THE GAUSS MAP FOR KÄHLERIAN SUBMANIFOLDS OF $\mathbb{R}^n$

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Abstract. We introduce a Gauss map for Kähler submanifolds of Euclidean space and study its geometry in relation to that of the given immersion. In particular we generalize a number of results of the classical theory of minimal surfaces in Euclidean space.

1. Introduction

Let $M$ be a Kähler manifold of (complex) dimension $s$, $f: M \rightarrow \mathbb{R}^n$ an immersion into the $n$-dimensional Euclidean space and indicate with $G_s(\mathbb{C}^n)$ the Grassmann manifold of complex $s$-planes in $\mathbb{C}^n$. We define the complex Gauss map

$$\gamma_f^C: M \rightarrow G_s(\mathbb{C}^n)$$

by assigning to each point $p \in M$ the complex $s$-space $df_p(T_pM^{(0,1)})$ where, as usual, $T_pM^{(0,1)}$ denotes the subspace of $(0, 1)$ vectors of the complexified tangent space of $M$ at $p$, and $df_p$ is linearly extended over $\mathbb{C}$.

The relevance of $\gamma_f^C$ in the study of the geometry of the submanifold $M$ relies on its manifest relation with the Kähler structure of $M$ itself and in what follows we analyze some of the aspects of the problem. Towards this aim the tensors defined below play a relevant role.

Let $f: M \rightarrow N$, $N$ a Riemann manifold, be a smooth map and interpret $df$ as a section of the bundle $TM^* \otimes f^{-1}TN$. Indicating with $\nabla$ the natural induced connection, let $\nabla df$ be the generalized second fundamental tensor of the map. Considering the complexified cotangent bundle of $M$, with the usual procedure, $\nabla df$ can be split into different components according to their types. We indicate with $\nabla df^{(p,q)}$, the $(p, q)$ component ($p+q=2$, $0 \leq p$, $q \leq 2$) and call the map $f$, $(p, q)$-geodesic if and only if $\nabla df^{(p,q)} \equiv 0$ on $M$.

The notion of $(1, 1)$-geodesic maps has been recently introduced in the literature under various names (pluriharmonic maps, circular maps [R, U, D-G, D-T, D-R]) and carefully studied, in case $N$ is a complex manifold, as a bridge condition between harmonicity, characterized by the equation $\text{tr}\ \nabla df = 0$, the trace being taken with respect to the metric on $M$, and holomorphicity. Indeed for $f$ an isometry, indicating with $J_M$ and $J_N$ the almost complex structures...
of $M$ and $N$ respectively, holomorphicity of $f$ is expressed by the system

$$\nabla df(X, Y) + \nabla df(JM X, JM Y) = 0,$$

$$\nabla df(X, Y) + JN \nabla df(X, JM Y) = 0$$

for each pair $X, Y$ of vector fields on $M$ [D-T]. Obviously the first equation is nothing but $\nabla df^{(1,1)} = 0$, that is, the definition of $(1, 1)$-geodesic map.

Clearly any $(1, 1)$-geodesic map is harmonic and, somehow surprisingly, the converse is also true under some circumstances. For instance, Dajczer and Rodriguez, [D-R] proved that

(1.1) for an isometric immersion $f: M \rightarrow \mathbb{R}^n$, $(1, 1)$-geodesic is equivalent to minimality of $f$.

(For another result in this direction see §3.)

On the other hand, we know the existence of minimal surfaces in $\mathbb{R}^{2m}$ which are not holomorphic curves with respect to any complex structure in $\mathbb{R}^{2m}$. Indeed the case where $M$ is a Riemann surface reveals itself to be special as the following further results of [D-T] show. Let $C^Q(C)$ be a complex space form of constant holomorphic sectional curvature $c$, $f: M \rightarrow C^Q(C)$ an isometric immersion $\dim_C M = s$ then:

(1.2) for $c < 0$, $s > 1$, $f$ is minimal if and only if $f$ is $\pm$ holomorphic.

(1.3) for $c > 0$, $s > 1$, $f$ is $(1, 1)$-geodesic if and only if $f$ is $\pm$ holomorphic.

where here and in the sequel with $+$ and $-$ holomorphic we respectively mean holomorphic and antiholomorphic.

With the above definition of complex Gauss map we prove

**Theorem 1.** Let $f: M \rightarrow \mathbb{R}^n$ be an immersion and $\gamma_f^C$ its complex Gauss map. If $f$ is $(1, 1)$-geodesic then $\gamma_f^C$ is $-$ holomorphic.

**Remarks.** 1. In the theorem we do not assume $f$ to be an isometry and in general harmonicity of $f$ does not imply $(1, 1)$-geodesic, for instance, let $f: \mathbb{C}^2 \rightarrow \mathbb{R}^2$ be defined by $f: (x, y, u, v) \rightarrow ((x^2 - y^2)uv, x, y, u, v)$.

2. For $s = 1$ clearly $f$ is $(1, 1)$-geodesic if and only if $f$ is harmonic and in this case the above result has been proven in [J-R].

Let $H_s(C^n)$ be the space of (complex) $s$-dimensional isotropic subspaces of $C^n$ or equivalently the space of $F$-structures of $\mathbb{R}^n$ of (real) rank $2s$. Having made the trivial observation that if $f$ is conformal then $\gamma_f^C$ factors through $H_s(C^n) \subset G_s(C^n)$ as a consequence of (1.1) and the fact that $H_s(C^n)$ is a Kähler holomorphic submanifold of $G_s(C^n)$ we have

**Corollary 2.** Let $f: M \rightarrow \mathbb{R}^n$ be an isometric immersion. Then $\gamma_f^C: M \rightarrow H_s(C^n)$ is $-$ holomorphic if and only if $f$ is minimal.

