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FREE ACTIONS ON $\mathbb{R}$-TREES

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ABSTRACT. We characterize the free minimal actions of finitely generated groups on $\mathbb{R}$-trees in terms of certain equivalence relations on compact metric graphs.

The results of this paper grew out of an effort to determine which groups act freely on $\mathbb{R}$-trees. An $\mathbb{R}$-tree is a certain kind of metric space which generalizes a tree. The definition of an $\mathbb{R}$-tree is reviewed in §1. It is well known that only free groups act freely on ordinary trees, but Morgan and Shalen [2] have shown that most surface groups act freely on $\mathbb{R}$-trees. In the course of my investigation I discovered a method for constructing all free minimal actions of finitely generated groups on $\mathbb{R}$-trees.

This method relies on two simple ideas.

Segment closure. Let T be a metric space. Let $\approx$ be an equivalence relation on T. Suppose that for all $\varepsilon > 0$ and all pairs of distance preserving maps $\iota_1: [0, \varepsilon] \to T$ and $\iota_2: [0, \varepsilon] \to T$ enjoying the property that for all t such that $0 \le t < \varepsilon$ we have $\iota_1(t) \approx \iota_2(t)$, it follows that $\iota_1(\varepsilon) \approx \iota_2(\varepsilon)$. In this case, we say that the equivalence relation $\approx$ is *segment closed.* Given any relation $D \subset T \times T$ on a metric space T, the *segment closure* $\approx$ of D is the intersection of all segment closed equivalence relations on T which contain D.

Notice the segment closure of a relation on a metric space exists and is segment closed.

Fold. Let T be a metric space. Let $\approx$ be an equivalence relation on T. Let $\varepsilon > 0$. Let $p, q \in T$ be such that $d(p, q) = 2\varepsilon$. Let $\iota: [-\varepsilon, \varepsilon] \to [p, q]$ be a distance preserving map. Then $[p, q]$ is a *fold of T with respect to* $\approx$ if and only if for all $t \in [0, \varepsilon]$ we have $\iota(-t) \approx \iota(t)$ (see Figure 1).

Observe that points equidistant from the midpoint of a fold are $\approx$-equivalent.

Suppose we are given a compact connected metric 1-complex Γ, i.e., a *graph.* Suppose $\approx$ is an equivalence relation on Γ which is segment closed and contains no fold in the interior of an edge of Γ. Let τ be the universal cover of Γ. We can lift the equivalence relation to τ by saying two points in τ are equivalent if and only if their projections in Γ are equivalent. This equivalence relation on τ may well have folds whose midpoints are vertices of τ. The idea of the paper is to collapse these folds by identifying points equidistant from the midpoint of the fold. Such a collapse can formally be defined via a pushout diagram. Once the original folds at the vertices of τ have been collapsed, new

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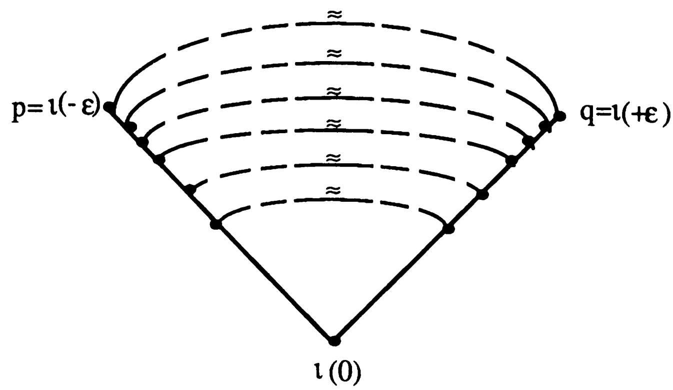

FIGURE 1

folds may appear. Collapse these new folds, and keep folding forever if necessary to remove all the folds. Formally, we define a category of folds with initial object τ and a unique terminal object T. It turns out that T is an $\mathbb{R}$-tree, the projection $\tau \to T$ is a $\pi_1(\Gamma)$-map, and $\pi_1(\Gamma)$ acts quasi-freely on T. Thus $\pi_1(\Gamma)$ modulo the stabilizer of a point of T is a group which acts freely on T. This theory is developed in §§2, 3, and 4.

In §5, we show that our theoretical construction for free actions on $\mathbb{R}$-trees from certain equivalence relations on compact graphs is completely general. This means that, given a finitely generated group G_1 acting freely and minimally by isometry on an $\mathbb{R}$-tree T_1, we may construct a certain compact metric graph with a segment closed equivalence relation. From this information we pass to a segment closed equivalence relation $\approx$ on the (metric) universal cover τ of this graph. No edge of τ contains a $\approx$-fold. By the above construction, we obtain a free action $G_* \times T \to T$ of a group G_* on an $\mathbb{R}$-tree T. Moreover, this action is equivalent to the action of G_1 on T_1, in that there is an isomorphism $h: G_* \to G_1$ and an isometry $\psi: T \to T_1$ such that for all $g \in G_*$ and $p \in T$, we have $\psi(gp) = h(g)\psi(p)$.

I would like to thank John Stallings, who gave me a great deal of encouragement, and unlimited use of his office and his Mac II during the year 1988–89 while I was at MSRI in Berkeley. He also uncovered a subtle error in the original version of this paper which led me to the concept of segment closure.

1. THE DEFINITION OF AN $\mathbb{R}$-TREE

In this section we review the definition of an $\mathbb{R}$-tree. Ordinary trees may be defined as $\mathbb{R}$-trees which admit an underlying simplicial structure. It is important to realize that the *metric* of an ordinary tree is not often stressed, in that all edges are imagined to have the same length. However, the possibility for different length edges contributes in an essential way to the richness of the subject of free actions on $\mathbb{R}$-trees.

Warning. Unless otherwise indicated, all group actions in this paper are by definition actions by isometry on some metric space, typically an $\mathbb{R}$-tree.

1.1 **Definitions.** A *simplicial tree* is a nonempty contractible simplicial one-complex with a specified metric and the induced metric topology. A simplicial tree is *locally finite* if each vertex lies in the boundary of at most a finite number of edges.

1.2 **Definitions.** A map $\iota\colon A \to B$ between metric spaces is *distance preserving* if for all $a, b \in A$ we have $d(a, b) = d(\iota(a), \iota(b))$, where $d(\cdot, \cdot)$ denotes the appropriate metric. A map between metric spaces is an *isometry* if it is both distance preserving and surjective. The isometry group of a metric space T is denoted $\mathrm{Iso}(T)$. Notice that every subgroup G of $\mathrm{Iso}(T)$ acts on T *a fortiori.*

In order to define an $\mathbb{R}$-tree, we shall adopt the point of view of Alperin and Bass [1]. We shall not be concerned, as Alperin and Bass were, with the more general concept of a Λ-tree. In this context, Λ refers to an ordered abelian group. For us this group will be $\mathbb{R}$ with its usual order. Thus an $\mathbb{R}$-metric space [1, p. 275] is just a metric space and an $\mathbb{R}$-metric morphism [1, p. 276] is a distance preserving map as defined above. We shall always use $d(\cdot, \cdot)$ to denote whatever metric is appropriate.

1.3 **Definition** (cf. [1, p. 277]). A metric space T is *geodesically linear* if, given $x, y \in T$, there is a unique distance preserving map $\iota\colon [0, d(p, q)] \to T$ such that $\iota(0) = p$ and $\iota(d(p, q)) = q$. We then denote by $[p, q]$ the image of ι, and call it the *closed interval* in T between p and q.

1.4 **Definition** (cf. [1, p. 278]). An $\mathbb{R}$-tree is a nonempty metric space T satisfying

(a) T is geodesically linear.

(b) If $p, q, r \in T$ then $[p, q] \cap [p, r] = [p, w]$ for some $w \in T$.

(c) If $p, q, r \in T$ and $[p, q] \cap [q, r] = \{q\}$, then $[p, q] \cup [q, r] = [p, r]$.

It is not hard to see that a simplicial tree is an $\mathbb{R}$-tree. Morgan and Shalen [2] show that most surface groups act freely on $\mathbb{R}$-trees; therefore, an $\mathbb{R}$-tree is not necessarily a simplicial tree.

1.5 **Definition.** An action of a group on an $\mathbb{R}$-tree T is *free* if no nonidentity group element leaves an element of T fixed. An action of a group G on T is *minimal* if the intersection of all subtrees S of T such that $GS = S$ is equal to T.

