_i) < a, \qquad G(u) \geq a, \qquad u \in M.$$

This resembles the situation of a traveler trying to cross a mountain range without climbing higher than necessary. If we can find a continuous path connecting the two points which does not take the traveler higher than any other such path, it is expected that this path will produce a critical point.

However, there is a difficulty which must be addressed. One must allow the competing paths to roam freely, and conceivably they can take the traveler to infinity while he is trying to cross some local mountains. For the mathematician this can make it extremely difficult for him to locate critical points. To deal with this problem most researchers use the Palais-Smale (PS) condition which requires the sequences $\{u_k\}$ satisfying $|G(u_k)| \leq C$, $G'(u_k) \to 0$ to have convergent subsequences. This has the effect of allowing one to deal with unbounded regions in a uniform way. However, there are many problems for which the (PS) condition is not satisfied. One approach is to require (PS) on bounded regions and control the growth of $\|G'(u)\|^{-1}$ near infinity (cf. [9, 7]). This has the same effect in that it allows one to deal with unbounded regions in a uniform way.

In this paper we consider some problems that do not yield to either approach. Our method is to restrict the competing paths to a bounded region. This can be done only if one can be assured that the paths will not leave the region as they approach the optimal one. We accomplish this by imposing a boundary

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condition. For the cases considered here we take the region to be the ball $\|u\| \le R$ and require that there be a $\Theta < 1$ such that

$$G'(u)u \ge -\Theta\|G'(u)\|\|u\|, \tag{1.1}$$

whenever u satisfies

$$c - \sigma \le G(u) \le c + \sigma, \qquad \|u\| = R. \tag{1.2}$$

(The constants c and σ depend on the problem.) If (1.1) holds for such u, we will be guaranteed that competing paths will remain in the ball if they are close to the optimal path. In order to apply (1.1) we were required to generalize the concept of pseudogradient (cf., e.g., [2, 3]). We need a mapping $Y(u)$, locally Lipschitz, such that for some $\alpha > 0$

$$\|Y(u)\| \le 1, \qquad G'(u)Y(u) \ge \alpha\|G'(u)\|, \qquad u \in X, \tag{1.3}$$

and

$$(Y(u), u) > 0, \qquad \|u\| = R, \qquad u \text{ satisfies (1.2)}. \tag{1.4}$$

(A Hilbert space framework is used.) We prove this under assumption (1.1) provided $\alpha < 1 - \Theta$. It is property (1.4) that keeps the competing paths inside the ball $\|u\| \le R$.

As an application for which the (PS) condition does not apply, we have

Theorem 1. *Let Ω be a smooth bounded domain in $\mathbf{R}^n$, and let $f(x, t)$ be a Carathéodory function satisfying $|f(x, t)| \le V_1(x)$, $|t| \le 1$, $V_1(x) \in L^1(\Omega)$, and*

$$F(x, t) := \int_0^t f(x, s)\, ds \le \frac{1}{2}b(x)t^2 + W(x), \qquad t \in \mathbf{R}, \tag{1.5}$$

where $b(x) \le b := \lambda_{l+1} - \lambda_l$, $W(x) \ge 0$ is in $L^1(\Omega)$. Here λ_l is an eigenvalue of the Dirichlet problem

$$-\Delta u = \lambda u \quad in\ \Omega, \qquad u = 0 \quad on\ \partial\Omega, \tag{1.6}$$

with eigenspace E_l, and λ_{l+1} is the next larger eigenvalue. Let

$$H(x, t) := F(x, t) - \frac{1}{2}tf(x, t), \tag{1.7}$$

and assume that $|H(x, t)| \le W_1(x) \in L^1(\Omega)$ and

$$H(x, t) \to H_\pm(x) \quad as\ t \to \pm\infty\ a.e., \tag{1.8}$$

with

$$\int_{v>0} H_+(x)\, dx + \int_{v<0} H_-(x)\, dx \ge B := \int W(x)\, dx, \tag{1.9}$$

for all solutions $v \not\equiv 0$ of

$$-\Delta u - \lambda_l u = b_+u_+ - b_-u_- \quad in\ \Omega, \qquad u = 0 \quad on\ \partial\Omega, \tag{1.10}$$

where

$$b_\pm(x) := \limsup_{t\to\pm\infty} 2t^{-2}F(x, t) \ge 0,$$

and

$$u_+(x) = \max[u(x), 0], \qquad u_-(x) = u_+(x) - u(x).$$

Finally, assume that there does not exist a $w \in E_{l+1} \backslash \{0\}$ *such that*

$$b_+(x) \equiv b, \qquad H_+(x) \equiv W(x) \quad \text{when } w > 0, \tag{1.11}$$
$$b_-(x) \equiv b, \qquad H_-(x) \equiv W(x) \quad \text{when } w < 0.$$

Then the nonlinear Dirichlet problem

$$-\Delta u - \lambda_l u = f(x, u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.12}$$

has a solution.

Another application is

Theorem 2. *Let* Ω *be a smooth bounded domain in* $\mathbf{R}^n$ *and let* $f(x, t)$ *be a Carathéodory function such that*

$$|f(x, t)| \le C|t|^\gamma + k_1(x), \qquad t \in \mathbf{R},$$

with $1 \le q = 1 + \gamma < 2$, $k_1 \in L^{q'}(\Omega)$. *Let* λ_l *be an eigenvalue of the problem* (1.6) *with eigenspace* E_l. *Assume that*

$$|t|^{-\gamma} f(x, t) \to h_\pm(x) \quad \text{a.e.} \quad \text{as } t \to \pm\infty, \tag{1.13}$$

and

$$B(v)B(-v) > 0, \qquad v \in E_l \backslash \{0\}, \tag{1.14}$$

where

$$B(v) = \int_{v>0} h_+(x)|v(x)|^q \, dx - \int_{v<0} h_-(x)|v(x)|^q \, dx.$$

Then the problem (1.12) *has a solution.*

Remark 1. One might be tempted to replace (1.14) by the seemingly weaker assumption

$$B(v) \ne -\mu B(-v), \qquad v \in E_l \backslash \{0\}, \ 0 \le \mu \le 1. \tag{1.15}$$

We shall show in §4 that this implies (1.14).

Remark 2. Hypothesis (1.14) implies that either

$$B(v) > 0, \qquad v \in E_l \backslash \{0\}, \tag{1.16}$$

or

$$B(v) < 0, \qquad v \in E_l \backslash \{0\}. \tag{1.17}$$

That is, $B(v)$ does not change sign on E_l. For $B(v)$ is a continuous function of v on E_l, and if $B(v_1) > 0$, $B(v_2) < 0$, then $(1-\theta)v_1 + \theta v_2 \ne 0$ for $0 < \theta < 1$. Otherwise there would be a θ such that $v_1 = -\theta v_2/(1-\theta)$. By (1.14)

$$B(v_1)B(\theta v_2/(1-\theta)) > 0.$$

Consequently

$$(\theta^q/(1-\theta)^q)B(v_1) < 0,$$

a contradiction. But $B([1-\theta]v_1 + \theta v_2)$ is positive for $\theta = 0$ and negative for $\theta = 1$. This would imply that $B(v) = 0$ for some $v \ne 0$.

As another application we have

Theorem 3. *Let Ω be a bounded smooth domain in $\mathbf{R}^n$, and let λ_l, b be as above. Let $f(x, t)$ be a Carathéodory function such that*

$$-W_1(x) \le F(x, t) \le \frac{1}{2}bt^2 + W(x), \tag{1.18}$$

$$|f(x, t)| \le V(x) \in L^2(\Omega), \quad tf(x, t) \le W_2(x), \tag{1.19}$$

where W, W_1, W_2 are in $L^1(\Omega)$. Assume also that

$$F(x, t) \to F_\pm(x) \quad a.e. \quad as\ t \to \pm\infty, \tag{1.20}$$

and

$$\int_{v>0} F_+(x)\,dx + \int_{v<0} F_-(x)\,dx > B := \int W(x)\,dx, \qquad v \in E_l\backslash\{0\}. \tag{1.21}$$

Then the problem (1.12) *has a solution.*

The roots of the mountain pass lemma go back to the "method of steepest descent", the "deformation theorem" and the "minimax principle" (cf. [1–3]). It was formulated in the present context by Ambrosetti and Rabinowitz [4] basing it on a proof by Clark [5]. Since then there have been several generalizations (cf. [6] for a survey). To the best of the author's knowledge, they all use either the (PS) condition or an estimate of the growth of $\|G'(u)\|^{-1}$ near infinity.