**Remark.** In case $s = 1$, $H_s(C^n)$ is the complex quadric $Q_{n-2}$ and Corollary 2 extends a result of Chern [C].

Somewhat dual to Theorem 1 is the following:

**Theorem 3.** Let $f: M \rightarrow \mathbb{R}^n$ be an immersion and $\gamma_f^C$ its complex Gauss map. If $f$ is $(2, 0)$-geodesic then $\gamma_f^C$ is holomorphic. In case $f$ is an isometry the two properties are in fact equivalent.
We observe that in case \( f \) is an isometry, \((2, 0)\)-geodesic has been analyzed by Ferus [F1], who has described \( f \) as a symmetric immersion. It is well known that in this case for \( s = 1 \), \( f: M \to \mathbb{R}^n \) is a totally umbilical surface.

The next result characterizes holomorphicity of \( f \) via \( \gamma_f^C \) and complements (1.2), and (1.3) in case \( c = 0 \) and \( s \geq 1 \).

**Theorem 4.** Let \( f: M \to \mathbb{R}^{2m} \) be a minimal isometric immersion and \( \gamma_f^C: M \to H_s(\mathbb{C}^{2m}) \) be its complex Gauss map. Then \( f \) is holomorphic with respect to some complex structure \( J \) on \( \mathbb{R}^{2m} \) if and only if \( \gamma_f^C(M) \) is contained in some complex Grassmannian of \( s \)-planes inside \( H_s(\mathbb{C}^{2m}) \).

**Remarks.**
1. For \( s = 1 \), Theorem 4 recovers the Calabi-Lawson result for minimal surfaces in \( \mathbb{R}^{2m} \) reported in [L].
2. From Theorem 1.1 of [D-R], a sufficient condition to guarantee that \( \gamma_f^C(M) \) is contained in some Grassmannian in \( H_s(\mathbb{C}^{2m}) \) is that the type number \( t(p) \) of \( f \) at \( p \) satisfies \( t(p) \geq 3 \) for all \( p \in M \).

The use of \( \gamma_f^C \) in the study of the geometry of \( f: M \to \mathbb{R}^n \) has also suggested the following Bernstein’s type result. Consider \( \gamma_f^C \) as a map into \( G_s(\mathbb{C}^n) \) and let \( A \) be a fixed \( s \)-plane in \( \mathbb{C}^n \). Let \( \langle \cdot , \cdot \rangle \) denote the \( \mathbb{C} \)-linear symmetric bilinear form from \( \mathbb{R}^n \).

**Theorem 5.** Let \( f: M \to \mathbb{R}^n \) be a minimal isometric immersion of a parabolic manifold such that its complex Gauss map \( \gamma_f^C \) satisfies \( |\langle \gamma_f^C, A \rangle|^2 \geq \varepsilon \) for some \( \varepsilon > 0 \). Then \( f(M) \) is contained in a \( 2s \)-plane of \( \mathbb{R}^n \).

Having analyzed the behaviour of \( \gamma_f^C \) with respect to holomorphicity it is natural to investigate the weaker property of harmonicity. In this case the guideline result is the Ruh-Vilms theorem, [R-V], asserting that for an isometric immersion \( f \) into \( \mathbb{R}^n \) the usual Gauss map \( \gamma_f: M \to G_{2s}(\mathbb{R}^n) \), (the real Grassmannian of \( 2s \)-planes in \( \mathbb{R}^n \)), is harmonic if and only if \( f \) has parallel mean curvature vector \( H \).

Assume that \( f: M \to \mathbb{R}^n \) is an isometric immersion so that \( \nabla df \) coincides with \( II \), the usual second fundamental tensor, and let \( II_H \) denote the inner product of \( II \) with \( H \).

**Theorem 6.** Let \( f: M \to \mathbb{R}^n \) be an isometric immersion and \( \gamma_f^C: M \to H_s(\mathbb{C}^n) \) its complex Gauss map. Then

(i) \( \gamma_f^C \) is harmonic as a map taking values in \( G_s(\mathbb{C}^n) \) if and only if \( H \) is parallel and \( II_H^{(0,2)} = 0 \).

(ii) \( \gamma_f^C \) is harmonic as a map taking values in \( H_s(\mathbb{C}^n) \) if and only if \( H \) is parallel and \( II_H^{(0,2)}(X, Y) = 0 \) for all pairs \( X, Y \) of vectors orthogonal with respect to the hermitian product in \( M \).

**Remarks.**
1. Observe the two different conclusions according to considering \( \gamma_f^C \) respectively as a map into \( G_s(\mathbb{C}^n) \) and into \( H_s(\mathbb{C}^n) \).
2. For \( s = 1 \), that is when \( M \) is a surface, the second condition in (ii) is vacuous. This agrees with the Ruh-Vilms theorem and the fact that \( H_1(\mathbb{C}^n) = G_2(\mathbb{R}^n) = Q_{n-2} \). If \( \gamma_f^C \) harmonic, from the work of Yau [Y] we have that either \( f: M \to \mathbb{R}^n \) is a minimal surface or it is a constant mean curvature surface in \( \mathbb{R}^3 \) or \( S^3 \) or a minimal surface in some sphere in \( \mathbb{R}^n \).
Analogously to Remark 2 above we have

**Corollary 7.** Let \( f: M \to \mathbb{R}^n \) be an isometric immersion of a Riemann surface and consider \( \gamma_f^C \) as a map into \( G_1(C^n) = \mathbb{C}P^{n-1} \). Then \( \gamma_f^C \) is harmonic if and only if either \( f \) is minimal or \( f \) is minimal in some sphere of \( \mathbb{R}^n \). In particular for \( n = 3 \) if \( f \) is not minimal, then \( f(M) \) is a piece of the standard 2-sphere in \( \mathbb{R}^3 \).

Some other strong consequences related to Theorem 7 are given in §3 in case \( M \) is a hypersurface.

