NONEXISTENCE OF NODAL SOLUTIONS OF ELLIPTIC EQUATIONS WITH CRITICAL GROWTH IN $\mathbb{R}^2$

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ABSTRACT. Let $f(t) = h(t)e^{bt^2}$ be a function of critical growth. Under a suitable assumption on $h$, we prove that

$$-\Delta u = f(u) \quad \text{in } B(R) \subset \mathbb{R}^2,$$
$$u = 0 \quad \text{on } \partial B(R),$$

does not admit a radial solution which changes sign for sufficiently small $R$.

1. Introduction

Let $B(R)$ denote the ball of radius $R$ in $\mathbb{R}^2$ with center at zero. Let $f(t) = h(t)e^{bt^2}$ be a function of critical growth (see Adimurthi-Yadava [1]). Consider the following problem

\begin{align*}
-\Delta u &= f(u) \quad \text{in } B(R), \\
u &= 0 \quad \text{on } \partial B(R).
\end{align*}

If $f$ satisfies the following condition

$$\lim_{t \to \infty} \frac{\log h(t)}{t} = \infty,$$

then (1.1) admits an infinite number of radial solutions which change sign (see Adimurthi-Yadava [1]).

In this note we show that the condition (1.2) is optimal for existence of infinitely many radial solutions which change sign by proving the following:

Theorem 1. Let $f(t) = t|t|^m e^{bt^2} + |t|^\beta$, $m \geq 0$, $b > 0$ and $0 < \beta \leq 1$. Then for every $\beta$ there exists $R(\beta) > 0$ such that for $0 < R < R(\beta)$, the problem

\begin{align*}
-\Delta u &= f(u) \quad \text{in } B(R), \\
u &= 0 \quad \text{on } \partial B(R),
\end{align*}

does not admit any radial solution which changes sign.

If $1 < \beta < 2$, then $f$ satisfies (1.2) and hence (1.2) is optimal.

In this connection similar results are available for critical exponent problems in $\mathbb{R}^n$, $n \geq 3$. There the dimension plays a role in the case of existence.
(see Cerami-Solomini-Struwe [5]) and nonexistence (see Atkinson-Brezis-Peletier [4]) of radial solutions which change sign.

2. PROOF OF THEOREM 1

Since we are looking for radial solutions, (1.3) becomes

\[
\begin{cases}
-\left( u'' + \frac{1}{r} u' \right) = f(u) & \text{in } (0, R), \\
u'(0) = u(R) = 0.
\end{cases}
\]  

(2.1)

By studying the following initial value problem we will prove the nonexistence of nodal solutions of (2.1) as in Atkinson-Brezis-Peletier [4]

\[
\begin{cases}
-\left( u'' + \frac{1}{r} u' \right) = f(u), \\
u'(0) = 0, \\
u(0) = \gamma > 0.
\end{cases}
\]  

(2.2)

Let \( R_k(\gamma) \), \( k = 1, 2, \ldots \), denote the \( k \)th zero of \( u \). Then by the similar argument as in Atkinson-Peletier [3] we have

\[
\lim_{\gamma \to 0} R_1(\gamma) = \begin{cases}
\infty & \text{if } m > 0, \\
C & \text{if } m = 0,
\end{cases}
\]  

(2.3)

where \( C \) is some positive constant. For the sake of completeness we will sketch the proof of (2.3) in Appendix 2. Now the proof of the theorem follows from the following:

Claim 1. For each \( 0 \leq \beta \leq 1 \), there exists a constant \( c(\beta) > 0 \) such that

\[
\lim_{\gamma \to \infty} R_2(\gamma) > c(\beta).
\]  

(2.4)

In order to prove Claim 1, make the standard substitution (as in Atkinson-Peletier [2]) by \( r = 2e^{-t/2} \) and \( u(r) = y(t) \), then (2.2) becomes

\[
\begin{cases}
y'' = e^{-t} f(y), \\
y(\infty) = \gamma, \\
y'(\infty) = 0.
\end{cases}
\]  

(2.5)

Let \( y(t, \gamma) \) be the corresponding solution and \( T_k(\gamma) \) the \( k \)th zero of \( y(t, \gamma) \). Then

\[
R_k(\gamma) = 2e^{-T_k(\gamma)/2}.
\]  

(2.6)

Now we have the following estimates on \( T_1(\gamma) \).

