TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 332, Number 1, July 1992

RATIONAL FIBRATIONS IN DIFFERENTIAL HOMOLOGICAL ALGEBRA

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ABSTRACT. In this paper, a result of [6] is generalized as follows: Given a fibration $F \to E \xrightarrow{\rho} B$ of simply connected spaces in which either, the fibre has finite dimensional rational cohomology, or, it has finite dimensional rational homotopy and ρ induces a surjection in rational homotopy, we construct an explicit isomorphism,

$$\varphi: \mathrm{Ext}_{C^*(B,\mathbf{Q})}(\mathbf{Q}, C^*(B;\mathbf{Q}))\hat{\otimes}\mathrm{Ext}_{C^*(F;\mathbf{Q})}(\mathbf{Q}, C^*(F;\mathbf{Q})) \xrightarrow{\cong} \mathrm{Ext}_{C^*(E;\mathbf{Q})}(\mathbf{Q}, C^*(E;\mathbf{Q})).$$

This is deduced from its "algebraic translation," a more general result in the environment of graded differential homological algebra.

INTRODUCTION

For 1-connected topological spaces S, $\mathrm{Ext}_{C^*(S;\mathbf{K})}(\mathbf{K}, C^*(S;\mathbf{K}))$ turns out to be a nice homotopy invariant whose study was developed from the analogous concept in local algebra, $\mathrm{Ext}_R(\mathbf{K}, R)$ for a local commutative ring R [2]. This invariant can be thought as the reduced homology of a "virtual Spivak fibre" and is a key notion in the establishment of interesting results on others, more classical, homotopy invariants [6].

Given a fibration $F \to E \to B$ of simply connected spaces in which $H^*(B;\mathbf{Q})$ has finite type (i.e., it is finite dimensional in each degree) and $H^*(F;\mathbf{Q})$ is finite dimensional, a result of Y. Felix, S. Halperin and J. C. Thomas [6, Theorem 4.3] asserts that

$$\mathrm{Ext}_{C^*(F;\mathbf{Q})}(\mathbf{Q}, C^*(F;\mathbf{Q}))\hat{\otimes}\mathrm{Ext}_{C^*(B;\mathbf{Q})}(\mathbf{Q}, C^*(B;\mathbf{Q})) \cong \mathrm{Ext}_{C^*(E;\mathbf{Q})}(\mathbf{Q}, C^*(E;\mathbf{Q})).$$

This statement was proved via duality and, since it is the result of composing several isomorphisms, it is not easy to work with to develop explicit applications. The purpose of this paper is to generalize this result and give a natural and explicit way of constructing this isomorphism. We begin by stating two algebraic facts (see §1 for notation and definitions):

1. Let R be a DGA and let M, N, U be R-modules. Define a natural map,

$$\psi: \mathrm{Ext}_R(M, N) \otimes \mathrm{Ext}_R(U, M) \to \mathrm{Ext}_R(U, N),$$

Received by the editors February 5, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55P62, 18G15.
Key words and phrases. Rational fibration, differential homological algebra, Sullivan models.
Partially supported by a DGICYT grant (PB88-0329).

as follows: choose semifree resolutions $P \xrightarrow{\simeq} M$ and $Q \xrightarrow{\simeq} U$ of M and U respectively and let $f: Q \to M$, $g: P \to N$ represent $[f] \in \mathrm{Ext}_R(U, M)$ and $[g] \in \mathrm{Ext}_R(M, N)$. Then, define

$$\psi([g] \otimes [f]) = [g \circ \overline{f}],$$

in which $\overline{f}$ is the homotopy lifting of f to P:

$$\begin{tikzcd} & P \arrow[d, "\simeq"] \\ Q \arrow[ur, "\overline{f}"] \arrow[r, "f"] & M \end{tikzcd}$$

2. On the other hand, given a sequence of DGA morphisms, $R \to S \to T$, there are two natural transformations, defined in the classical way:

$$\eta: \mathrm{Ext}_R(-, -) \to \mathrm{Ext}_S(- \otimes_R S; - \otimes_R S), \qquad \nu: \mathrm{Ext}_T(-, -) \to \mathrm{Ext}_S(-, -).$$

Next, we combine 1 and 2 as follows: Let $A \to (A \otimes \Lambda Y, d) \to (\Lambda Y, \overline{d})$ be a KS-extension of the commutative differential graded algebra (CDGA) A defined over a field $\mathbf{K}$ of characteristic zero. Given an A-module N and a $(\Lambda Y, \overline{d})$-module U, we define a map

$$\varphi: \mathrm{Ext}_A(\mathbf{K}, N) \hat{\otimes} \mathrm{Ext}_{\Lambda Y}(U, \Lambda Y) \to \mathrm{Ext}_{A \otimes \Lambda Y}(U, N \otimes \Lambda Y)$$

by $\varphi(\sum_{i=n}^{\infty} \alpha_i \otimes \beta_i) = \sum_{i=n}^{\infty} \psi(\eta(\alpha_i) \otimes \nu(\beta_i))$.

Observe that φ is well defined. In fact, by definition of semicomplete tensor product (see §1 for notation and definitions), the degree of β_i is bounded above, so is the degree of $\nu(\beta_i)$. Therefore, since $(\Lambda Y, \overline{d})$ is positively graded, given $P \xrightarrow{\cong} U$ a semifree resolution of U, $f_i: P \to (\Lambda Y, \overline{d})$ representing $\nu(\beta_i)$ and $\Phi \in P$, there is only a finite number of i for which $f_i(\Phi) \neq 0$.

Our aim is to prove

Theorem A. *Let $(A \otimes \Lambda Y, d)$ be a KS-extension of the connected CDGA A. Let N be an A-module and let $U = U^{\geq r}$ be a $(\Lambda Y, \overline{d})$-module of finite type for some $r \in \mathbf{Z}$. If $H^*(\Lambda Y, \overline{d})$ is finite dimensional, then*

$$\varphi: \mathrm{Ext}_A(\mathbf{K}, N) \hat{\otimes} \mathrm{Ext}_{\Lambda Y}(U, \Lambda Y) \xrightarrow{\simeq} \mathrm{Ext}_{A \otimes \Lambda Y}(U, N \otimes \Lambda Y)$$

is an isomorphism. [1]

We prove that there is another set of conditions which makes φ an isomorphism:

For any minimal KS-extension $(A \otimes \Lambda Y, d)$ in which A or $(\Lambda Y, \overline{d})$ is 1-connected and $H^*(\Lambda Y, \overline{d})$ has finite type, there is another KS-extension $(\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d) \to (\Lambda Y, \overline{d})$ and "quisms" (morphisms inducing cohomology isomorphisms) α and γ, such that the diagram

$$\begin{tikzcd} A \arrow[r] & (A \otimes \Lambda Y, d) \arrow[r] & (\Lambda Y, \overline{d}) \\ (\Lambda X, d) \arrow[u, "\alpha", "\simeq"'] \arrow[r] & (\Lambda X \otimes \Lambda Y, d) \arrow[u, "\gamma", "\simeq"'] \arrow[r, "ot"] & (\Lambda Y, \overline{d}) \arrow[u, equal] \end{tikzcd}$$

[1] In this case and as we shall prove

$$\mathrm{Ext}_A(\mathbf{K}, N) \hat{\otimes} \mathrm{Ext}_{\Lambda Y}(U, \Lambda Y) \cong \mathrm{Ext}_A(\mathbf{K}, N) \hat{\otimes} \mathrm{Ext}_{\Lambda Y}(U, \Lambda Y).$$

commutes, and $(\Lambda X, d)$ is the minimal model of A [7, §6]. If $(\Lambda X \otimes \Lambda Y, d)$ is minimal, we shall say that $(A \otimes \Lambda Y, d)$ is a *intrinsic* KS-extension [9, Definition 4.13].

