EXAMPLES OF CAPACITY FOR SOME ELLIPTIC OPERATORS

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ABSTRACT. We study $L$-capacities for uniformly elliptic operators of nondivergence form

$$L = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_j a_j(x) \frac{\partial}{\partial x_j};$$

and construct examples of large sets having zero $L$-capacity for some $L$, and small sets having positive $L$-capacity. The relations between ellipticity constants of the coefficients and the sizes of these sets are also considered.

A compact set $S \subseteq \{|x| < 1\} \subseteq \mathbb{R}^n$, $n \geq 2$, has zero capacity for the Laplacian if and only if it is a removable set for the class of bounded subharmonic functions on $\{|x| < 1\}$; equivalently, there exists a positive superharmonic function $(\neq +\infty)$ on $\{|x| < 1\} \setminus S$ which approaches $+\infty$ continuously on $S$.

In this note, we study $L$-capacities for uniformly elliptic operators of nondivergence form

$$L = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_j a_j(x) \frac{\partial}{\partial x_j};$$

and construct examples of large sets having zero $L$-capacity for some $L$, and small sets having positive $L$-capacity. We also study the relations between ellipticity constants of the coefficients and the sizes of these sets.

When $(a_{ij})$ are Dini continuous and $(a_j)$ are bounded, sets of $L$-capacity zero are precisely those of capacity zero for the Laplacian; this follows from the growth of the Green function for these operators. (See [1] and [7].) In general, there are elliptic operators $L$ with continuous coefficients for $n = 2$, bounded coefficients for $n \geq 3$, for which a single point has positive $L$-capacity; again this reflects the behavior of the Green function. (See [2, 4, 5 and 12].)

For uniformly elliptic operators of divergence form, the growth of Green function near its pole is comparable to that for the Laplacian [10]. Therefore sets of capacity zero are exactly those for the Laplacian.

Let $L$ be the above operator and coefficients of $L$ be continuous in a domain $\Omega \subseteq \mathbb{R}^n$. We consider strong solutions of $L = 0$ in $W^{2,n}_{\text{loc}}(\Omega)$ and call them $L$-solutions. The maximum principle, the existence and uniqueness of the solution to the Dirichlet problem and the Harnack principle are well known. A lower semicontinuous function $v$ is called an $L$-supersolution on $\Omega$, if for any closed ball $B \subseteq \Omega$ and any $L$-solution $u$ in $B$ with $u$ continuous on $\overline{B}$, the...
inequality \( v \geq u \) on \( \partial B \) implies that \( v \geq u \) in \( B \). A function \( v \in C^2(\Omega) \) is an \( L \)-supersolution if and only if \( L v \leq 0 \). A function \( v \) is called an \( L \)-subsolution if \( -v \) is an \( L \)-supersolution. (See [6, Chapter 9].)

Let \( D = \{ |x| < 1 \}, \overline{D} = (\frac{1}{2}, 0, 0, \ldots 0) \) and \( S \) be a compact set in \( \{ |x| \leq \frac{1}{4} \} \). We define the \( L \)-capacity of \( S \) as

\[
L\text{-}\text{cap} \ S = \inf \left\{ v(x) : v \text{ is a positive \( L \)-supersolution on } D \text{ and } v \geq 1 \text{ on } S \right\},
\]

when the coefficients of \( L \) are bounded continuous in \( D \). When the coefficients of \( L \) are only known to be bounded continuous on \( D \setminus S \), we say \( L\text{-}\text{cap} \ S = 0 \) provided that

\[
\inf_{x \rightarrow x_0} \{ v(x) : v \text{ is a positive \( L \)-supersolution on } D \setminus S, \ v(x) \geq 1 \text{ at each } x_0 \in S \} = 0;
\]

otherwise we say \( L\text{-}\text{cap} \ S > 0 \). Both definitions of \( L \)-capacity zero agree when the coefficients of \( L \) are continuous on \( D \). We note that if there exists a positive \( L \)-supersolution on \( D \setminus S \) which approaches \( +\infty \) continuously on \( S \), then \( L\text{-}\text{cap} \ S = 0 \); and that if there exists a bounded positive \( L \)-subsolution on \( D \setminus S \) which approaches \( 0 \) continuously on \( \partial D \), then \( L\text{-}\text{cap} \ S > 0 \).