Some of our applications have been considered by others using different methods. In particular we mention Landesman and Lazer [1], De Figueiredo [14], Gaines and Mawhin [15], Tarafdar [16].

Our mountain pass lemma is stated and proved in §3. Applications are stated in §§1 and 2 and proved in §4.

We give two more applications of the method. Let

$$f(x, t) = f_+(x)t_+ - f_-(x)t_- + W(x)\psi'(t), \tag{1.22}$$

where $W(x) \ge 0$ is a function in $L^1(\Omega)$ and $\psi(t) \in C^1(\mathbf{R})$ satisfies

$$|\psi(t)| + |\psi'(t)| \le C, \qquad \psi(t) < 1, \qquad \psi(0) = 0, \tag{1.23}$$

$$\varphi(t) := \psi - \frac{1}{2}t\psi' \to 1 \quad \text{as } |t| \to \infty. \tag{1.24}$$

We let λ_l be an eigenvalue of the Dirichlet problem (1.6) with eigenspace E_l. We have

Theorem 4. *Assume in addition that $0 \le f_\pm(x) \le \lambda_{l+1} - \lambda_l$, and that*

$$\int_{v>0} f_+(x)v(x)^2\,dx + \int_{v<0} f_-(x)v(x)^2\,dx > 0, \qquad v \in E_l\backslash\{0\}. \tag{1.25}$$

Then (1.12) *has a solution.*

In our next theorem we do not require (1.25) to hold on the whole of E_l. We can allow E_l to be split up into the direct sum of two orthogonal subspaces with (1.25) holding on one of them and the opposite inequality holding on the other. We can do this in the following way. Let $E_l = N_0 \oplus N_1$, where the N_i are orthogonal subspaces. Let

$$N' = \bigoplus_{j<l} E_j, \qquad N = N' \oplus N_0.$$

We have

Theorem 5. *Under hypotheses* (1.22)–(1.24) *assume that* $\lambda_{l-1} - \lambda_l \leq f_\pm(x) \leq \lambda_{l+1} - \lambda_l$ *and that*

$$(1.26)\quad -([\Delta + \lambda_l]v\,,\, v) < \int_{v>0} f_+(x)v(x)^2 + \int_{v<0} f_-(x)v(x)^2\,, \qquad v \in N\backslash\{0\}\,,$$

and

$$(1.27)\quad -([\Delta + \lambda_l]w\,,\, w) \geq \int_{w>0} f_+(x)w(x)^2 + \int_{w<0} f_-(x)w(x)^2\,, \qquad w \perp N.$$

Then (1.12) *has a solution.*

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2. Semilinear equations

Let A be a selfadjoint operator on $L^2(\Omega)$, and let $f(x\,,\,t)$ be a function from $\Omega \times \mathbf{R}$ to $\mathbf{R}$. We are concerned with finding solutions to the equation

$$(2.1)\qquad Au = f(x\,,\,u)\,, \qquad u \in D(A).$$

A function $u \in D := D(|A|^{1/2})$ will be called a semistrong solution of (2.1) if

$$(2.2)\qquad 2a(u\,,\,v) = (f(x\,,\,u)\,,\,v)\,, \qquad v \in D\,,$$

where

$$a(u\,,\,v) = \frac{1}{2}(Au\,,\,v)\,, \qquad a(u) = a(u\,,\,u)\,,$$

and

$$(u\,,\,v) = \int_\Omega u(x)v(x)\,dx\,, \qquad \|u\|^2 = (u\,,\,u).$$

It is clear that any semistrong solution u for which $f(x\,,\,u)$ is in $L^2(\Omega)$ is an actual (strong) solution. We call a semistrong solution a solution.

We make the following assumptions.

I. The essential spectrum $\sigma_e(A)$ of A, if any, is positive. Thus any non-positive point of the spectrum $\sigma(A)$ of A is an isolated eigenvalue of finite multiplicity.

II. We assume that $f(x\,,\,t)$ is a Carathéodory function. This means that it is measurable in x for every t and continuous in t for almost every x. Also for some $q \geq 1$ such that

$$(2.3)\qquad \|h\|_q^2 \leq C_1 a(h) + C_2\|h\|^2\,, \qquad h \in D\,,$$

we assume that

$$|f(x\,,\,t)| \leq V_0(x)^q|t|^{q-1} + V_0(x)k_1(x)\,,$$

where $V_0(x) > 0$ is a compact operator from D to $L^q(\Omega)$ and k_1 is in $L^{q'}(\Omega)$. Here $q' := q/(q-1)$ and

$$\|h\|_q := \left(\int_\Omega |h(x)|^q\,dx\right)^{1/q}.$$

III. We assume that there is a subspace N_0 of $N(A)$ (possibly empty or the whole of $N(A)$) such that

$$(2.4)\qquad c_1 := \inf_M G > -\infty\,,$$

$$(2.5)\qquad c_0 := \limsup\{G(v)\,,\ \|v\| \to \infty\,,\ v \in N\} < c_1\,,$$

where

$$N := N' \oplus N_0, \qquad N' := \bigoplus_{\lambda<0} N(A-\lambda), \qquad M := D \cap N^\perp,$$

$$G(u) := a(u) - \int_\Omega F(x, u)\,dx, \tag{2.6}$$

$$F(x, t) := \int_0^t f(x, s)\,ds. \tag{2.7}$$

IV. For each $c \geq c_1$ for which the set $c \leq G(u) \leq c+1$ is unbounded, there is a $\Theta < 1$ such that

$$b(c) := \liminf_{\substack{\|u\|\to\infty \\ c\leq G(u)\leq c+1}} \left\{ \int_\Omega H(x, u)\,dx + \frac{\Theta}{2}\|G'(u)\|\|u\| \right\} > -c, \tag{2.8}$$

where

$$H(x, t) := F(x, t) - \frac{1}{2}tf(x, t). \tag{2.9}$$

Theorem 6. *Under hypotheses* I–IV, *equation* (2.1) *has a solution.*

Theorem 7. *Hypothesis* IV *can be replaced by*

IV$'$. *Assume that* $q < 2$ *and that functions in* $N(A)$ *are either nonzero a.e., or vanish identically. Assume also that there are functions* $W_0(x)$, $W_1(x)$ *in* $L^1(\Omega)$ *such that*

$$\liminf_{|t|\to\infty} H(x, t) \geq W_0(x) \quad a.e., \tag{2.10}$$

$$H(x, t) \geq W_1(x) \quad a.e., \qquad t \in \mathbf{R}, \tag{2.11}$$

and

$$b_0 + c_1 > 0, \tag{2.12}$$

where

$$b_i := \int_\Omega W_i(x)\,dx, \qquad i = 0, 1, \tag{2.13}$$

and $H(x, t)$ *is given by* (2.9).

Proposition 1. *Hypothesis* III *will be fulfilled if* $N_0 = N(A)$,

$$F(x, t) < \frac{1}{2}\lambda_+ t^2 + W_0(x), \quad t \in \mathbf{R}, \tag{2.14}$$

and

$$\liminf_{|t|\to\infty} F(x, t) \geq W_2(x), \qquad F(x, t) \geq W_3(x) \quad a.e., \tag{2.15}$$

where λ_+ *is the smallest positive point in* $\sigma(A)$, *the* $W_i(x)$ *are in* $L^1(\Omega)$ *with*

$$b_2 := \int_\Omega W_2(x)\,dx > -b_0 := -\int_\Omega W_0(x)\,dx, \tag{2.16}$$

and functions in N *which vanish on a set of positive measure vanish identically.*

Theorem 8. *If there is a function* $V_1(x)$ *in* $L^2(\Omega)$ *such that*

$$|f(x, t)| \le V_1(x) \quad a.e., \qquad t \in \mathbf{R}, \tag{2.17}$$

then hypothesis IV *can be replaced by*

IV$''$. *Functions in* $N(A)$ *not identically zero are nonzero a.e. There is a real number* Θ, $|\Theta| < 1$, *and functions* $W_4(x)$, $W_5(x)$ *in* $L^1(\Omega)$ *such that*

$$\liminf_{|t|\to\infty} H_\Theta(x, t) \ge W_4(x), \qquad H_\Theta(x, t) \ge W_5(x) \quad a.e., \tag{2.18}$$

and

$$b_4 + c_1 > 0, \tag{2.19}$$

where

$$H_\Theta(x, t) := F(x, t) - \frac{1}{2}(1+\Theta)tf(x, t), \tag{2.20}$$

and

$$b_i := \int_\Omega W_i(x)\,dx. \tag{2.21}$$

Theorem 9. *If* $f(x, t)$ *satisfies* (2.17) *for some* $V_1 \in L^2(\Omega)$, *then* (2.4) *and* (2.5) *are implied by*

$$\int_\Omega F(x, v)\,dx \to \infty \quad as\ \|v\| \to \infty, \qquad v \in N_0, \tag{2.22}$$

and

$$\alpha_1 := \sup_{N_1} \int_\Omega F(x, v)\,dx < \infty, \tag{2.23}$$

where $N(A) = N_0 \oplus N_1$.