Then, we prove

Theorem B. *Let $(A \times \Lambda Y, d)$ be an intrinsic KS-extension and let $N = N^{\geq r}$ be an A-module. If Y is finite dimensional, then*

$$\varphi\colon \mathrm{Ext}_Z(\mathbf{Q}, N) \otimes \mathrm{Ext}_{\Lambda Y}(\mathbf{Q}, \Lambda Y) \xrightarrow{\simeq} \mathrm{Ext}_{A \otimes \Lambda Y}(\mathbf{Q}, N \otimes \Lambda Y)$$

is an isomorphism. [2]

Now, given a fibration $F \to E \xrightarrow{\rho} E$ of simply connected spaces, we can consider its associated sequence of differential forms $A(B) \to A(E) \to A(F)$, [12] or [8, Chapter 20]. A classical result on Sullivan's theory of minimal models [9, §4] asserts the existence of a KS-extension $A(B) \to (A(B) \otimes \Lambda Y, d) \to (\Lambda Y, \overline{d})$ and quisms ϕ and α, such that the following diagram commutes:

$$\begin{array}{ccccc} A(B) & \longrightarrow & A(E) & \longrightarrow & A(F) \\ \| & & \phi\uparrow\simeq & & \simeq\uparrow\alpha \\ A(B) & \longrightarrow & (A(B) \otimes \Lambda Y, d) & \longrightarrow & (\Lambda Y, \overline{d}) \end{array}$$

This KS-extension is intrinsic if and only if $\pi_*(\rho) \otimes \mathbf{Q}$ is surjective [9, §4]. It is also known [3, Chapter 11] that the vector space Y can be identified to the rational homotopy of the fibre, $\pi_*(F) \otimes \mathbf{Q}$. On the other hand, for any space S, the DGAs $A(S)$ and $C^*(S; \mathbf{Q})$ have the same weak homotopy type.

Considering these facts, and applying Theorems A and B to the KS-extension $A(B) \to (A(B) \otimes \Lambda Y, d) \to (\Lambda Y, \overline{d})$, we deduce

Theorem C. *Let $F \to E \xrightarrow{\rho} B$ be a fibration of simply connected spaces in which B has finite $\mathbf{Q}$-type, and where either* (i) *$H^*(F; \mathbf{Q})$ is finite dimensional or* (ii) *$\pi_*(F) \otimes \mathbf{Q}$ is finite dimensional and $\pi_*(\rho) \otimes \mathbf{Q}$ is surjective. Then, there exists an isomorphism*

$$\begin{aligned} \varphi\colon \mathrm{Ext}_{C^*(B;\mathbf{Q})}(\mathbf{Q}, C^*(B; \mathbf{Q})) \hat{\otimes} \mathrm{Ext}_{C^*(F;\mathbf{Q})}(\mathbf{Q}, C^*(F; \mathbf{Q})) \\ \xrightarrow{\cong} \mathrm{Ext}_{C^*(E;\mathbf{Q})}(\mathbf{Q}, C^*(E; \mathbf{Q})). \end{aligned}$$

Example. Consider the fibration

$$\Omega(S^3) \to * \to S^3.$$

A model for this map is given by

$$(\Lambda x, 0) \to (\Lambda(x, y), d) \to (\Lambda y, 0),$$

where x has degree 3, y has degree 2 and $dy = x$.

On the other hand $\mathrm{Ext}_{\Lambda x}(\mathbf{Q}, \Lambda x)$ and $\mathrm{Ext}_{\Lambda y}(\mathbf{Q}, \Lambda y)$ are 1-dimensional vector spaces [6, §3] and basis for such spaces are represented respectively by the morphisms

$$g\colon (\Lambda(x, \overline{x}), d) \to (\Lambda x, 0), \quad g(1) = x, \quad g(\overline{x}^n) = 0, \qquad n \geq 1 \ (d\overline{x} = x),$$
$$f\colon (\Lambda(y, \overline{y}), d) \to (\Lambda y, 0), \quad f(1) = 0, \quad f(\overline{y}) = 1 \qquad (d\overline{y} = y).$$

[2] Observe that via [6, §3] and since Y is finite dimensional, $(\Lambda Y, \overline{d})$ is Gorenstein, i.e., $\mathrm{Ext}_{\Lambda Y}(\mathbf{K}, \Lambda Y)$ is a one dimensional $\mathbf{K}$-vector space. Hence the semicomplete tensor product coincides with the classical one.

A short computation shows that in this case $\varphi \equiv 0$, even though (see [6, §3]),

$$\mathrm{Ext}_{\Lambda x}(\mathbf{Q}, \Lambda x) \otimes \mathrm{Ext}_{\Lambda y}(\mathbf{Q}, \Lambda y) \cong \mathrm{Ext}_{\Lambda(x,y)}(\mathbf{Q}, \Lambda(x, y)) \cong \mathbf{Q}.$$

Observe that in this example the rational cohomology of the fiber is not finite dimensional and, although the rational homotopy of the fibre is finite dimensional, this fibration is not intrinsic, i.e. it does not induce a surjection in homotopy.

This paper is organized as follows: In the next section we give notation, definitions and some basic facts. §2 is devoted to the proof of Theorem A; in §3, we shall prove Theorem B. We finish with a remark where we give a slight generalization of Theorems A and B.

This paper is part of my thesis, carried out under the supervision of Professor Stephen Halperin of the University of Toronto whom I would like to express my gratitude for his infinite patience and help.

Also, I would like to thank the referee for his helpful comments.

1. Algebraic preliminaries

For definitions and basic facts from Sullivan's theory of minimal models and its connection with rational homotopy theory, standard references are [12, 8 and 5]. The notation we shall use is the one in [8] or [5] where a very good and brief algebraic summary can be found.

We shall work over a field $\mathbf{K}$ of characteristic 0 unless explicitly stated otherwise. In a graded object, the degree of an element x will be denoted by $|x|$.