Claim 2. For every \( \beta \), \( 0 \leq \beta \leq 1 \), there exist constants \( C_\beta > 0 \) and \( \gamma_0 > 0 \) such that for all \( \gamma \geq \gamma_0 \),

\[
\gamma y'(T_1(\gamma), \gamma) \leq C_\beta,
\]  

(2.7)

\[
\frac{T_1(\gamma)}{\gamma} \leq C_\beta,
\]  

(2.8)

\[
\lim_{\gamma \to \infty} T_1(\gamma) = \infty.
\]  

(2.8')
Proof of Claim 1. Assuming Claim 2 we will complete the proof of Claim 1. Without loss of generality we may assume
\[ \lim_{\gamma \to \infty} T_2(\gamma) \geq 1. \] (2.9)

By using the convexity of \( y \) on \([T_2(\gamma), T_1(\gamma)]\) together with (2.7) and (2.8) we have for all \( \gamma \geq \gamma_0 \) and \( t \in [T_2(\gamma), T_1(\gamma)]\),
\[ |y(t, \gamma)| \leq |T_1(\gamma)y'(T_1(\gamma), \gamma)| \leq \frac{T_1(\gamma)}{\gamma} y y'(T_1(\gamma), \gamma) \leq C_\beta^2. \] (2.10)

Let
\[ K(\beta) = \sup \left\{ \frac{f(y)}{y} : 0 \leq y \leq C_\beta^2 \right\} \] (2.11)
and choose \( t_0(\beta) > 0 \) such that for \( t \geq t_0(\beta) \),
\[ 4t^2 e^{-t} K(\beta) < 1. \] (2.12)

From (2.8)', we can choose a \( \gamma_1 > \gamma_0 \) such that for all \( \gamma \geq \gamma_1 \),
\[ t_0(\beta) < T_1(\gamma). \] (2.13)

Hence from (2.10), (2.11) and (2.12) for all \( t \geq t_0(\beta), t \in [T_2(\gamma), T_1(\gamma)] \), \( \gamma \geq \gamma_1 \), we have
\[ 4t^2 e^{-t} \frac{f(y(t, \gamma))}{y(t, \gamma)} < 1. \] (2.14)

Let \( Z = t^{1/2} \), then \( Z \) satisfies
\[ Z'' + \frac{1}{4t^2} Z = 0 \] (2.15)
and
\[ y'' + \frac{1}{4t^2} \left( 4t^2 e^{-t} \frac{f(y)}{y} \right) y = 0. \] (2.16)

Hence from (2.14) and by Sturm's Comparison Theorem we have for all \( \gamma \geq \gamma_1 \),
\[ T_2(\gamma) < t_0(\beta). \] (2.17)

Now (2.4) follows from (2.6) and (2.17). This completes the proof of Claim 1 and hence Theorem 1.

In order to prove Claim 2 we need the following proposition.

Let \( F: \mathbb{R}_+ \to \mathbb{R}_+ \) be a locally Lipschitz continuous function and \( s_0 \geq 0 \) such that
\[ F(s) \text{ is strictly increasing for } s \geq s_0. \] (2.18)

Let \( G(s) = \log F(s) \) be \( C^2 \) and convex for \( s \geq s_0 \).
\[ (\gamma G'(\gamma))^2 e^{-\left\{ G(\gamma) - \frac{1}{2}(\gamma - s_0) G'(\gamma) \right\}} = O(1) \quad \text{as } \gamma \to \infty. \] (2.19)

\[ \lim_{\gamma \to \infty} \frac{\gamma G^{(p+1)}(\gamma)}{G^{(p)}(\gamma)} = L_p \neq 0 \quad \text{for } p = 0, 1, \] (2.20)
where $G^{(p)}$ denotes the $p$th derivative of $G$.