3. The mountain pass without the Palais-Smale condition

In this section we prove a version of the mountain pass lemma in which the competing curves or surfaces are restricted to a bounded region. We do this by imposing "boundary conditions". In order to obtain the most general boundary conditions, we generalize the notion of pseudogradient (cf. [3]). For this purpose we have:

Theorem 10. *Let* $\mathscr{H}$ *be a Hilbert space, and let* $X(u)$ *be a continuous mapping of* $\mathscr{H}$ *into itself such that* $X(u) \ne 0$ *for all* u. *Let* $v_i(u)$ *be continuous mappings such that* $v_i(u)$ *does not vanish on a closed set* Q_i. *Assume that*

$$(v_i(u), v_j(u)) = 0, \qquad i \ne j, \tag{3.1}$$

and that there are numbers $\Theta_i \ge 0$ *such that*

$$\Theta^2 = \sum \Theta_i^2 < 1, \tag{3.2}$$

and

$$(X(u), v_i(u)) \le \Theta_i \|X(u)\|\|v_i(u)\|, \qquad u \in Q_i. \tag{3.3}$$

If $\alpha < 1 - \Theta$, *then there is a locally Lipschitz map* $Y(u)$ *such that* $\|Y(u)\| \leq 1$ *and*

$$(X(u), Y(u)) \geq \alpha \|X(u)\|, \qquad u \in \mathscr{H}, \tag{3.4}$$

$$(Y(u), v_i(u)) < 0, \qquad u \in Q_i. \tag{3.5}$$

We shall give the proof of Theorem 10 at the end of this section. Now we shall use it in proving

Theorem 11. *Let* $G(u) \in C'(\mathscr{H}, \mathbf{R})$ *satisfy*

$$\begin{aligned} u_k \to u \quad \textit{weakly}, \qquad |G(u_k)| \leq C, \\ G'(u_k) \to 0 \quad \textit{imply that } G'(u) = 0. \end{aligned} \tag{3.6}$$

assume that $\mathscr{H} = N \oplus M$, *where* N, M *are orthogonal subspaces with* $\dim N < \infty$. *Assume that there are constants* $R \geq R_0 > 0$ *such that*

$$c_0 := \max_{\partial B_0} G < c_1 := \inf_B G \leq c_2 := \max_{B_0} G, \tag{3.7}$$

where

$$B_0 := \{v \in N \mid \|v\| \leq R_0\}, \qquad B := \{w \in M \mid \|w\| \leq R\}, \tag{3.8}$$

and that for each c *satisfying* $c_1 \leq c \leq c_2$ *there are constants* $\sigma > 0$, $\Theta < 1$ *such that*

$$(G'(u), u) \geq -\Theta R \|G'(u)\|, \tag{3.9}$$

holds for all u *satisfying*

$$c - \sigma \leq G(u) \leq c + \sigma, \qquad \|u\| = R. \tag{3.10}$$

Then there is a $u \in \mathscr{H}$ *satisfying* $G'(u) = 0$.

Proof. Let Q be the ball $\|u\| \leq R$, and assume that $G'(u) \neq 0$ for all u. Let $\mathscr{S}$ denote the set of all continuous mappings $\varphi(v)$ of B_0 into Q such that

$$\varphi(v) = v, \qquad v \in \partial B_0. \tag{3.11}$$

It is clear that

$$\varphi(B_0) \cap B \neq \phi, \qquad \varphi \in \mathscr{S}. \tag{3.12}$$

Let

$$c = \inf_{\varphi \in \mathscr{S}} \sup_{v \in B_0} G(\varphi(v)). \tag{3.13}$$

Then $c_1 \leq c \leq c_2$, and there exist constants $\sigma > 0$, $\Theta < 1$ such that (3.9) holds for all u satisfying (3.10). Let $\varepsilon > 0$ be such that $3\varepsilon < c_1 - c_0$ and $3\varepsilon < \sigma$. Let

$$\begin{aligned} Q_1 &= \{u \in Q \mid c - \varepsilon \leq G(u) \leq c + \varepsilon\}, \\ Q_2 &= \{u \in Q \mid G(u) \leq c - 2\varepsilon \text{ or } G(u) \geq c + 2\varepsilon\}. \end{aligned}$$

There is a constant $a > 0$ such that $\|G'(u)\| \geq a$ for all $u \in Q$ satisfying (3.10). For otherwise there would be a sequence $\{u_k\}$ of such elements such that $G'(u_k) \to 0$. A subsequence would converge weakly to an element $u \in Q$, and (3.6) would imply that $G'(u) = 0$. Let

$$\eta(u) = d(u, Q_2)/[d(u, Q_1) + d(u, Q_2)].$$

Then η is a Lipschitz continuous function which equals 1 on Q_1, vanishes on Q_2 and satisfies

$$\eta(u) \neq 0 \quad \text{for } u \notin Q_2, \qquad 0 \leq \eta(u) \leq 1. \tag{3.14}$$

By Theorem 10 there is a locally Lipschitz continuous mapping $Y(u)$ such that

$$(G'(u), Y(u)) \geq \alpha \|G'(u)\|, \qquad \|Y(u)\| \leq 1, \qquad u \in \mathscr{H}, \tag{3.15}$$

and

$$(Y(u), u) > 0, \quad u \text{ satisfies } (3.10), \tag{3.16}$$

where $\alpha < 1 - \Theta$. (Here there is only one $v_i(u)$, which we take to be $-u$. Q_i is taken as the set of those u satisfying (3.10).) By (3.13) there is a $\varphi \in \mathscr{S}$ such that

$$G(\varphi(v)) < c + \varepsilon, \qquad v \in B_0. \tag{3.17}$$

We let $\rho(t) = \rho(t, v)$ be the solution of

$$d\rho/dt = -\eta(\rho(t))Y(\rho(t))/\|Y(\rho(t))\|, \tag{3.18}$$

$$\rho(0) = \varphi(v). \tag{3.19}$$

Since $Y(u)$ is Lipschitz continuous and does not vanish, $\rho(t, v)$ exists for all $t > 0$ and $v \in B_0$. Note that $\partial B_0 \subset Q_2$. Then $\rho(t, v) = v$ for all $v \in \partial B_0$. We claim that $\rho(t)$ never leaves Q. $\rho(t)$ cannot leave via a point which is in Q_2 because $\eta = 0$ in Q_2. Moreover, $\rho(t)$ cannot approach points of ∂Q which are not in Q_2 because $\eta > 0$ at such points and $\rho(t)$ is directed inward at them and consequently in the neighborhood of such points by (3.15) and (3.18). Now

$$d\|\rho(t) - \varphi(v)\|/dt \leq \|d\rho/dt\| \leq 1,$$

and consequently

$$\|\rho(t) - \varphi(v)\| \leq t. \tag{3.20}$$

Moreover

$$\begin{aligned} d(G(\rho(t))/dt &= -\eta(\rho(t))G'(\rho(t))Y(\rho(t))/\|Y(\rho(t))\| \\ &\leq -\eta(\rho(t))\alpha\|G'(\rho(t))\|. \end{aligned} \tag{3.21}$$

Thus we have

$$G(\rho(t_2)) \leq G(\rho(t_1)) \leq G(\varphi(v)), \qquad t_1 < t_2. \tag{3.22}$$

Let T satisfy $2\varepsilon < \alpha a T$. If $\rho(t)$ does not leave Q_1 for $0 \leq t \leq T$, then

$$\begin{aligned} G(\rho(T)) - G(\varphi(v)) &\leq -\alpha \int_0^T \|G'(\rho(t))\|\, dt \\ &\leq -\alpha a T < -2\varepsilon. \end{aligned} \tag{3.23}$$