Given two differential graded vector spaces V and W, we define their *complete tensor product* $V \hat{\otimes} W$ as follows:

$$(V \hat{\otimes} W)^p = \Pi_i (V^i \otimes W^{p-i}), \qquad (d\Phi)_i = d_V \Phi_{i-1} + d_W \Phi_i,$$
$$\Phi = \{\Phi_i\} \in (V \hat{\otimes} W)^p.$$

Observe that an element $\Phi \in (V \hat{\otimes} W)^p$ can be written as

$$\Phi = \sum_{i=-\infty}^{\infty} v_i \otimes w_i,$$

where $|v_i| + |w_i| = p$, $|v_i| \leq |v_{i+1}|$, and for each $q \in \mathbf{Z}$ the set $\{v_i, |v_i| = q\}$ is finite. Also we can define the *semicomplete tensor product* $V \tilde{\otimes} W$ as the subspace of $V \hat{\otimes} W$ of the elements Φ, in the form just described, satisfying: $\Phi = \sum_{i=n_\Phi}^{\infty} v_i \otimes w_i$, for some $n_\Phi \in \mathbf{Z}$. In both cases, $H(V \hat{\otimes} W) = H(V) \hat{\otimes} H(W)$, $H(V \tilde{\otimes} W) = H(V) \tilde{\otimes} H(W)$.

Observe that the semicomplete tensor product is not a symmetric object: $V \hat{\otimes} W$ is not isomorphic to $W \hat{\otimes} V$!

A KS-*extension* of a CDGA (commutative differential graded algebra) A is a CDGA $(A \otimes \Lambda X, d)$ where ΛX is the free commutative graded algebra over the graded vector space X, satisfying:

(i) The inclusion $\mathrm{id}_A \otimes 1 : A \to (A \otimes \Lambda X, d)$ is a morphism of CDGAs.

(ii) X admits a well-ordered basis $\{x_\alpha\}$ such that $dx_\alpha \in \Lambda X_{<\alpha}$.

We usually write a KS-extension as a sequence

$$A \to (A \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, \overline{d}),$$

with $\rho: A \otimes \Lambda X \to A \otimes \Lambda X / A^+ \otimes \Lambda X = (\Lambda X, \overline{d})$, the projection.

If $dX \subset A^+ \otimes \Lambda X + \mathbf{K} \otimes \Lambda^{\geq 2} X$ we shall say that the KS-extension is *minimal*.

A KS-*complex* (resp. *minimal* KS-*complex*) is a KS-extension (resp. minimal KS-extension of $\mathbf{K}$.

Given a space S and its associated CDGA consisting of the polynomials forms on S, $A(S)$ [12, §7], the minimal model of S is a minimal KS-complex $(\Lambda X, d)$ together with a morphism $\gamma: (\Lambda X, d) \xrightarrow{\simeq} A(S)$, inducing homology isomorphism. Any space S admits a minimal model unique up to isomorphism [8, Chapter 6].

With respect to basic facts we need in the context of differential homological algebra, introduced by Eilenberg and J. C. Moore [11], we shall follow literally the approach in [6, Appendix] or [1]:

1.1. **Definition** [6, A.2]. Let A be a DGA and let M be an A-module.

(i) M is *A-semifree* if there exists a filtration of A-submodules $0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots$ such that $M = \bigcup_i F_i$ and for each i, F_i / F_{i+1} is $A^{\#}$-free on a basis of cycles. If the differential in $\mathbf{K} \otimes_A M = 0$ we shall say that M is *minimal*.

(ii) A quism of A-modules $P \xrightarrow{\simeq} M$ is called a *semifree resolution* (resp. *minimal resolution*) of M if P is semifree (resp. minimal and semifree).

1.2. **Example.** Let $\gamma: A \to B$ be a morphism of augmented c-connected CDGAs. Observe that a Sullivan model for γ, $\psi: (A \otimes \Lambda X, d) \xrightarrow{\simeq} B$, is a semifree resolution of B (as A-modules) [6, A.9]. In particular, a sullivan model of the augmentation $A \to (\mathbf{K}, 0)$, $(A \otimes \Lambda X, d) \xrightarrow{\simeq} (\mathbf{K}, 0)$, is a semifree resolution of $\mathbf{K}$. We shall say that $(A \otimes \Lambda X, d)$ is an *acyclic closure* of A.

1.3. **Proposition** [6, A.3]. (i) *Any A module M, has semifree resolutions and, if A is 1-connected and $M = M^{\geq r}$, M has minimal resolutions.*

(ii) *Given a diagram of A-modules*

$$\begin{array}{ccc} & & Q \\ & & \Big\downarrow{\scriptstyle g}\ \simeq \\ P & \xrightarrow{\ f\ } & M \end{array}$$

where P is semifree and g is a quism, there exists a homomorphism $h: P \to Q$, unique up to homotopy, such that $g \circ h \sim f$. In particular, $H(g) \circ H(h) = H(f)$.

Let M, N, P be A-modules. Define a morphism of differential vector spaces,

$$\psi: \mathrm{Hom}_A(M, N) \hat{\otimes} P \to \mathrm{Hom}_A(M, N \hat{\otimes} P)$$

by

$$\psi\left(\sum_{i=-\infty}^{\infty} f_i \otimes p_i\right)(m) = \sum_{i=-\infty}^{\infty} (-1)^{|p_i||m|} f_i(m) \otimes p_i,$$

with $\sum_{i=-\infty}^{\infty} f_i \otimes p_i \in \mathrm{Hom}_A(M, N) \hat{\otimes} P$ and $m \in M$.

Then we have

1.4. **Proposition.** *If $M^{\#}$ is $A^{\#}$-free and P or M has finite type,*

$$\psi: \mathrm{Hom}_A(M, N) \hat{\otimes} P \xrightarrow{\simeq} \mathrm{Hom}_A(M, N \hat{\otimes} P)$$

is an isomorphism.

Proof. First, it is easy to check that in fact ψ commutes with the differentials so it is a map of differential vector spaces.

Next, observe that since $M^{\#}$ is $A^{\#}$-free, it is of the form,

$$M^{\#} = A^{\#} \otimes V$$

for some vector space V. Then, as vector spaces, we have

$$\begin{aligned}
[\mathrm{Hom}_A(M, N)\hat{\otimes} P]^s &= \Pi_i(\mathrm{Hom}_A^{s-i}(M, N) \otimes P^i) \\
&= \Pi_i(\mathrm{Hom}_A^{s-i}(A \otimes V, N) \otimes P^i) \\
&= \Pi_i(\mathrm{Hom}_{\mathbf{K}}^{s-i}(V, N) \otimes P^i) \\
&= \Pi_{i,j}(\mathrm{Hom}_{\mathbf{K}}(V^j, N^{s+j-i}) \otimes P^i) \\
&\overset{3}{\cong} \Pi_{i,j}\mathrm{Hom}_{\mathbf{K}}(V^j, N^{s+j-i} \otimes P^i) \\
&= \Pi_j\mathrm{Hom}_{\mathbf{K}}(V^j, \Pi_i(N^{s+j-i} \otimes P^i)) \\
&= \Pi_j\mathrm{Hom}_{\mathbf{K}}(V^j, (N\hat{\otimes} P)^{s+j}) \\
&= \mathrm{Hom}_{\mathbf{K}}^s(V, N\hat{\otimes} P) = \mathrm{Hom}_A^s(M, N\hat{\otimes} P).
\end{aligned}$$

Now, it is a straightforward computation to prove that this chain of isomorphisms coincides with ψ. □

1.5. **Definition** [6, A.2]. Let M, N be A-modules and $P \xrightarrow{\simeq} M$ a semifree resolution. Define $\mathrm{Ext}_A(M, N) = H(\mathrm{Hom}_A(P, N))$.