1.6 **Theorem** (cf. [1, Theorem 3.17, p. 310]). *Let T be a metric space with a basepoint $b \in T$. Define $\bigwedge\colon T \times T \to \mathbb{R}$ by $p \wedge q = \frac{1}{2}(d(p, b) + d(q, b) - d(p, q))$. Then T is an $\mathbb{R}$-tree if and only if conditions* (1) *and* (2) *hold:*

(1) *For all $p, q, r \in T$, $p \wedge r \geq \min(p \wedge q, q \wedge r)$, and*

(2) *For all $p \in T$, there is a distance preserving map $\iota\colon [0, d(p, b)] \to T$ such that $\iota(0) = b$ and $\iota(d(p, b)) = p$.*

1.7 **Lemma.** *Let T be an $\mathbb{R}$-tree and let $p_0, p_1, \ldots, p_n$ be points of T. Then there exist points $s_0, s_1, \ldots, s_k$ such that*

(i) $s_0 = p_0$, $s_k = p_n$,

(ii) *for i such that $0 \leq i < k$, there exists j, $0 \leq j < n$, such that $[s_i, s_{i+1}] \subset [p_j, p_{j+1}]$,*

(iii) *for i such that $0 \leq i < k - 1$, we have $[s_i, s_{i+1}] \cap [s_{i+1}, s_{i+2}] = \{s_{i+1}\}$,*

(iv) $[p_0, p_n] = \bigcup_{i=0}^{k-1} [s_i, s_{i+1}]$, *and*

(v) $d(p_0, p_n) = \sum_{i=0}^{k-1} d(s_i, s_{i+1})$.

Sketch of proof. The proof is by induction on n, using parts (b) and (c) of Definition 1.4. This lemma is a more detailed version of [1, Piecewise geodesic proposition, p. 280]. □

The referee pointed out the following result to me (in a slightly different form).

1.8 **Theorem** [4, Remark 1.2(1)]. *A connected metric space T is an $\mathbb{R}$-tree if and only if T is* 0*-hyperbolic, i.e., if and only if condition* (1) *of Theorem* 1.6 *holds.*

Theorem 1.6 was used in the original proof of Theorem 2.17. Using Theorem 1.8 instead vastly simplifies the proof of 2.17 as the principal difficulty in the proof lay in constructing the requisite map $\iota\colon [0, d(p, b)] \to T$.

(An alternative definition for 0-hyperbolic is as follows: for all w, x, y, z

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w),\ d(x, w) + d(y, z)\}.$$

This condition is implied by condition (1) of Theorem 1.6, see [4, Corollary 2.4].)

In §5 we shall need the notion of a *translation axis.* Let G be a group which acts freely on an $\mathbb{R}$-tree T. Then every element $g \in G$ such that $g \neq 1$ has a translation axis $A(g)$ (see [1, Theorem 6.6]). The translation axis of g is the unique subspace $A(g) \subset T$ of T such that $gA(g) = A(g)$ and $A(g)$ is isometric to $\mathbb{R}$. Moreover, for all $g \in G$ such that $g \neq 1$, there is a number $|g| \in \mathbb{R}$ greater than zero such that for all $p \in A(g)$ we have $d(p, gp) = |g|$. The number $|g|$ is the *translation length* of g.

2. The category of $\mathbb{R}$-trees with equivalence relations

In this section we set up a convenient category and prove some preliminary results. After becoming familiar with the objects and morphisms in this category, one may wish to read the rest of §2 after examining the proof of the main theorem in §4.

In this section and the following sections we shall use the notions of segment closure and fold defined precisely in the introduction of the paper. For convenience, we repeat these definitions here.

2.1 **Definition.** Let T be a metric space. Let $\approx$ be an equivalence relation on T. Suppose that for all $\varepsilon > 0$ and all pairs of distance preserving maps $\iota_1\colon [0, \varepsilon] \to T$ and $\iota_2\colon [0, \varepsilon] \to T$ enjoying the property that for all t such that $0 \leq t < \varepsilon$ we have $\iota_1(t) \approx \iota_2(t)$, it follows that $\iota_1(\varepsilon) \approx \iota_2(\varepsilon)$. In this case, we say that the equivalence relation $\approx$ is *segment closed.* Given any relation $D \subset T \times T$ on a metric space T, the *segment closure* $\approx$ of D is the intersection of all segment closed equivalence relations on T which contain D.

Notice the segment closure of a relation on a metric space always exists and is segment closed.

2.2 **Definition.** Let T be a metric space. Let $\approx$ be an equivalence relation on T. Let $\varepsilon > 0$. Let $p, q \in T$ be such that $d(p, q) = 2\varepsilon$. Let $\iota\colon [-\varepsilon, \varepsilon] \to [p, q]$ be a distance preserving map. Then $[p, q]$ is a *fold of T with respect to* $\approx$ if and only if for all $t \in [0, \varepsilon]$ we have $\iota(-t) \approx \iota(t)$.

2.3 **Definition.** The *objects* of the category are $\mathbb{R}$-trees with some extra structure. We use the notation $\mathbb{C}(T)$ to denote an object of the category whose underlying space is the $\mathbb{R}$-tree T. Specifically, $\mathbb{C}(T) \equiv (T, D(T), \mathscr{E}(T))$ is a triple consisting of

(i) an $\mathbb{R}$-tree T,

(ii) an equivalence relation $D(T) \subset T \times T$, and

(iii) a collection $\mathscr{E}(T)$ of closed intervals in T of nonzero length, such that every closed interval of T is contained in the union of a finite number of elements of $\mathscr{E}(T)$.

In the event of a subscripted $\mathbb{R}$-tree T_n, we use the more expedient notation $\mathbb{C}(T_n) \equiv (T_n, D_n, \mathscr{E}_n)$.

Notice property (iii) rules out the possibility of an object whose underlying space is a single point.

2.4 **Definition.** A *morphism* in the category is a map $\varphi: \tau \to T$ of $\mathbb{R}$-trees which transfers the structure of $\mathbb{C}(\tau)$ to $\mathbb{C}(T)$. The notation $\mathbb{C}(\varphi)$ denotes a morphism of the category whose underlying map is φ. A morphism $\mathbb{C}(\varphi): \mathbb{C}(\tau) \to \mathbb{C}(T)$ satisfies

(i) the underlying map $\varphi: \tau \to T$ is continuous and surjective,

(ii) for all $E \in \mathscr{E}(\tau)$, φ is distance preserving on E and $\mathscr{E}(T) = \{\varphi(E) \subset T : E \in \mathscr{E}(\tau)\}$, and

(iii) for all $p, q \in \tau$, we have $(p, q) \in D(\tau) \Leftrightarrow (\varphi(p), \varphi(q)) \in D(T)$.

2.5 **Lemma.** *Let $\mathbb{C}(\varphi_1): \mathbb{C}(T) \to \mathbb{C}(T_1)$ and $\mathbb{C}(\varphi_2): \mathbb{C}(T) \to \mathbb{C}(T_2)$ be morphisms.*

(i) *If there exists a morphism $\mathbb{C}(\psi): \mathbb{C}(T_1) \to \mathbb{C}(T_2)$ such that $\mathbb{C}(\varphi_2) = \mathbb{C}(\varphi_1) \circ \mathbb{C}(\psi)$, then this morphism is unique.*

(ii) *Suppose there exists a bijection $\psi: T_1 \to T_2$ such that the following diagram of $\mathbb{R}$-trees commutes.*

$$\begin{tikzcd} & T_1 \arrow[dd, "\psi"] \\ T \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \\ & T_2 \end{tikzcd}$$

Then ψ is the underlying map of an equivalence $\mathbb{C}(\psi): \mathbb{C}(T_1) \to \mathbb{C}(T_2)$.

Proof. (i) This follows from the fact that φ_1 is surjective.

(ii) Suppose first that $p, q \in T_1$ lie in $\varphi_1(E)$ for some $E \in \mathscr{E}(T)$. By (ii) of Definition 2.4, there are unique points $p', q' \in E$ such that $\varphi_1(p') = p$ and $\varphi_1(q') = q$ and $d(p', q') = d(p, q)$. As $\psi(p) = \varphi_2(p')$ and $\psi(q) = \varphi_2(q')$, another application of 2.4(ii) implies that $d(\psi(p), \psi(q)) = d(p', q') = d(p, q)$. In general, (ii) of Definition 2.4 and (iii) of Definition 2.3 imply that arbitrary points $p, q \in T_1$ are such that $[p, q]$ is contained in some union of a finite number of elements $\{\varphi(E_1), \dots, \varphi(E_n) |$ each $E_i \in \mathscr{E}(T)\}$. By Lemma 1.7, we may choose n to be minimal and then order $E_1, \dots, E_n$ such that there is a sequence of points $t_0 = p, t_1, \dots, t_n = q$ in T_1 satisfying $[t_{i-1}, t_i] \subset \varphi_1(E_i)$ for $i = 1, \dots, n$ and $d(p, q) = \sum_{i=1}^n d(t_{i-1}, t_i)$. By the first case, we know that $d(\psi(t_{i-1}), \psi(t_i)) = d(t_{i-1}, t_i)$ for $i = 1, \dots, n$. As ψ is injective, we deduce that $d(p', q') = \sum_{i=1}^n d(t_{i-1}, t_i) = d(p, q)$. As ψ