On the other hand, if there is a t_1 such that $0 \leq t_1 \leq T$ and $\rho(t_1)$ is not in Q_1, then we must have

$$G(\rho(t_1)) < c - \varepsilon,$$

since we cannot have $G(\rho(t_1)) < c + \varepsilon$. Thus

$$G(\rho(T)) \leq G(\rho(t_1)) < c - \varepsilon. \tag{3.24}$$

Let $\varphi_1(v) = \rho(T, v)$. Since $\rho(t)$ never leaves Q, we see that φ_1 is in $\mathscr{S}$. Moreover, (3.23) and (3.24) show that

$$G(\varphi_1(v)) < c - \varepsilon, \qquad v \in B_0.$$

This contradicts (3.13), showing that the assumption $G'(u) \neq 0$ is incorrect. □

In proving Theorem 10 we shall make use of

Lemma 1. *Let* $\Theta_i \geq 0$ *be such that*

$$\Theta^2 = \sum \Theta_i^2 < 1, \tag{3.25}$$

and let $\alpha > 0$ *satisfy* $\alpha < 1 - \Theta$. *Then for any elements* $u \neq 0$, $v_i \neq 0$, *such that*

$$(u, v_i) \leq \Theta_i \|u\| \|v_i\|, \qquad (v_i, v_j) = 0, \qquad i \neq j, \tag{3.26}$$

there is an element h *such that*

$$(u, h) \geq \alpha \|u\| \|h\|, \qquad (h, v_i) < 0 \text{ for each } i. \tag{3.27}$$

Proof. We may assume that u and the v_i are unit vectors. We take h of the form

$$h = u - \sum \beta_i v_i, \qquad \beta_i \geq 0.$$

If $\beta^2 = \sum \beta_i^2$, then

$$\|h\| \leq 1 + \beta, \qquad (h, v_i) \leq \Theta_i - \beta_i,$$

and

$$(u, h) \geq 1 - \sum \beta_i \Theta_i \geq 1 - \beta\Theta.$$

If we take $\beta_i > \Theta_i$ such that $\alpha(1 + \beta) \leq 1 - \beta\Theta$, (3.27) will be satisfied. This can be done because of the assumptions on Θ and α. □

Proof of Theorem 10. Let α' be any number satisfying $\alpha < \alpha' < 1 - \Theta$. By Lemma 1 for each u there is an element $h(u)$ such that

$$(X(u), h(u)) \geq \alpha' \|X(u)\|, \qquad \|h(u)\| = 1, \qquad u \in \mathscr{H}, \tag{3.28}$$

$$(h(u), v_i(u)) < 0, \qquad u \in Q_i. \tag{3.29}$$

(If u is not in Q_i, (3.3) does not hold and (3.29) is not needed. We merely take $\beta_i = 0$ in the construction of h in Lemma 1.) By the continuity of $X(u)$ and the $v_i(u)$, there is a neighborhood $N(u)$ of each u such that

$$(X(g), h(u)) \geq \alpha \|X(g)\|, \qquad g \in N(u), \tag{3.30}$$

$$(h(u), v_i(g)) < 0, \qquad u \in Q_i, \ g \in N(u). \tag{3.31}$$

If u is not in Q_i, we require that $N(u)$ not intersect Q_i. The set of all such neighborhoods covers $\mathscr{H}$. There is a locally finite, locally Lipschitz partition of unity $\{\psi_\tau\}$ subordinate to a refinement of this covering (cf., e.g., [3]). Let

$$Y(g) = \sum \psi_\tau(g) h_\tau(u_\tau).$$

$Y(g)$ is locally Lipschitz continuous since u_τ is constant on the support of ψ_τ. Now

$$\|Y(g)\| \leq \sum \psi_\tau(g) \|h_\tau\| \leq 1,$$

and for $g \in Q_i$

$$(Y(g), v_i(g)) = \sum \psi_\tau(g)(h_\tau(u_\tau), v_i(g)) < 0.$$

Also for any g

$$(X(g), Y(g)) = \sum \psi_\tau(g)(X(g), h_\tau(u_\tau)) \geq \sum \psi_\tau(g)\alpha\|X(g)\| = \alpha\|X(g)\|.$$

Thus (3.4) and (3.5) hold. □

4. The reduction

Now we show how the mountain pass lemma (Theorem 11) can be used to prove the theorems of §2. First we give the

Proof of Theorem 6. We apply Theorem 11 to $G(u)$ given by (2.6). We take $\mathscr{H} = D$ with norm

$$|||u||| := [a(u) + K\|u\|^2]^{1/2} \geq \|u\|, \tag{4.1}$$

where K is sufficiently large. We must show that G has a continuous derivative. Let u, h be any two functions in D. Then

$$t^{-1}\int_\Omega [F(x, u+th) - F(x, u)]\,dx = \int_\Omega \int_0^1 f(x, u + \Theta th)h\,d\Theta\,dx. \tag{4.2}$$

The integrand converges a.e. to $f(x, u)h(x)$ as $t \to 0$. By hypothesis II, it is bounded in absolute value by

$$(CV_0h)[V_0^{q-1}(|u|^{q-1} + |h|^{q-1}) + k_1]. \tag{4.3}$$

Since u, h are in D, the second factor is in $L^{q'}(\Omega)$ while the first is in $L^q(\Omega)$. Thus the expression (4.3) is in $L^1(\Omega)$ and majorizes the integrand of (4.2). Hence the right-hand side of (4.2) converges to

$$\int_\Omega f(x, u)h(x)\,dx.$$

This shows that

$$(G'(u), h) = 2a(u, h) - \int_\Omega f(x, u)h\,dx. \tag{4.4}$$

Next we check that (3.6) holds. If u_k converges weakly to u in D, then

$$a(u_k - u, h) \to 0, \qquad h \in D, \tag{4.5}$$

and there is a subsequence for which V_0u_k converges strongly in $L^q(\Omega)$ and another which converges a.e. But

$$|f(x, u_k)h(x)| \leq |V_0u_k|^{q-1}|V_0h| + k_1|V_0h|, \tag{4.6}$$

and V_0u_k converges to V_0u in $L^q(\Omega)$. Thus $|V_0u_k|^{q-1}$ converges to $|V_0u|^{q-1}$ in $L^{q'}(\Omega)$, and consequently the right-hand side of (4.6) converges to $|V_0u|^{q-1}|V_0h| + k_1|V_0h|$ in $L^1(\Omega)$. Hence

$$\int_\Omega f(x, u_k)h(x)\,dx \to \int_\Omega f(x, u)h(x)\,dx. \tag{4.7}$$

By (4.5) and (4.6)

$$(G'(u_k), h) \to (G'(u), h). \tag{4.8}$$

Thus if $G'(u_k) \to 0$, we must have $G'(u) = 0$.

Next we take R_0 so large that

$$\max\{G(v), \|v\| = R_0, \ v \in N\} < c_1. \tag{4.9}$$

This is possible by (2.5). Next, let c be any number satisfying $c_1 \leq c$, and let $c' = \max(c_1, c - \sigma)$, where $0 < \sigma < 1$. Let $b = b(c')$, and take $\varepsilon < b + c$. If $G(u)$ satisfies (3.10), then

$$\frac{1}{2}(G'(u), u) \geq c' + \int_\Omega H(x, u)\, dx. \tag{4.10}$$

Take R so large that

$$\int_\Omega H(x, u)\, dx + \frac{\Theta}{2}\|G'(u)\|\|u\| > b - \varepsilon, \qquad \|u\| \geq R, \tag{4.11}$$

and $R \geq R_0$. Then for u satisfying (3.10) we have

$$\frac{1}{2}(G'(u), u) + \frac{\Theta}{2}\|G'(u)\|\|u\| > c' + b - \varepsilon > 0. \tag{4.12}$$

Thus (3.9) holds. Theorem 11 tells us that there is a $u \in D$ such that $G'(u) = 0$. This is precisely what we want in view of (4.4). □

Proof of Theorem 7. We shall show that for any $\Theta > 0$ and $\sigma < b_0 + c_1$ there is an R such that

$$(G'(u), u) + \Theta\|G'(u)\|\|u\| > 0 \tag{4.13}$$

holds for all u satisfying (3.10). If this were not so, there would be a sequence $\{u_k\} \subset D$ such that

$$\|u_k\| \to \infty, \qquad \|G'(u_k)\|\|u_k\| \leq C, \tag{4.14}$$

and

$$\lim_{k\to\infty}(G'(u_k), u_k) \leq 0. \tag{4.15}$$

(Recall that $\frac{1}{2}(G'(u), u) \geq c - \sigma + b_1$ when (3.10) holds by (4.10) and (2.11).) Let $M' = [N' \oplus N(A)]^\perp \cap D$. We write $u_k = v_{0k} + v_k + w_k$, where $v_{0k} \in N(A)$, $v_k \in N'$ and $w_k \in M'$. In view of (4.14) we have

$$|2a(v_k) - (f(u_k), v_k)| \leq C, \qquad |2a(w_k) - (f(u_k), w_k)| \leq C. \tag{4.16}$$