1.6. *Remarks.* (1) This definition is independent of the particular resolution chosen because of Proposition 1.3.

(2) $\mathrm{Ext}_A(M, N)$ is a covariant functor in N and contravariant in the variables A and M.

(3) Moreover, if $\varphi: A \xrightarrow{\simeq} B$ is a quism of DGAs, M and N A-modules, M' and N' B-modules and $f: M \xrightarrow{\simeq} M'$, $g: N \xrightarrow{\simeq} N'$ are quisms of A-modules (M', N' regarded as A-modules via φ), we have the following natural isomorphisms,

$$\mathrm{Ext}_A(M, N) \xrightarrow{\alpha\cong} \mathrm{Ext}_A(M, N') \xleftarrow{\beta\cong} \mathrm{Ext}_A(M', N') \xleftarrow{\gamma\cong} \mathrm{Ext}_B(M', N').$$

More explicitly, if $P \xrightarrow{\simeq} M$ is a semifree resolution,

$$\alpha: H(\mathrm{Hom}_A(P, N)) \xrightarrow{\cong} H(\mathrm{Hom}_A(P, N')), \qquad \alpha[h] = [g \circ h].$$

On the other hand, note that $P \xrightarrow{\simeq} M \xrightarrow{\simeq} M'$ is a semifree resolution of M' as A-modules and thus, β can be regarded as the identity in $H(\mathrm{Hom}_A(P, N'))$.

Finally, consider $P' \xrightarrow{\simeq} M'$ a semifree resolution of M' as B-module (which is also a quism of A-modules!) and apply Proposition 1.3 to get the following homotopy commutative diagram of A-modules:

$$\begin{tikzcd}
 & P' \arrow[d, "\simeq"] \\
P \arrow[ur, "\eta"] \arrow[r, "\simeq"] & M'.
\end{tikzcd}$$

[3] Either, V^j or P^i is finite dimensional.

Then,

$$\gamma: H(\mathrm{Hom}_B(P', N')) \xrightarrow{\cong} H(\mathrm{Hom}_A(P, N')), \qquad \gamma[h] = [h \circ \eta].$$

2. φ AT THE Hom LEVEL

This section is devoted to the proof of Theorem A, so from now on, fix a KS-extension $A \to (A \otimes \Lambda Y, d) \to (\Lambda Y, \overline{d})$, an A-module N and a $(\Lambda Y, \overline{d})$-module $U = U^{\geq r}$, for some $r \in \mathbf{Z}$. Also, choose an A-semifree resolution of $\mathbf{K}$, $(A \otimes \Lambda X, d) \xrightarrow{\simeq} \mathbf{K}$, given by Example 1.2, and a $(\Lambda Y, \overline{d})$-semifree resolution of U, $M \xrightarrow{\simeq} U$. Observe that, since $U = U^{\geq r}$, M can be also chosen bounded below.

2.1. **Lemma.** *The inclusion,*

$$\begin{aligned} i: &\mathrm{Hom}_A(A \otimes \Lambda X, N) \tilde{\otimes} \mathrm{Hom}_{\Lambda Y}(M, \Lambda Y) \\ &\xrightarrow{\simeq} \mathrm{Hom}_A(A \otimes \Lambda X, N) \hat{\otimes} \mathrm{Hom}_{\Lambda Y}(M, \Lambda Y) \end{aligned}$$

is a quism.

Proof. Let n be an integer such that $H^{>n}(\Lambda Y, \overline{d}) = 0$. Consider the finite dimensional CDGA, $B = (\Lambda Y, \overline{d})/I$, where $I = (\Lambda Y)^{>n} \oplus J$, with J a complement of the cocycles in degree n. Clearly, I is acyclic so the projection $\Lambda Y \xrightarrow{\simeq} B$ is a quism.

Next, consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(A \otimes \Lambda X, N) \tilde{\otimes} \mathrm{Hom}_{\Lambda Y}(M, \Lambda Y) & \overset{i}{\hookrightarrow} & \mathrm{Hom}_A(A \otimes \Lambda X, N) \hat{\otimes} \mathrm{Hom}_{\Lambda Y}(M, \Lambda Y) \\ \cong \downarrow & & \downarrow \simeq \\ \mathrm{Hom}_A(A \otimes \Lambda X, N) \tilde{\otimes} \mathrm{Hom}_{\Lambda Y}(M, B) & \overset{j}{\hookrightarrow} & \mathrm{Hom}_A(A \otimes \Lambda X, N) \hat{\otimes} \mathrm{Hom}_{\Lambda Y}(M, B) \end{array}$$

Observe that because M is semifree, the vertical arrows are quisms. Also, since B is finite dimensional and $M = M^{\geq r}$, $[\mathrm{Hom}_{\Lambda Y}(M, B)]^{>n-r} = 0$. Hence,

$$\mathrm{Hom}_A(A \otimes \Lambda X, N) \tilde{\otimes} \mathrm{Hom}_{\Lambda Y}(M, B) = \mathrm{Hom}_A(A \otimes \Lambda X, N) \hat{\otimes} \mathrm{Hom}_{\Lambda Y}(M, B),$$

and j is the identity. Therefore, i is a quism. □

Next, consider the CDGA $(A \otimes \Lambda X, d) \otimes_A (A \otimes \Lambda Y, d) = (A \otimes \Lambda X \otimes \Lambda Y, d)$. Since $A(A \otimes \Lambda X, d)$ is acyclic, the projection $\rho: (A \otimes \Lambda X \otimes \Lambda Y, d) \xrightarrow{\cong} (\Lambda Y, \overline{d})$ is a quism, so there exists a section $\sigma: (\Lambda Y, \overline{d}) \xrightarrow{\simeq} (A \otimes \Lambda X \otimes \Lambda Y, d)$ of ρ, which is also a quism.

Observe that, since

$$\mathrm{Hom}_A(A \otimes \Lambda X, N \otimes_A (A \otimes \Lambda Y)) = \mathrm{Hom}_A(A \otimes \Lambda X, N \otimes \Lambda Y)$$

has structures of $(A \otimes \Lambda X, d)$- and $(A \otimes \Lambda Y, d)$-modules, it is also a $(A \otimes \Lambda X \otimes \Lambda Y, d)$-module and therefore we can regard it as $(\Lambda Y, \overline{d})$-module via the section σ.

On the other hand, a homomorphism $g \in \mathrm{Hom}_A(A \otimes \Lambda X, N)$ can be identified to $\tilde{g} \in \mathrm{Hom}_A(A \otimes \Lambda X, N \otimes \Lambda Y)$ by $\tilde{g}(\Psi) = g(\Psi) \otimes 1$.

Then, the key fact in the proof of Theorem A is the following lemma:

2.2. **Lemma.** *A quism of* $(\Lambda Y, \overline{d})$*-modules,*

$$\alpha: \Lambda Y \otimes \mathrm{Hom}_A(A \otimes \Lambda X, N) \xrightarrow{\simeq} \mathrm{Hom}_A(A \otimes \Lambda X, N \otimes \Lambda Y),$$

is given by $\alpha(\Phi \otimes g) = \Phi \cdot \tilde{g}$.