2. Preliminaries and first properties of \( \gamma_f^C \)

To describe the geometry of \( \mathbb{R}^n \) we consider the transitive action of the group of rigid motions \( E(n) = SO(n) \times \mathbb{R}^n \) on it and describe the Euclidean space as the homogeneous manifold \( E(n)/SO(n) \), where the isotropy subgroup is computed at the origin 0. From now on we fix the index conventions \( 1 \leq i, j, \cdots \leq s, 1 \leq u, v, \cdots \leq 2s, 2s+1 \leq \alpha, \beta, \cdots \leq n, 1 \leq A, B, \cdots \leq n \).

Indicating with \( (\varphi, \theta) \) the Maurer-Cartan form of \( E(n) \), its components \( \varphi_A^A, \theta_A^A \) satisfy

\[
(2.1) \quad \varphi_A^A + \varphi_B^B = 0
\]

and the structure equations

\[
(2.2) \quad d\theta^A = -\varphi_B^A \wedge \theta^B, \quad d\varphi_B^A = -\varphi_C^A \wedge \varphi_B^C.
\]

Thus given any local section \( \sigma \) of the bundle

\[
(2.3) \quad \pi: E(n) \to \mathbb{R}^n,
\]

the metric \( ds_{\mathbb{R}^n}^2 \) on \( \mathbb{R}^n \) can be written as

\[
(2.4) \quad ds_{\mathbb{R}^n}^2 = \sum_A \sigma^*(\theta^A)^2
\]

where from now on we will systematically drop the pull-back notation it being clear from the context where forms have to be considered. Thus from (2.1) and (2.2) we deduce that the \( \varphi_B^A \)'s are the Levi-Civita connection forms corresponding to the orthonormal coframe \( \{\theta^A\} \).

Let \( M \) be a Kahler manifold of (complex) dimension \( s \). Then the Kahler structure of \( M \) is naturally described by a unitary coframe \( \{\varphi^i\} \) of \((1, 0)\)-type 1-forms giving the metric

\[
(2.5) \quad ds_M^2 = \sum_j \varphi^j \overline{\varphi}^j
\]

with \( - \) denoting complex conjugation, and the corresponding Kahler connection forms \( \omega^j_i \) characterized by the property

\[
(2.6) \quad \omega^j_i + \overline{\omega}^j_i = 0
\]

and by the structure equations

\[
(2.7) \quad d\varphi^j = -\omega^j_k \wedge \varphi^k.
\]
The Kähler curvature forms $\Omega^j_k$ are determined by the second structure equations
\begin{equation}
(2.8) \quad d\omega^j_k = -\omega^j_k \wedge \omega^j_k + \Omega^j_k
\end{equation}
and satisfy the symmetry relations
\begin{equation}
(2.9) \quad \Omega^j_k + \Omega^k_j = 0.
\end{equation}
In what follows we will be interested in the Riemannian structure determined by the metric (2.5), underlying the Kähler one. Thus if we set
\begin{align}
(2.10) \quad \rho^j &= \mu^j + i\mu^{s+j} \\
(2.11) \quad \omega^j_k &= \mu^j_k + i\mu_k^{s+j}, \\
(2.12) \quad \mu^j_k &= \mu_k^{s+j}, \quad \mu_k^{s+j} = -\mu_k^{s+j},
\end{align}
the $\mu^j$, $\mu^{s+j}$'s give an orthonormal coframe for (2.5) whose corresponding Levi-Civita connection forms are determined by (2.11), (2.12) and by (2.6)-(2.8). Analogously setting
\begin{align}
(2.13) \quad \Omega^k_j &= M^k_j + iM^{s+k}_j, \\
(2.14) \quad M^k_j &= M^{s+k}_j, \quad M^{s+k}_j = -M^k_j.
\end{align}
The $M^u$'s defined in (2.13), (2.14) together with skew symmetry, coincide with the corresponding curvature forms. Thus letting $R^u_{vwz}$ be the coefficients of the Riemann curvature tensor determined by
\begin{equation}
(2.15) \quad M^u = \frac{1}{2} R^u_{vwz} \mu^w \wedge \mu^z
\end{equation}
we have that, in addition to the usual symmetry relations, they have to obey those derived from (2.14). Observe that from (2.8) the complex structure $J_M$ on $M$ is determined by the requirements
\begin{align}
(2.16) \quad J_M \mu^k &= -\mu^{s+k}, \quad J_M^2 = -\text{id}
\end{align}
and (2.12) are equivalent to the parallelism of $J_M$ with respect to the Levi-Civita connection.

Let $f: M^{2s} \rightarrow \mathbb{R}^n$ be an immersion and let $(e, f)$ be a Darboux frame along $f$, that is, $(e, f)$ is a smooth function $(e, f): U \subset M \rightarrow E(n), U$ open, with the property
\begin{equation}
(2.17) \quad (e, f)^* \theta^a \equiv 0.
\end{equation}
We set
\begin{equation}
(2.18) \quad (e, f)^* \theta^A = B_u^A \mu^u
\end{equation}
for some smooth, locally defined, functions $B_u^A$ so that from (2.17) we have
\begin{equation}
(2.19) \quad B_u^a \equiv 0.
\end{equation}
Observe that since $f$ is an immersion the matrix $(B_u^a)$ is nonsingular.

With respect to the considered Darboux frame the coefficients, $B_{uv}^A$, of the generalized second fundamental tensor $\nabla d\bar{f}$ are defined by
\begin{equation}
(2.20) \quad dB_u^A - B_v^A \mu_u^v + B_u^B \phi_A^B = B_{uv}^A \mu^v, \quad B_{uv}^A = B_{vu}^A,
\end{equation}
and remark that using (2.19) in (2.20) we obtain

\[(2.21) \quad B_{uv}^A \mu^v = B_u^v \phi_u^A.\]

The coefficients $B_{uvw}^A$ of the covariant derivative of $\nabla df$ are given by the formula

\[(2.22) \quad dB_{uv}^A - B_{wuv}^A \mu_w^v - B_{uvw}^A \mu_u^v + B_{uv}^B \phi_B^A = B_{uvw}^A \mu^w.\]

Formula (2.20), its exterior derivative, and use of the structure equations give

\[(2.23) \quad B_{uvw}^A = B_{vuw}^A, \quad B_{uvw}^A = B_{uvw}^A + B_z^A R_z^w,\]

which can be considered as generalized Codazzi equations.