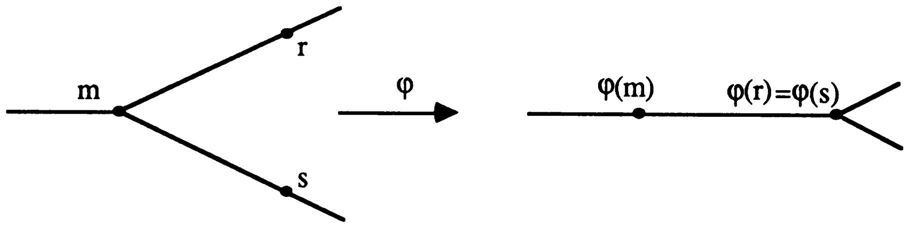

FIGURE 2

is bijective, it follows that ψ is an isometry. Conditions (i), (ii), and (iii) are now easily verified for $\mathbb{C}(\psi)\colon \mathbb{C}(T_1) \to \mathbb{C}(T_2)$. □

2.6 *Remark.* In particular, if $T_1 = T_2$ and ψ is the identity map on T_1, then $\mathbb{C}(T_1) = \mathbb{C}(T_2)$. We can thus paraphrase Lemma 2.5 by saying that an object $\mathbb{C}(T)$ together with a continuous surjective map $\varphi\colon T \to T_1$ determine at most one object $\mathbb{C}(T_1)$ such that $\mathbb{C}(\varphi)\colon \mathbb{C}(T) \to \mathbb{C}(T_1)$ is a morphism.

2.7 **Definition.** Suppose $\mathbb{C}(\varphi)\colon \mathbb{C}(\tau) \to \mathbb{C}(T)$ is a morphism such that there exists an interval $[r, s] \subset \tau$, $r \neq s$, with midpoint m such that for all distinct $p, q \in T$, we have $\varphi(p) = \varphi(q) \Leftrightarrow p, q \in [r, s]$ and $d(p, m) = d(m, q)$. Then $\mathbb{C}(\varphi)\colon \mathbb{C}(\tau) \to \mathbb{C}(T)$ is an (r, m, s)-*elementary morphism* with *midpoint* m (see Figure 2).

2.8 *Remark.* From part (iii) of Definition 2.4, we see that if $\mathbb{C}(\varphi)\colon \mathbb{C}(\tau) \to \mathbb{C}(T)$ is (r, m, s)-elementary, then for all $p, q \in [r, s]$ such that $d(p, m) = d(m, q)$, we have $(p, q) \in D(\tau)$.

2.9 **Lemma.** *Suppose $\mathbb{C}(\tau)$ is an object. Suppose $[r, s] \subset \tau, r \neq s$, is an interval in τ with midpoint m such that*

(i) *no neighborhood of m in $[r, s]$ is contained in an element of $\mathscr{E}(\tau)$, and*

(ii) *for all $p, q \in [r, s]$ such that $d(p, m) = d(m, q)$ we have $(p, q) \in D(\tau)$.*

Then there exists an (r, m, s)-elementary morphism with domain $\mathbb{C}(\tau)$.

Proof. Define an equivalence relation $\sim$ on τ by $x \sim y \Leftrightarrow x = y$ or $x, y \in [r, s]$ and $d(x, m) = d(m, y)$. Let $T = \tau/\sim$ and let $\varphi\colon \tau \to T$ be the natural projection. It is easy to verify directly from Definition 1.4 that T is an $\mathbb{R}$-tree. By (i), φ is distance preserving on each element of $\mathscr{E}(\tau)$. Set $\mathscr{E}(T) = \{\varphi(E) \subset T\colon E \in \mathscr{E}(\tau)\}$. Set

$$D(T) = \{(\varphi(p), \varphi(q)) \in T \times T\colon (p, q) \in D(\tau)\}.$$

By property (iii), we deduce that $D(T)$ is an equivalence relation. Thus $\mathbb{C}(T) = (T, D(T), \mathscr{E}(T))$ is an object. It is now easy to verify conditions (i), (ii), and (iii) of Definition 2.4, whence $\mathbb{C}(\varphi)\colon \mathbb{C}(\tau) \to \mathbb{C}(T)$ is an (r, m, s)-elementary morphism. □

2.10 *Remark.* Let $\mathbb{C}(\varphi)\colon \mathbb{C}(\tau) \to \mathbb{C}(T)$ be an (r, m, s)-elementary morphism. Then $r, m, s \in \tau$ are unique up to permutation of r and s, and $[r, s]$ is a $D(\tau)$-fold of τ. However, not every $D(\tau)$-fold gives rise to an elementary morphism. In fact, if $[r, s]$ is a $D(\tau)$-fold with midpoint m, then there exists an (r, m, s)-elementary morphism if and only if no neighborhood of m in $[r, s]$ is contained in the interior of some element of $\mathscr{E}(\tau)$.

2.11 **Definition.** A map $\varphi\colon \tau \to T$ of metric spaces is *distance decreasing* if for all $p, q \in \tau$, we have $d(p, q) \geq d(\varphi(p), \varphi(q))$.

2.12 **Lemma.** *If* $\mathbb{C}(\varphi)\colon \mathbb{C}(\tau) \to \mathbb{C}(T)$ *is a morphism, then the underlying map* $\varphi\colon \tau \to T$ *is distance decreasing.*

Proof. Let $p, q \in \tau$. By part (iii) of Definition 2.3, $[p, q]$ is covered by a finite number of elements $\{E_i\} \subset \mathscr{E}_1$. From Lemma 1.7, there exist a finite number of closed intervals $\{I_j\}$, such that each I_j is contained in some E_i, $[p, q] = \bigcup\{I_j\}$, and $d(p, q) = \sum \mathrm{length}(I_j)$. As φ is distance preserving on each I_j, another application of Lemma 1.7 implies that $[\varphi(p), \varphi(q)] \subset \varphi[p, q]$ and that $d(\varphi(p), \varphi(q)) \leq d(p, q)$. □

2.13 **Theorem.** *If* $\mathbb{C}(\varphi)\colon \mathbb{C}(\tau) \to \mathbb{C}(T)$ *is a morphism, and the underlying map* $\varphi\colon \tau \to T$ *is not an isometry, then* $\mathbb{C}(\varphi)$ *factors through an elementary morphism.*

Proof. Suppose φ is not an isometry. By Lemma 2.12, there exist $p, q \in \tau$ such that $d(\varphi(p), \varphi(q)) < d(p, q)$. By Lemma 1.7, there exist distinct points $p_1, \ldots, p_n \in \tau$ such that

(i) $p_1 = p$, $p_n = q$,

(ii) for $i = 1, \ldots, n-1$, we have $[p_i, p_{i+1}]$ is contained in some element of $\mathscr{E}(\tau)$,

(iii) for $i = 2, \ldots, n-1$, we have $[p_{i-1}, p_i] \cap [p_i, p_{i+1}] = \{p_i\}$, and

(iv) $[p, q] = \bigcup_{i=1}^{n-1} [p_i, p_{i+1}]$.

As φ is distance preserving on each $[p_i, p_{i+1}]$, we conclude that for some i such that $1 < i < n$, we have $r \in [p_{i-1}, p_i]$ and $s \in [p_i, p_{i+1}]$, such that φ factors through an (r, p_i, s)-elementary morphism. □

2.14 **Definition.** An object $\mathbb{C}(T)$ is *segment closed* if T is segment closed with respect to $D(T)$.

2.15 **Lemma.** *Let* $\mathbb{C}(\varphi)\colon \mathbb{C}(\tau) \to \mathbb{C}(T)$ *be a morphism. If* $\mathbb{C}(\tau)$ *is segment closed, then so is* $\mathbb{C}(T)$.

Proof. Suppose $\mathbb{C}(T)$ is not segment closed. Thus there exist distinct points $p, p', q, q' \in T$ such that

(i) $(p, q) \notin D(T)$,

(ii) $[p', p]$ and $[q', q]$ are both contained in elements of $\mathscr{E}(T)$,

(iii) $d(p', p) = \varepsilon = d(q', q)$ for some $\varepsilon > 0$, and

(iv) there exist distance preserving maps $\iota_1\colon [0, \varepsilon] \to [p', p]$ and $\iota_2\colon [0, \varepsilon] \to [q', q]$ such that $\iota_1(\varepsilon) = p$, $\iota_2(\varepsilon) = q$, and $(\iota_1(t), \iota_2(t)) \in D(T)$ for all t such that $0 \leq t < \varepsilon$. By properties (ii) and (iii) of Definition 2.4, we deduce that $\mathbb{C}(\tau)$ is not segment closed. We have therefore established the contrapositive form of the lemma. □

2.16 **Lemma.** *Direct limits exist in the category of metric spaces and surjective distance decreasing maps.*

Sketch of proof. Suppose $(A, \leq)$ is a directed set and $\{\varphi(\alpha, \beta)\colon T_\alpha \to T_\beta\}_{\alpha \leq \beta \in A}$ is a direct system in this category. Let T' be a set such that $\{\varphi_\alpha\colon T_\alpha \to T'\}$ is the direct limit of this direct system in the category of sets. Verify that the formula

$$d(p, q) = \inf\{d(p_\alpha, q_\alpha) \in \mathbb{R}\colon \varphi_\alpha(p_\alpha) = p,\ \varphi_\alpha(q_\alpha) = q\}$$

defines a pseudometric on T'. Notice that in T', the relation $p \sim q$ iff $d(p, q) = 0$ is an equivalence relation. Let $T = T' \bmod \sim$. Let $\psi: T' \to T$ be the map which collapses each $\sim$-class to a single point. Verify that $\{\psi \circ \varphi_\alpha: T_\alpha \to T\}$ satisfies the universal property for the direct limit in the category of metric spaces and surjective distance decreasing maps.