There exist positive constants $C_1, C_2, l$ and $\gamma_1$ such that

$$C_1 \gamma^l \leq G(\gamma) \leq C_2 \gamma^l.$$  

Let $Y(t, \gamma)$ denote the solution of

$$-Y'' = e^{-t}F(Y),$$

$$Y(\infty) = \gamma,$$

$$Y'(\infty) = 0,$$

and $S(\gamma)$ the first zero of $Y(t, \gamma)$. Let $S_0(\gamma)$ be such that $Y(S_0(\gamma), \gamma) = s_0$. Note that $S(\gamma) \leq S_0(\gamma)$. Then we have the following:

**Proposition 2.** We have, as $\gamma \to \infty$,

$$Y'(S_0(\gamma), \gamma) = \frac{2}{G'(\gamma)} \left[ 1 + O\left( \frac{(\log \gamma)^2}{G(\gamma)} \right) + O(\gamma G'(\gamma)e^{-\left(\frac{1}{2}G(\gamma) - \frac{1}{2}(\gamma - s_0)G'(\gamma)\right)}) \right],$$

$$S_0(\gamma) = \left( G(\gamma) - \frac{1}{2} \gamma G'(\gamma) \right) + s_0 \left( \frac{G'(\gamma)}{2} \right) + \log \frac{G'(\gamma)}{2}$$

$$+ O((\log \gamma)^2) + O(\gamma G'(\gamma)^2 e^{-\left(\frac{1}{2}G(\gamma) - \frac{1}{2}(\gamma - s_0)G'(\gamma)\right)})],$$

$$S(\gamma) \geq \left( G(\gamma) - \frac{1}{2} \gamma G'(\gamma) \right) + \log \frac{G'(\gamma)}{2} + O(1).$$

Proof of this proposition follows exactly as in Atkinson-Peletier [2] (see Lemma 10 and Theorem 4). Since the hypotheses here on $G$ are little bit different from those in Atkinson-Peletier [2] we shall for completeness sketch the proof in Appendix 1.

**Proof of Claim 2.** Let $F(s) = s|s|^me^{bs^2 + |s|^\beta}$, then for $s \geq 0$, we have

$$G(s) = bs^2 + s^\beta + (m + 1) \log s,$$

$$G'(s) = 2bs + \beta s^{\beta - 1} + \frac{m + 1}{s},$$

$$G''(s) = 2b + \beta(\beta - 1)s^{\beta - 2} - \frac{m + 1}{s^2},$$

$$G(s) - \frac{1}{2}sG'(s) = \left( 1 - \frac{\beta}{2} \right)s^\beta + (m + 1) \log s - \frac{m + 1}{2},$$

$$\lim_{s \to \infty} \frac{sG'(s)}{G(s)} = 2, \quad \lim_{s \to \infty} \frac{sG''(s)}{G'(s)} = 1,$$