Using the definitions given in §1 we have that $f$ is $(1, 1)$-geodesic or $(2, 0)$-geodesic respectively when

\[(2.24) \quad B_{ij} + B_{s+i+s+j}^A = 0, \quad B_{is+j}^A - B_{s+ij}^A = 0,\]

or

\[(2.25) \quad B_{ij} - B_{s+i+s+j}^A = 0, \quad B_{is+j}^A + B_{s+ij}^A = 0.\]

By (2.22) and (2.24), if $f$ is $(1, 1)$-geodesic then,

\[(2.26) \quad B_{ijw}^A + B_{s+i+s+jw}^A = 0, \quad B_{is+jw}^A - B_{s+ijw}^A = 0;\]

and analogously for $f$ $(2, 0)$-geodesic from (2.22) and (2.25).

Let $G_s(C^n)$ be the complex Grassmannian of $s$-planes in $C^n$. Then the complex Gauss map $\gamma_f^C : M \to G_s(C^n)$ can be defined by

\[(2.27) \quad \gamma_f^C : p \to [(B_{1}^u + iB_{u+1}^u)e_u, \ldots, (B_{s}^u + iB_{2s}^u)e_u] \text{ at } p.\]

Let $\tilde{\gamma}_f^C$ indicate the homogeneous representation of $\gamma_f^C$ given by

\[\tilde{\gamma}_f^C = (B_{1}^u + iB_{u+1}^u)e_u \wedge \cdots \wedge (B_{s}^u + iB_{2s}^u)e_u\]

and set

\[
\tilde{\gamma}_f^C(k) = (B_{1}^u + iB_{u+1}^u)e_u \wedge \cdots \wedge (B_{u-1}^u + iB_{s+k-1}^u)e_u \wedge (B_{u-k+1}^u + iB_{s+k+1}^u)e_u \wedge \cdots \wedge (B_{s}^u + iB_{2s}^u)e_u.
\]

Then using (2.21), (2.20), (2.19) and (2.12) we compute

\[(2.28) \quad d\tilde{\gamma}_f^C = i\mu_{s+k}^A \tilde{\gamma}_f^C + (-1)^{k+1} (B_{k}^u + iB_{s+k}^u) \mu^v e_A \wedge \tilde{\gamma}_f^C(k)\]

and therefore from (2.10)

\[(2.29) \quad d\tilde{\gamma}_f^C = i\mu_{s+k}^A \tilde{\gamma}_f^C + \frac{1}{2} (-1)^{k+1} \]

\[
\cdot \{[B_{kj}^A + B_{s+k}^A + i(B_{s+k}^A - B_{s+k}^A)] \phi^j + [B_{kj}^A - B_{s+k}^A + i(B_{s+k}^A + B_{s+k}^A)] \overline{\phi^j}\} e_A \wedge \tilde{\gamma}_f^C(k).
\]

Since $f$ is an immersion, (2.24), (2.25) and (2.29) prove Theorems 1 and 3.
Lemma 2.1. Let $f: M \to \mathbb{R}^n$ be a $(1, 1)$-geodesic immersion of the Kähler manifold $M$ into $\mathbb{R}^n$ and let $\gamma_f^C: M \to G_2(\mathbb{C}^n)$ be its complex Gauss map. Fix an $s$-plane $A$ in $\mathbb{C}^n$ and consider the smooth function $|\langle \gamma_f^C, A \rangle|^2$. Then the following formula holds on the open set where $\langle \gamma_f^C, A \rangle \neq 0$.

\begin{equation}
(2.30) \quad \Delta \log |\langle \gamma_f^C, A \rangle|^2 = 2R_{ks+kjs+j}.
\end{equation}

Proof. First of all recall that given a real function $a$ on $M$, the unitary coframe of (2.5) and the corresponding Kähler connection forms, the Laplace-Beltrami operator on $a$, $\Delta a$, is computed as follows. Set

$$
da = a_j \varphi^j + a^{-j} \bar{\varphi}^j \quad (a_j = \bar{a}_j)$$

and define $a_{j\bar{k}}$ via the formula

$$
da - a_k \omega^k_j = a_{j\bar{k}} \varphi^k + a_{\bar{k}j} \bar{\varphi}^k$$

then

$$
\Delta a = 4a_{j\bar{k}}.
$$

For $a > 0$, to compute $\Delta \log a$ we make use of the formula

\begin{equation}
(2.31) \quad \Delta \log a = \frac{1}{a} \Delta a - \frac{1}{a^2} |\nabla a|^2.
\end{equation}

Observe that the function $|\langle \gamma_f^C, A \rangle|^2$ is defined independently of the homogeneous representatives $\gamma_f^C$ and $\bar{A}$ respectively of $\gamma_f^C$ and $A$. From (2.29), and since $f$ is $(1, 1)$-geodesic, (that is, (2.24) holds), we have

$$
d\langle \gamma_f^C, \bar{A} \rangle = (-1)^{k+1} \langle e_A \wedge \gamma_f^C(k), \bar{A} \rangle (B_{kj}^A + iB_{s+kj}^A) \bar{\varphi}^j + i\langle \gamma_f^C, \bar{A} \rangle \mu_s^{k}$$

from which we immediately deduce

\begin{equation}
(2.32) \quad d\langle \gamma_f^C, A \rangle^2 \equiv (-1)^{k+1} \langle e_A \wedge \gamma_f^C(k), \bar{A} \rangle \langle \gamma_f^C, \bar{A} \rangle (B_{kj}^A - iB_{s+kj}^A) \varphi^j \quad \text{mod} (\bar{\varphi}^l).
\end{equation}

