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THE GAUSS MAP FOR KÄHLERIAN SUBMANIFOLDS OF $\mathbf{R}^n$

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ABSTRACT. We introduce a Gauss map for Kähler submanifolds of Euclidean space and study its geometry in relation to that of the given immersion. In particular we generalize a number of results of the classical theory of minimal surfaces in Euclidean space.

1. INTRODUCTION

Let M be a Kähler manifold of (complex) dimension s, $f: M \to \mathbf{R}^n$ an immersion into the n-dimensional Euclidean space and indicate with $G_s(\mathbf{C}^n)$ the Grassmann manifold of complex s-planes in $\mathbf{C}^n$. We define the complex Gauss map

$$\gamma_f^{\mathbf{C}}: M \to G_s(\mathbf{C}^n)$$

by assigning to each point $p \in M$ the complex s-space $df_p(T_pM^{(0,1)})$ where, as usual, $T_pM^{(0,1)}$ denotes the subspace of $(0, 1)$ vectors of the complexified tangent space of M at p, and df_p is linearly extended over $\mathbf{C}$.

The relevance of $\gamma_f^{\mathbf{C}}$ in the study of the geometry of the submanifold M relies on its manifest relation with the Kähler structure of M itself and in what follows we analyze some of the aspects of the problem. Towards this aim the tensors defined below play a relevant role.

Let $f: M \to N$, N a Riemann manifold, be a smooth map and interpret df as a section of the bundle $TM^* \otimes f^{-1}TN$. Indicating with ∇ the natural induced connection, let ∇df be the generalized second fundamental tensor of the map. Considering the complexified cotangent bundle of M, with the usual procedure, ∇df can be split into different components according to their types. We indicate with $\nabla df^{(p,q)}$, the (p, q) component $(p+q = 2,\ 0 \le p,\ q \le 2)$ and call the map f, (p, q)-geodesic if and only if $\nabla df^{(p,q)} \equiv 0$ on M.

The notion of $(1, 1)$-geodesic maps has been recently introduced in the literature under various names (pluriharmonic maps, circular maps [R, U, D-G, D-T, D-R]) and carefully studied, in case N is a complex manifold, as a bridge condition between harmonicity, characterized by the equation $\operatorname{tr}\nabla df = 0$, the trace being taken with respect to the metric on M, and holomorphicity. Indeed for f an isometry, indicating with J_M and J_N the almost complex structures

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of M and N respectively, holomorphicity of f is expressed by the system

$$\nabla df(X, Y) + \nabla df(J_M X, J_M Y) = 0,$$
$$\nabla df(X, Y) + J_N \nabla df(X, J_M Y) = 0$$

for each pair X, Y of vector fields on M [D-T]. Obviously the first equation is nothing but $\nabla df^{(1,1)} = 0$, that is, the definition of $(1, 1)$-geodesic map.

Clearly any $(1, 1)$-geodesic map is harmonic and, somehow surprisingly, the converse is also true under some circumstances. For instance, Dajczer and Rodriguez, [D-R] proved that

(1.1) *for an isometric immersion* $f\colon M \to \mathbf{R}^n$, $(1, 1)$*-geodesic is equivalent to minimality of* f.

(For another result in this direction see §3.)

On the other hand, we know the existence of minimal surfaces in $\mathbf{R}^{2m}$ which are not holomorphic curves with respect to any complex structure in $\mathbf{R}^{2m}$. Indeed the case where M is a Riemann surface reveals itself to be special as the following further results of [D-T] show. Let $\mathbf{C}Q(C)$ be a complex space form of constant holomorphic sectional curvature c, $f\colon M \to \mathbf{C}Q(C)$ an isometric immersion $\dim_{\mathbf{C}} M = s$ then:

(1.2) *for* $c < 0$, $s > 1$, f *is minimal if and only if* f *is* $\pm$ *holomorphic.*

(1.3) *for* $c > 0$, $s > 1$, f *is* $(1, 1)$*-geodesic if and only if* f *is* $\pm$ *holomorphic.*
where here and in the sequel with $+$ and $-$ holomorphic we respectively mean holomorphic and antiholomorphic.

With the above definition of complex Gauss map we prove

Theorem 1. *Let* $f\colon M \to \mathbf{R}^n$ *be an immersion and* $\gamma_f^{\mathbf{C}}$ *its complex Gauss map. If* f *is* $(1, 1)$*-geodesic then* $\gamma_f^{\mathbf{C}}$ *is* $-$ *holomorphic.*

Remarks. 1. In the theorem we do not assume f to be an isometry and in general harmonicity of f does not imply $(1, 1)$-geodesic, for instance, let $f\colon \mathbf{C}^2 \to \mathbf{R}^5$ be defined by $f\colon (x, y, u, v) \to ((x^2 - y^2)uv, x, y, u, v)$.

2. For $s = 1$ clearly f is $(1, 1)$-geodesic if and only if f is harmonic and in this case the above result has been proven in [J-R].

Let $H_s(\mathbf{C}^n)$ be the space of (complex) s-dimensional isotropic subspaces of $\mathbf{C}^n$ or equivalently the space of F-structures of $\mathbf{R}^n$ of (real) rank 2s. Having made the trivial observation that if f is conformal then $\gamma_f^{\mathbf{C}}$ factors through $H_s(\mathbf{C}^n) \subset G_s(\mathbf{C}^n)$ as a consequence of (1.1) and the fact that $H_s(\mathbf{C}^n)$ is a Kähler holomorphic submanifold of $G_s(\mathbf{C}^n)$ we have

Corollary 2. *Let* $f\colon M \to \mathbf{R}^n$ *be an isometric immersion. Then* $\gamma_f^{\mathbf{C}}\colon M \to H_s(\mathbf{C}^n)$ *is* $-$ *holomorphic if and only if* f *is minimal.*

Remark. In case $s = 1$, $H_s(\mathbf{C}^n)$ is the complex quadric Q_{n-2} and Corollary 2 extends a result of Chern [C].