$$bs^2 \leq G(s) \leq \left( b + 1 + \frac{m + 1}{2e} \right)s^2 \quad \text{for} \ s \geq 1.$$
from (2.18) to (2.22). Hence from Proposition 2, (2.28) and (2.30) we have as \( \gamma \to \infty \),
(2.33) \[ Y'(S_0(\gamma), \gamma) = O(1/\gamma), \]
(2.34) \[ S_0(\gamma) = s_0 b \gamma + O(\gamma^{\beta}), \]
(2.35) \[ T_1(\gamma) = S(\gamma) \geq (1 - \beta/2)\gamma^{\beta} + O(\log \gamma). \]
Hence \( T_1(\gamma) \to \infty \) as \( \gamma \to \infty \). This proves (2.8)' . Now from (2.34) and using
0 \leq \beta \leq 1 \) we have for \( \gamma \) large,
(2.36) \[ \frac{T_1(\gamma)}{\gamma} \leq \frac{S_0(\gamma)}{\gamma} \leq s_0 b + O(\gamma^{\beta-1}) \leq C_3 \]
for some constant \( C_3 \geq 0 \). This proves (2.8).
Let \( C_4 = \sup_{0 < s \leq s_0} F(s) \), then from (2.23) we have for \( t \in [S(\gamma), S_0(\gamma)] \),
(2.37) \[ -Y'' \leq C_4 e^{-t}. \]
Integrating (2.37) from \( T_1(\gamma) (= S(\gamma)) \) to \( S_0(\gamma) \) we have
(2.38) \[ Y'(T_1(\gamma), \gamma) \leq Y'(S_0(\gamma), \gamma) + C_4 (e^{-T_1(\gamma)} - e^{-S_0(\gamma)}). \]
Now from (2.33), (2.34) and (2.35), we can choose a constant \( C_5 > 0 \) such that for all \( \gamma \) large, (2.38) implies \( \gamma Y''(T_1(\gamma), \gamma) \leq C_5 \). This proves (2.7) and hence Claim 2.

Remark. The above proof shows that Theorem 1 can be stated in a more general form as follows.
Let \( f: \mathbb{R}^+ \to \mathbb{R}^+ \) be a \( C^1 \) function and let \( s_0 \geq 0 \) be such that
(2.39) \[ f \text{ is strictly increasing for } s \geq s_0, \]
(2.40) \[ g(s) = \log f(s) \text{ is } C^2 \text{ convex for } s \geq s_0, \]
\[ g(s) = bs^2 + g_1(s) \text{ with } b > 0 \text{ such that} \]
(2.41) \[ \lim_{s \to \infty} \frac{g_1(s)}{s^2} = 0, \quad \lim_{s \to \infty} |g_1'(s)| < \infty, \]
\[ \lim_{s \to \infty} g_1''(s) = 0, \quad \lim_{s \to \infty} \left| \frac{g_1(s) - \frac{1}{2} s g_1'(s)}{s} \right| < \infty. \]

Then we have the following

Theorem 1'. Let \( f \) satisfy (2.39), (2.40) and (2.41). Further assume that \( f(0) = 0 \) and extend \( f \) as an odd function on \( \mathbb{R} \). Then there exists an \( R_0 > 0 \) such that for \( 0 < R < R_0 \), (1.1) does not admit any radial solution which changes sign.

3. APPENDIX

Appendix 1. Let \( F: \mathbb{R}^+ \to \mathbb{R}^+ \) be a locally Lipschitz continuous function and \( G(s) = \log F(s) \) satisfies (2.18) to (2.22). Following the same notations as in Proposition 2 and denoting \( G(\gamma) = G \), \( G'(\gamma) = G' \), \( Y(t, \gamma) = Y(t) \), we have the following
Lemma 3.1. For $S_0 \leq t < \infty$ we have

\begin{align}
(3.1) \quad Y(t) &\leq \gamma - \frac{2}{G'} \log \left(1 + \frac{1}{2} G' e^{G' t}\right), \\
(3.2) \quad G(Y(t)) &\geq G - 2 \log \left(1 + \frac{1}{2} G' e^{G' t}\right), \\
(3.3) \quad t &> G - \frac{1}{2} (\gamma - Y(t)) G' + \log \frac{G'}{2}, \\
(3.4) \quad t \leq \frac{1}{2} \left(G + G(Y(t))\right) + \log \frac{G'}{2} - \log \left(1 - e^{(G(Y(t)) - G)/2}\right), \\
(3.5) \quad Y'(t) &\leq e^{\left(G + G(Y(t))\right)/2} - t, \\
(3.6) \quad Y''(t) &\geq e^{(G - (\gamma - Y(t))) G'/2 - t}, \\
(3.7) \quad S(y) &\geq G - \frac{1}{2} G y G' + \log \frac{G'}{2} + O(1) \quad \text{as } y \to \infty.
\end{align}

For the proof of this lemma we refer to Atkinson-Peletier [2]. In fact (3.1), (3.2), (3.4), (3.5), (3.6) and (3.7) of the above lemma correspond to (4.4), (4.5), (4.16), (4.18), (4.21), (4.22) and (3.5) of Atkinson-Peletier [2].