According to our procedure we have to compute the $(0, 1)$ part of

$$
\Lambda = d\{(-1)^{k+1} \langle e_A \wedge \gamma_f^C(k), \bar{A} \rangle \langle \gamma_f^C, \bar{A} \rangle (B_{kj}^A - iB_{s+kj}^A)\}
\quad - (-1)^{k+1} \langle e_A \wedge \gamma_f^C(k), \bar{A} \rangle \langle \gamma_f^C, \bar{A} \rangle (B_{k\bar{l}}^A - iB_{s+k\bar{l}}^A) \omega^l.
$$

Using (2.11), (2.12), (2.22), (2.21), (2.20), (2.24), (2.23), (2.26) and the extra symmetries of the Riemann tensor of $M$ due to (2.14), after a long computation, we obtain

$$
\Lambda \equiv \{\frac{1}{2}|\langle \gamma_f^C, A \rangle|^2 (R_{ks+kjs+t} - iR_{ks+kjt})
\quad + (-1)^{k+1}(-1)^{l+1} \langle e_A \wedge \gamma_f^C(k), \bar{A} \rangle \langle e_B \wedge \gamma_f^C(l), \bar{A} \rangle
\quad \cdot (B_{kj}^A - iB_{s+kj}^A)(B_{l\bar{t}}^B + iB_{s+l\bar{t}}^B) \}\bar{\varphi}^l \quad \text{mod} (\varphi^l).
$$

Therefore

$$
\Delta|\langle \gamma_f^C, A \rangle|^2 = 2|\langle \gamma_f^C, A \rangle|^2 R_{ks+kjs+j} + 4 \left| \sum_{A,k,j} \langle e_A \wedge \gamma_f^C(k), \bar{A} \rangle (B_{kj}^A - iB_{s+kj}^A) \right|^2
$$

and (2.30) follows from (2.32) and (2.31).
Remark. The function \( R_{k+j+k} \) is an average of the holomorphic bisectional curvatures.

Observe that in case \( f \) is an isometry we can choose the coefficients \( B^3 \) in (2.18) to be \( \delta^3 \) so that from (2.21), the \( B^3 \) are precisely the coefficients of the second fundamental tensor \( \Pi \). In this case the Riemann curvature of \( M \) is related to \( \Pi \) by the Ricci equations, that is,

\[
R_{uvwx} = B^3_{uv} B^3_{wz} - B^3_{uw} B^3_{vz} \tag{2.33}
\]

From (2.33), in case \( f \) is \((1, 1)\)-geodesic, it follows immediately that

\[
2R_{k+j+k+j} = -|\Pi|^2. \tag{2.34}
\]

Hence under the assumptions of Theorem 5 from (2.30) we have that \( \log |\gamma^\mathcal{C}, \mathcal{A})|^2 \) is a superharmonic function bounded below and therefore constant. From (2.30) and again (2.34) we conclude that \( \Pi \equiv 0 \), that is, \( f \) is totally geodesic completing the proof of Theorem 5.

3. Isometric immersions

Let \( H_s(C^n) \) be the space of \( s \) (complex) dimensional isotropic planes of \( C^n \).

We now briefly describe its geometry. First of all observe that given any point \( q \in H_s(C^n) \), that is, given any \( s \) dimensional isotropic subspace of \( C^n \) we can find a basis for it of vectors of the form \( a_k + i a_{s+k} \) with the \( a_u \)'s orthonormal vectors of \( R^n \). Then \( SO(n) \) transitively acts on \( H_s(C^n) \) in an obvious way. Fix as an origin in \( H_s(C^n) \) the point

\[
0 = [\epsilon_1 + i \epsilon_{s+1}, \ldots, \epsilon_s + i \epsilon_{2s}]
\]

where \( \{\epsilon_u\} \) is the canonical basis of \( R^n \). Then \( H_s(C^n) \) is realized as the homogeneous space \( SO(n)/U(s) \times SO(n-2s) \) where the isotropy subgroup is computed at 0. Let \( \varphi \) be the Maurer-Cartan form of \( SO(n) \) (consistent with the notation for the Maurer-Cartan form of \( E(n) \) in §2). Then the quadratic form

\[
Q = \sum_{\alpha, \beta} (\varphi^\alpha_{u})^2 + \frac{1}{4} \sum_{j<k} (\varphi^j_{k} - \varphi^{s+j}_{s+k})^2 + (\varphi^j_{s+k} + \varphi^{s+j}_{k})^2
\]

descend to a Riemannian metric \( ds_H^2 \) on \( H_s(C^n) \) via local sections of the bundle

\[
\tilde{\pi}: SO(n) \to H_s(C^n). \tag{3.2}
\]

In particular a (local) orthonormal coframe on \( H_s(C^n) \) is given by the forms

\[
\omega^\alpha_{u} = \varphi^\alpha_{u}, \quad \omega^{jk-} = \frac{1}{2} (\varphi^j_{k} - \varphi^{s+j}_{s+k}), \quad \omega^{jk+} = \frac{1}{2} (\varphi^j_{s+k} + \varphi^{s+j}_{k}), \quad j < k.
\]

With the aid of (3.3) we introduce an almost complex structure \( J_H \) by defining as a local basis for the \((1, 0)\) forms

\[
\rho^{ja} = \omega^{ja} + i \omega^{*+ja}, \quad \rho^{jk} = \omega^{jk-} + i \omega^{jk+}, \quad k < k. \tag{3.4}
\]

Proposition 3.1. The almost complex structure \( J_H \) is symplectic and integrable so that \( H_s(C^n) \) is a Kähler manifold.