Now by (4.6), (4.1) and hypothesis II

$$|(f(u), h)| \leq (\|V_0 u\|_q^{q-1} + \|k_1\|_{q'})\|h\|_q \leq C_1(|||u|||^{q-1} + 1)|||h|||.$$

Thus if we put $t_k = |||u_k|||$, (4.16) implies

$$|a(v_k)| + a(w_k) \leq C + C_1(t_k^q + t_k). \tag{4.17}$$

Let $\tilde{u}_k = u_k/t_k$. Then $|||\tilde{u}_k||| = 1$ and (4.17) implies

$$|||\tilde{v}_k + \tilde{w}_k||| \to 0.$$

Hence

$$|||\tilde{v}_{0k}||| = K\|\tilde{v}_{0k}\| \to 1.$$

Thus there is a subsequence for which $\tilde{v}_{0k} \to \tilde{v}_0$, $\tilde{v}_k \to 0$, $\tilde{w}_k \to 0$ in $L^2(\Omega)$ and a.e. Since $\tilde{v}_0 \not\equiv 0$, $\tilde{v}_0 \neq 0$ a.e., and

$$|u_k|/t_k = |\tilde{v}_{0k} + \tilde{v}_k + \tilde{w}_k| \to |\tilde{v}_0| \quad \text{a.e.}$$

Hence $|u_k| \to \infty$ a.e. This implies

$$\liminf_{k\to\infty} H(x, u_k) \geq W_0(x), \qquad H(x, u_k) \geq W_1(x),$$

and consequently

$$\liminf_{k\to\infty} (G'(u_k), u_k) \geq c - \sigma + b_0 > 0,$$

contradicting (4.15). The result now follows from Theorem 11. □

Proof of Proposition 1. If w is in M,

$$G(w) \geq \frac{1}{2}\lambda_+\|w\|^2 - \int_\Omega F(x, w)\,dx \geq -b_0.$$

Thus (2.4) holds with $c_1 \geq -b_0$. On the other hand for $v \in N$

$$G(v) \leq -\int_\Omega F(x, v)\,dx.$$

But

$$\liminf_{\|v\|\to\infty} \int_\Omega F(x, v)\,dx \geq b_2. \tag{4.18}$$

For otherwise there would be a sequence $\{v_k\} \subset N$ and an $\varepsilon > 0$ such that $\|v_k\| \to \infty$ and

$$\int_\Omega F(x, v_k)\,dx \to b_2 - \varepsilon. \tag{4.19}$$

Let $t_k = \|v_k\|$ and $\tilde{v}_k = v_k/t_k$. Then $\|\tilde{v}_k\| = 1$, and consequently there is a subsequence which converges to some $\tilde{v} \in N$. Thus $\tilde{v} \not\equiv 0$ and hence $\tilde{v} \neq 0$ a.e. This means that $|v_k| = t_k|\tilde{v}_k| \to \infty$ a.e. From this we see that

$$\liminf \int_\Omega F(x, v_k)\,dx \geq b_2,$$

contradicting (4.19). □

In proving Theorem 8 we shall make use of

Lemma 2. *Suppose there are functions $W_6(x)$, $W_7(x)$ in $L^1(\Omega)$ such that*

$$\liminf_{|t|\to\infty} M(x, t) \geq W_6(x), \qquad M(x, t) \geq W_7(x) \quad \text{a.e.} \tag{4.20}$$

Then

$$\liminf \int_\Omega M(x, u)\,dx \geq b_6 := \int_\Omega W_6(x)\,dx, \tag{4.21}$$

as $\|u\| \to \infty$ provided $\hat{a}(u) := \frac{1}{2}(|A|u, u) \leq C$.

Before proving the lemma we show how it can be used in giving the

Proof of Theorem 8. Let R_0 be chosen as in the proof of Theorem 6. For $u \in D$ we write $u = v_0 + v + w$, where $v_0 \in N(A)$, $v \in N'$ and $w \in M'$. We note that

$$\begin{aligned}(4.22)\quad |\Theta|(G'(u), w - v + (\operatorname{sgn}\Theta)v_0)&\\ &= |\Theta|(2a(w) - 2a(v) - (\operatorname{sgn}\Theta)(f, v_0))\\ &= |\Theta|(2a(w) - 2a(v) - (\operatorname{sgn}\Theta)(f, u) + (\operatorname{sgn}\Theta)(f, v + w))\\ &= -\Theta(f, u) + |\Theta|(2a(w) - 2a(v) + (\operatorname{sgn}\Theta)(f, v + w)).\end{aligned}$$

Since

$$\|f\| \le \|V_1\|,$$

$$K(u) := a(w) - a(v) + \tfrac{1}{2}(\operatorname{sgn}\Theta)(f, v + w) \to \infty, \tag{4.23}$$

as $\hat{a}(u) = a(w) - a(v) \to \infty$. By (4.10) and (4.22)

$$\begin{aligned}(4.24)\quad \frac{1}{2}(G'(u), u) + \frac{1}{2}|\Theta|\|G'(u)\|\|u\| &\ge c - \sigma + \int_\Omega H_\Theta(x, u)\,dx + |\Theta|K(u)\\ &\ge c - \sigma + b_5 + |\Theta|K(u).\end{aligned}$$

We take $\sigma < b_4 + c_1$ and take C_2 so large that the right-hand side of (4.24) and $K(u)$ are positive for $\hat{a}(u) \ge C_2$. This can be done by (4.23). We can now apply Lemma 2 to conclude that

$$\liminf \int_\Omega H_\Theta(x, u)\,dx \ge b_4,$$

as $\|u\| \to \infty$ provided $\hat{a}(u) \le C_2$. Thus we can take R so large that the left-hand side of (4.24) is positive when $\|u\| \ge R$ and $\hat{a}(u) \le C_2$. We see that the left-hand side of (4.24) is positive when $\|u\| = R$. The conclusion now follows from Theorem 11. □

Proof of Lemma 2. We shall show that for each c, $\varepsilon > 0$ there is an $R > 0$ such that

$$\int_\Omega M(x, u)\,dx > b_6 - \varepsilon \tag{4.25}$$

holds whenever $\hat{a}(u) \le c$ and $\|u\| > R$. If (4.25) did not hold, there would be constants, C, $\varepsilon > 0$ and a sequence $\{u_k\}$ such that $\hat{a}(u_k) \le C$, $\|u_k\| \to \infty$ and

$$\int_\Omega M(x, u_k)\,dx \le b_6 - \varepsilon \quad \text{for each } k. \tag{4.26}$$

We write $u_k = v_{0k} + v_k + w_k$, where $v_{0k} \in N(A)$, $v_k \in N'$, $w_k \in M'$. By hypothesis II, there is a subsequence (also denoted $\{u_k\}$) such that $V_0 v_k \to V_0 v$, $V_0 w_k \to V_0 w$ in $L^q(\Omega)$, and we may assume that $v_k \to v$, $w_k \to w$ a.e. Thus $t_k = \|v_{0k}\| \to \infty$. Put $v'_{0k} = v_{0k}/t_k$. Then $\|v'_{0k}\| = 1$. Since $\dim N(A) < \infty$, there is a subsequence (renamed) such that $v'_{0k} \to v'_0$ in norm and a.e. Since $v'_0 \not\equiv 0$, we have $v'_0 \ne 0$ a.e. Consequently, $|v_{0k}| = t_k|v'_{0k}| \to \infty$ a.e. Since v and w are finite a.e., we see that $|u_k| \to \infty$ a.e. Thus

$$\liminf_{k\to\infty} M(x, u_k) \ge W_6(x), \qquad M(x, u_k) \ge W_7(x), \quad \text{a.e.}$$

This implies

$$\liminf_{k\to\infty} \int_\Omega M(x, u_k)\,dx \geq b_6,$$

contradicting (4.26). □

Proof of Theorem 9. For $w \in M$, write $w = w' + v_1$, where $w' \in M'$ and $v_1 \in N_1$. Then we have

$$\begin{aligned} G(w) = a(w') - \int F(x, w' + v_1)\,dx &\geq \frac{1}{2}\lambda_+ \|w'\|^2 - \|V_1\|\|w'\| - \alpha_1 \\ &\geq -(\|V_1\|^2/2\lambda_+) - \alpha_1. \end{aligned}$$