Somewhat dual to Theorem 1 is the following:

Theorem 3. *Let* $f\colon M \to \mathbf{R}^n$ *be an immersion and* $\gamma_f^{\mathbf{C}}$ *its complex Gauss map. If* f *is* $(2, 0)$*-geodesic then* $\gamma_f^{\mathbf{C}}$ *is holomorphic. In case* f *is an isometry the two properties are in fact equivalent.*

We observe that in case f is an isometry, $(2, 0)$-geodesic has been analyzed by Ferus [F1], who has described f as a symmetric immersion. It is well known that in this case for $s = 1$, $f: M \to \mathbf{R}^n$ is a totally umbilical surface.

The next result characterizes holomorphicity of f via $\gamma_f^{\mathbf{C}}$ and complements (1.2), and (1.3) in case $c = 0$ and $s \geq 1$.

Theorem 4. *Let $f: M \to \mathbf{R}^{2m}$ be a minimal isometric immersion and $\gamma_f^{\mathbf{C}}: M \to H_s(\mathbf{C}^{2m})$ be its complex Gauss map. Then f is holomorphic with respect to some complex structure J on $\mathbf{R}^{2m}$ if and only if $\gamma_f^{\mathbf{C}}(M)$ is contained in some complex Grassmannian of s-planes inside $H_s(\mathbf{C}^{2m})$.*

Remarks. 1. For $s = 1$, Theorem 4 recovers the Calabi-Lawson result for minimal surfaces in $\mathbf{R}^{2m}$ reported in [L].

2. From Theorem 1.1 of [D-R], a sufficient condition to guarantee that $\gamma_f^{\mathbf{C}}(M)$ is contained in some Grassmannian in $H_s(\mathbf{C}^{2m})$ is that the type number $t(p)$ of f at p satisfies $t(p) \geq 3$ for all $p \in M$.

The use of $\gamma_f^{\mathbf{C}}$ in the study of the geometry of $f: M \to \mathbf{R}^n$ has also suggested the following Bernstein's type result. Consider $\gamma_f^{\mathbf{C}}$ as a map into $G_s(\mathbf{C}^n)$ and let A be a fixed s-plane in $\mathbf{C}^n$. Let $\langle\ ,\ \rangle$ denote the $\mathbf{C}$-linear symmetric bilinear form from $\mathbf{R}^n$.

Theorem 5. *Let $f: M \to \mathbf{R}^n$ be a minimal isometric immersion of a parabolic manifold such that its complex Gauss map $\gamma_f^{\mathbf{C}}$ satisfies $|\langle \gamma_f^{\mathbf{C}}, A\rangle|^2 \geq \varepsilon$ for some $\varepsilon > 0$. Then $f(M)$ is contained in a $2s$-plane of $\mathbf{R}^n$.*

Having analyzed the behaviour of $\gamma_f^{\mathbf{C}}$ with respect to holomorphicity it is natural to investigate the weaker property of harmonicity. In this case the guideline result is the Ruh-Vilms theorem, [R-V], asserting that for an isometric immersion f into $\mathbf{R}^n$ the usual Gauss map $\gamma_f: M \to G_{2s}(\mathbf{R}^n)$, (the real Grassmannian of $2s$-planes in $\mathbf{R}^n$), is harmonic if and only if f has parallel mean curvature vector H.

Assume that $f: M \to \mathbf{R}^n$ is an isometric immersion so that ∇df coincides with II, the usual second fundamental tensor, and let II_H denote the inner product of II with H.

Theorem 6. *Let $f: M \to \mathbf{R}^n$ be an isometric immersion and $\gamma_f^{\mathbf{C}}: M \to H_s(\mathbf{C}^n) \subset G_s(\mathbf{C}^n)$ its complex Gauss map. Then*

(i) *$\gamma_f^{\mathbf{C}}$ is harmonic as a map taking values in $G_s(\mathbf{C}^n)$ if and only if H is parallel and $\mathrm{II}_H^{(0,2)} = 0$.*

(ii) *$\gamma_f^{\mathbf{C}}$ is harmonic as a map taking values in $H_s(\mathbf{C}^n)$ if and only if H is parallel and $\mathrm{II}_H^{(0,2)}(X, Y) = 0$ for all pairs X, Y of vectors orthogonal with respect to the hermitian product in M.*

Remarks. 1. Observe the two different conclusions according to considering $\gamma_f^{\mathbf{C}}$ respectively as a map into $G_s(\mathbf{C}^n)$ and into $H_s(\mathbf{C}^n)$.

2. For $s = 1$, that is when M is a surface, the second condition in (ii) is vacuous. This agrees with the Ruh-Vilms theorem and the fact that $H_1(\mathbf{C}^n) = G_2(\mathbf{R}^n) = Q_{n-2}$. If $\gamma_f^{\mathbf{C}}$ harmonic, from the work of Yau [Y] we have that either $f: M \to \mathbf{R}^n$ is a minimal surface or it is a constant mean curvature surface in $\mathbf{R}^3$ or S^3 or a minimal surface in some sphere in $\mathbf{R}^n$.

Analogously to Remark 2 above we have

Corollary 7. *Let $f: M \to \mathbf{R}^n$ be an isometric immersion of a Riemann surface and consider $\gamma_f^{\mathbf{C}}$ as a map into $G_1(\mathbf{C}^n) = \mathbf{C}P^{n-1}$. Then $\gamma_f^{\mathbf{C}}$ is harmonic if and only if either f is minimal or f is minimal in some sphere of $\mathbf{R}^n$. In particular for $n = 3$ if f is not minimal, then $f(M)$ is a piece of the standard 2-sphere in $\mathbf{R}^3$.*

Some other strong consequences related to Theorem 7 are given in §3 in case M is a hypersurface.

2. Preliminaries and first properties of $\gamma_f^{\mathbf{C}}$

To describe the geometry of $\mathbf{R}^n$ we consider the transitive action of the group of rigid motions $\mathbf{E}(n) = SO(n) \times \mathbf{R}^n$ on it and describe the Euclidean space as the homogeneous manifold $\mathbf{E}(n)/SO(n)$, where the isotropy subgroup is computed at the origin 0. From now on we fix the index conventions $1 \leq i, j, \dots \leq s$, $1 \leq u, v, \dots \leq 2s$, $2s+1 \leq \alpha, \beta, \dots \leq n$, $1 \leq A, B, \dots \leq n$.