Proof. This amounts to showing that the Kähler form corresponding to the unitary coframe (3.4) is closed and that the ideal (3.4) generates is a differential ideal. This is immediately verified with the use of the structure equations (2.3).
It is not hard to see that with this complex structure the inclusion
\[ i: H_S(C^n) \rightarrow G_S(C^n) \]
is a holomorphic isometric immersion. Corollary 2 follows from this and the assumption that \( f \) is an isometry.

Let us now consider the isometric immersion \( f: M \rightarrow \mathbb{R}^n \) so that
\[ B_{u}^{A} = \delta_{u}^{A}, \quad B_{uv}^{u} = 0 \]
and in standard notation
\[ B_{uv}^{a} = h_{uv}^{a} \]
are the coefficients of the second fundamental tensor \( \Pi \).

We now characterize, via \( \gamma_{f}^{C} \), those isometric immersions \( f: M \rightarrow \mathbb{R}^n \), with \( n = 2m \), which are holomorphic with respect to some complex structure \( J \) on \( \mathbb{R}^{2m} \), that is, such that
\[ J \circ df = df \circ J_{M} \]
Fix holomorphic (local) coordinates \( z_{j} = x_{j} + iy_{j} \) on \( M \) such that \( J_{M} \) is the canonical complex structure associated to the complex manifold \( M \) (Newlander and Nirenberg [NN]), that is,
\[ J_{M} \begin{pmatrix} \frac{\partial}{\partial x_{j}} \\ \frac{\partial}{\partial y_{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad J_{M} \begin{pmatrix} \frac{\partial}{\partial y_{j}} \\ \frac{\partial}{\partial x_{j}} \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

Considering the canonical coordinates on \( \mathbb{R}^{2m} \), (3.7) is then equivalent to
\[ J \frac{\partial f}{\partial x_{j}} = \frac{\partial f}{\partial y_{j}}. \]

As a consequence \( f \) is holomorphic with respect to \( J \) on \( \mathbb{R}^{2m} \) if and only if \( \gamma_{f}^{C}: M \rightarrow H_S(C^n) \) can be written in the form
\[ \gamma_{f}^{C}: p \rightarrow \begin{pmatrix} \frac{\partial f}{\partial x_{1}} + iJ \frac{\partial f}{\partial x_{2}}, \ldots, \frac{\partial f}{\partial x_{s}} + iJ \frac{\partial f}{\partial x_{s}} \end{pmatrix} \text{ at } p. \]

Let \( H_{J} \) be the subset of \( H_S(C^{2m}) \) of all the elements of the form \( [v_{1} + iJv_{1}, \ldots, v_{s} + iJv_{s}] \) with \( v_{k} \perp v_{j}, Jv_{j}, k \neq j, v_{k} \neq 0 \), for each \( k \).

Clearly \( f \) is holomorphic with respect to \( J \) if and only if
\[ \gamma_{f}^{C}(M) \subseteq H_{J}. \]

To give \( H_{J} \) a differentiable structure, we fix the almost complex structure \( J_{0} \) on \( \mathbb{R}^{2m} \) whose matrix representation in the canonical basis \( \{e_{A}\} \) of \( \mathbb{R}^{2m} \) is given by
\[
\begin{pmatrix}
0 & -I_{s} & 0 & 0 \\
I_{s} & 0 & 0 & 0 \\
0 & 0 & 0 & -I_{m-s} \\
0 & 0 & I_{m-s} & 0
\end{pmatrix}
\]
with \( I_{r} \) the \( r \) by \( r \) identity matrix. Observe that if we let \( O(2m) \) act on \( H_{J} \) in the obvious way, then there exists \( A \in O(2m) \) such that \( AH_{J} = H_{J_{0}} \). Indeed let \( A \) be an element of \( O(2m) \) such that
\[ A^{-1}J_{0}A = J \]
whose existence is guaranteed by the fact that the almost complex structures on \( \mathbb{R}^{2m} \) are parametrized by the homogeneous space \( O(2m)/U(m) \), then
\[
A[v_1 + iJv_1, \ldots, v_s + iJv_s] = [Av_1 + iAJ(A^{-1}A)v_1, \ldots, Av_s + iAJ(A^{-1}A)v_s] = [Av_1 + iJ_0 Av_1, \ldots, Av_s + iJ_0 Av_s].
\]

It is therefore enough to give a differentiable structure to \( H_{J_0} \). Towards this aim observe that if \( A \in O(2m) \) then \( AJ_0 = J_0 A \) if and only if \( A \in U(m) \) so that \( AH_{J_0} = H_{J_0} \) if and only if \( A \in U(m) \). One verifies that the action of \( U(m) \) on \( H_{J_0} \) is transitive. Fix as an origin in \( H_{J_0} \) the point \( O' \) given by the isotropic \( s \)-plane
\[
[e_1 + iJ_0 e_1, \ldots, e_s + iJ_0 e_s] = [et + i_{s+1}, \ldots, es + i_{2s}]
\]
then the isotropy subgroup of \( O' \) is given by \( U(s) \times U(m-s) \) and \( H_{J_0} = U(m)/U(s) \times U(m-s) \) is the Grassmannian of complex \( s \)-planes in \( \mathbb{C}^m \) providing a proof of Theorem 4.