Proof. Since A is 1-connected, by Proposition 1.3.(i), we can build a minimal resolution of $(A \otimes \Lambda Y, d)$, $\psi: P \xrightarrow{\simeq} (A \otimes \Lambda Y, d)$. The procedure to get such a resolution is similar to the construction of the minimal model of a CDGA, that is to say, we build a graded vector space V, a differential d in $A \otimes V$ which extends the one in A, and a quism $\psi: (A \otimes V, d) \xrightarrow{\simeq} (A \otimes \Lambda Y, d)$, such that the differential induced by d in $\mathbf{K} \otimes_A (A \otimes V)$ is zero. Then, ψ induces a quism

$$(\mathbf{K} \otimes_A (A \otimes V), 0) \cong (V, 0) \xrightarrow{\simeq} (\Lambda Y, \overline{d}).$$

Therefore for certain differential d, $P = (A \otimes H(\Lambda Y), d)$.

Write $\Lambda Y = C \oplus \overline{d}C \oplus H$ where $\overline{d}H = 0$. Then, an A-linear isomorphism

$$\gamma: A \otimes (C \oplus \overline{d}C \oplus H) \xrightarrow{\cong} (A \otimes \Lambda Y, d)$$

is given by $\gamma(c) = c$, $\gamma(\overline{d}c) = dc$, $\gamma(h) = h$. Since $C \oplus \overline{d}C$ is acyclic, the projection

$$\pi: (A \otimes (C \oplus \overline{d}C \oplus H), d) \xrightarrow{\simeq} (A \otimes H(\Lambda Y), d)$$

is a quism. Consider the quism $\phi = \pi \circ \gamma^{-1}$ and observe that, given a Φ cycle of degree p in $(\Lambda Y, \overline{d})$, $\gamma^{-1}(\Phi) = \gamma^{-1}(\overline{d}c + h) = \overline{d}c + h + \Psi$, with $\Psi \in A \otimes (\Lambda Y)^{<p}$. Therefore,

$$\phi(a \otimes \Phi) = a \otimes [\Phi] + \Omega, \qquad \Omega \in A \otimes H^{<p}(\Lambda Y). \tag{1}$$

Now, since ϕ is a quism of A-semifree modules,

$$1 \otimes \phi: N \otimes \Lambda Y = N \otimes_A (A \otimes \Lambda Y) \xrightarrow{\simeq} N \otimes_A (A \otimes H(\Lambda Y)) = N \otimes H(\Lambda Y)$$

is a quism of A-modules. Therefore, since $A \otimes \Lambda X$ is A-semifree, it induces a quism

$$\mathrm{Hom}_A(A \otimes \Lambda X, N \otimes \Lambda Y) \xrightarrow{\simeq} \mathrm{Hom}_A(A \otimes \Lambda X, N \otimes H(\Lambda Y)). \tag{2}$$

Composing (2) with α and the quism $\mathrm{Hom}_A(A \otimes \Lambda X, N) \otimes H(\Lambda Y) \xrightarrow{\simeq} \mathrm{Hom}_A(A \otimes \Lambda X, A) \otimes \Lambda Y$ we have

$$\begin{aligned} \mathrm{Hom}_A(A \otimes \Lambda X, N) \otimes H(\Lambda Y) &\xrightarrow{\simeq} \mathrm{Hom}_A(A \otimes \Lambda X, N) \otimes \Lambda Y \\ &\xrightarrow{\alpha} \mathrm{Hom}_A(A \otimes \Lambda X, N \otimes \Lambda Y) \\ &\xrightarrow{\simeq} \mathrm{Hom}_A(A \otimes \Lambda X, N \otimes H(\Lambda Y)). \end{aligned} \tag{3}$$

To show that α is a quism it is sufficient to prove that (3) is an isomorphism. On the other hand, since $H^*(\Lambda Y, \overline{d})$ is finite dimensional, we have an isomorphism of graded vector spaces

$$\mathrm{Hom}_A(A \otimes \Lambda X, N \otimes H(\Lambda Y)) \cong \mathrm{Hom}_A(A \otimes \Lambda X, N) \otimes H(\Lambda Y). \tag{4}$$

Composing (3) and (4), we get a morphism $\overline{\alpha}$,

$$\overline{\alpha}: \mathrm{Hom}_A(A \otimes \Lambda X, N) \otimes H(\Lambda Y) \to \mathrm{Hom}_A(A \otimes \Lambda X, N) \otimes H(\Lambda Y).$$

Finally, in view of equation (1), $\overline{\alpha}$ maps $f \otimes \Phi$ to $f \otimes \Phi + \Omega$, $\Omega \in \mathrm{Hom}_A(A \otimes \Lambda X, N) \otimes H^{<p}(\Lambda Y)$, $p = |\Phi|$, and hence, since $H^*(\Lambda Y)$ is finite dimensional, is an isomorphism.

Therefore, (3) is an isomorphism and then, α is a quism. □

Proof of Theorem A. The inclusion i of Lemma 2.1 composed with the isomorphism β of Proposition 1.5, gives a quism

$$\begin{aligned}\beta \circ i\colon &\mathrm{Hom}_A(A \otimes \Lambda X, N)\hat{\otimes}\mathrm{Hom}_{\Lambda Y}(M, \Lambda Y)\\ &\xrightarrow{\simeq} \mathrm{Hom}_{\Lambda Y}(M, \Lambda Y\hat{\otimes}\mathrm{Hom}_A(A \otimes \Lambda X, N)).\end{aligned}$$

Explicitly, $[(\beta \circ i)(\sum_{i=0}^{\infty} g_i \otimes f_i)](m) = \sum_{i=0}^{\infty}(-1)^{|g_i|\,|f_i(m)|} f_i(m) \otimes g_i$.

Since $|f_i| \to -\infty$ as $i \to \infty$, we have $f_i(m) \in (A \otimes \Lambda X)^{<0} = 0$, for i large. This shows that $\beta \circ i$ factors as

$$\begin{tikzcd}\mathrm{Hom}_A(A \otimes \Lambda X, N)\hat{\otimes}\mathrm{Hom}_{\Lambda Y}(M, \Lambda Y) \arrow[r, "\beta\circ i"] \arrow[dr, "\hat{\beta}"'] & \mathrm{Hom}_{\Lambda Y}(M, \Lambda Y\hat{\otimes}\mathrm{Hom}_A(A \otimes \Lambda X, N)) \\ & \mathrm{Hom}_{\Lambda Y}(M, \Lambda Y \otimes \mathrm{Hom}_A(A \otimes \Lambda X, N)) \arrow[u, "\hat{\gamma}=\mathrm{Hom}_{\Lambda Y}(M,\gamma)"']\end{tikzcd}$$

where

$$\gamma\colon \Lambda Y \otimes \mathrm{Hom}_A(A \otimes \Lambda X, N) \xhookrightarrow{\simeq} \Lambda Y\hat{\otimes}\mathrm{Hom}_A(A \otimes \Lambda X, N)$$

is the canonical inclusion and, since $H(\Lambda Y, \overline{d})$ is finite dimensional, is a quism.