Indicating with (φ, θ) the Maurer-Cartan form of $\mathbf{E}(n)$, its components φ_B^A, θ^A satisfy

$$\varphi_B^A + \varphi_A^B = 0 \tag{2.1}$$

and the structure equations

$$d\theta^A = -\varphi_B^A \wedge \theta^B, \qquad d\varphi_B^A = -\varphi_C^A \wedge \varphi_B^C. \tag{2.2}$$

Thus given any local section σ of the bundle

$$\pi: \mathbf{E}(n) \to \mathbf{R}^n, \tag{2.3}$$

the metric $ds^2_{\mathbf{R}^n}$ on $\mathbf{R}^n$ can be written as

$$ds^2_{\mathbf{R}^n} = \sum_A \sigma^*(\theta^A)^2 \tag{2.4}$$

where from now on we will systematically drop the pull-back notation it being clear from the context where forms have to be considered. Thus from (2.1) and (2.2) we deduce that the φ_B^A's are the Levi-Civita connection forms corresponding to the orthonormal coframe $\{\theta^A\}$.

Let M be a Kähler manifold of (complex) dimension s. Then the Kähler structure of M is naturally described by a unitary coframe $\{\varphi^i\}$ of $(1, 0)$-type 1-forms giving the metric

$$ds^2_M = \sum_j \varphi^j \overline{\varphi}^j \tag{2.5}$$

with $^-$ denoting complex conjugation, and the corresponding Kähler connection forms ω_j^i characterized by the property

$$\omega_j^i + \overline{\omega}_i^j = 0 \tag{2.6}$$

and by the structure equations

$$d\varphi^j = -\omega_k^j \wedge \varphi^k. \tag{2.7}$$

The Kähler curvature forms Ω^j_k are determined by the second structure equations

$$d\omega^j_k = -\omega^j_i \wedge \omega^i_k + \Omega^j_k \tag{2.8}$$

and satisfy the symmetry relations

$$\Omega^j_k + \overline{\Omega}^k_j = 0 . \tag{2.9}$$

In what follows we will be interested in the Riemannian structure determined by the metric (2.5), underlying the Kähler one. Thus if we set

$$\varphi^j = \mu^j + i\mu^{s+j} \tag{2.10}$$

$$\omega^j_k = \mu^j_k + i\mu^{s+j}_k , \tag{2.11}$$

$$\mu^j_k = \mu^{s+j}_{s+k} , \qquad \mu^j_{s+k} = -\mu^{s+j}_k , \tag{2.12}$$

the μ^j, μ^{s+j}'s give an orthonormal coframe for (2.5) whose corresponding Levi-Civita connection forms are determined by (2.11), (2.12) and by (2.6)–(2.8). Analogously setting

$$\Omega^k_j = M^k_j + iM^{s+k}_j , \tag{2.13}$$

$$M^k_j = M^{s+k}_{s+j} , \qquad M^j_{s+k} = -M^{s+j}_k . \tag{2.14}$$

The M^u_v's defined in (2.13), (2.14) together with skew symmetry, coincide with the corresponding curvature forms. Thus letting R^u_{vwz} be the coefficients of the Riemann curvature tensor determined by

$$M^u_v = \tfrac{1}{2} R^u_{vwz} \mu^w \wedge \mu^z \tag{2.15}$$

we have that, in addition to the usual symmetry relations, they have to obey those derived from (2.14). Observe that from (2.8) the complex structure J_M on M is determined by the requirements

$$J_M \mu^k = -\mu^{s+k} , \qquad J^2_M = -\mathrm{id} \tag{2.16}$$

and (2.12) are equivalent to the parallelism of J_M with respect to the Levi-Civita connection.

Let $f\colon M^{2s} \to \mathbf{R}^n$ be an immersion and let (e, f) be a Darboux frame along f, that is, (e, f) is a smooth function $(e, f)\colon U \subset M \to \mathbf{E}(n)$, U open, with the property

$$(e, f)^* \theta^\alpha \equiv 0 . \tag{2.17}$$

We set

$$(e, f)^* \theta^A = B^A_u \mu^u \tag{2.18}$$

for some smooth, locally defined, functions B^A_u so that from (2.17) we have

$$B^\alpha_u \equiv 0 . \tag{2.19}$$

Observe that since f is an immersion the matrix (B^u_v) is nonsingular.

With respect to the considered Darboux frame the coefficients, B^A_{uv}, of the generalized second fundamental tensor ∇df are defined by

$$dB^A_u - B^A_v \mu^v_u + B^B_u \varphi^A_B = B^A_{uv} \mu^v , \qquad B^A_{uv} = B^A_{vu} , \tag{2.20}$$

and remark that using (2.19) in (2.20) we obtain

$$B^{\alpha}_{uv}\mu^{v} = B^{v}_{u}\varphi^{\alpha}_{v}. \tag{2.21}$$

The coefficients B^{A}_{uvw} of the covariant derivative of ∇df are given by the formula

$$dB^{A}_{uv} - B^{A}_{wv}\mu^{w}_{u} - B^{A}_{uw}\mu^{w}_{v} + B^{B}_{uv}\varphi^{A}_{B} = B^{A}_{uvw}\mu^{w}. \tag{2.22}$$

Formula (2.20), its exterior derivative, and use of the structure equations give

$$B^{A}_{uvw} = B^{A}_{vuw}, \qquad B^{A}_{uvw} = B^{A}_{uwv} + B^{A}_{z}R^{z}_{uvw} \tag{2.23}$$

which can be considered as generalized Codazzi equations.

Using the definitions given in §1 we have that f is $(1, 1)$-geodesic or $(2, 0)$-geodesic respectively when

$$B^{A}_{ij} + B^{A}_{s+i\,s+j} = 0, \qquad B^{A}_{is+j} - B^{A}_{s+ij} = 0, \tag{2.24}$$

or

$$B^{A}_{ij} - B^{A}_{s+i\,s+j} = 0, \qquad B^{A}_{is+j} + B^{A}_{s+ij} = 0. \tag{2.25}$$

By (2.22) and (2.24), if f is $(1, 1)$-geodesic then,

$$B^{A}_{ijw} + B^{A}_{s+i\,s+jw} = 0, \qquad B^{A}_{is+jw} - B^{A}_{s+ijw} = 0; \tag{2.26}$$

and analogously for f $(2, 0)$-geodesic from (2.22) and (2.25).