Given the isometric immersion \( f: M \to \mathbb{R}^n \) and the Darboux frame \((e, f)\) along \( f \) observe that we have the commutative diagram

\[
\begin{array}{ccc}
SO(n) & \xrightarrow{\pi} & \mathbb{E}(n) \\
\downarrow{\gamma_f} \ & \ & \downarrow{\pi} \\
H_s(\mathbb{C}^n) & \xrightarrow{(e, f)} & \mathbb{R}^n \\
\end{array}
\]

where the maps \( \pi \) and \( \hat{\pi} \) have been defined above and \( F \) means forget the \( \mathbb{R}^n \) bit, that is \( F: (e, v) \to e \). It therefore follows from (3.3), (2.12), (3.6), (2.21), (2.33) that
\[
(3.11) \quad \gamma_f^*(ds^2_H) = -\text{Ric}(M) + 2s\Pi_H
\]
where \( \text{Ric}(M) \) is the symmetric Ricci 2-form of \( M \) and \( \Pi_H = \langle \Pi, H \rangle \) for \( H = \frac{1}{2s} \text{tr} \Pi \), the mean curvature vector of the isometric immersion. From (3.11) we therefore obtain the following:

**Proposition 3.2.** Let \( f: M \to \mathbb{R}^n \) be an isometric immersion and \( \gamma_f^c: M \to H_s(\mathbb{C}^n) \) is complex Gauss map. Then any two of the following properties imply the third

(i) \( M \) is Einstein,
(ii) \( f \) is pseudo-umbilical, that is, \( \Pi_H \) is a multiple of \( ds^2_M \).
(iii) \( \gamma_f^c \) is weakly conformal.

**Remarks.** 1. Proposition 3.2 is the analogue of Theorem 1 in Obata [O] relative to the usual Gauss map \( \gamma_f: M \to G_{2s}(\mathbb{R}^n) \).

2. From (3.11) by the definition of the third fundamental form \( \Pi_M \) of \( f \), we have \( \Pi_M \equiv \gamma_f^c*(ds^2_H) \). Define the volume of \( \gamma_f^c \) at \( p \in M \) to be
\[
\sigma(p, f) = \frac{2}{c_{2s}} \left\{ \det(h_{uv}^a h_{uv}^a) \right\}^{1/2}
\]
and let $\tau(p,f)$ be the Chern-Lashof, [C-L], total curvature at $p$. Then, using the work of Ferus [F2], $\tau(p,f) \leq \sigma(p,f)$ equality holding if and only if at least one of the following conditions is satisfied:

1. the first normal space of $f$ at $p$ is of (real) dimension $\leq 1$,
2. $s = 1$, and $H(p) = 0$,
3. $H(p)$ is singular at $p$,
4. $\gamma^C_f$ is not regular at $p$.

We recall that realizing the real Grassmannian $G_{2s}(\mathbb{R}^n)$ as

$$SO(n)/S(0(2s) \times O(n - 2s)),$$

where the isotropy subgroup is computed at the origin $O = [e_1, \ldots, e_{2s}]$ of $G_{2s}(\mathbb{R}^n)$, then for the usual Gauss map $\gamma_f: M \to G_{2s}(\mathbb{R}^n)$, with respect to a Darboux frame $(e, f)$ along $f$, we have

$$\nabla d\gamma_f = h_{uvw}^\alpha \theta^v \theta^w \otimes E_{u\alpha}$$

where $\{E_{u\alpha}\}$ is dual to the coframe $\{\varphi^u\}$ realizing the Riemannian structure of $G_{2s}(\mathbb{R}^n)$ and $h_{uvw}^\alpha$ are the coefficients of the covariant derivative of $H$. In the isometric case, the $h_{uvw}^\alpha$ coincide with the $B_{uvw}^\alpha$ of (2.24). As a consequence $\gamma_f$ is $(1, 1)$-geodesic if and only if

$$h_{uij}^\alpha + h_{us+is+j}^\alpha = 0 = h_{us+j}^\alpha - h_{us+ij}^\alpha$$

and this is immediately verified to be equivalent to

$$\nabla \perp \Pi^{(1,1)} = 0.$$

We have therefore proved

**Proposition 3.3.** Let $f: M \to \mathbb{R}^n$ be an isometric immersion of a Kähler manifold into $\mathbb{R}^n$ and let $\gamma_f: M \to G_{2s}(\mathbb{R}^n)$ be its usual Gauss map. Then $\Pi^{(1,1)}$ is parallel in the normal bundle if and only if $\gamma_f$ is $(1, 1)$-geodesic.

**Corollary 3.4.** Let $f: M \to \mathbb{R}^n$ be an isometric immersion of a Kähler manifold into $\mathbb{R}^n$. If $\gamma_f$ is $(1, 1)$-geodesic and the mean curvature vector $H$ of $f$ is zero at one point, then $f$ is minimal and $\gamma_f^C$ is holomorphic.

Let $f: M \to \mathbb{R}^n$ be an isometric immersion and $(e, f)$ a Darboux frame along $f$. To simplify notation we set

$$E_k = e_k + ie_{s+k}, \quad E_{-k} = e_k - ie_{s+k}$$

so that the homogeneous representation of $\gamma_f^C$ given in §2 becomes

$$\gamma^C_f = E_1 \wedge \cdots \wedge E_s$$

and (2.28) can be rewritten as

$$d\gamma^C_f(X) = \sum_k E_1 \wedge \cdots \wedge E_{k-1} \wedge \Pi(X, E_k) \wedge E_{k+1} \wedge \cdots \wedge E_s.$$

This can be immediately checked observing that $(dE_k, E_{-j}) = 0$ for each $k, j$, where $(\ , \ )$ is the Hermitian inner product; that is, the derivatives of $(0, 1)$-vectors are of the same type. To compute the tension field of $\gamma_f^C$ considered
as a map into $G_s(C^n)$ we introduce the following notation. For $v \in C^n$ let $v^k$ denote
\[ v^k = E_1 \wedge \cdots \wedge E_{k-1} \wedge v \wedge E_{k+1} \wedge \cdots \wedge E_s. \]