2.17 **Theorem.** *Let $\mathbb{C}(\tau)$ be a segment closed object. Let $\mathbb{C}$ be the full subcategory generated by the objects with domain $\mathbb{C}(\tau)$. Suppose $(A, \leq)$ is a directed set with smallest element* 1. *Then direct limits directed by the set A exist in the category $\mathbb{C}$, and any such direct limit is segment closed.*

Proof. Suppose $\{\mathbb{C}(\varphi(\alpha, \beta)): \mathbb{C}(T_\alpha) \to \mathbb{C}(T_\beta)\}_{\alpha \leq \beta \in A}$ is a direct system in $\mathbb{C}$. Without loss of generality we assume that $\tau = T_1$, where, by hypothesis, 1 is the smallest element of A. By Lemmas 2.12 and 2.16, there exists a metric space T such that $\{\varphi_\alpha: T_\alpha \to T\}$ is the direct limit of this direct system in the category of metric spaces and distance decreasing maps. In particular, we have $\varphi_1: \tau \to T$ as $\tau = T_1$ according to the notation we have set up. We simplify this notation by setting $\varphi \equiv \varphi_1$, so that $\varphi: \tau \to T$ is the canonical projection of τ into the direct limit. Choose a basepoint $b_1 \in \tau$ and set $b_\alpha = \varphi(1, \alpha)(b_1)$.

Define $\bigwedge: T_\alpha \times T_\alpha \to \mathbb{R}$ by the formula

$$p_\alpha \wedge q_\alpha = \tfrac{1}{2}(d(p_\alpha, b_\alpha) + d(q_\alpha, b_\alpha) - d(p_\alpha, q_\alpha)).$$

By Theorem 1.6, we have for all $p_\alpha, q_\alpha, r_\alpha \in T_\alpha$,

$$p_\alpha \wedge r_\alpha \geq \min(p_\alpha \wedge q_\alpha, q_\alpha \wedge r_\alpha).$$

Define $\bigwedge: T \times T \to \mathbb{R}$ by the formula

$$p \wedge q = \inf\{p_\alpha \wedge q_\alpha: \varphi_\alpha(p_\alpha) = p, \varphi_\alpha(q_\alpha) = q, p_\alpha, q_\alpha \in T_\alpha\}.$$

A standard limit argument reveals that $p \wedge q = \frac{1}{2}(d(p, b) + d(q, b) - d(p, q))$ and for all $p, q, r \in T$, $p \wedge r \geq \min(p \wedge q, q \wedge r)$. Thus T and $\wedge$ satisfy condition (1) of Theorem 1.6. Certainly T is connected. By Theorem 1.8, T is an $\mathbb{R}$-tree. Thus $\mathbb{C}(\varphi): \mathbb{C}(\tau) \to \mathbb{C}(T)$ is a morphism. It is routine to verify that $\mathbb{C}(\varphi): \mathbb{C}(\tau) \to \mathbb{C}(T)$ satisfies the universal property of a direct limit. By Lemma 2.15, $\mathbb{C}(T)$ is segment closed. This concludes the proof. □

3. Quasi-free actions on $\mathbb{R}$-trees

In this section we review some more or less standard terminology for group actions and also prove an important result (Theorem 3.6) about actions on $\mathbb{R}$-trees which are not quasi-free.

3.1 **Definition.** Suppose G acts on the sets τ and T. Then $\varphi: \tau \to T$ is a *G-map* if for all $\alpha \in G$ and $p \in \tau$, we have $\varphi(\alpha p) = \alpha\varphi(p)$.

3.2 **Definition.** Suppose G acts on a set T. Suppose that there exists a normal subgroup N of G such that for all $p \in T$, we have N is the stabilizer of p. Then G acts *quasi-freely* on T with stabilizer N.

3.3 **Definition.** Suppose G acts quasi-freely on $\mathbb{R}$-trees T_1 and T_2 with stabilizers N_1 and N_2 respectively. Suppose $\varphi: T_1 \to T_2$ is a G-map. Then $\varphi: T_1 \to T_2$ is a *free* G-map if $N_1 \subset N_2$.

3.4 **Lemma.** *The following are equivalent.*

(i) *G acts quasi-freely on an $\mathbb{R}$-tree T.*

(ii) *If $\alpha \in G$ fixes an element of T, then $\alpha p = p$ for all $p \in T$.*

(iii) *Let $*: G \to \mathrm{Iso}(T)$ be the map defined by $\alpha_*(p) = \alpha p$, for all $p \in T$ and $\alpha \in G$. Let G_* be the image of $*$ in $\mathrm{Iso}(T)$. Then G_* acts freely on T.*

Proof. Easy. □

3.5 *Remark.* Suppose G acts quasi-freely on T, say with stabilizer N. The kernel of $*: G \to G_*$ is equal to N. Thus G/N acts freely on T.

3.6 **Theorem.** *Suppose G is a countable group which acts on an $\mathbb{R}$-tree T. Suppose $[r, s] \subset T$, $r \neq s$, is an interval of T with midpoint m such that for all $p, q \in [r, s]$ such that $d(p, m) = d(m, q)$, there exists $\alpha \in G$ such that $\alpha p = q$. Then G does not act quasi-freely on T.*

Proof. For each $\alpha \in G$, let $I_\alpha = \{p \in [r, m]: \alpha p \in [m, s] \text{ and } d(p, m) = d(m, \alpha p)\}$. It is easy to see that each I_α is a closed interval contained in $[r, m]$. By hypothesis, we have $\bigcup_\alpha I_\alpha = [r, m]$. Notice that $I_1 = \{m\}$. Let $\mathcal{I} = \{I_\alpha \subset [r, m]: I_\alpha \neq \varnothing,\ \alpha \in G\}$.

Case 1. There exist distinct $\alpha, \beta \in G$ such that I_α and I_β are both in the collection $\mathcal{I}$ and there exists a point $p \in I_\alpha \cap I_\beta$. Then $d(m, \alpha p) = d(m, p) = d(m, \beta p)$, whence $\alpha p = \beta p$. It follows that $\beta^{-1}\alpha p = p$. As $\beta^{-1}\alpha \neq 1$, part (ii) of Lemma 3.4 implies that G does not act quasi-freely on T.

Case 2. The elements of $\mathcal{I}$ are pairwise disjoint. It follows that $\mathcal{I}$ is a countable covering of $[r, m]$ by disjoint closed sets. As $[r, m]$ is a continuum, it follows that $\mathcal{I}$ contains exactly one element (see, for example, Willard [5, exercise 28E.2, p. 209]). But $I_1 \in \mathcal{I}$ and $I_1 = \{m\}$, whence $[r, m] = \{m\}$, contradicting the fact that r and m are distinct. Thus this case is impossible, establishing the theorem. □

4. The Main Theorem

In this section, we prove the Main Theorem. This theorem yields the construction for free actions on $\mathbb{R}$-trees discussed in the introduction.

4.1 **Main Theorem.** *Let G be a countable group which acts on the locally finite simplicial tree τ. Let $\mathcal{E}$ be the set of edges of τ. Let $\approx$ be a segment closed equivalence relation on τ. Suppose* (I) *for all p and q in τ, if $p \approx q$, then $p \approx \alpha q$ for all $\alpha \in G$ and* (II) *for all $E \in \mathcal{E}$, the edge E contains no $\approx$-fold. Then there exist a quasi-free action of G on an $\mathbb{R}$-tree T and a continuous surjective G-map $\varphi: \tau \to T$ satisfying the following property. Suppose G acts quasi-freely on T_1. Suppose $\varphi_1: \tau \to T_1$ is a continuous surjective G-map such that*

(a) *for all $E \in \mathcal{E}$, φ_1 is distance preserving on $\mathcal{E}$,*

(b) *for all $p \in \tau$ and for all $\alpha \in G$, $\varphi_1(\alpha p) = \alpha\varphi_1(p)$, and*

(c) *for all $p, q \in \tau$, if $p \approx q$, then there exists $\alpha \in G$ such that $\alpha\varphi_1(p) = \varphi_1(q)$.*

Then there exists a free G-map $\psi: T \to T_1$ such that $\varphi_1 = \psi \circ \varphi$.