Let $G_s(\mathbf{C}^n)$ be the complex Grassmannian of s-planes in $\mathbf{C}^n$. Then the complex Gauss map $\gamma^{\mathbf{C}}_f\colon M \to G_s(\mathbf{C}^n)$ can be defined by

$$\gamma^{\mathbf{C}}_f\colon p \to [(B^{u}_{1} + iB^{u}_{s+1})e_u, \dots, (B^{u}_{s} + iB^{u}_{2s})e_u] \quad \text{at } p. \tag{2.27}$$

Let $\tilde{\gamma}^{\mathbf{C}}_f$ indicate the homogeneous representation of $\gamma^{\mathbf{C}}_f$ given by

$$\tilde{\gamma}^{\mathbf{C}}_f = (B^{u}_{1} + iB^{u}_{s+1})e_u \wedge \cdots \wedge (B^{u}_{s} + iB^{u}_{2s})e_u$$

and set

$$\begin{aligned}\tilde{\gamma}^{\mathbf{C}}_f(k) = {} & (B^{u}_{1} + iB^{u}_{s+1})e_u \wedge \cdots \wedge (B^{u}_{k-1} + iB^{u}_{s+k-1})e_u \\ & \wedge (B^{u}_{k+1} + iB^{u}_{s+k+1})e_u \wedge \cdots \wedge (B^{u}_{s} + iB^{u}_{2s})e_u.\end{aligned}$$

Then using (2.21), (2.20), (2.19) and (2.12) we compute

$$d\tilde{\gamma}^{\mathbf{C}}_f = i\mu^{k}_{s+k}\tilde{\gamma}^{\mathbf{C}}_f + (-1)^{k+1}(B^{A}_{kv} + iB^{A}_{s+kv})\mu^{v}e_A \wedge \tilde{\gamma}^{\mathbf{C}}_f(k) \tag{2.28}$$

and therefore from (2.10)

$$\begin{aligned} d\tilde{\gamma}^{\mathbf{C}}_f = {} & i\mu^{k}_{s+k}\tilde{\gamma}^{\mathbf{C}}_f + \tfrac{1}{2}(-1)^{k+1} \\ & \cdot \{[B^{A}_{kj} + B^{A}_{s+k\,s+j} + i(B^{A}_{s+kj} - B^{A}_{ks+j})]\varphi^{j} \\ & + [B^{A}_{kj} - B^{A}_{s+k\,s+j} + i(B^{A}_{s+kj} + B^{A}_{ks+j})]\overline{\varphi}^{j}\}e_A \wedge \tilde{\gamma}^{\mathbf{C}}_f(k). \end{aligned} \tag{2.29}$$

Since f is an immersion, (2.24), (2.25) and (2.29) prove Theorems 1 and 3.

Lemma 2.1. *Let* $f: M \to \mathbf{R}^n$ *be a* $(1, 1)$*-geodesic immersion of the Kähler manifold* M *into* $\mathbf{R}^n$ *and let* $\gamma_f^{\mathbf{C}}: M \to G_s(\mathbf{C}^n)$ *be its complex Gauss map. Fix an* s*-plane* A *in* $\mathbf{C}^n$ *and consider the smooth function* $|\langle \gamma_f^{\mathbf{C}}, A\rangle|^2$. *Then the following formula holds on the open set where* $\langle \gamma_f^{\mathbf{C}}, A\rangle \neq 0$.

$$\Delta \log |\langle \gamma_f^{\mathbf{C}}, A\rangle|^2 = 2R_{ks+kj\,s+j}. \tag{2.30}$$

Proof. First of all recall that given a real function a on M, the unitary coframe of (2.5) and the corresponding Kähler connection forms, the Laplace-Beltrami operator on a, Δa, is computed as follows. Set

$$da = a_j\varphi^j + a_{\bar{j}}\overline{\varphi}^j \qquad (a_{\bar{j}} = \overline{a_j})$$

and define $a_{j\bar{k}}$ via the formula

$$da_j - a_k\omega_j^k = a_{jk}\varphi^k + a_{j\bar{k}}\overline{\varphi}^k$$

then

$$\Delta a = 4a_{k\bar{k}}.$$

For $a > 0$, to compute $\Delta \log a$ we make use of the formula

$$\Delta \log a = \frac{1}{a}\Delta a - \frac{1}{a^2}|\nabla a|^2. \tag{2.31}$$

Observe that the function $|\langle \gamma_f^{\mathbf{C}}, A\rangle|^2$ is defined independently of the homogeneous representatives $\tilde{\gamma}_f^{\mathbf{C}}$ and $\widetilde{A}$ respectively of $\gamma_f^{\mathbf{C}}$ and A. From (2.29), and since f is $(1, 1)$-geodesic, (that is, (2.24) holds), we have

$$d\langle \tilde{\gamma}_f^{\mathbf{C}}, \widetilde{A}\rangle = (-1)^{k+1}\langle e_A \wedge \tilde{\gamma}_f^{\mathbf{C}}(k), \widetilde{A}\rangle(B_{kj}^A + iB_{s+kj}^A)\overline{\varphi}^j + i\langle \tilde{\gamma}_f^{\mathbf{C}}, \widetilde{A}\rangle\mu_{s+k}^k$$

from which we immediately deduce

$$d|\langle \gamma_f^{\mathbf{C}}, A\rangle|^2 \equiv (-1)^{k+1}\langle e_A \wedge \overline{\tilde{\gamma}_f^{\mathbf{C}}(k)}, \overline{\widetilde{A}}\rangle\langle \tilde{\gamma}_f^{\mathbf{C}}, \widetilde{A}\rangle(B_{kj}^A - iB_{s+kj}^A)\varphi^j \mod(\overline{\varphi}^t). \tag{2.32}$$