Hence, because M is ΛY-semifree, $\hat{\gamma}$ is also a quism so is $\hat{\beta}$. For the same reason, α of Lemma 2.2 induces a quism $\hat{\alpha} = \mathrm{Hom}_{\Lambda Y}(M, \alpha)$.

Therefore, we have the following chain of quisms:

$$\begin{aligned}&\mathrm{Hom}_A(A \otimes \Lambda X, N)\hat{\otimes}\mathrm{Hom}_{\Lambda Y}(M, \Lambda Y)\\ &\xrightarrow{\hat{\beta}} \mathrm{Hom}_{\Lambda Y}(M, \Lambda Y \otimes \mathrm{Hom}_A(A \otimes \Lambda X, N))\\ &\xrightarrow{\hat{\alpha}} \mathrm{Hom}_{\Lambda Y}(M, \mathrm{Hom}_A(A \otimes \Lambda X, N \otimes \Lambda Y))\\ &\xrightarrow{\simeq} \mathrm{Hom}_{\Lambda Y}(M, \mathrm{Hom}_{A\otimes\Lambda Y}(A \otimes \Lambda X \otimes \Lambda Y, N \otimes \Lambda Y))\\ &\xrightarrow{\cong} \mathrm{Hom}_{A\otimes\Lambda Y}(A \otimes \Lambda X \otimes \Lambda Y \otimes_{\Lambda Y} M, N \otimes \Lambda Y)\end{aligned} \tag{5}$$

Since M is ΛY-semifree, the projection $(A \otimes \Lambda X \otimes \Lambda Y, d) \xrightarrow{\simeq} (\Lambda Y, \overline{d})$ induces a quism of $(A \otimes \Lambda Y, d)$-modules $A \otimes \Lambda X \otimes \Lambda Y \otimes_{\Lambda Y} M \xrightarrow{\simeq} \Lambda Y \otimes_{\Lambda Y} M = M$. This map composed with $M \xrightarrow{\simeq} U$ gives a $A \otimes \Lambda Y$-semifree resolution of U. Hence, the sequence (5) induces an isomorphism:

$$\mathrm{Ext}_A(\mathbf{K}, N)\hat{\otimes}\mathrm{Ext}_{\Lambda Y}(U, \Lambda Y) \xrightarrow{\cong} \mathrm{Ext}_{A\otimes\Lambda Y}(U, N \otimes \Lambda Y).$$

Finally, it is a straightforward computation to see that this isomorphism is the map φ of the theorem. □

2.3. *Remark.* In the case that $(A \otimes \Lambda Y, d) = A \otimes (\Lambda Y, \overline{d})$ is a trivial KS-extension and $N = N^{\geq s}$, for some $s \in \mathbf{Z}$, Theorem A also holds even without assuming $H^*(\Lambda Y, \overline{d})$ finite dimensional.

In fact, in this special case, $N \otimes (\Lambda Y, \overline{d}) = N\hat{\otimes}(\Lambda Y, \overline{d})$, and so, by Proposition 1.4,

$$\Lambda Y\hat{\otimes}\mathrm{Hom}_A(A \otimes \Lambda X, N) \xrightarrow{\cong} \mathrm{Hom}_A(A \otimes \Lambda X, N \otimes \Lambda Y).$$

Again, the composition,

$$\begin{aligned}\varphi\colon &\mathrm{Hom}_A(A\otimes\Lambda X\,,\,N)\hat{\otimes}\mathrm{Hom}_{\Lambda Y}(M\,,\,\Lambda Y)\\ &\xrightarrow{\cong}\mathrm{Hom}_{\Lambda Y}(M\,,\,\Lambda Y\hat{\otimes}\mathrm{Hom}_A(A\otimes\Lambda X\,,\,N))\\ &\xrightarrow{\cong}\mathrm{Hom}_{\Lambda Y}(M\,,\,\mathrm{Hom}_A(A\otimes\Lambda X\,,\,N\otimes\Lambda Y))\\ &\xrightarrow{\cong}\mathrm{Hom}_{\Lambda Y}(M\,,\,\mathrm{Hom}_{A\otimes\Lambda Y}(A\otimes\Lambda X\otimes\Lambda Y\,,\,N\otimes\Lambda Y))\\ &\xrightarrow{\cong}\mathrm{Hom}_{A\otimes\Lambda Y}(A\otimes\Lambda X\otimes\Lambda Y\otimes_{\Lambda Y}M\,,\,N\otimes\Lambda Y)\,,\end{aligned}$$

induces in homology the map φ of Theorem A and therefore, it is an isomorphism.

3. Associativity of φ

Theorem B will be established once we prove

3.1. **Theorem.** *Let* $(\Lambda X\,,\,d)\to(\Lambda X\otimes\Lambda Y\,,\,d)\to(\Lambda Y\,,\,\overline{d})$ *be a minimal KS-extension of the minimal KS-complex* $(\Lambda X\,,\,d)$ *in which* $(\Lambda X\otimes\Lambda Y\,,\,d)$ *is also minimal. Let* $N=N^{\geq r}$ *be an* $(\Lambda X\,,\,d)$*-module for some* $r\in\mathbf{Z}$. *If* Y *is finite dimensional, then*

$$\varphi\colon\mathrm{Ext}_{\Lambda X}(\mathbf{Q}\,,\,N)\otimes\mathrm{Ext}_{\Lambda Y}(\mathbf{Q}\,,\,\Lambda Y)\xrightarrow{\cong}\mathrm{Ext}_{\Lambda X\otimes\Lambda Y}(\mathbf{Q}\,,\,N\otimes\Lambda Y)$$

is an isomorphism.

For that, we need the following result.

3.2. **Proposition.** *Let* $(\Lambda X\,,\,d)$ *be a* 1*-connected KS-complex of finite type and let* $\Psi\in\Lambda^{\geq 2}X$ *be a decomposable cycle. Then, there exists a KS-extension of* $(\Lambda X\,,\,d)$, $(\Lambda X\,,\,d)\to(\Lambda X\otimes\Lambda Y\,,\,d)\to(\Lambda Y\,,\,\overline{d})$ *such that:*

(i) Y *is finite dimensional and oddly graded.*
(ii) *In* $(\Lambda X\otimes\Lambda Y\,,\,d)$, Ψ *is a boundary, that is to say, for some* $\Phi\in\Lambda X\otimes\Lambda Y$, $\Psi=d\Phi$.

Proof. Let $|\Psi|=n+1$. Consider the projection of $(\Lambda X\,,\,d)$ onto its space of odd indecomposables

$$\pi\colon(\Lambda X\,,\,d)\to(\mathbf{K}\oplus X^{\mathrm{odd}}\,,\,0).$$

Then take the minimal model for π:

$$\begin{tikzcd}(\Lambda X\,,\,d)\arrow[r] \arrow[dr, "\pi"'] & (\Lambda X\otimes\Lambda Y'\,,\,d)\arrow[r] \arrow[d, "\varphi", "\simeq"'] & (\Lambda Y'\,,\,\overline{d}) \\ & (\mathbf{K}\oplus X^{\mathrm{odd}}\,,\,0) & \end{tikzcd}$$

Next, we consider the linear part $d_1\colon Y'\to X$ of the differential d in $\Lambda X\otimes\Lambda Y'$. If $d_1(y)\neq 0$, then $d_1(y)$ belongs to X^{even} and the degree of y is odd. If $d_1(y)=0$, then y defines an element in the minimal model of $(\mathbf{K}\oplus X^{\mathrm{odd}}\,,\,0)$ and therefore the degree of y is also odd.