We recall that for the Laplacian, a set has positive capacity if it has positive \( h \)-Hausdorff measure for some \( h > 0 \) satisfying \[
\int_0^1 h(r)/r^{n-1} \, dr < \infty;
\]
and a set has zero capacity if it has finite \((n-2)\)-dimensional Hausdorff measure when \( n \geq 3 \), or finite logarithmic measure when \( n = 2 \). Therefore \( n - 2 \) is the critical dimension for studying sets of capacity zero.

We shall prove the following:

**Theorem 1.** Let \( n \geq 2 \) and \( n - 2 < \alpha < n \). Then there exist a constant \( \Lambda_{n, \alpha} > 1 \), a compact set \( S \subseteq D \) of Hausdorff dimension \( \alpha \), an operator \( L = \sum a_{ij} \partial^2/\partial x_i \partial x_j \) with coefficients bounded smooth in \( D \setminus S \), satisfying

\[
|\xi|^2 \leq \sum a_{ij}(x)|\xi_i \xi_j| \leq \Lambda_{n, \alpha}|\xi|^2, \quad x, \xi \in \mathbb{R}^n,
\]

so that \( S \) has zero \( L \)-capacity in the sense (0.1). In fact, there is a positive \( L \)-supersolution \( v \) (\( \neq +\infty \)) in \( D \setminus S \) approaching \( +\infty \) continuously on \( S \). Moreover,

\[
\Lambda_{n, \alpha} = 1 + O(1) (\alpha - n + 2) \quad \text{as } \alpha \to n - 2,
\]

and

\[
\Lambda_{n, \alpha} = O(1) (n - \alpha)^{-1} \quad \text{as } \alpha \to n,
\]

with the \( O(1) \) terms positive and independent of \( n \) and \( \alpha \).

We believe that (0.3) is sharp, and do not know whether (0.4) can be improved.

**Theorem 2.** Let \( n \geq 2 \), \( a > 0 \), and \( h(r) = r^{n-2} (\log \frac{1}{r})^{-1-a} \). Then there exist a constant \( \beta > 0 \), a compact set \( S \subseteq D \) of dimension \( n - 2 \), positive Hausdorff \( h \)-measure, an operator \( L = \sum a_{ij} \partial^2/\partial x_i \partial x_j \) with coefficients continuous in \( \mathbb{R}^n \), smooth off \( S \) satisfying

\[
|\xi|^2 \leq \sum a_{ij}(x)|\xi_i \xi_j| \leq \left( 1 + \beta \left( \log \frac{1}{\text{dist}(x, S)} \right)^{-1} \right) |\xi|^2,
\]
for all $x, \xi \in \mathbb{R}^n$, and a positive $L$-supersolution $v$ on $D$, which is an $L$-solution on $D \setminus S$ and approaches $+\infty$ continuously on $S$. In particular, $S$ has zero $L$-capacity, and positive capacity for the Laplacian.

**Theorem 3.** Let $n \geq 3$, $a > 0$, and $h(r) = r^{n-2}(\log \frac{1}{r})^a$. Then there exist a compact set $S \subseteq D$ of dimension $n-2$, vanishing $h$-measure, a constant $\beta > 0$, an operator $L = \sum a_{ij} \partial^2 / \partial x_i \partial x_j$ with coefficients continuous in $\mathbb{R}^n$, smooth off $S$ and satisfying

$$\left\{1 - \beta \left( \log \frac{1}{\text{dist}(x, S)} \right)^{-1} \right\} |\xi|^2 \leq \sum a_{ij}(x) \xi_i \xi_j \leq |\xi|^2$$

for $x, \xi \in \mathbb{R}^n$, and a bounded positive $L$-supersolution $w$ on $D$, which is an $L$-solution on $D \setminus S$. Thus $S$ has zero capacity for the Laplacian, and positive capacity for the operator $L$.

**Theorem 4.** Let $n \geq 3$ and $0 < \alpha < n - 2$. Then there exist a positive constant $\lambda_{n, \alpha} < 1$, a compact set $S \subseteq D$ of dimension $\alpha$, an operator $L = \sum a_{ij} \partial^2 / \partial x_i \partial x_j$ with $a_{ij}$ bounded smooth off $S$, satisfying

$$\lambda_{n, \alpha} |\xi|^2 \leq \sum a_{ij}(x) \xi_i \xi_j \leq |\xi|^2, \quad x, \xi \in \mathbb{R}^n,$$

so that $S$ has positive $L$-capacity. In fact, there exists a bounded positive $L$-subsolution $w$ on $D \setminus S$ which vanishes continuously on $\partial D$. Moreover

\begin{equation}
\lambda_{n, \alpha} = (1 - 2\alpha)(n - 1)^{-1}, \quad 0 < \alpha < 1/4,
\end{equation}

and

\begin{equation}
\lambda_{n, \alpha} = 1 - O(1)(n - 2 - \alpha) \quad \text{as } \alpha \to n - 2,
\end{equation}

with the $O(1)$ term positive and independent of $n$ and $\alpha$.