According to our procedure we have to compute the $(0, 1)$ part of

$$\begin{aligned}\Lambda = {} & d\{(-1)^{k+1}\langle e_A \wedge \overline{\tilde{\gamma}_f^{\mathbf{C}}(k)}, \overline{\widetilde{A}}\rangle\langle \tilde{\gamma}_f^{\mathbf{C}}, \widetilde{A}\rangle(B_{kj}^A - iB_{s+kj}^A)\} \\ & - (-1)^{k+1}\langle e_A \wedge \overline{\tilde{\gamma}_f^{\mathbf{C}}(k)}, \overline{\widetilde{A}}\rangle\langle \gamma_f^{\mathbf{C}}, A\rangle(B_{kt}^A - iB_{s+kt}^A)\omega_j^t.\end{aligned}$$

Using (2.11), (2.12), (2.22), (2.21), (2.20), (2.24), (2.23), (2.26) and the extra symmetries of the Riemann tensor of M due to (2.14), after a long computation, we obtain

$$\begin{aligned}\Lambda \equiv {} & \{\tfrac{1}{2}|\langle \gamma_f^{\mathbf{C}}, A\rangle|^2(R_{ks+kj\,s+t} - iR_{ks+kjt}) \\ & + (-1)^{k+1}(-1)^{l+1}\langle e_A \wedge \overline{\tilde{\gamma}_f^{\mathbf{C}}(k)}, \overline{\widetilde{A}}\rangle\langle e_B \wedge \tilde{\gamma}_f^{\mathbf{C}}(l), \widetilde{A}\rangle \\ & \cdot (B_{kj}^A - iB_{s+kj}^A)(B_{lt}^B + iB_{s+lt}^B)\}\overline{\varphi}^t \mod (\varphi^t).\end{aligned}$$

Therefore

$$\Delta|\langle \gamma_f^{\mathbf{C}}, A\rangle|^2 = 2|\langle \gamma_f^{C}, A\rangle|^2 R_{ks+kjs+j} + 4\left|\sum_{A,k,j}\langle e_A \wedge \overline{\tilde{\gamma}_f^{\mathbf{C}}(k)}, \overline{A}\rangle(B_{kj}^A - iB_{s+kj}^A)\right|^2$$

and (2.30) follows from (2.32) and (2.31).

Remark. The function $R_{ks+kjs+j}$ is an average of the holomorphic bisectional curvatures.

Observe that in case f is an isometry we can choose the coefficients B_u^A in (2.18) to be δ_u^A so that from (2.21), the B_{uv}^α are precisely the coefficients of the second fundamental tensor II. In this case the Riemann curvature of M is related to II by the Ricci equations, that is,

$$R_{uvwz} = B_{uw}^\alpha B_{vz}^\alpha - B_{uz}^\alpha B_{vw}^\alpha \tag{2.33}$$

From (2.33), in case f is $(1, 1)$-geodesic, it follows immediately that

$$2R_{ks+kjs+j} = -|\mathrm{II}|^2 . \tag{2.34}$$

Hence under the assumptions of Theorem 5 from (2.30) we have that $\log|\langle\gamma_f^{\mathbf{C}}, A\rangle|^2$ is a superharmonic function bounded below and therefore constant. From (2.30) and again (2.34) we conclude that $\mathrm{II} \equiv 0$, that is, f is totally geodesic completing the proof of Theorem 5.

3. Isometric immersions

Let $H_s(\mathbf{C}^n)$ be the space of s (complex) dimensional isotropic planes of $\mathbf{C}^n$.

We now briefly describe its geometry. First of all observe that given any point $q \in H_s(\mathbf{C}^n)$, that is, given any s dimensional isotropic subspace of $\mathbf{C}^n$ we can find a basis for it of vectors of the form $a_k + ia_{s+k}$ with the a_u's orthonormal vectors of $\mathbf{R}^n$. Then $SO(n)$ transitively acts on $H_s(\mathbf{C}^n)$ in an obvious way. Fix as an origin in $H_s(\mathbf{C}^n)$ the point

$$0 = [\varepsilon_1 + i\varepsilon_{s+1}, \dots, \varepsilon_s + i\varepsilon_{2s}]$$

where $\{\varepsilon_A\}$ is the canonical basis of $\mathbf{R}^n$. Then $H_s(\mathbf{C}^n)$ is realized as the homogeneous space $SO(n)/U(s) \times SO(n-2s)$ where the isotropy subgroup is computed at 0. Let φ be the Maurer-Cartan form of $SO(n)$ (consistent with the notation for the Maurer-Cartan form of $\mathbf{E}(n)$ in §2). Then the quadratic form

$$Q = \sum_{\alpha, u}(\varphi_u^\alpha)^2 + \frac{1}{4}\sum_{j<k}(\varphi_k^j - \varphi_{s+k}^{s+j})^2 + (\varphi_{s+k}^j + \varphi_k^{s+j})^2 \tag{3.1}$$

descend to a Riemannian metric ds_H^2 on $H_s(\mathbf{C}^n)$ via local sections of the bundle

$$\tilde{\pi} \colon SO(n) \to H_s(\mathbf{C}^n) . \tag{3.2}$$

In particular a (local) orthonormal coframe on $H_s(\mathbf{C}^n)$ is given by the forms

(3.3)

$$\omega^{u\alpha} = \varphi_u^\alpha , \qquad \omega^{jk-} = \tfrac{1}{2}(\varphi_k^j - \varphi_{s+k}^{s+j}) , \qquad \omega^{jk+} = \tfrac{1}{2}(\varphi_{s+k}^j + \varphi_k^{s+j}) , \qquad j < k .$$

With the aid of (3.3) we introduce an almost complex structure J_H by defining as a local basis for the $(1, 0)$ forms

$$\rho^{j\alpha} = \omega^{j\alpha} + i\omega^{s+j\alpha} , \qquad \rho^{jk} = \omega^{jk-} + i\omega^{jk+} , \qquad k < k . \tag{3.4}$$

Proposition 3.1. *The almost complex structure J_H is symplectic and integrable so that $H_S(\mathbf{C}^n)$ is a Kähler manifold.*

Proof. This amounts to showing that the Kähler form corresponding to the unitary coframe (3.4) is closed and that the ideal (3.4) generates is a differential ideal. This is immediately verified with the use of the structure equations (2.3).