Moreover, T contains no $D(T)$-fold, where $D(T) = \{(\varphi(p), \varphi(q)): p \approx q,\ p, q \in \tau\}$.

4.2 *Remark.* The substance of the theorem is that T contains no $D(T)$-fold. Notice that τ certainly may contain $\approx$-folds centered about a vertex. The basic idea of the proof is to "fold up" the action of G on τ to obtain the action of G on T. See [3] for other characterizations of free actions on $\mathbb{R}$-trees based on Theorem 4.1.

4.3 *Remark.* As G acts quasi-freely on T, we have a free action of G/N on T, where N is the stabilizer of the action of G on T. E. Rips has recently proved that G/N is a free product of free groups and surface groups.

4.4 *Proof of the Main Theorem.* Let $\mathbb{C}(\tau)$ be the object $(\tau, \approx, \mathscr{E})$, as defined in 2.3. Let $\mathbb{C}$ be the full subcategory generated by objects whose domain is $\mathbb{C}(\tau)$. Define a relation $\leq$ on the objects of $\mathbb{C}$ by $\mathbb{C}(T_1) \leq \mathbb{C}(T_2)$ if and only if there exist morphisms $\mathbb{C}(\varphi_1): \mathbb{C}(\tau) \to \mathbb{C}(T_1)$, $\mathbb{C}(\varphi_2): \mathbb{C}(\tau) \to \mathbb{C}(T_2)$, and $\mathbb{C}(\varphi(1, 2)): \mathbb{C}(T_1) \to \mathbb{C}(T_2)$ such that $\varphi_2 = \varphi(1, 2) \circ \varphi_1$. It is easy to check that $\leq$ is a partial order. By Theorem 2.17, all $\leq$-chains are bounded above. Clearly $\mathbb{C}$ is nonempty, as $\mathbb{C}(\tau) \in \mathbb{C}$. Thus Zorn's lemma implies that $\mathbb{C}$ contains a $\leq$-maximal element.

We claim that all $\leq$-maximal elements are equivalent. Suppose $\mathbb{C}(T_1)$ and $\mathbb{C}(T_2)$ are two $\leq$-maximal elements of $\mathbb{C}$. Let $\mathbb{C}(\varphi_1): \mathbb{C}(\tau) \to \mathbb{C}(T_1)$ and $\mathbb{C}(\varphi_2): \mathbb{C}(\tau) \to \mathbb{C}(T_2)$ be morphisms in $\mathbb{C}$. Define $\varphi(1, 2) \subset T_1 \times T_2$ to be the relation

$$\varphi(1, 2) = \{(\varphi_1(p), \varphi_2(p)) \in T_1 \times T_2 : p \in \tau\},$$

and set $\varphi(1, 2)^{-1} = \{(q, p) \in T_2 \times T_1 : (p, q) \in \varphi(1, 2)\}$.

Case 1: $\varphi(1, 2): T_1 \to T_2$ and $\varphi(1, 2)^{-1}: T_2 \to T_1$ are inverse functions. It follows from Lemma 2.5(ii) that $\mathbb{C}(\varphi(1, 2)): \mathbb{C}(T_1) \to \mathbb{C}(T_2)$ and $\mathbb{C}(\varphi(1, 2)^{-1}): \mathbb{C}(T_2) \to \mathbb{C}(T_1)$ are inverse morphisms, establishing the claim.

Case 2: Either $\varphi(1, 2)$ or $\varphi(1, 2)^{-1}$ is not a function. By interchanging T_1 and T_2 if necessary, we may assume that $\varphi(1, 2)$ is not a function. Choose $p, q \in \tau$ such that $\varphi_1(p) = \varphi_1(q)$ and $\varphi_2(p) \neq \varphi_2(q)$. Let $\mathbb{C}(T_3)$ be an object of $\mathbb{C}$ which is $\leq$-maximal among all objects in $\mathbb{C}$ with the following property: there exist morphisms $\mathbb{C}(\varphi_3): \mathbb{C}(\tau) \to \mathbb{C}(T_3)$, $\mathbb{C}(\varphi(3, 1)): \mathbb{C}(T_3) \to \mathbb{C}(T_1)$, and $\mathbb{C}(\varphi(3, 2)): \mathbb{C}(T_3) \to \mathbb{C}(T_2)$ such that $\varphi_1 = \varphi(3, 1) \circ \varphi_3$ and $\varphi_2 = \varphi(3, 2) \circ \varphi_3$. By Theorem 2.13 there exist distinct points $r, m, s \in T_3$ such that $\mathbb{C}(\varphi(3, 1))$ factors through an (r, m, s)-elementary morphism. On the other hand, maximality of T_3 implies that $\varphi(3, 2)$ is distance-preserving on $[r, s]$. By Remark 2.10 and $\leq$-maximality of T_2, we deduce that some neighborhood of $\varphi(3, 2)(m)$ in $\varphi(3, 2)[r, s]$ is contained in an element of $\mathscr{E}(T_2)$. As there exists a morphism $\mathbb{C}(\varphi_2): \mathbb{C}(\tau) \to \mathbb{C}(T_2)$, it follows from parts (ii) and (iii) of Definition 2.4 that some edge of τ contains a $\approx$-fold.

This contradicts hypothesis (II) of the theorem. Thus Case 2 is impossible, establishing the claim that all $\leq$-maximal elements are equivalent.

Thus there exists a morphism $\mathbb{C}(\varphi): \mathbb{C}(\tau) \to \mathbb{C}(T)$ with the property that $\mathbb{C}(\varphi)$ factors through all the morphisms of $\mathbb{C}$. By Remark 2.10 and hypothesis (II) of the Main Theorem, T contains no fold with respect to $D(T)$. Thus $\varphi: \tau \to T$ satisfies the final assertion of the Main Theorem. We now construct an action of G on T and show that $\varphi: \tau \to T$ is a G-map.

For each $\alpha \in G$, hypothesis (I) implies that there exist morphisms $\mathbb{C}(\alpha)$: $\mathbb{C}(\tau) \to \mathbb{C}(\tau)$ and $\mathbb{C}(\alpha^{-1}): \mathbb{C}(\tau) \to \mathbb{C}(\tau)$. By construction, $\mathbb{C}(\varphi)$ factors through the maps $\mathbb{C}(\varphi) \circ \mathbb{C}(\alpha): \mathbb{C}(\tau) \to \mathbb{C}(T)$ and $\mathbb{C}(\varphi) \circ \mathbb{C}(\alpha^{-1}): \mathbb{C}(\tau) \to \mathbb{C}(T)$. By Lemma 2.5, there exists a morphism $\mathbb{C}(\alpha_*): \mathbb{C}(T) \to \mathbb{C}(T)$ such that $\alpha_* \varphi =$

$\varphi\alpha^{-1}$. Similarly there exists a morphism $\mathbb{C}((\alpha^{-1})_*)\colon \mathbb{C}(T) \to \mathbb{C}(T)$ such that $(\alpha^{-1})_*\varphi = \varphi\alpha$. Hence we have the following diagram.

$$\begin{tikzcd}
\tau \arrow[r, "\alpha"] \arrow[d, "\varphi"] & \tau \arrow[r, "\alpha^{-1}"] \arrow[d, "\varphi"] & \tau \arrow[d, "\varphi"] \\
T \arrow[r, "\alpha_*"'] & T \arrow[r, "(\alpha^{-1})_*"'] & T
\end{tikzcd}$$

It follows that we have a commutative diagram

$$\begin{tikzcd}
& \tau \arrow[dl, "\varphi"'] \arrow[dr, "\varphi"] & \\
T \arrow[rr, "(\alpha^{-1})_* \cdot \alpha_*"'] & & T
\end{tikzcd}$$

By Lemma 2.5(i), $(\alpha^{-1})_*\alpha_* = 1_T$, and similarly, $\alpha_*(\alpha^{-1})_* = 1_T$. It follows that $\alpha_*\colon T \to T$ is a bijection. By part (ii) of Lemma 2.5, we deduce that α_* is an isometry.