On the other hand, since φ is a quism, there exists an element $\Phi\in\Lambda X\otimes\Lambda Y'$ such that $d\Phi=\Psi$.

To finish define $Y=Y'^{\leq n}$. $\square$

We also need to prove, and this is the key fact in the proof of Theorem 3.1, that φ is "associative": Consider KS-extensions, $A \to (A \otimes \Lambda Y, d) \to (\Lambda Y, \overline{d})$, $A \to (A \otimes \Lambda Z, d) \to (\Lambda Z, \overline{d})$ of the connected CDGA A. Let N be an A-module. Then, we have

3.3. **Proposition.** *The following diagram commutes*:

$$\begin{array}{ccc}
\mathrm{Ext}_A(\mathbf{K}, N) \otimes \mathrm{Ext}_{\Lambda Y}(\mathbf{K}, \Lambda Y) \otimes \mathrm{Ext}_{\Lambda Z}(\mathbf{K}, \Lambda Z) & \xrightarrow{\varphi \otimes 1} & \mathrm{Ext}_{A \otimes \Lambda Y}(\mathbf{K}, N \otimes \Lambda Y) \otimes \mathrm{Ext}_{\Lambda Z}(\mathbf{K}, \Lambda Z) \\
\varphi \otimes 1 \downarrow & & \downarrow \varphi \\
\mathrm{Ext}_{A \otimes \Lambda Z}(\mathbf{K}, N \otimes \Lambda Z) \otimes \mathrm{Ext}_{\Lambda Y}(\mathbf{K}, \Lambda Y) & \xrightarrow{\varphi} & \mathrm{Ext}_{A \otimes \Lambda Y \otimes \Lambda Z}(\mathbf{K}, N \otimes \Lambda Y \otimes \Lambda Z)
\end{array}$$

Notes. (1) In this case, φ is the restriction to the classical tensor product.

(2) By the left vertical arrow we mean

$$\begin{aligned}
&\mathrm{Ext}_A(\mathbf{K}, N) \otimes \mathrm{Ext}_{\Lambda Y}(\mathbf{K}, \Lambda Y) \otimes \mathrm{Ext}_{\Lambda Z}(\mathbf{K}, \Lambda Z) \\
&\qquad \cong \mathrm{Ext}_A(\mathbf{K}, N) \otimes \mathrm{Ext}_{\Lambda Z}(\mathbf{K}, \Lambda Z) \otimes \mathrm{Ext}_{\Lambda Y}(\mathbf{K}, \Lambda Y) \\
&\qquad \xrightarrow{\varphi \otimes 1} \mathrm{Ext}_{A \otimes \Lambda Z}(\mathbf{K}, N \otimes \Lambda Z) \otimes \mathrm{Ext}_{\Lambda Y}(\mathbf{K}, \Lambda Y).
\end{aligned}$$

Proof. Consider a Sullivan model of the projection $(A \otimes \Lambda Y \otimes \Lambda Z, d) \to (\mathbf{K}, 0)$,

$$(A \otimes \Lambda X \otimes \Lambda Y \otimes \Lambda \overline{Y} \otimes \Lambda Z \otimes \Lambda \overline{Z}, d) \xrightarrow{\simeq} (\mathbf{K}, 0),$$

which constitutes a $(A \otimes \Lambda Y \otimes \Lambda Z, d)$-semifree resolution of $(\mathbf{K}, 0)$. Observe that, for example, $(\Lambda Y \otimes \Lambda \overline{Y}, \overline{d})$, $(A \otimes \Lambda X \otimes \Lambda Y, d)$ are resolution of $(\mathbf{K}, 0)$ and $(\Lambda Y, \overline{d})$ as $(\Lambda Y, \overline{d})$-and $(A \otimes \Lambda Y, d)$-modules respectively.

Choosing those particular resolutions, it is a straightforward computation to show that the diagram above commutes. □

Proof of Theorem 3.1. We shall proceed by induction on $n = \dim Y$.

Suppose first $n = 1$. Then $(\Lambda Y, \overline{d}) = (\Lambda y, 0)$. If $|y|$ is odd $H^*(\Lambda Y)$ is finite dimensional so we apply Theorem A and therefore φ is an isomorphism.

Suppose $|y|$ even. Then, by Proposition 3.2, there exists a KS-extension $(\Lambda X \otimes \Lambda Z, d)$ such that Z is finite dimensional and oddly graded, and $dy = d\Phi$ for some $\Phi \in \Lambda X \otimes \Lambda Z$. Then, we "redefine" y by $u - \Phi$ in $A \otimes \Lambda Z \otimes \Lambda y$ to get an isomorphism

$$(1) \qquad (\Lambda X \otimes \Lambda Z \otimes \Lambda y, d) \cong (\Lambda X \otimes \Lambda Z, d) \otimes (\Lambda y, 0).$$

Now, apply Proposition 3.3 to the KS-extensions $(\Lambda X \otimes \Lambda Z, d)$ and $(\Lambda X \otimes \Lambda y, d)$ to obtain the following diagram:

$$\begin{array}{ccc}
\mathrm{Ext}_{\Lambda X}(\mathbf{K}, N) \otimes \mathrm{Ext}_{\Lambda y}(\mathbf{K}, \Lambda y) \otimes \mathrm{Ext}_{\Lambda Z}(\mathbf{K}, \Lambda Z) & \xrightarrow{\varphi \otimes 1} & \mathrm{Ext}_{\Lambda X \otimes \Lambda y}(\mathbf{K}, N \otimes \Lambda y) \otimes \mathrm{Ext}_{\Lambda Z}(\mathbf{K}, \Lambda Z) \\
\varphi \otimes 1 \downarrow & & \downarrow \varphi \\
\mathrm{Ext}_{\Lambda X \otimes \Lambda Z}(\mathbf{K}, N \otimes \Lambda Z) \otimes \mathrm{Ext}_{\Lambda y}(\mathbf{K}, \Lambda y) & \xrightarrow{\varphi} & \mathrm{Ext}_{\Lambda X \otimes \Lambda Z \otimes \Lambda y}(\mathbf{K}, N \otimes \Lambda Z \otimes \Lambda y).
\end{array}$$

Since Z is oddly graded, $H^*(\Lambda Z)$ is finite dimensional. Hence, by Theorem A, the two vertical arrows are isomorphisms. On the other hand, considering (1) and Remark 2.3, the bottom map is also an isomorphism and therefore,

$$\varphi\colon \mathrm{Ext}_{\Lambda X}(\mathbf{K}, N) \otimes \mathrm{Ext}_{\Lambda y}(\mathbf{K}, \Lambda y) \xrightarrow{\simeq} \mathrm{Ext}_{\Lambda X \otimes \Lambda y}(\mathbf{K}, N \otimes \Lambda y)$$

is an isomorphism.