Since it is known that a point can have positive $L$-capacity, the only new part of Theorem 4 is the relation between the ellipticity constants and the dimension.

In the proofs of all four theorems, we start with the Laplace operator, then modify the coefficients on a sequence of rings, accumulating on a Cantor set $S$, so that on the rings all eigenvalues are greater than 1 (or less than 1). When all are chosen properly, it will produce an $L$-supersolution which grows faster than (or slower than) the fundamental solution of the Laplacian near each point in $S$. This explains the relation between the normalization of the ellipticity constants and the size of the set $S$.

A related subject, the boundary regularity problem for the operator $L$, has been studied by many. A partial list includes [4, 7, 8, 9, 10, 11, 12, and 13].

### 1. Preliminary Lemmas

Let $\Delta$ be the Laplace's operator and $r = |x|$ for $x \in \mathbb{R}^n$.

**Lemma 1.** Let $B(x)$ be positive continuous in a domain $\Omega$, and let

$$L = \frac{B(x)}{n-1} \sum \frac{\partial^2}{\partial x_i^2} - \left( \frac{B(x)}{n-1} - 1 \right) \sum \frac{x_i x_j}{|x|^2} \frac{\partial^2}{\partial x_i \partial x_j}$$
on $\Omega \setminus \{0\}$. Then the coefficients $a_{ij}$ of $L$ are continuous, symmetric on $\Omega \setminus \{0\}$, satisfying

\begin{equation}
|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq \frac{B(x)}{n-1}|\xi|^2 \quad \text{when } B(x) \geq n-1,
\end{equation}

and

\begin{equation}
\frac{B(x)}{n-1}|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq |\xi|^2 \quad \text{when } B(x) \leq n-1.
\end{equation}

The coefficients of $L$ can be extended to be continuous on $\Omega$ if $B(0) = n-1$. Moreover, $|x|^{-B+1}$ is a solution of $L = O(x \neq 0)$, when $B(x) \equiv a$ constant $B$.

The characteristic values of $(a_{ij}(x))$ are $1, B(x)/(n-1), B(x)/(n-1), \ldots, B(x)/(n-1)$; and for $x \neq 0$,

\[ L = \frac{B(x)}{n-1} \Delta - \frac{B(x)}{n-1} \frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial r^2} + \frac{B(x)}{r} \frac{\partial}{\partial r} + \frac{B(x)}{n-1} r^{-2}\delta, \]

where $\delta$ is the Beltrami operator in the spherical coordinates. Whence the lemma follows.

Denote by $D(x, a)$ the closed ball centered at $x$ of radius $a$, and recall that $D = D(0, 1)$. When $U$ is a ball, denote by $cU$ the ball concentric to $U$ of radius $c$ times that of $U$.

**Lemma 2.** Let $0 < \delta < \frac{1}{16}$ and $D(a, r) \subseteq D(0, \delta)$. Then there exists a diffeomorphism $y = Tx$ from $\mathbb{R}^n$ onto $\mathbb{R}^n$, which fixes every point in $\mathbb{R}^n \setminus D(0, \frac{9}{16})$, maps each point $x$ in $D(a, r)$ to $x - a$, and satisfies on $D(0, \delta)$:

\begin{equation}
\frac{\partial y_i}{\partial x_j} - \delta_{ij} = c(x)a_i(x_j - a_j),
\end{equation}

\begin{equation}
\left| \sum \frac{\partial^2 y_i}{\partial x_l \partial x_m} \xi_l \right| \leq 272\delta|\xi|,
\end{equation}

where $0 < c(x) < 32$, $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$. Moreover if $(a_{ij}(x))$ is symmetric positive definite with all its eigenvalues bounded above by $\Lambda$, and

\[ b_{ij}(x) = \sum_{l,m} a_{lm}(x) \frac{\partial y_i}{\partial x_l} \frac{\partial y_j}{\partial x_m}, \]

then

\begin{equation}
\left| \sum_{i,j} b_{ij} \xi_i\xi_j - \sum_{i,j} a_{ij} \xi_i\xi_j \right| \leq 128\delta\Lambda|\xi|^2,
\end{equation}

and

\begin{equation}
\left| \sum b_{ii} - \sum a_{ii} \right| \leq 128\delta\Lambda.
\end{equation}