It is not hard to see that with this complex structure the inclusion

$$i\colon H_S(\mathbf{C}^n) \to G_S(\mathbf{C}^n)$$

is a holomorphic isometric immersion. Corollary 2 follows from this and the assumption that f is an isometry.

Let us now consider the isometric immersion $f\colon M \to \mathbf{R}^n$ so that

$$B_u^A = \delta_u^A\,, \qquad B_{vw}^u = 0 \tag{3.5}$$

and in standard notation

$$B_{uv}^\alpha = h_{uv}^\alpha \tag{3.6}$$

are the coefficients of the second fundamental tensor II.

We now characterize, via $\gamma_f^{\mathbf{C}}$, those isometric immersions $f\colon M \to \mathbf{R}^n$, with $n = 2m$, which are holomorphic with respect to some complex structure J on $\mathbf{R}^{2m}$, that is, such that

$$J \circ df = df \circ J_M \tag{3.7}$$

Fix holomorphic (local) coordinates $z_j = x_j + iy_j$ on M such that J_M is the canonical complex structure associated to the complex manifold M (Newlander and Niremberg [NN]), that is,

$$J_M\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}\,, \qquad J_M\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}\,.$$

Considering the canonical coordinates on $\mathbf{R}^{2m}$, (3.7) is then equivalent to

$$J\frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial y_j}\,. \tag{3.8}$$

As a consequence f is holomorphic with respect to J on $\mathbf{R}^{2m}$ if and only if $\gamma_f^{\mathbf{C}}\colon M \to H_s(\mathbf{C}^n)$ can be written in the form

$$\gamma_f^{\mathbf{C}}\colon p \to \left[\frac{\partial f}{\partial x_1} + iJ\frac{\partial f}{\partial x_1}\,, \dots\,, \frac{\partial f}{\partial x_s} + iJ\frac{\partial f}{\partial x_s}\right] \quad \text{at } p\,. \tag{3.9}$$

Let H_J be the subset of $H_s(\mathbf{C}^{2m})$ of all the elements of the form $[v_1 + iJv_1\,, \dots\,, v_s + iJv_s]$ with $v_k \perp v_j\,, Jv_j\,,\ k \neq j\,,\ v_k \neq 0$, for each k.

Clearly f is holomorphic with respect to J if and only if

$$\gamma_f^{\mathbf{C}}(M) \subseteq H_J\,. \tag{3.10}$$

To give H_J a differentiable structure, we fix the almost complex structure J_0 on $\mathbf{R}^{2m}$ whose matrix representation in the canonical basis $\{\varepsilon_A\}$ of $\mathbf{R}^{2m}$ is given by

$$\left(\begin{array}{c|c|c|c} 0 & -I_s & 0 & 0 \\ \hline I_s & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -I_{m-s} \\ \hline 0 & 0 & I_{m-s} & 0 \end{array}\right)$$

with I_r the r by r identity matrix. Observe that if we let $O(2m)$ act on H_J in the obvious way, then there exists $A \in O(2m)$ such that $AH_J = H_{J_0}$. Indeed let A be an element of $O(2m)$ such that

$$A^{-1}J_0A = J$$

whose existence is guaranteed by the fact that the almost complex structures on $\mathbf{R}^{2m}$ are parametrized by the homogeneous space $O(2m)/U(m)$, then

$$\begin{aligned} A[v_1 + iJv_1, \dots, v_s + iJv_s] &\\ = [Av_1 + iAJ(A^{-1}A)v_1, \dots, Av_s + iAJ(A^{-1}A)v_s] &\\ = [Av_1 + iJ_0Av_1, \dots, Av_s + iJ_0Av_s]. & \end{aligned}$$

It is therefore enough to give a differentiable structure to H_{J_0}. Towards this aim observe that if $A \in O(2m)$ then $AJ_0 = J_0A$ if and only if $A \in U(m)$ so that $AH_{J_0} = H_{J_0}$ if and only if $A \in U(m)$. One verifies that the action of $U(m)$ on H_{J_0} is transitive. Fix as an origin in H_{J_0} the point O' given by the isotropic s-plane

$$[\varepsilon_1 + iJ_0\varepsilon_1, \dots, \varepsilon_s + iJ_0\varepsilon_s] = [\varepsilon_1 + i\varepsilon_{s+1}, \dots, \varepsilon_s + i\varepsilon_{2s}]$$

then the isotropy subgroup of O' is given by $U(s) \times U(m-s)$ and $H_{J_0} = U(m)/U(s) \times U(m-s)$ is the Grassmannian of complex s-planes in $\mathbf{C}^m$ providing a proof of Theorem 4.

Given the isometric immersion $f\colon M \to \mathbf{R}^n$ and the Darboux frame (e, f) along f observe that we have the commutative diagram

$$\begin{tikzcd} & SO(n) \arrow[dl, "\tilde{\pi}"'] & \mathbf{E}(n) \arrow[l, "F"'] \arrow[dd, "\pi"] \\ H_S(\mathbf{C}^n) & & \\ & M \arrow[ul, "\gamma_f^{\mathbf{C}}"] \arrow[uur, "{(e,f)}"] \arrow[r, "f"'] & \mathbf{R}^n \end{tikzcd}$$

where the maps π and $\tilde{\pi}$ have been defined above and F means *forget the* $\mathbf{R}^n$ *bit*, that is $F\colon (e, v) \to e$. It therefore follows from (3.3), (2.12), (3.6), (2.21), (2.33) that

$$\gamma_f^{\mathbf{C}*}(ds_H^2) = -\operatorname{Ric}(M) + 2s\mathrm{II}_H \tag{3.11}$$

where $\operatorname{Ric}(M)$ is the symmetric Ricci 2-form of M and $\mathrm{II}_H = \langle \mathrm{II}, H\rangle$ for $H = \frac{1}{2s}\operatorname{tr}\mathrm{II}$, the mean curvature vector of the isometric immersion. From (3.11) we therefore obtain the following:

Proposition 3.2. *Let* $f\colon M \to \mathbf{R}^n$ *be an isometric immersion and* $\gamma_f^{\mathbf{C}}\colon M \to H_s(\mathbf{C}^n)$ *is complex Gauss map. Then any two of the following properties imply the third*

(i) M *is Einstein,*

(ii) f *is pseudo-umbilical, that is,* II_H *is a multiple of* ds_M^2,

(iii) $\gamma_f^{\mathbf{C}}$ *is weakly conformal.*

Remarks. 1. Proposition 3.2 is the analogue of Theorem 1 in Obata [O] relative to the usual Gauss map $\gamma_f\colon M \to G_{2s}(\mathbf{R}^n)$.