This gives (2.4). For $v \in N$, we write $v = v' + v_0$, $v' \in N'$, $v_0 \in N_0$. Then

$$\begin{aligned} G(v) &= a(v') - \int F(x, v' + v_0)\,dx \\ &\leq a(v') - \int F(x, v_0)\,dx + \|V_1\|\|v'\| \\ &\leq \frac{1}{2}\lambda_- \|v'\|^2 + \|V_1\|\|v'\| - \int F(x, v_0)\,dx, \end{aligned}$$

where λ_- is the largest negative point in $\sigma(A)$. By (2.22),

$$K_0 = \inf_{N_0} \int F(x, v_0)\,dx > -\infty.$$

Thus

$$G(v) \leq \frac{1}{2}\lambda_- \|v'\|^2 + \|V_1\|\|v'\| - K_0.$$

On the other hand

$$G(v) \leq (\|V_1\|^2/2\lambda_-) - \int F(x, v_0)\,dx.$$

Thus there is an R such that $G(v) < c_1$ will hold if either $2\|v'\| \geq R$ or $2\|v_0\| \geq R$. It will therefore hold if $\|v' + v_0\| \geq R$. This proves (2.5). □

Proof of Theorem 1. We apply Theorem 6. Hypothesis I is satisfied with $A = -\Delta - \lambda_l$. We take $N_0 = N(A) = E_l$ and note that by (1.5)

$$G(w) \geq a(w) - \frac{1}{2}b\|w\|^2 - B \geq -B, \quad w \in M.$$

Hence $c_1 \geq -B$. Thus (2.4) holds. Next we note that if $u \equiv 0$ is not a solution of (1.12), then $c_1 > -B$. To see this we make use of the formulas

$$\begin{aligned} F(x, t) &= \frac{1}{2}b_+(x)t^2 + F_0(x, t), \qquad t > 0, \\ &= \frac{1}{2}b_-(x)t^2 + F_0(x, t), \qquad t < 0. \end{aligned} \tag{4.27}$$

$$\begin{aligned} F_0(x, t) &= 2t^2 \int_t^\infty s^{-3} H(x, s)\,ds, \qquad t > 0, \\ &= -2t^2 \int_{-\infty}^t s^{-3} H(x, s)\,ds, \qquad t < 0. \end{aligned} \tag{4.28}$$

These formulas are easily derived from the fact that

$$\partial(Ft^{-2})/\partial t = -2t^{-3}H(x, t).$$

Now suppose $c_1 = -B$. Then there is a sequence $\{w_k\} \subset M$ such that $G(w_k) \to -B$. If the $a(w_k)$ are bounded, then there is a subsequence (renamed) which converges weakly in D, $w_k \to w$ in $L^2(\Omega)$ and a.e. in Ω. Thus

$$\begin{aligned}\int_\Omega F(x, w_k)\,dx &= \frac{1}{2}\int b_+(x)w_{k+}(x)^2\,dx + \frac{1}{2}\int b_-(x)w_{k-}(x)^2\,dx \\ &\quad + \int_\Omega F_0(x, w_k)\,dx \\ &\to \frac{1}{2}\int b_+w_+(x)^2\,dx + \frac{1}{2}\int b_-w_-(x)^2\,dx + \int_\Omega F_0(x, w)\,dx.\end{aligned}$$

Since $a(w) \le \liminf a(w_k)$, we have $G(w) = c_1$. Thus

$$a(w) + B \le \int F(x, w)\,dx \le \frac{1}{2}b\|w\|^2 + B \le a(w) + B.$$

Thus $a(w) = \frac{1}{2}b\|w\|^2$, implying that $w \in E_{l+1}$. Also

$$F(x, w) \equiv \frac{1}{2}bw^2 + W(x).$$

In view of (1.5), this implies $[b-b(x)]w(x)^2 \le 0$. By hypothesis, $b(x) \le b$ and $b(x) \not\equiv b$. The only way we can obtain ≤ 0 is if $w(x) = 0$ on a set of positive measure. But then $w(x) \equiv 0$. This implies $W(x) \equiv 0$, $F(x, t) \le \frac{1}{2}bt^2$, $f(x, 0) \equiv 0$. On the other hand, if $t_k^2 = a(w_k) \to \infty$, let $\tilde{w}_k = w_k/t_k$. Again there is a subsequence (renamed) such that $\tilde{w}_k \to \tilde{w}$ weakly in D, $\tilde{w}_k \to \tilde{w}$ in $L^2(\Omega)$ and a.e. in Ω. Thus

$$\int b_+\tilde{w}_{k+}^2 + \int b_-\tilde{w}_{k-}^2 \to \int b_+\tilde{w}_+^2 + \tilde{b}_-\tilde{w}_-^2 \le b\|\tilde{w}\|^2 \le 2. \tag{4.29}$$

If this limit does not equal 2, then

$$G(w_k) = t_k^2\left[1 - \frac{1}{2}\int b_+\tilde{w}_{k+}^2 - \frac{1}{2}\int b_-\tilde{w}_{k-}^2\right] - \int F_0(x, t_k\tilde{w}_k)\,dx$$

will converge to ∞ as $k \to \infty$. Thus we must have equality in (4.29). In particular, we see that $\tilde{w} \not\equiv 0$. Thus $\tilde{w} \ne 0$ a.e. (the unique continuation property). Consequently

$$b_+(x) \equiv b \quad \text{when } \tilde{w} > 0, \qquad b_-(x) \equiv b \quad \text{when } \tilde{w} < 0. \tag{4.30}$$

Since by (1.15)

$$\frac{1}{2}b_+w_k^2 + F_0(x, w_k) \le \frac{1}{2}bw_k^2 + W(x),$$

we have

$$F_0(x, w_k) \le W(x) \quad \text{when } w_k > 0,\ \tilde{w} > 0.$$

Consequently $H_+(x) \le W(x)$ when $\tilde{w} > 0$. Similar reasoning gives $H_-(x) \le W(x)$ when $\tilde{w} < 0$. But

$$-B \leftarrow G(w_k) \ge -\int F_0(x, t_k\tilde{w}_k)\,dx \to -\int_{\tilde{w}>0} H_+(x)\,dx - \int_{\tilde{w}<0} H_-(x)\,dx.$$

Thus

$$B \leq \int_{\tilde{w}>0} H_+(x)\,dx + \int_{\tilde{w}<0} H_-(x)\,dx \leq B.$$

This means that

$$H_+(x) \equiv W(x) \quad \text{when } \tilde{w} > 0, \qquad H_-(x) \equiv W(x) \quad \text{when } \tilde{w} < 0. \tag{4.31}$$

But (4.30) and (4.31) contradict the hypotheses of the theorem. Thus $c_1 > -B$.

Next we show that $c_0 \leq -B$. Let $\{v_k\} \subset N$ be such that $G(v_k) \to c_0$, $t_k = \|v_k\| \to \infty$. Let $\tilde{v}_k = v_k/t_k$. Then $\|\tilde{v}_k\| = 1$, and consequently there is a subsequence (renamed) such that $\tilde{v}_k \to \tilde{v}$ in N. Since

$$G(v_k) = t_k^2\left[a(\tilde{v}_k) - \frac{1}{2}\int_\Omega b_+\tilde{v}_{k+}^2\,dx - \frac{1}{2}\int_\Omega b_-\tilde{v}_{k-}^2\,dx\right] - \int_\Omega F_0(x\,,\,t_k\tilde{v}_k)\,dx\,,$$

and $a(\tilde{v}_k) \to a(\tilde{v})$, we will have $G(v_k) \to -\infty$ unless

$$a(\tilde{v}) = 0, \qquad b_+\tilde{v}_+^2 \equiv 0, \qquad b_-\tilde{v}_-^2 \equiv 0.$$