**Proof.** Let

\[ \psi(s) = \begin{cases} 
10s^3 - 15s^4 + 6s^5, & 0 \leq s \leq 1, \\
0, & s < 0, \\
1, & s > 1;
\end{cases} \]
and note that $\psi$ is $C^2$, $0 \leq \psi \leq 1$, $0 \leq \psi' \leq 15/8$ and $|\psi''| \leq 10/\sqrt{3}$. Let
\[
\phi(t) = 1 - \psi \left( \frac{t - \delta^2}{\frac{5}{4} - \delta^2} \right).
\]
Thus $\phi = 0$ for $t \geq \frac{5}{4}$, $\phi = 1$ for $t \leq \delta^2$, $0 \leq \phi \leq 1$, $-8 \leq \phi' \leq 0$ and $|\phi''| < 160$. Then $Tx = x - \phi(|x - a|^2)a$ is a diffeomorphism on $\mathbb{R}^m$ that fixes every point in $\mathbb{R}^m \setminus D(0, \frac{9}{16})$ and maps $x \in D(a, r)$ to $x - a$. Moreover, on $D(0, \frac{9}{16})$, $T$ satisfies (1.3) and (1.4) with $c(x) = -2\phi'(|x - a|^2)$.

To show (1.5), we let $x \in D(0, \frac{9}{16})$, and note that
\[
b_{ij} - a_{ij} = \sum_{l, m} a_{lm} \left[ \delta_{il} + c(x) a_i (x_l - a_l) \right] \left[ \delta_{jm} + c(x) a_j (x_m - a_m) \right] - a_{ij}
\]
\[
= c(x) \sum_m a_{lm} a_j (x_m - a_m) + c(x) \sum_l a_{lj} a_i (x_l - a_l)
\]
\[
+ c(x)^2 \sum_{l, m} a_{lm} a_j a_i (x_l - a_l) (x_m - a_m).
\]
Thus
\[
\left| \sum_{i, j} b_{ij} \xi_i \xi_j - \sum_{i, j} a_{ij} \xi_i \xi_j \right|
\]
\[
\leq c(x) \left| \sum_j \sum_{i, m} a_{im} \xi_i (x_m - a_m) a_j \xi_j \right| + c(x) \left| \sum_i \sum_{j, l} a_{ij} \xi_j (x_l - a_l) a_i \xi_i \right|
\]
\[
+ c(x)^2 \left| \sum_{i, j, l, m} a_{im} (x_l - a_l) (x_m - a_m) a_i a_j \xi_i \xi_j \right|.
\]
Since $|a| < \delta$, $|x - a| < 1$ and eigenvalues of $(a_{ij})$ are bounded above by $\Lambda$, we conclude that
\[
\left| \sum_{i, j} b_{ij} \xi_i \xi_j - \sum_{i, j} a_{ij} \xi_i \xi_j \right| \leq 2c(x) \Lambda \delta + c(x)^2 \Lambda \delta^2 \leq 128\Lambda \delta.
\]
Similarly,
\[
\left| \sum_i b_{ii} - \sum a_{ii} \right| \leq c(x) \left| \sum_{i, m} a_{im} a_i (x_m - a_m) \right| + c(x) \left| \sum_{i, l} a_{il} a_i (x_l - a_l) \right|
\]
\[
+ c(x)^2 \left| \sum_i a_i^2 \sum_{l, m} a_{lm} (x_l - a_l) (x_m - a_m) \right| \leq 128\Lambda \delta.
\]

2. The construction

Given $B^* \geq n - 1$, integer $k_0 > 0$, let $\{\delta_k\}$, $\{r_k\}$, and $\{N_k\}$ be sequences satisfying $0 < \delta_k < (2400B^*)^{-1}$, $0 < r_k < r_{k-1} < r_1 \leq \frac{1}{2}$ and $16\sqrt{n}/\delta_k < N_k < r_{k-1}/r_k$, for $k \geq k_0$. Then $r_{k+1} < \delta_{k+1} N_{k+1} r_{k+1} < N_{k+1} r_{k+1} < r_k$ for $k \geq k_0$.

Let $[\ ]$ be the greatest integer function, $I_{k_0} = 1$ and $I_k = \prod_{j=k_0}^{k} [\delta_j N_j/16\sqrt{n}]^n$ for $k > k_0$. 