2. From (3.11) by the definition of the third fundamental form III of f, we have $\mathrm{III} \equiv \gamma_f^{\mathbf{C}*}(ds_H^2)$. Define the volume of $\gamma_f^{\mathbf{C}}$ at $p \in M$ to be

$$\sigma(p, f) = \frac{2}{c_{2s}}\{\det(h_{uv}^{\alpha}h_{vw}^{\alpha})\}^{1/2}$$

and let $\tau(p, f)$ be the Chern-Lashof, [C-L], total curvature at p. Then, using the work of Ferus [F2], $\tau(p, f) \leq \sigma(p, f)$ equality holding if and only if at least one of the following conditions is satisfied:

(1) the first normal space of f at p is of (real) dimension ≤ 1,
(2) $s = 1$, and $H(p) = 0$,
(3) III is singular at p,
(4) $\gamma_f^{\mathbf{C}}$ is not regular at p.

We recall that realizing the real Grassmannian $G_{2s}(\mathbf{R}^n)$ as

$$SO(n)/S(O(2s) \times O(n-2s)),$$

where the isotropy subgroup is computed at the origin $\widetilde{O} = [\varepsilon_1, \dots, \varepsilon_{2s}]$ of $G_{2s}(\mathbf{R}^n)$, then for the usual Gauss map $\gamma_f \colon M \to G_{2s}(\mathbf{R}^n)$, with respect to a Darboux frame (e, f) along f, we have

$$\nabla\, d\gamma_f = h^\alpha_{uvw}\theta^v\theta^w \otimes E_{u\alpha}$$

where $\{E_{u\alpha}\}$ is dual to the coframe $\{\varphi^\alpha_u\}$ realizing the Riemannian structure of $G_{2s}(\mathbf{R}^n)$ and h^α_{uvw} are the coefficients of the covariant derivative of II. In the isometric case, the h^α_{uvw} coincide with the B^α_{uvw} of (2.24). As a consequence γ_f is $(1, 1)$-geodesic if and only if

$$h^\alpha_{uij} + h^\alpha_{us+is+j} = 0 = h^\alpha_{uis+j} - h^\alpha_{us+ij}$$

and this is immediately verified to be equivalent to

$$\nabla^\perp \mathrm{II}^{(1,1)} = 0.$$

We have therefore proved

Proposition 3.3. *Let $f \colon M \to \mathbf{R}^n$ be an isometric immersion of a Kähler manifold into $\mathbf{R}^n$ and let $\gamma_f \colon M \to G_{2s}(\mathbf{R}^n)$ be its usual Gauss map. Then $\mathrm{II}^{(1,1)}$ is parallel in the normal bundle if and only if γ_f is $(1, 1)$-geodesic.*

Corollary 3.4. *Let $f \colon M \to \mathbf{R}^n$ be an isometric immersion of a Kähler manifold into $\mathbf{R}^n$. If γ_f is $(1, 1)$-geodesic and the mean curvature vector H of f is zero at one point, then f is minimal and $\gamma_f^{\mathbf{C}}$ is $-$ holomorphic.*

Let $f \colon M \to \mathbf{R}^n$ be an isometric immersion and (e, f) a Darboux frame along f. To simplify notation we set

$$E_k = e_k + ie_{s+k}, \qquad E_{-k} = e_k - ie_{s+k}$$

so that the homogeneous representation of $\gamma_f^{\mathbf{C}}$ given in §2 becomes

$$\tilde{\gamma}_f^{\mathbf{C}} = E_1 \wedge \cdots \wedge E_s$$

and (2.28) can be rewritten as

$$d\tilde{\gamma}_f^{\mathbf{C}}(X) = \sum_k E_1 \wedge \cdots \wedge E_{k-1} \wedge \mathrm{II}(X, E_k) \wedge E_{k+1} \wedge \cdots \wedge E_s.$$

This can be immediately checked observing that $(dE_k, E_{-j}) = 0$ for each k, j, where $(\ ,\ )$ is the Hermitian inner product; that is, the derivatives of $(0, 1)$-vectors are of the same type. To compute the tension field of $\gamma_f^{\mathbf{C}}$ considered as a map into $G_s(\mathbf{C}^n)$ we introduce the following notation. For $v \in \mathbf{C}^n$ let v^k denote

$$v^k = E_1 \wedge \cdots \wedge E_{k-1} \wedge v \wedge E_{k+1} \wedge \cdots \wedge E_s .$$

Then the covariant derivative $\nabla\, d\gamma_f^{\mathbf{C}}$ is given by

$$\begin{aligned}
(\nabla_X\, d\gamma_f^{\mathbf{C}})(Y) &= \nabla_X(d\gamma_f^{\mathbf{C}}(Y)) - d\gamma_f^{\mathbf{C}}(\nabla_X Y) \\
&= \sum_{k=1}^{s} \nabla_X(\mathrm{II}(E_k , Y))^k - d\gamma_f^{\mathbf{C}}(\nabla_X Y) \\
&= \sum_{k=1}^{s} \{\nabla_X^{\perp}\mathrm{II}(E_k , Y) + (\nabla_X \mathrm{II}(E_k , Y) , E_{-i})E_{-i}\}^k \\
&\quad + \sum_{j,k=1}^{s} \{(\nabla_X E_j , E_k)\mathrm{II}(E_k , Y)\}^j - d\gamma_f^{\mathbf{C}}(\nabla_X Y)
\end{aligned}$$