Define an action of G on T by $\alpha p = \alpha_*(p)$, for all $\alpha \in G$ and $p \in T$. To see that this action is well defined, consider the diagram

$$\begin{tikzcd}
\tau \arrow[r, "\beta"] \arrow[d, "\varphi"] \arrow[rr, bend left, "\alpha\beta"] & \tau \arrow[r, "\alpha"] \arrow[d, "\varphi"] & \tau \arrow[d, "\varphi"] \\
T \arrow[r, "\beta_*"'] \arrow[rr, bend right, "\alpha_* \cdot \beta_*"'] & T \arrow[r, "\alpha_*"'] & T
\end{tikzcd}$$

By 2.5(i) we deduce that $(\alpha\beta)_* = \alpha_* \circ \beta_*$ as desired. To verify that $\varphi\colon \tau \to T$ is a G-map, suppose $\alpha \in G$ and $p \in \tau$. Then $\alpha\varphi(p) = \alpha_*\varphi(p) = \varphi(\alpha p)$ as desired.

We claim that G acts quasi-freely on T. Notice that for each $\alpha \in G$, we have $\alpha_* \in \mathrm{Iso}(T)$. Let $G_* = \{\alpha_* \in \mathrm{Iso}(T)\colon \alpha \in G\}$. We have shown above that $*\colon G \to G_*$ is an epimorphism. By Lemma 3.4 it suffices to prove that G_* acts freely on T. To see this, let $\alpha \in G$. Suppose there exists $p_0 \in T$ such that $\alpha_* p_0 = p_0$. Let $Z = \{z \in \tau\colon \alpha_*\varphi(z) = \varphi(z)\}$. Then Z is nonempty because

$\varphi^{-1}(p_0) \subset Z$ and Z is closed by continuity of φ and α_*. We claim that Z is open. Let $p \in Z$. Recall from the statement of the Main Theorem that τ is assumed to be locally finite. This means that each point of τ lies in at most a finite number of edges of τ. We deduce that there is an open neighborhood $N \subset \tau$ of p in τ such that if $E \subset \tau$ is a component of $N - \{p\}$, then E is contained in some edge of τ. Fix such a neighborhood N and let E be a component of $N - \{p\}$. Notice E and αE are contained in the set of edges $\mathscr{E}$ of τ, whence $\varphi\colon \tau \to T$ is distance preserving on E and αE. Thus, for all $q \in E$, $d(\varphi(p), \varphi(q)) = d(\varphi(\alpha p), \varphi(\alpha q)) = d(\alpha_*\varphi(p), \alpha_*\varphi(q))$. Since $p \in Z$, we have $d(\varphi(p), \varphi(q)) = d(\varphi(p), \alpha_*(\varphi(q)))$. By hypothesis (I) of the Main Theorem, we know that $(\varphi(q), \varphi(\alpha q)) \in D(T)$. By $\leq$-maximality of T, we deduce that $\varphi(q) = \alpha_*\varphi(q)$, whence $q \in Z$. It follows that Z is open. Since τ is connected, $Z = \tau$. Thus α_* is the identity element of T, proving that G_* acts freely on T.

We show that $\varphi\colon \tau \to T$ satisfies properties (a) and (b) of the Main Theorem. Let G_1 act freely on an $\mathbb{R}$-tree T_1. Suppose $\varphi_1\colon \tau \to T_1$ is a continuous surjective G-map satisfying properties (a), (b), and (c) of the Main Theorem. Observe that T_1 may not inherit a structure from φ_1 which makes φ_1 the underlying map of a morphism. The reason for this is that there could exist p and q in τ which are $\approx$-inequivalent but $\varphi_1(p) = \varphi_2(q)$ in violation of part (iii) of Definition 2.4. Nevertheless, we must speak of maps into T_1 which preserve all the available structure. Let $\mathbb{C}(T_2)$ be an object of $\mathbb{C}$. Say a continuous map $\varphi(2,1)\colon T_2 \to T_1$ is *good* if for all $E \in \mathscr{E}_2$, $\varphi(2,1)$ is distance preserving on E. In particular, property (a) of the Main Theorem implies that $\varphi_1\colon \tau \to T_1$ is good. Define a partial order $\leq_1$ on the morphisms of $\mathbb{C}$ with domain $\mathbb{C}(\tau)$ as follows. Let $\mathbb{C}(\varphi_3)\colon \mathbb{C}(\tau) \to \mathbb{C}(T_3)$ and $\mathbb{C}(\varphi_4)\colon \mathbb{C}(\tau) \to \mathbb{C}(T_4)$. Then $\mathbb{C}(\varphi_3) \leq_1 \mathbb{C}(\varphi_4)$ if there exists a commutative diagram of surjective maps

$$\begin{tikzcd}
& T_3 \arrow[dd, "\varphi(3,4)"] \arrow[dr, "\varphi(3,1)"] & \\
\tau \arrow[ur, "\varphi_3"] \arrow[dr, "\varphi_4"'] & & T_1 \\
& T_4 \arrow[ur, "\varphi(4,1)"'] &
\end{tikzcd}$$

such that $\varphi(3,1)$ and $\varphi(4,1)$ are good maps and there is an induced commutative morphism diagram as follows:

$$\begin{tikzcd}
& \mathbb{C}(T_3) \arrow[dd, "\mathbb{C}(\varphi(3,4))"] \\
\mathbb{C}(\tau) \arrow[ur, "\mathbb{C}(\varphi_3)"] \arrow[dr, "\mathbb{C}(\varphi_4)"'] & \\
& \mathbb{C}(T_4)
\end{tikzcd}$$

By Theorem 2.17, we have a $\leq_1$-maximal morphism $\mathbb{C}(\varphi_2)\colon \mathbb{C}(\tau) \to \mathbb{C}(T_2)$, together with a good map $\varphi(2,1)\colon T_2 \to T_1$ and commutative diagrams

$$\begin{tikzcd}
 & & \tau \arrow[dll, "\varphi_1"'] \arrow[d, "\varphi_2"] \arrow[drr, "\varphi"] & & \\
 & & T_2 \arrow[dll, "\varphi(2,1)"] \arrow[drr, "\Phi"'] & & \\
T_1 & & & & T
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C}(\tau) \arrow[d, "\mathbb{C}(\varphi_2)"'] \arrow[dr, "\mathbb{C}(\varphi)"] & \\
\mathbb{C}(T_2) \arrow[r, "\mathbb{C}(\Phi)"] & \mathbb{C}(T)
\end{tikzcd}$$

Notice that $\mathbb{C}(\Phi)\colon \mathbb{C}(T_2) \to \mathbb{C}(T)$ exists by construction of $\mathbb{C}(T)$.

We claim that $\Phi\colon T_2 \to T$ is an isometry. Suppose not. By Theorem 2.13, there exist distinct points $r, m, s \in T_2$ such that Φ factors through an (r, m, s)-elementary morphism. By $\leq_1$-maximality of $\mathbb{C}(\varphi_2)$, we know that $\varphi(2, 1)\colon T_2 \to T_1$ is distance preserving on $[r, s]$. It follows from part (iii) of Definition 2.4 and property (c) of the Main Theorem that for all $p \in \varphi(2, 1)[r, m]$, there exists $\alpha \in G$ such that

(i) $\alpha p \in \varphi(2, 1)[m, s]$, and

(ii) $d(p, m) = d(m, \alpha p)$.

By Theorem 3.6, we deduce that G does not act quasi-freely on T_1, a contradiction. Thus Φ must be an isometry.

Define $\psi\colon T \to T_1$ by $\psi = \Phi(2, 1) \circ \Phi^{-1}$. Notice that

$$\psi \circ \varphi = \Phi(2, 1) \circ \Phi^{-1} \circ \varphi = \Phi(2, 1) \circ \varphi_2 = \varphi_1 .$$

It remains only to show that ψ is a free G-map. Let $\alpha \in G$ and $p \in \tau$. Then

$$\alpha\psi(\varphi(p)) = \alpha\varphi_1(p) = \varphi_1(\alpha p) = \psi\varphi(\alpha p) = \psi\alpha\varphi(p) .$$

As φ is onto, we see that ψ is a G-map. Let N be the stabilizer of the action of G on T. Suppose $\alpha \in N$. Thus for all $p \in \tau$, $\alpha\varphi(p) = \varphi(p)$. Fix some $p \in \tau$. Then $\alpha\varphi_1(p) = \psi\alpha\varphi(p) = \psi\varphi(p) = \varphi_1(p)$. As φ is onto, it follows that $\alpha \in N_1$. Thus $N \subset N_1$, and hence $\psi\colon T \to T_1$ is a free G-map. □

5. The generality of the construction for free actions on ℝ-trees

Every free minimal action of a finitely generated group on an ℝ-tree gives rise to a segment closed equivalence relation on a graph which contains no fold in an edge of the graph. Moreover, passing to the universal cover and applying the Main Theorem yields the original action. In order to express this fact concisely, we introduce some terminology.

5.1 **Definition.** A *metric graph* is a connected simplicial one-complex with a specified metric and the induced metric topology.

Notice that a metric graph is compact if and only if it has a finite number of edges.

5.2 **Definition.** A *universal covering map* of a metric graph Γ is a universal covering map $\pi\colon \tau \to \Gamma$ in the topological sense together with an ℝ-tree structure for τ. The metric on τ must be chosen so that π is a local isometry.