This implies $\tilde{v} \in E_l$, and it is a solution of (1.10). By unique continuation $\tilde{v} \neq 0$ a.e. Thus $|v_k| \to \infty$ a.e., and we have

$$\begin{aligned}\limsup G(v_k) &\leq -\liminf \int_\Omega F_0(x\,,\,t_k\tilde{v}_k)\,dx \\ &= -\int_{\tilde{v}>0} H_+(x)\,dx - \int_{\tilde{v}<0} H_-(x)\,dx \leq -B\,,\end{aligned}$$

by hypothesis. Thus (2.5) holds. Finally we verify (2.8). An examination of (4.27) (4.28) shows that

$$|f(x\,,\,t)| \leq C(|t| + W_1(x)|t|^{-1})\,, \tag{4.32}$$

and

$$f(x\,,\,t)/t \to b_\pm(x) \quad \text{as } t \to \pm\infty \quad \text{a.e.}$$

Thus $f(x\,,\,t)$ satisfies hypothesis I with $q = 2$. Let $\{u_k\}$ be a sequence such that (4.14) holds. Let $t_k = \|u_k\|$ and $\tilde{u}_k = u_k/t_k$. Then (4.16) holds. By (4.32) we have

$$a(w_k) \leq C(\|u_k\|\|w_k\| + 1)\,, \qquad |a(v_k)| \leq C(\|u_k\|\|v_k\| + 1).$$

This shows that $|||\tilde{u}_k||| \leq C$. Since the embedding of D into $L^2(\Omega)$ is compact, there is a subsequence (renamed) such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$ and a.e. in Ω. Since

$$|a(\tilde{u}_k\,,\,h) - \tfrac{1}{2}t_k^{-1}(f(x\,,\,u_k)\,,\,h)| \leq C\|h\|/t_k \quad \forall h \in D\,,$$

we have in the limit

$$a(\tilde{u}\,,\,h) = \frac{1}{2}\int(b_+\tilde{u}_+ - b_-\tilde{u}_-)h\,dx\,, \qquad h \in D.$$

Thus $\tilde{u}$ is a solution of

$$Au = b_+u_+ - b_-u_- \quad \text{in } \Omega\,, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{4.33}$$

By the unique continuation property, $\tilde{u}$ cannot vanish on a set of positive measure since $\|\tilde{u}\| = 1$. Thus $|u_k| = t_k|\tilde{u}_k| \to \infty$ a.e., as $k \to \infty$.

Now suppose $\{u_k\}$ is a sequence satisfying $G(u_k) \geq c_1$, $\|u_k\| \to \infty$. Then we claim that for any $\Theta > 0$

$$\liminf_{k\to\infty}[(G'(u_k), u_k) + \Theta\|G'(u_k)\|\|u_k\|] > 0. \tag{4.34}$$

For

$$\frac{1}{2}(G'(u_k), u_k) = G(u_k) + \int H(x, u_k)\,dx, \tag{4.35}$$

is bounded from below. The only way (4.34) can fail to hold is if (4.14) holds. But then $|u_k| \to \infty$ a.e. In this case

$$\int H(x, u_k)\,dx \to \int_{\tilde{u}>0} H_+(x)\,dx + \int_{\tilde{u}<0} H_-(x)\,dx.$$

Hence

$$\liminf \frac{1}{2}(G'(u_k), u_k) \geq c_1 + B > 0. \quad \square$$

Remark. Note that if there exists a $w \in E_{l+1}\backslash\{0\}$ satisfying (1.11), then

$$G(tw) = -\int_\Omega F_0(x, tw)\,dx \to -B \quad \text{as } t \to \infty.$$

In this case we have $c_1 = -B$. Instead of assuming $b(x) \not\equiv b$ we can assume that there does not exist a $w \in E_{l+1}\backslash\{0\}$ such that

$$F(x, w) \equiv \frac{1}{2}bw^2 + W(x).$$

If we insist on a strict inequality in (1.9) we can dispense with the assumptions $b(x) \not\equiv b$ and the nonexistence of a $w \in E_{l+1}\backslash\{0\}$ satisfying (1.11).

Proof of Theorem 2. Clearly hypotheses I and II of Theorem 6 are satisfied. Note also that

$$|F(x, t)| \leq q^{-1}C|t|^q + k_1|t|.$$

Assume first that (1.16) holds. We take $A = \Delta - \lambda_l$, $N_0 = N(A) = E_l$, $b = \lambda_{l+1} - \lambda_l$. Thus

$$qG(w) \geq \int_\Omega \left(\frac{q}{2}bw^2 - C|w|^q - qk_1|w|\right)dx, \qquad w \in M.$$

Thus $G(w) \to \infty$ as $\|w\| \to \infty$, $w \in M$. This shows that (2.4) holds. On the other hand

$$q|t|^{-q}F(x, t) \to \pm h_\pm(x) \quad \text{a.e.}, \quad \text{as } t \to \pm\infty, \tag{4.36}$$

$$q|t|^{-q}H(x, t) \to \pm(1 - q/2)h_\pm(x) \quad \text{a.e.}, \quad \text{as } t \to \pm\infty, \tag{4.37}$$

by (1.13). Let $\{v_k\} \subset N$ be such that $t_k = \|v_k\| \to \infty$. Let $\tilde{v}_k = v_k/t_k$. Then $\|\tilde{v}_k\| = 1$ and there is a renamed subsequence such that $\tilde{v}_k \to \tilde{v}$ as $k \to \infty$. If $a(\tilde{v}_k) \leq -c < 0$, then

$$G(v_k) = t_k^2\left[a(\tilde{v}_k) - t_k^{-2}\int F(x, v_k)\,dx\right] \to -\infty.$$

If $a(\tilde{v}_k) \to 0$, then $a(\tilde{v}) = 0$ and $\tilde{v} \in E_l$. Since $\tilde{v} \not\equiv 0$, $\tilde{v} \neq 0$ a.e., and $|v_k(x)| = t_k|\tilde{v}_k(x)| \to \infty$ a.e. But by (4.36)

$$qt_k^{-q} \int F(x, v_k)\,dx = q \int [F(x, t_k\tilde{v}_k)/t_k^q|\tilde{v}_k|^q]|\tilde{v}_k|^q\,dx$$
$$\to B(\tilde{v}) \quad \text{as } k \to \infty.$$

Thus

$$\int F(x, v_k)\,dx \to \infty \quad \text{as } k \to \infty,$$

by (1.16). Hence

$$G(v_k) \leq -\int F(x, v_k)\,dx \to -\infty \quad \text{as } k \to \infty.$$

This shows that $c_0 = -\infty$ and (2.5) is satisfied. Finally, suppose $\{u_k\}$ is a sequence such that

$$(4.38) \qquad |G(u_k)| \leq C, \qquad t_k = \|u_k\| \to \infty.$$

The only way (4.34) can fail to hold is if (4.14) holds. But this implies $|u_k(x)| \to \infty$ a.e. (see the proof of Theorem 7). But then (4.37) implies

$$qt_k^{-q} \int H(x, u_k)\,dx \to (1 - q/2)B(\tilde{u}) \quad \text{as } k \to \infty,$$

(see (4.38)). Hence

$$\int H(x, u_k)\,dx \to \infty \quad \text{as } k \to \infty.$$

If we now make use of (4.35) and (4.38), we see that

$$(G'(u_k), u_k) \to \infty \quad \text{as } k \to \infty,$$

and the proof for this case is complete. If (1.17) holds, we take $N_0 = \{0\}$. Thus $E_l \subset M$. Let $\{w_k\}$ be a sequence in M such that $t_k = \|w_k\| \to \infty$. If $a(\tilde{w}_k) \geq c > 0$, then

$$G(w_k) = t_k^2 \left[a(\tilde{w}_k) - t_k^{-2} \int F(x, w_k)\,dx\right] \to \infty.$$

Otherwise $a(\tilde{w}_k) \to 0$. There is a $\tilde{w} \in M$ such that $\tilde{w}_k \to \tilde{w}$ weakly in D, strongly in $L^2(\Omega)$ and a.e. in Ω. Thus $\tilde{w} \in E_l$. Since $\|\tilde{w}\| = 1$, $\tilde{w} \neq 0$ a.e. Consequently, $|w_k(x)| \to \infty$ a.e. In view of (4.36)

$$q|t_k|^{-q} \int F(x, w_k)\,dx$$
$$= q \int [F(x, t_k\tilde{w}_k)/t_k^q|\tilde{w}_k|^q]|\tilde{w}_k|^q\,dx \to B(\tilde{w}) \quad \text{as } k \to \infty.$$