where with $\nabla^{\perp}$ we have indicated the connection in the normal bundle of the isometric immersion f. Choose now the Darboux frame (e , f) and the vector field Y on M such that at the point $p \in M$, $\nabla e_k = 0$ and $\nabla Y = 0$. Then at p we have

$$(3.12) \qquad (\nabla_X\, d\gamma_f^{\mathbf{C}})(Y) = \sum_k \{\nabla_X^{\perp}\mathrm{II}(E_k , Y) + (\nabla_X \mathrm{II}(E_k , Y) , E_{-i})E_{-i}\}^k$$

so that,

$$(3.13) \qquad \tau(\gamma_f^{\mathbf{C}}(p)) = \sum_{u=1}^{2s} (\nabla_{e_u}\, d\gamma_f^{\mathbf{C}})e_u = 0$$

if and only if the following two conditions are satisfied

$$\nabla_{e_u}^{\perp}\mathrm{II}(E_k , e_u) = 0 \quad \text{for each } k ,$$

$$\sum_{t=1}^{s} \langle \mathrm{II}(E_k , E_t) , \mathrm{II}(E_{-t} , E_j)\rangle = 0 \quad \text{for each } k , j .$$

Using Codazzi equations the first is easily seen to be equivalent to

$$(3.14) \qquad \nabla^{\perp} H = 0$$

while the second, using Gauss equations, is equivalent to

$$(3.15) \qquad \sum_{t=1}^{s} \langle R(E_k , E_t)E_{-t} , E_j\rangle = \mathrm{II}_H(E_k , E_j) .$$

Observing that for a Kähler manifold $R(E_k , E_t) \equiv 0$ we have achieved the proof of part (i) of Theorem 6. To show (ii) observe that since $H_s(\mathbf{C}^n)$ is isometrically immersed into $G_s(\mathbf{C}^n)$ the projection of the tension field (3.13) in the tangent space of $H_s(\mathbf{C}^n)$ will give the tension field of $\gamma_f^{\mathbf{C}}$ considered as a map into $H_s(\mathbf{C}^n)$. On the other hand the tangent space of $H_s(\mathbf{C}^n)$ at some point p is generated by all vectors of the form v^k where either $v = e_\alpha$ or

$v = E_{-i}$, $i \neq k$. We therefore conclude that $\gamma_f^{\mathbf{C}}$ is harmonic in $H_s(\mathbf{C}^n)$ if and only if $\nabla^\perp H = 0$ and

$$\sum_{t=1}^{s}\langle R(E_u, E_t)E_{-t}, E_i\rangle = \mathrm{II}_H(E_k, E_i), \qquad k \neq i,$$

from which we easily deduce the validity of (ii) completing the proof of Theorem 6.

To prove Corollary 7 from Theorem 6 we have that $\gamma_f^{\mathbf{C}}: M \to \mathbf{C}P^n$ is harmonic if and only if $\nabla^\perp H = 0$ and

$$\langle \mathrm{II}(E_1, E_1), \mathrm{II}(E_1, E_{-1})\rangle = 0.$$

If $H \neq 0$, since $\mathrm{II}(E_1, E_{-1})$ is a nonzero real multiple of H, we have $\mathrm{II}(E_1, E_1) \perp H$. Therefore II_H is a multiple of the metric of the surface and thus, from [Y] or [R-T], f is minimal in some sphere of $\mathbf{R}^n$. In particular for $n = 3$ and $H \neq 0$, since II_H is a multiple of the metric, $f(M)$ has to be a piece of the standard 2-sphere.

Theorem 3.5. *Let $f: M \to \mathbf{R}^n$ be an isometric immersion of a Kähler manifold and $\gamma_f^{\mathbf{C}}: M \to G_s(\mathbf{C}^n)$ be its complex Gauss map. Then $\gamma_f^{\mathbf{C}}$ is (1,1)-geodesic if and only if the following two conditions are satisfied.*

(i) $\nabla^\perp \mathrm{II}^{(1,1)} = 0$,

(ii) $\langle \mathrm{II}^{(0,2)}, \mathrm{II}^{(1,1)}\rangle = 0$.

Proof. By definition $\gamma_f^{\mathbf{C}}$ is $(1, 1)$-geodesic if and only if

$$(\nabla_X d\gamma_f^{\mathbf{C}})Y + (\nabla_{J_M X} d\gamma_f^{\mathbf{C}})J_M Y = 0$$

for each pair of vector fields X and Y on M. From (3.12) this is equivalent to

$$\nabla_X^\perp \mathrm{II}(E_k, Y) + \nabla_{J_M X}^\perp \mathrm{II}(E_k, J_M Y) = 0, \tag{3.16}$$

$$\langle \mathrm{II}(E_k, X), \mathrm{II}(Y, E_i)\rangle + \langle \mathrm{II}(E_k, J_M X), \mathrm{II}(J_M Y, E_i)\rangle = 0. \tag{3.17}$$

Using Codazzi equations and Gauss equations similarly to Theorem 6 it is easy to see that (3.16) and (3.17) are respectively equivalent to (i) and (ii) of the theorem.

Corollary 3.6. *Let $f: M \to \mathbf{R}^n$ be a Kähler isometrically immersed hypersurface and assume that $\gamma_f^{\mathbf{C}}: M \to G_s(\mathbf{C}^n)$ is $(1, 1)$-geodesic. Then either f is $(1, 1)$-geodesic or $(0, 2)$-geodesic.*

Proof. Observe that from Theorem 3.5 (ii)

$$\langle \mathrm{II}(E_k, E_i), \mathrm{II}(E_{-j}, E_r)\rangle = 0 \quad \text{for each } k, i, j, r.$$

Therefore if f is not $(1, 1)$-geodesic for some j, r the real vector $\mathrm{II}(E_{-j}, E_r) + \mathrm{II}(E_j, E_{-r})$ is nonzero at each point $p \in M$ (since $\nabla^\perp \mathrm{II}^{(1,1)} = 0$) and as a consequence $\mathrm{II}(E_k, E_i) \equiv 0$.

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