Notice that if τ is the metric universal cover of Γ, then $\pi_1(\Gamma)$ acts on τ by isometries. Moreover, τ is locally finite if Γ is compact.

5.3 **Definition.** Let $D(\Gamma) \subset \Gamma \times \Gamma$ be an equivalence relation on a metric graph Γ. Let $\pi: \tau \to T$ be a universal covering map. The *lift* of $D(\Gamma)$ is the equivalence relation $\approx$ on τ defined by the formula $p \approx q \Leftrightarrow \pi(p) \approx \pi(q)$.

Observe that $\approx$ is segment closed if $D(\Gamma)$ is.

5.4 *Remark.* Let Γ be a compact metric graph. Let $D(\Gamma)$ be a segment closed relation on Γ such that each edge of Γ contains no $D(\Gamma)$-fold. Let $\pi: \tau \to \Gamma$ be a universal covering map. Let G be the fundamental group of Γ. Then the action of G on τ, together with the lift $\approx$ of $D(\Gamma)$, satisfy hypotheses (I) and (II) of the Main Theorem.

By the above remark, we are entitled to make the following definition.

5.5 **Definition.** Let Γ, $D(\Gamma)$, τ, and $\approx$ be as in Remark 5.4. Let $G \times T \to T$ be the quasi-free action of G on an $\mathbb{R}$-tree T which satisfies the conclusions of the main theorem as applied to τ and $\approx$. Let N be the stabilizer of the action of G on T and let $G_* = G/N$. Then the induced action of G_* on T is called the $(\Gamma, D(\Gamma))$*-action.*

5.6 **Definition.** Let $G_1 \times T_1 \to T_1$ and $G_2 \times T_2 \to T_2$ be free actions on $\mathbb{R}$-trees. These two actions are *equivalent* if there exists an isomorphism $\rho: G_1 \to G_2$ and an isometry $\psi: T_1 \to T_2$ such that for all $\alpha \in G_1$ and $p \in T_1$ we have $\psi(\alpha p) = \rho(\alpha)\psi(p)$.

5.7 **Theorem.** *Let G_1 be a finitely generated nontrivial group which acts freely and minimally on an $\mathbb{R}$-tree T_1. Then there exists a compact metric graph Γ with an equivalence relation $D(\Gamma)$ such that*

(i) *$D(\Gamma)$ is segment closed,*

(ii) *no edge of Γ contains a $D(\Gamma)$-fold, and*

(iii) *the action of G_1 on T_1 is equivalent to the $(\Gamma, D(\Gamma))$-action.*

Proof. Let $X \subset G_1$ be a finite set of generators for G_1, such that $1 \notin X$. As G_1 is nontrivial, X is nonempty. For each $x \in X$, let $A(x)$ be the translation axis of x and let $|x|$ be the translation length of x. (See the end of §1 for a discussion of translation axes and translation lengths.) Fix a point $b \in T_1$. For each $x \in X$, define $b(x) \in T_1$ to be the point in $A(x)$ closest to b. Thus $[b, b(x)] \cap A(x) = \{b(x)\}$ and $b = b(x) \Leftrightarrow b \in A(x)$. For each $x \in X$, let M_x be a closed interval isometric to $[0, d(b, b(x))]$. Let $M_x(0) \in M_x$ correspond to 0. Let E_x be isometric to the half open interval $[0, |x|)$. Let $E_x(0) \in E_x$ correspond to 0. Given $t \in [0, d(b, b(x))]$, let $M_x(t)$ be the point of M_x such that $d(M_x(0), M_x(t)) = t$. Similarly, given $t \in [0, |x|)$, let $E_x(t)$ be the point of E_x such that $d(E_x(0), E_x(t)) = t$. Let D_Γ be a certain disjoint union modulo certain point identifications, as detailed in the following formula.

$$D_\Gamma = \bigcup_{x \in X} M_x \cup E_x \bmod \left\{ \begin{array}{ll} M_y(0) = M_z(0) & y, z \in X \\ M_y(d(b, b(y))) = E_y(0) & y \in X \end{array} \right\}.$$

Notice that for all $x \in X$, we have a distance preserving map of M_x into T_1 via $M_x(0) = b$, $M_x(d(b, b(x))) = b(x)$. Similarly, for all $x \in X$, we have a distance preserving map of E_x into T_1 via $E_x(t) = p$ such that $p \in A(x)$, $d(b(x), p) = t$, and $d(p, xb(x)) = |x| - t$.

These embeddings induce a map $\varphi_1: D_\Gamma \to T_1$. Let D^+ be the metric completion of D_Γ. Notice D^+ is obtained from D_Γ by adding a single point

$E_x(|x|)$ to the half open interval $E_x \subset D_\Gamma$, for each $x \in X$. Thus D^+ is a contractible simplicial tree with edges contained in the set $\{M_x : x \in X\} \cup \{E_x \cup \{E_x(|x|)\} : x \in X\}$. Give D^+ the obvious metric structure such that $\varphi_1 : D^+ \to T_1$ is distance preserving on each edge of D^+.

Let $\pi : D^+ \to \Gamma$ be the identification map defined by identifying $E_x(0)$ and $E_x(|x|)$ for each $x \in X$. Thus Γ is a compact one-complex. Give Γ the metric such that π extends to the universal cover $\pi : \tau \to \Gamma$ in the sense of Definition 5.2. Notice $D_\Gamma \subset \tau$. Let G be the fundamental group of Γ acting by covering translations on τ. Notice D_Γ is a fundamental domain for the action of G on τ. Let $\zeta \in D_\Gamma$ be the point $\{M_x(0) : x \in X\}$. Let $z \in \Gamma$ be the point $z = \pi(\zeta)$. Regard z as the basepoint of Γ. For each $x \in X$, let $\pi(x) \in G$ be the homotopy class obtained by travelling upstairs in τ from ζ to $E_x(0)$ to $E_x(|x|)$, projecting this path to Γ, and then travelling backwards along πM_x to z.

Notice G is free on the set of generators $\pi(x) \subset G$. Let $\rho : G \to G_1$ be the epimorphism defined by $\pi(x) \to x$ for all $x \in X$. Extend the map $\varphi_1 : D_\Gamma \to T_1$ to all of τ by $\varphi_1(\alpha p) = \rho(\alpha)\varphi_1(p)$ for all $\alpha \in G$ and $p \in D_\Gamma$. As φ_1 is distance preserving on the edges of D^+, it is distance preserving on every edge of τ.

Certainly φ_1 is continuous at every point of τ which is in the complement of $G(D^+ - D_\Gamma)$ in τ. Let $E_x(|x|) \in D^+ - D_\Gamma$ for some $x \in X$. Note $E_x(|x|) = \pi_x(x)E_x(0)$ by construction. As $\varphi_1(\pi(x)E_x(0)) = x\varphi_1(E_x(0)) = xb(x)$, we have

$$\varphi_1\left(\lim_{t \to |x|} E_x(t)\right) = \varphi_1(E_x(|x|)) = xb(x) = \lim_{t \to |x|} \varphi_1(E_x(t)).$$

Thus φ_1 is continuous at $E_x(|x|)$ for all $x \in X$. We deduce that φ_1 is continuous on all of τ.

Define the equivalence relation $\approx$ on τ by $p \approx q \Leftrightarrow \alpha\varphi_1(p) = \varphi_1(q)$ for some $\alpha \in G_1$. Let $\mathscr{E}$ be the set of edges of τ. We must verify hypotheses (I) and (II) of the Main Theorem for the object $\mathbb{C}(\tau) = (\tau, \approx, \mathscr{E})$.

We claim that $\approx$ is a segment closed relation. Suppose that $\varepsilon > 0$, and $\iota_1 : [0, \varepsilon] \to \tau$ and $\iota_2 : [0, \varepsilon] \to \tau$ are such that $\iota_1(t) \approx \iota_2(t)$ for all $t \in [0, \varepsilon)$. We show that $\iota_1(\varepsilon) \approx \iota_2(\varepsilon)$. We may suppose that $\operatorname{im} \iota_1 \subset E_1$ and $\operatorname{im} \iota_2 \subset E_2$ for some $E_1, E_2 \in \mathscr{E}$. Choose $\alpha \in G$ such that $\rho(\alpha)\varphi_1\iota_1(0) = \varphi_1\iota_2(0)$. Define $\iota_3 : [0, \varepsilon] \to \tau$ by $\iota_3 = \alpha \circ \iota_1$. Thus for all $t \in [0, \varepsilon)$, we have $\iota_3(t) \approx \iota_2(t)$.