Consequently

$$G(w_k) \geq -\int F(x, w_k)\,dx \to \infty \quad \text{as } k \to \infty,$$

and (2.4) holds. On the other hand, if $\{v_k\} \subset N$ and $t_k = \|v_k\| \to \infty$, then $a(\tilde{v}_k) \leq -c < 0$ and

$$G(v_k) = t_k^2 \left[a(\tilde{v}_k) - t_k^{-2} \int F(x, v_k)\,dx\right] \to -\infty,$$

as $k \to \infty$. Thus (2.5) holds. Finally, if the sequence $\{u_k\}$ satisfies (4.38) and (4.14) holds, then $|u_k(x)| \to \infty$ a.e., as before, and (4.37) implies

$$\int H(x, u_k)\,dx \to -\infty \quad \text{as } k \to \infty.$$

It then follows from (4.35) and (4.38) that

$$|(G'(u_k), u_k)| \to \infty,$$

contradicting (4.14). Thus (4.38) implies

$$\|G'(u_k)\|\|u_k\| \to \infty,$$

and the result follows. □

Proof of Remark 1. Assumption (1.15) says that for each $v \in E_l \backslash \{0\}$ we have either

$$\text{(4.39)} \qquad \text{(a)}\ -\frac{B(v)}{B(-v)} < 0, \quad \text{or} \quad \text{(b)}\ -\frac{B(v)}{B(-v)} > 1.$$

If $B(v) > 0$, this implies either

$$\text{(4.40)} \qquad \begin{aligned} &\text{(a) } B(-v) > 0, \quad \text{or} \\ &\text{(b) } B(-v) < 0, \quad \text{and} \quad B(v) > -B(-v). \end{aligned}$$

If $B(v) < 0$, (4.39) implies either

$$\text{(4.41)} \qquad \begin{aligned} &\text{(a) } B(-v) < 0, \quad \text{or} \\ &\text{(b) } B(-v) > 0, \quad \text{and} \quad -B(v) > B(-v). \end{aligned}$$

In particular, if $B(v)B(-v) < 0$, then we must have $|B(v)| > |B(-v)|$. By symmetry, we must also have $|B(-v)| > |B(v)|$, which is a contradiction. Thus only option (a) is possible.

Proof of Theorem 3. We apply Theorems 6 and 8. Hypotheses I and II are clearly satisfied. In view of (1.18) we see that $c_1 \geq -B$. If $\{v_k\} \subset N$ and $t_k = \|v_k\| \to \infty$, let $\tilde{v}_k = v_k/t_k$. Then $\|\tilde{v}_k\| = 1$ and there is a renamed subsequence such that $\tilde{v}_k \to \tilde{v}$ in N. If $a(\tilde{v}) \neq 0$, then

$$G(v_k) = t_k^2 \left[a(\tilde{v}_k) - t_k^{-2} \int F(x, v_k)\,dx\right] \to -\infty.$$

If $a(\tilde{v}) = 0$, $\tilde{v} \in E_l \backslash \{0\}$. Hence $\tilde{v}(x) \neq 0$ a.e. Thus $|v_k(x)| \to \infty$ a.e., and

$$G(v_k) \leq -\int F(x, v_k)\,dx \to -\int_{\tilde{v}>0} F_+(x)\,dx - \int_{\tilde{v}<0} F_-(x)\,dx < -B,$$

by (1.21). Let $\varepsilon > 0$ be so small that

$$2\varepsilon b_2 < \int_{v>0} F_+(x)\,dx + \int_{v<0} f_-(x)\,dx - B, \qquad v \in E_l \backslash \{0\},$$

where $b_2 := \int W_2(x)\,dx$. If $\{u_k\}$ is a sequence satisfying (4.14), we have shown that $|u_k(x)| \to \infty$ a.e. If $\Theta < 1$ is such that $1 - \Theta < 2\varepsilon$, then

$$H_\Theta = F - \frac{1}{2}(1 - \Theta)tf \geq F - \varepsilon W_2.$$

Thus

$$\liminf_{t\to\pm\infty} H_\Theta \geq F_\pm(x) - \varepsilon W_2,$$

and

$$\liminf \int H_\Theta(x, u_k)\,dx \geq B + \varepsilon b_2.$$

We can now apply Theorem 8 to conclude that hypothesis IV is satisfied. □

Proof of Theorem 5. We apply Theorem 6 taking $q = 2$, $A = -\Delta - \lambda_l$, and

$$F(x, t) = \frac{1}{2}f_+(x)t_+^2 + \frac{1}{2}f_-(x)t_-^2 + W(x)\psi(t).$$

By (1.27), for $w \perp N$

$$G(w) \geq -\int W(x)\psi(w)\,dx > -\|W\|_1 = -B.$$

Thus $c_1 > -B$. On the other hand we claim

$$G(v) \to -\infty \quad \text{as } \|v\| \to \infty, \qquad v \in N. \tag{4.42}$$

To see this let $\{v_k\} \subseteq N$ be a sequence such that $t_k = \{v_k\} \to \infty$. Let $\tilde{v}_k = v_k/t_k$. Then $\|\tilde{v}_k\| = 1$ and there is a subsequence (renamed) for which $\tilde{v}_k \to \tilde{v}$ in N. Thus

$$\begin{aligned} t_k^{-2}G(v_k) &= a(\tilde{v}_k) - \frac{1}{2}\int (f_+\tilde{v}_{k+}^2 + f_-\tilde{v}_{k-}^2)\,dx - t_k^{-2}\int W(x)\psi(v_k)\,dx \\ &\to a(\tilde{v}) - \frac{1}{2}\int (f_+\tilde{v}_+^2 + f_-\tilde{v}_-^2)\,dx < 0, \end{aligned}$$

by (1.26) since $\tilde{v} \not\equiv 0$. Thus $G(v_k) \to -\infty$ and (4.42) holds. Hence hypothesis III is verified. We now turn to (2.8). Let $\{u_k\}$ be a sequence such that (4.14) holds. Let $\tilde{u}_k = u_k/t_k$ where $t_k = \|u_k\|$. Then (4.16) holds. Since

$$\begin{gathered} |f(x, t)| \leq C(|t| + 1), \quad a(w_k) \leq C(\|u_k\|\,\|w_k\| + 1), \\ |a(v_k)| \leq C(\|u_k\|\,\|v_k\| + 1). \end{gathered}$$

This shows that $|||\tilde{u}_k||| \leq C$. Since the embedding of $H_0'(\Omega) = D(|A|^{1/2})$ into $L^2(\Omega)$ is compact, there is a subsequence (renamed) such that $\tilde{u}_k \to \tilde{u}$ weakly in $H_0'(\Omega)$, strongly in $L^2(\Omega)$ and a.e. in Ω. Since

$$|a(\tilde{u}_k, h) - \tfrac{1}{2}t_k^{-1}(f(x, u_k), h)| \leq C\|h\|/t_k \quad \forall h$$

for each k, we have in the limit

$$a(\tilde{u}, h) = \frac{1}{2}\int [f_+\tilde{u}_+ - f_-\tilde{u}_-]h\,dx \quad \forall h.$$

Hence $\tilde{u}$ is a solution of (4.33). By the unique continuation property $\tilde{u} \neq 0$ a.e., since $\|\tilde{u}\| = 1$. Consequently, $|u_k| = t_k|\tilde{u}_k| \to \infty$ a.e.

Now suppose $\{u_k\}$ is a sequence such that

$$G(u_k) \geq c_1, \qquad \|u_k\| \to \infty.$$

Then we claim that for any $\Theta > 0$

$$\liminf_{k\to\infty}[(G'(u_k), u_k) + \Theta\|G'(u_k)\|\,\|u_k\|] > 0. \tag{4.43}$$

For

$$\frac{1}{2}(G'(u_k), u_k) = G(u_k) + \int W(x)\varphi(u_k)\,dx, \tag{4.44}$$

which is bounded from below. Thus the only way (4.43) can fail to hold is if (4.14) holds. But then we showed that $|u_k| \to \infty$ a.e. In this case by (1.24)

$$\liminf_{k\to\infty} \frac{1}{2}(G'(u_k), u_k) \geq c_1 + B. \quad \square$$

Proof of Theorem 4. This is a special case of Theorem 5. We have $N_0 = E_l$, $N = N' \oplus E_l$. Inequality (1.27) clearly holds since the left-hand side is $\geq (\lambda_{l+1} - \lambda_l)\|w\|^2$ while the right-hand side is bounded above by this quantity. Inequality (1.26) also holds. For the right-hand side is never negative. If $v \in N$ is not in E_l, then the left-hand side of (1.26) is negative. Thus (1.26) holds for such v. On the other hand, if $v \not\equiv 0$ is in E_l, then the left-hand side vanishes and the right-hand side is positive by (1.25). Thus all of the hypotheses of Theorem 5 are satisfied.

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