Let $k$ be a large positive (but fixed) number and define

\begin{align}
(3.8) \quad \delta &= k \log \gamma, \\
(3.9) \quad S_1 &= G + \log \frac{G'}{2} - \delta.
\end{align}

Then we have the following

Lemma 3.2. As $\gamma \to \infty$, we have

\begin{align}
(3.10) \quad Y(S_1) &= \gamma - \frac{2}{G'} \delta + O \left(\frac{\delta^2}{G}\right), \\
(3.11) \quad G(Y(S_1)) &= G - 2 \delta + O \left(\frac{\delta^2}{G}\right), \\
(3.12) \quad Y''(S_1) &= \frac{2}{G'} \left[1 + O \left(\frac{\delta^2}{G}\right)\right].
\end{align}

Proof. Taking $t = S_0$ in (3.4) we have for large $\gamma$,

\begin{align}
S_0 &\leq G + \log \frac{G'}{2} - \frac{1}{2} G + O(1) \\
&= S_1 - \left(\frac{1}{2} G - \delta\right) + O(1) < S_1.
\end{align}

Hence from (3.1) and (3.2), we have

\begin{align}
(3.14) \quad Y(S_1) &\leq \gamma - \frac{2}{G'} \log(1 + e^\delta) \leq \gamma - \frac{2}{G'} \delta + O \left(\frac{2}{G'} e^{-\delta}\right).
\end{align}
and
\begin{equation}
G(Y(S_1)) \geq G - 2\delta + O(e^{-\delta}).
\end{equation}

Since $G$ is an increasing function, we have from (3.14)
\begin{equation}
G(Y(S_1)) \leq G \left( \gamma - \frac{2\delta}{G'} + O \left( \frac{2}{G} e^{-\delta} \right) \right)
\end{equation}
\begin{equation}
= G - \left[ \frac{2\delta}{G} + O \left( \frac{2}{G} e^{-\delta} \right) \right] G' + \left[ \frac{2\delta}{G} + O \left( \frac{2}{G} e^{-\delta} \right) \right]^2 \frac{G''(\xi)}{2}
\end{equation}

for some $\xi$ in the interval $[\gamma - \frac{2\delta}{G'} + O(2e^{-\delta}/G'), \gamma]$. Now from (2.21), we have $G''(\xi)/(G')^2 = O(1/G)$ and hence (3.16) implies
\begin{equation}
G(Y(S_1)) \leq G - 2\delta + O(\delta^2/G).
\end{equation}

Therefore from (3.15) we have
\begin{equation}
G(Y(S_1)) = G - 2\delta + O(\delta^2/G).
\end{equation}

This proves (3.11).

From (3.15) we have
\begin{equation}
Y(S_1) \geq G^{-1}(G - 2\delta + O(e^{-\delta}))
\end{equation}
\begin{equation}
= \gamma - \frac{(2\delta + O(e^{-\delta}))}{G'} - \frac{(2\delta + O(e^{-\delta}))^2}{2} \frac{G''(\eta)}{(G'(\eta))^3}
\end{equation}
for some $\eta$ such that
\begin{equation}
G - 2\delta + O(e^{-\delta}) \leq G(\eta) \leq G.
\end{equation}

Now from (2.22) it follows that there exists a constant $C_1 > 0$ such that $C_1 \gamma \leq \eta \leq \gamma$. Therefore from (3.18) and (3.14) we have
\begin{equation}
Y(S_1) = \gamma - \frac{2\delta}{G'} + O \left( \frac{\delta^2}{G} \right).
\end{equation}

This proves (3.10).