Let $r, m, s \in T_1$ be defined by $r = \varphi_1\iota_3(\varepsilon)$, $m = \varphi_1\iota_3(0) = \varphi_1\iota_2(0)$, and $s = \varphi_1\iota_2(\varepsilon)$. Notice that $\varphi_1 \circ \iota_3$ and $\varphi_1 \circ \iota_2$ are both distance preserving maps on $[0, \varepsilon]$. Moreover, for each $t \in [0, \varepsilon)$ there exists $\beta \in G_1$ such that $\beta\varphi_1\iota_3(t) = \varphi_1\iota_2(t)$. As G_1 acts freely on T_1, Theorem 3.6 implies that $\varphi_1\iota_3(t) = \varphi_1\iota_2(t)$ for all $t \in [0, \varepsilon)$. By continuity, $\varphi_1\iota_3(\varepsilon) = \varphi_1\iota_2(\varepsilon)$, whence $\iota_3(\varepsilon) \approx \iota_2(\varepsilon)$. Thus $\approx$ is segment closed.

We claim that no edge $E \in \mathscr{E}$ contains a $\approx$-fold. For suppose E is such an edge. Then there exists distinct points $r, m, s \in \varphi_1 E$ such that $d(r, m) = d(m, s)$ and for each pair $p, q \in [r, s]$ such that $d(p, m) = d(m, q)$ there exists $\beta \in G_1$ such that $\beta p = q$. As G_1 acts freely on T_1, this contradicts Theorem 3.6.

We have shown that $\mathbb{C}(\tau)$ satisfies the hypotheses of the Main Theorem. Let $\varphi : \tau \to T$ be the surjective G-map satisfying the conclusion of the Main Theorem. Define a map $\psi : T \to \operatorname{im} \varphi_1$ according to the formula $\psi(\varphi(p)) = \varphi_1(p)$ for all $p \in \tau$.

We claim that ψ is well defined. To see this, let $\mathbb{C}$ be the full subcategory generated by the morphisms with domain $\mathbb{C}(\tau)$. Let $\mathbb{C}(\varphi_2)\colon \mathbb{C}(\tau) \to \mathbb{C}(T_2)$ be a morphism of $\mathbb{C}$. Say that $\mathbb{C}(\varphi_2)$ is *good* if the map $\psi_2\colon T_2 \to \operatorname{im}\varphi_1$ defined by the formula $\psi_2(\varphi_2(p)) = \varphi_1(p)$ for all $p \in \tau$ is in fact well defined. Notice that $\mathbb{C}(1_\tau)$ is good.

We claim that a direct limit of good morphisms is good. For suppose $\mathbb{C}(\varphi(1, \alpha))\colon \mathbb{C}(\tau) \to \mathbb{C}(T_\alpha)$ is a good morphism. Since $\varphi_1\colon \tau \to \operatorname{im}\varphi_1$ and $\varphi(1, \alpha)\colon \tau \to T_\alpha$ are both distance preserving on the edges of τ, we deduce that $\psi_\alpha\colon T_\alpha \to \operatorname{im}\varphi_1$ must also have this property. The argument of Lemma 2.12 now applies to show that ψ_α is distance decreasing. Now suppose $\{\mathbb{C}(\varphi(\alpha, \beta))\colon \mathbb{C}(T_\alpha) \to \mathbb{C}(T_\beta)\}_{\alpha \le \beta \in A}$ is a direct system of good morphisms. By 2.17, this system has a direct limit in $\mathbb{C}$, say $\mathbb{C}(T_\gamma)$. Suppose that there exist $p, q \in \tau$ such that $\varphi(1, \gamma)(p) = \varphi(1, \gamma)(q)$ in T_γ, but $\varphi_1(p) \neq \varphi_1(q)$. Clearly $d(\varphi_1(p), \varphi_1(q)) > 0$. Thus for some $\alpha \in A$, $d(\varphi(1, \alpha)(p), \varphi(1, \alpha)(q)) < d(\varphi_1(p), \varphi_1(q))$. This contradicts the fact that $\psi_\alpha\colon T_\alpha \to \operatorname{im}\varphi_1$ is distance decreasing.

Given good morphisms $\mathbb{C}(\varphi_2)$ and $\mathbb{C}(\varphi_3)$, say $\mathbb{C}(\varphi_2) \le \mathbb{C}(\varphi_3)$ if there exists a morphism $\mathbb{C}(\varphi_4)$ such that $\mathbb{C}(\varphi_3) = \mathbb{C}(\varphi_4) \circ \mathbb{C}(\varphi_2)$. Thus every chain of good morphisms has an upper bound which is again a good morphism. Therefore there is a $\le$-maximal morphism $\mathbb{C}(\varphi_2)\colon \mathbb{C}(\tau) \to \mathbb{C}(T_2)$.

By the proof of the Main Theorem, every morphism factors through $\mathbb{C}(\varphi)$. Thus there exists a morphism $\mathbb{C}(\Phi)\colon \mathbb{C}(T_2) \to \mathbb{C}(T)$ such that $\mathbb{C}(\varphi) = \mathbb{C}(\Phi) \circ \mathbb{C}(\varphi_2)$. Suppose that $\Phi\colon T_2 \to T$ is not an isometry. By Theorem 2.13, there exist $r, m, s \in T_2$ such that $\mathbb{C}(\Phi)$ factors through an (r, m, s)-elementary morphism. Thus for all $p, q \in [r, s]$ such that $d(p, m) = d(m, q)$, we have $(p, q) \in D(T_2)$. As $\mathbb{C}(\varphi_2)$ is good, the map $\psi_2\colon T_2 \to \operatorname{im}\varphi_1$ defined by $\psi_2(\varphi(p)) = \varphi_1(p)$ for all $p \in \tau$ is well defined. By $\le$-maximality of $\mathbb{C}(\varphi_2)$, we have ψ_2 is distance preserving on $[r, s]$. Applying Theorem 3.6 to $\psi_2[r, s]$, we deduce that G_1 does not act freely on T_1, a contradiction. Thus $\Phi\colon T_2 \to T$ is in fact an isometry. We deduce that $\psi = \psi_2 \circ \Phi^{-1}$, whence $\psi\colon T \to \operatorname{im}\varphi_1$ is well defined.

Clearly ψ is surjective. We claim that ψ is injective. By construction ψ is distance preserving on the elements of $\mathscr{E}(T)$. Moreover, given $p, q \in T$, if $\psi(p) = \psi(q)$, we deduce that $(p, q) \in D(T)$. Thus, by the proof of Theorem 2.13, if ψ is not injective, then ψ factors through the underlying map of an elementary morphism. This contradicts the fact that T contains no $D(T)$-fold. Thus ψ is bijective. As φ_1 is distance preserving on the elements of $\mathscr{E}$, it is easy to verify that ψ is an isometry. We conclude that $\operatorname{im}\varphi_1$ is an $\mathbb{R}$-tree. By definition, $G(\operatorname{im}\varphi_1) = \operatorname{im}\varphi_1$, so clearly $G_1(\operatorname{im}\varphi_1) = \operatorname{im}\varphi_1$ as $\rho\colon G \to G_1$ is surjective. As G_1 acts minimally on T_1, we deduce that $T_1 = \operatorname{im}\varphi_1$. Thus $\varphi_1\colon \tau \to T_1$ is a surjective G-map satisfying conditions (a), (b), and (c) of the Main Theorem. Evidently $\psi\colon T \to T_1$ is the free G-map produced by the Main Theorem. As ψ is an isometry, the stabilizer N of the action of G on T equals the stabilizer of the action of G on T_1. Thus $G_* = G/N$ is isomorphic to G_1. We conclude that the action of G_* on T is equivalent to the action of G_1 on T_1.

Recall the covering map $\pi\colon \tau \to \Gamma$. Define a relation of $D(\Gamma)$ by

$$(\pi(p), \pi(q)) \in D(\Gamma) \Leftrightarrow p \approx q$$

for all $p, q \in \tau$. It is clear from the definition of $\approx$ that $D(\Gamma)$ is an equivalence relation. Moreover, it is clear that $\approx$ is the lift of $D(\Gamma)$ and $D(\Gamma)$ satisfies properties (i) and (ii) of the theorem. Thus the action of G_1 on T_1 is equivalent to the $(\Gamma, D(\Gamma))$-action, i.e., the action of G_* on T. This concludes the proof. $\square$

References

1. Roger Alperin and Hyman Bass, *Length functions of group actions on* Λ*-trees*, Ann. of Math. Studies, no. 111, Princeton Univ. Press, Princeton, N. J., 1987.
2. J. Morgan and P. Shalen, *Valuations, trees, and degenerations of hyperbolic structures*. I, Ann. of Math. (2) **120** (1984), 401–476.
3. F. Rimlinger, *Two-complexes with similar foliations*, submitted to Proc. Amer. Math. Soc. Preprint available from author.
4. H. Short, ed., *Notes on word hyperbolic groups*, Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, and A. Verjovsky, eds.), World Scientific, Singapore, 1991, pp. 3–63.
5. S. Willard, *General topology*, Addison-Wesley, 1970.

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