Let $\dim Y = n$ and suppose that our assertion is true for $\dim Y \leq n-1$.

Write $(\Lambda Y, d) = (\Lambda(y_1, \dots, y_n), d)$ and choose a model for the projection $(\Lambda(y_1, \dots, y_n), d) \to (\mathbf{K}, 0)$ of the form

$$(\Lambda(y_1, \dots, y_n, \overline{y}_1, \dots, \overline{y}_n), d) \xrightarrow{\simeq} (\mathbf{K}, 0),$$

with $d\overline{y}_i = y_i + \Omega$, $\omega \in \Lambda Y_{<i} \otimes \Lambda \overline{Y}_{<i}$. Note that the KS-complexes

$$(\Lambda Y_{<n} \otimes \Lambda \overline{Y}_{<n}, d) = (\Lambda(y_1, \dots, y_{n-1}, \overline{y}_1, \dots, \overline{y}_{n-1}), d), \quad (\Lambda(y_n, \overline{y}_n), \overline{d}),$$

are semifree resolution of $\mathbf{K}$ as $\Lambda(y_1, \dots, y_{n-1})$- and Λy_n-modules respectively.

Choosing those particular resolutions, it is straightforward to verify that the diagram

$$\begin{array}{ccc}
\mathrm{Ext}_{\Lambda X}(\mathbf{K}, N) \otimes \mathrm{Ext}(\Lambda Y_{<n}) \otimes \mathrm{Ext}(\Lambda y_n) & \xrightarrow{\varphi \otimes 1} & \mathrm{Ext}_{\Lambda X \otimes \Lambda Y_{<n}}(\mathbf{K}, N \otimes \Lambda Y_{<n}) \otimes \mathrm{Ext}(\Lambda y_n) \\
\downarrow{\scriptstyle 1 \otimes \varphi} & & \downarrow{\scriptstyle \varphi} \\
\mathrm{Ext}_{\Lambda X}(\mathbf{K}, N) \otimes \mathrm{Ext}_{\Lambda Y}(\mathbf{K}, \Lambda Y) & \xrightarrow{\varphi} & \mathrm{Ext}_{\Lambda X \otimes \Lambda Y}(\mathbf{K}, N \otimes \Lambda Y)
\end{array}$$

where $\mathrm{Ext}(\Lambda Y_{<n})$ and $\mathrm{Ext}(\Lambda y_n)$ denote $\mathrm{Ext}_{\Lambda Y_{<n}}(\mathbf{K}, \Lambda Y_{<n})$ and $\mathrm{Ext}_{\Lambda y_n}(\mathbf{K}, \Lambda y_n)$ respectively, commutes.

By induction hypothesis, the top arrows and the two vertical arrows are isomorphisms. Hence,

$$\varphi\colon \mathrm{Ext}_{\Lambda X}(\mathbf{K}, N) \otimes \mathrm{Ext}_{\Lambda Y}(\mathbf{K}, \Lambda Y) \xrightarrow{\cong} \mathrm{Ext}_{\Lambda X \otimes \Lambda Y}(\mathbf{K}, N \otimes \Lambda Y)$$

is an isomorphism. □

4. A final remark

Theorems A and B can be generalized as follows:

Let $(A \otimes \Lambda X, d)$ and $(A \otimes \Lambda Y, d)$ be two KS-extensions of the 1-connected CDGA A. Suppose that the projection, $(A \otimes \Lambda X) \otimes_A (A \otimes \Lambda Y) = (A \otimes \Lambda X \otimes \Lambda Y, d) \xrightarrow{\rho} (\Lambda Y, \overline{d})$, admits a section $\sigma\colon (\Lambda Y, \overline{d}) \to (A \otimes \Lambda X \otimes \Lambda Y, d)$. Then the map,

$$(1) \qquad (A \otimes \Lambda X, d) \otimes (\Lambda Y, \overline{d}) \xrightarrow{\cong} (A \otimes \Lambda X \otimes \Lambda Y, d), \qquad \Phi \otimes \Psi \mapsto \Psi \cdot \sigma(\Psi)$$

is an isomorphism.

Then we define two functors (which depend on σ!),

$$A\text{-modules} \xrightarrow{\prime} A \otimes \Lambda Y\text{-modules}, \qquad N' = N \otimes_A (A \otimes \Lambda Y) = N \otimes \Lambda Y,$$

$$\Lambda Y\text{-modules} \xrightarrow{\prime\prime} A \otimes \Lambda X \otimes \Lambda Y\text{-modules}, \qquad M'' = (A \otimes \Lambda X) \otimes M$$

(M'' with the structure of an $(A \otimes \Lambda X \otimes \Lambda Y)$-module induced by the isomorphism (1)).

Both functors preserve semifree modules and quisms. Then, given an A-module N and a $(\Lambda Y, \overline{d})$-module M, we can define a map

$$\varphi\colon \mathrm{Ext}_A(A \otimes \Lambda X, N) \tilde{\otimes} \mathrm{Ext}_{\Lambda Y}(M, \Lambda Y) \to \mathrm{Ext}_{A \otimes \Lambda Y}(A \otimes \Lambda X \otimes M, N \otimes \Lambda Y)$$

by $\varphi(\sum_{i=0}^{\infty}[g_i] \otimes [f_i]) = \sum_{i=0}^{\infty}[g_i' \circ f_i'']$.

Then, we have

4.1. **Theorem A′.** *Let N be an A-module and let $M = M^{\geq r}$ be a $(\Lambda Y, \overline{d})$-module of finite type for some $r \in \mathbf{Z}$. If $H^*(\Lambda Y, \overline{d})$ is finite dimensional, then*

$$\varphi: \mathrm{Ext}_A(A \otimes \Lambda X, N) \hat{\otimes} \mathrm{Ext}_{\Lambda Y}(M, \Lambda Y) \xrightarrow{\cong} \mathrm{Ext}_{A \otimes \Lambda Y}(A \otimes \Lambda X \otimes M, N \otimes \Lambda Y)$$

is an isomorphism.

4.2. **Theorem B′.** *Let $N = N^{\geq r}$ be an A-module for some $r \in \mathbf{Z}$. If the KS-extension $(A \otimes \Lambda Y, d)$ is intrinsic and Y is finite dimensional, then*

$$\varphi: \mathrm{Ext}_A(A \otimes \Lambda X, N) \otimes \mathrm{Ext}_{\Lambda Y}(\mathbf{K}, \Lambda Y) \xrightarrow{\cong} \mathrm{Ext}_{A \otimes \Lambda Y}(A \otimes \Lambda X, N \otimes \Lambda Y)$$

is an isomorphism.

Proofs for those results are analogous to the ones of Theorems A and B.

Note. In the work *Bass series of local ring homomorphisms of finite flat dimension*, a preprint by L. L. Avramov, H. Foxby and J. Lescot, Theorem 4.1 seems to be the analogue of Theorem A of this paper, in the environment of local algebra.

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