Let $t = S_1$ in (3.5). Then using (3.11) we have
\begin{equation}
Y'(S_1) \leq e^{-\log G'/2 + O(\delta^2/G)} = \frac{2}{G'} \left[ 1 + O \left( \frac{\delta^2}{G} \right) \right].
\end{equation}

Similarly from (3.6) we obtain
\begin{equation}
Y'(S_1) \geq \frac{2}{G'} \left[ 1 + O \left( \frac{\delta^2}{G} \right) \right].
\end{equation}

Combining (3.20) and (3.21) we get (3.12). This completes the proof of the lemma.

**Lemma 3.3.** For $S_0 \leq t \leq S_1$, we have
\begin{equation}
Y'(t) = \frac{2}{G'} \left[ 1 + O \left( \frac{\delta^2}{G} \right) + O(\gamma G'e^{-\frac{1}{2}(G-\psi(Y))}) \right].
\end{equation}

**Proof.** From (3.3) we have
\begin{equation}
G(Y(t)) - t \leq \left\{ G(Y(t)) - \frac{1}{2} Y(t) G' \right\} - \left\{ G - \frac{1}{2} \gamma G' \right\} - \log \frac{G'}{2} = \psi(Y).
\end{equation}
Since $\psi''(Y) \geq 0$, it follows that

(3.23) \quad G(Y(t)) - t \leq \max \{\psi(Y(S_1)), \psi(Y(S_0))\}.

Using (3.10) and (3.11), the above implies that

(3.24) \quad G(Y(t)) - t \leq -\log \frac{G'}{2} + \max \left\{-\delta, -\left(1 - \frac{1}{2}(\gamma - s_0)G'\right)\right\} + O(1).

Hence from (3.12) and (3.24) we have for any $t \in [S_0, S_1]$

Y'(t) = Y'(S_1) + \int_t^{S_1} e^{G(Y(s)) - s} \, ds

(3.25)

= \frac{2}{G'} \left[1 + O\left(\frac{\delta^2}{G}\right)\right] + O\left[\frac{S_1 - S_0}{G'} \max\left\{e^{-\delta}, e^{-\left(1 - \frac{1}{2}(\gamma - s_0)G'\right)}\right\}\right].

From (2.20), $G - \frac{1}{2}(\gamma - s_0)G' > 0$ for $\gamma$ large, hence we have from (3.3), $S_0 \geq \log G'/2$ which implies that $S_1 - S_0 \leq G$. Hence (3.25) implies

(2.25)\quad S_0 = S_1 - \frac{Y(S_1) - s_0}{Y'(t)}.

This proves the lemma.

Proof of Proposition 2. (2.24) follows from Lemma 3.3. (2.26) follows from (3.7) of Lemma 3.1. Now from the mean value theorem, there exists a $t \in [S_0, S_1]$ such that

(3.26)\quad S_0 = S_1 - \frac{Y(S_1) - s_0}{Y'(t)}.

From (3.10) and (3.22), (3.26) implies that

\begin{align*}
S_0 &= S_1 - \frac{\gamma - 2\delta/G' + O(\delta^2/G) - s_0}{(2/G')[1 + O(\delta^2/G) + O(\gamma G' e^{-\left(1 - \frac{1}{2}(\gamma - s_0)G'\right)})]} \\
&= G - \frac{1}{2}\gamma G' + \frac{s_0 G'}{2} + \log \frac{G'}{2} + O\left(\frac{\gamma G'}{G}\delta^2\right) \\
&\quad + O((\gamma G')^2 e^{-\left(1 - \frac{1}{2}(\gamma - s_0)G'\right)}).
\end{align*}

(3.27)

Since $O(\gamma G' \delta^2/G) = O((\log \gamma)^2)$, (3.27) implies (2.25). This proves the proposition.