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Denote by $D_{k_0, 1} = D(0, r_{k_0})$. After $\{D_{k, l}: 1 \leq l \leq I_k\}$ are selected for some $k \geq k_0$ we let $\mathcal{D}_{k, l}$ be the ball $\delta_{k+1}N_{k+1}r_{k+1}^{-1}D_{k, l}$ of radius $\delta_{k+1}N_{k+1}r_{k+1}$; and choose from each $\mathcal{D}_{k, l}$ a number of $[\delta_{k+1}N_{k+1}/16\sqrt{n}]^n$ balls of radius $r_{k+1}$ to form the collection $\{D_{k+1, l}: 1 \leq l \leq I_{k+1}\}$. Moreover, we require their doublings $\{2D_{k+1, l}\}$ to be mutually disjoint and contained in $\bigcup \mathcal{D}_{k, l}$. Let $S$ be the Cantor set defined by $S = \bigcap_{k=k_0}^{\infty} \bigcup_{l=1}^{I_k} D_{k, l}$). And let $\mu$ be the continuous measure on $S$, defined by $\mu(D_{k, l}) = I_k^{-1}$ for all $k \geq k_0$ and $1 \leq l \leq I_k$. For $k \geq k_0$, denote by $P_{k, l}$ the center of $D_{k, l}$,\[ R_{k, l} = \{N_{k+1}r_{k+1} \leq |x - P_{k, l}| \leq r_k\}, \]and\[ R'_{k, l} = \{\frac{3}{4}N_{k+1}r_{k+1} \leq |x - P_{k, l}| \leq \frac{5}{4}r_k\}, \]and note that $\{R'_{k, l}: k \geq k_0, 1 \leq l \leq I_k\}$ are mutually disjoint.

Let $B(r)$ be a smooth function for $r > 0$, satisfying $n - 1 \leq B(r) \leq B^*$, with
\[
B(r) \equiv n - 1 \text{ on } \{r > \frac{5}{4}r_{k_0}\} \cup \bigcup_{k=k_0}^{\infty} \left[\frac{3}{4}r_k, \frac{5}{4}N_kr_k\right],
\]
and $B(r)$ monotone in each of the remaining intervals. Define on $\mathbb{R}^n$ an elliptic operator
\[
L = \begin{cases} \Delta, & \text{on } \mathbb{R}^n \setminus \bigcup_{k, l} R'_{k, l}, \\ \frac{B(r)}{n - 1}\Delta - \frac{B(r)}{n - 1} - 1 \frac{\partial^2}{\partial r^2}, & \text{at } x + P_{k, l} \in R'_{k, l}, \end{cases}
\]
where $r = |x|$. Rewrite $L$ in the standard form $\sum a_{ij}\partial^2/\partial x_i \partial x_j$. We note from Lemma 1 and properties of $B(r)$ that the coefficients $a_{ij}$ are symmetric and are smooth off $S$; and that $a_{ij}$ are continuous on $\mathbb{R}^n$ if $\lim_{r \to 0} B(r) = n - 1$. Let
\[
B_k = \sup\{B(r): 0 < r \leq N_kr_k\},
\]
from (1.1) it follows that
\[
|\xi|^2 \leq \sum a_{ij}\xi_i\xi_j \leq \frac{B_k}{n - 1}|\xi|^2 \text{ on } R'_{k, l}.
\]

Next, we construct positive $L$-supersolutions.
Fix a point $x_0 \in S$ and rearrange the indices if necessary, we may assume that $x_0 \in \bigcap_k D_{k, 1}$. Let
\[
D_{k, 1} = \{|x - P_{k, 1}| \leq \frac{5}{4}r_k\}, \\
D''_{k, 1} = \{|x - P_{k, 1}| \leq \frac{3}{4}N_{k+1}r_{k+1}\}, \\
S_{k, 1} = D''_{k-1, 1} \setminus D'_{k, 1};
\]
and note that $D''_k, 1 \subseteq D_k, 1 \subseteq D'_k, 1 \subseteq D''_k, 1$ and $D'_k, 1 \subseteq D(P_k-1, \delta N_k r_k)$. Observe also that

$$\mathbb{R}^n \setminus \{x_0\} = \bigcup_{k \geq k_0} R'_k, 1 \cup \bigcup_{k \geq k_0 + 1} S_k, 1 \cup \{|x| \geq \frac{5}{4} r_k\};$$

and that $\bigcup_k R'_k, 1$ and $\bigcup_k S_k, 1$ meet on the boundaries only. Denote by

$$a_k = (a_k^1, a_k^2, \ldots, a_k^n) = P_k, 1 - P_{k-1}, 1,$$

$$x_k = (x_k^1, x_k^2, \ldots, x_k^n) = x - P_{k-1}, 1,$$

then $|a_k| < \delta_k N_k r_k$ and $|x_k - a_k| \leq N_k r_k$ if $x_k \in S_k, 1$.