Then the covariant derivative $\nabla d\gamma^C_f$ is given by
\[
(\nabla X d\gamma^C_f)(Y) = \nabla X(d\gamma^C_f(Y)) - d\gamma^C_f(\nabla_X Y)
\]
\[
= \sum_{k=1}^{s} \nabla X(II(E_k, Y))^k - d\gamma^C_f(\nabla_X Y)
\]
\[
= \sum_{k=1}^{s} \{ \nabla_X^\perp II(E_k, Y) + (\nabla X II(E_k, Y), E_{-i})E_{-i} \}^k
\]
\[
+ \sum_{j,k=1}^{s} \{ (\nabla X E_j, E_k)II(E_k, Y) \}^j - d\gamma^C_f(\nabla_X Y)
\]

where with $\nabla^\perp$ we have indicated the connection in the normal bundle of the isometric immersion $f$. Choose now the Darboux frame $(e, f)$ and the vector field $Y$ on $M$ such that at the point $p \in M$, $\nabla e_k = 0$ and $\nabla Y = 0$. Then at $p$ we have
\[
(3.12) \quad (\nabla X d\gamma^C_f)(Y) = \sum_{k=1}^{s} \{ \nabla_X^\perp II(E_k, Y) + (\nabla X II(E_k, Y), E_{-i})E_{-i} \}^k
\]
so that,
\[
(3.13) \quad \tau(\gamma^C_f(p)) = \sum_{u=1}^{2s} (\nabla_{e_u} d\gamma^C_f)e_u = 0
\]
if and only if the following two conditions are satisfied
\[
\nabla_{e_u}^\perp II(E_k, e_u) = 0 \quad \text{for each } k,
\]
\[
\sum_{t=1}^{s} \langle II(E_k, E_t), II(E_{-i}, E_j) \rangle = 0 \quad \text{for each } k, j.
\]

Using Codazzi equations the first is easily seen to be equivalent to
\[
(3.14) \quad \nabla^\perp H = 0
\]
while the second, using Gauss equations, is equivalent to
\[
(3.15) \quad \sum_{t=1}^{s} \langle R(E_k, E_t)E_{-i}, E_j \rangle = II_H(E_k, E_j).
\]

Observing that for a Kähler manifold $R(E_k, E_t) \equiv 0$ we have achieved the proof of part (i) of Theorem 6. To show (ii) observe that since $H_s(C^n)$ is isometrically immersed into $G_s(C^n)$ the projection of the tension field (3.13) in the tangent space of $H_s(C^n)$ will give the tension field of $\gamma^C_f$ considered as a map into $H_s(C^n)$. On the other hand the tangent space of $H_s(C^n)$ at some point $p$ is generated by all vectors of the form $v^k$ where either $v = e_a$ or
v = E_{-i}, i \neq k. \text{ We therefore conclude that } γ^C_j \text{ is harmonic in } H_\delta(C^n) \text{ if and only if } \nabla^{\perp} H = 0 \text{ and }

\sum_{i=1}^{s}\langle R(E_i, E_i)E_{-i}, E_i \rangle = \Pi_H(E_k, E_i), \quad k \neq i,

\text{from which we easily deduce the validity of (ii) completing the proof of Theorem 6.}

To prove Corollary 7 from Theorem 6 we have that γ^C_j: M \to CP^n \text{ is harmonic if and only if } \nabla^{\perp} H = 0 \text{ and }

\langle \Pi(E_1, E_1), \Pi(E_1, E_{-1}) \rangle = 0.

If H \neq 0, since \Pi(E_1, E_{-1}) \text{ is a nonzero real multiple of } H, \text{ we have } \Pi(E_1, E_1) \perp H. \text{ Therefore } \Pi_H \text{ is a multiple of the metric of the surface and thus, from [Y] or [R-T], } f \text{ is minimal in some sphere of } R^n. \text{ In particular for } n = 3 \text{ and } H \neq 0, \text{ since } \Pi_H \text{ is a multiple of the metric, } f(M) \text{ has to be a piece of the standard 2-sphere.}

**Theorem 3.6.** Let \( f: M \to R^n \) be an isometric immersion of a Kähler manifold and \( γ^C_j: M \to G_\delta(C^n) \) be its complex Gauss map. Then \( γ^C_j \) is \((1,1)\)-geodesic if and only if the following two conditions are satisfied.

(i) \( \nabla^{\perp} \Pi^{(1,1)} = 0 \),

(ii) \( \langle \Pi^{(0,2)}, \Pi^{(1,1)} \rangle = 0 \).

**Proof.** By definition \( γ^C_j \) is \((1,1)\)-geodesic if and only if

\((\nabla_X dγ^C_j)Y + (\nabla_JX dγ^C_j)JM Y = 0\)

for each pair of vector fields \( X \) and \( Y \) on \( M \). From (3.12) this is equivalent to

\begin{align*}
(3.16) & \quad \nabla_X^{\perp} \Pi(E_k, Y) + \nabla_JX^{\perp} \Pi(E_k, JM Y) = 0, \\
(3.17) & \quad \langle \Pi(E_k, X), \Pi(Y, E_i) \rangle + \langle \Pi(E_k, JM X), \Pi(JM Y, E_i) \rangle = 0.
\end{align*}

Using Codazzi equations and Gauss equations similarly to Theorem 6 it is easy to see that (3.16) and (3.17) are respectively equivalent to (i) and (ii) of the theorem.

**Corollary 3.6.** Let \( f: M \to R^n \) be a Kähler isometrically immersed hypersurface and assume that \( γ^C_j: M \to G_\delta(C^n) \) is \((1,1)\)-geodesic. Then either \( f \) is \((1,1)\)-geodesic or \((0,2)\)-geodesic.

**Proof.** Observe that from Theorem 3.5 (ii)

\( \langle \Pi(E_k, E_i), \Pi(E_{-j}, E_r) \rangle = 0 \) for each \( k, i, j, r \).

Therefore if \( f \) is not \((1,1)\)-geodesic for some \( j, r \) the real vector \( \Pi(E_{-j}, E_r) + \Pi(E_r, E_{-j}) \) is nonzero at each point \( p \in M \) (since \( \nabla^{\perp} \Pi^{(1,1)} = 0 \)) and as a consequence \( \Pi(E_k, E_i) \equiv 0 \).

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