Appendix 2. Let $f: \mathbb{R}_+ \to \mathbb{R}_+$ be a $C^1$ function such that $f(0) = 0$. Let $y(t, \gamma)$ be the solution of

\begin{align*}
\frac{d^2y}{dt^2} &= e^{-t} f(y), \\
y(\infty) &= \gamma > 0, \\
y'(\infty) &= 0,
\end{align*}

(3.28)

and $T_1(\gamma)$ the first zero of $y(t, \gamma)$. Then there exists a real number $C$ such that

(3.29) \quad \lim_{\gamma \to 0} T_1(\gamma) = \begin{cases} -\infty & \text{if } f'(0) = 0, \\ C & \text{if } f'(0) \neq 0. \end{cases}
Proof. (i) Let \( f'(0) = 0 \). Then integrating (3.28) from \( T_1(y) \) to \( \infty \), we obtain
\[
\gamma = \int_{T_1(y)}^{\infty} (s - T_1(y)) e^{-s} f(y(s)) \, ds
\]
\[
\leq \gamma e^{-T_1(y)} \sup_{0 \leq y \leq y} \frac{f(y)}{y}.
\]
This implies that
\[
e^{T_1(y)} \leq \sup_{0 \leq y \leq y} \frac{f(y)}{y} \to 0 \quad \text{as} \quad \gamma \to 0.
\]
Hence \( T_1(y) \to -\infty \) as \( y \to 0 \).

(ii) Let \( f'(0) > 0 \). Let \( \phi \) be the solution of
\[
\begin{cases}
-\phi'' = f'(0)e^{-t} \phi, \\
\phi(\infty) = 1, \\
\phi'(\infty) = 0,
\end{cases}
\]
and \( T \) the first zero of \( \phi \).

For any two nonnegative continuous functions \( \rho_1 \) and \( \rho_2 \) defined on \( \mathbb{R} \) with \( \rho_1 \geq \rho_2 \), consider the following problem \( \pi_i \) \( (i = 1, 2) \).
\[
(\pi_i): \begin{cases}
-\psi'' = \rho_i e^{-t} \psi \quad \text{in} \quad \mathbb{R}, \\
\psi(\infty) > 0, \\
\psi'(\infty) = 0.
\end{cases}
\]
Let \( \psi_i \) be a solution of \( (\pi_i) \) and \( T_i \) the first zero of \( \psi_i \). Then we claim that
\[
T_2 \leq T_1.
\]
Suppose not, then \( T_1 < T_2 \) and let \( W(t) = \psi_1 \psi_2' - \psi_2 \psi_1' \). Then \( W'(t) = \psi_1 \psi_2 e^{-t}(\rho_1 - \rho_2) \) and hence integrating \( W'(t) \) from \( T_2 \) to \( \infty \), we have
\[
-\psi_1(T_2)\psi_2'(T_2) = \int_{T_2}^{\infty} \psi_1 \psi_2 e^{-t}(\rho_1 - \rho_2) \, dt \geq 0
\]
which is a contradiction. This proves (3.32).

Now for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for \( 0 < y < \delta \),
\[
(1 - \varepsilon)f'(0) \leq f(y)/y \leq (1 + \varepsilon)f'(0),
\]
Let \( 0 < y < \delta \) and by taking
\[
\rho_2 = (1 - \varepsilon)f'(0), \quad \rho_1 = \frac{f(y)}{y}, \quad \psi_2(t) = \phi \left( t + \log \frac{1}{1 - \varepsilon} \right),
\]
we obtain from (3.32) and (3.33),
\[
T + \log(1 - \varepsilon) \leq T(y).
\]
Similarly by taking
\[
\rho_1 = (1 + \varepsilon)f'(0), \quad \psi_1 = \phi \left( t + \log \frac{1}{1 + \varepsilon} \right), \quad \rho_2 = \frac{f(y)}{y},
\]
we obtain
\[
T(y) \leq T + \log(1 + \varepsilon).
\]
Since \( \varepsilon \) is arbitrary, from (3.34) and (3.35) we obtain \( \lim_{y \to 0} T(y) = T \). This proves (3.29).
REFERENCES


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