Applying Lemma 2 to $D'_k, 1$ and $D''_k, 1$ instead of $D(0, \frac{3}{4})$ and $D(a, r)$ for each $k \geq 1 + k_0$ in succession, we obtain, after a scale change, a diffeomorphism $T$ from $\mathbb{R}^n \setminus \{x_0\}$ onto $\mathbb{R}^n \setminus \{0\}$ so that $T$ fixes every point in $\{|x| > \frac{3}{4} N_k r_{k+1}\}$, and is a translation on $R'_k, 1$ for each $k \geq 1 + k_0$ with

$$T(R'_k, 1) = \{\frac{3}{4} N_{k+1} r_{k+1} \leq |y| \leq \frac{5}{4} r_k\};$$

and that for $x \in S_k, 1,$

$$\left| \frac{\partial y_i}{\partial x_j} - \delta_{ij} \right| \leq 32|a_k^i| |x_k^j - a_k^j|(N_k r_k)^{-2} \leq 32 \delta_k,$$

$$\sum_j \left| \frac{\partial^2 y_j}{\partial x_i \partial x_m} \xi_j \right| \leq 272 \delta_k |\xi|(N_k r_k)^{-1},$$

and

$$T(S_k, 1) = \{\frac{5}{4} r_k < |y| \leq \frac{3}{4} N_k r_k\}.$$

Let $T(x_0) = 0$ and note that $T$ is homeomorphic on $\mathbb{R}^n$.

Let $M$ be the operator on $\mathbb{R}^n \setminus \{0\}$ defined by $Mv(y) = L(v \circ T)(x)$ when $y = Tx$; that is,

$$M = \sum_{i,j} b_{ij} \frac{\partial^2}{\partial y_i \partial y_j} + \sum_j b_j \frac{\partial}{\partial y_j},$$

with

$$b_{ij} = \sum_{l,m} a_{lm} \frac{\partial y_i}{\partial x_l} \frac{\partial y_j}{\partial x_m} \text{ and } b_j = \sum_{l,m} a_{lm} \frac{\partial^2 y_j}{\partial x_l \partial x_m}.$$

Thus $M$ is the Laplacian on $\{|y| \geq \frac{3}{4} r_{k_0}\}$. Since $T$ is a translation on $R'_k, 1$,

$$M = \frac{B(\rho)}{\rho^{n-1}} \Delta - \left( \frac{B(\rho)}{\rho^{n-1}} - 1 \right) \frac{\partial^2}{\partial \rho^2} \text{ on } T(R'_k, 1),$$

where $\rho = |y|$. In view of (1.5) and (2.5), we obtain after a scale change that for $x \in S_k, 1$,

$$\left| \sum_{i,j} b_{ij}(Tx) \xi_i \xi_j - \sum a_{ij}(x) \xi_i \xi_j \right| \leq 128 \delta_k \sup_{S_k, 1} \sum a_{ij}(x) \xi_i \xi_j \leq \frac{128}{n-1} B_k \delta_k |\xi|^2.$$
The last inequality follows from (2.3) and the fact that $S_{k,1}$ contains rings from $\{R'_{k,l}\}$ but none from the larger ones $\{R'_{k-1,l}\}$. Similarly it follows from (1.6) and (2.5) that

$$\sum b_{ii}(Tx) - \sum a_{ii}(x) \leq \frac{128}{n-1} B_k \delta_k \quad \text{on } S_{k,1}.$$  

Let $f$ be a smooth function on $r > 0$, bounded above by $B(r)$, with values $f(r) \equiv 0$ for $r \geq \frac{3}{4} r_k$, $f(r) = B(r)$ on $\bigcup_{k \geq k_0} [N_{k+1}r_{k+1}, r_k]$, 

$$f(r) = \left( n - 2 + \frac{n-1}{B_k} \right) \left( 1 - 1200 B^* \delta_k \right) \quad \text{on } \left[ \frac{3}{4} r_k, \frac{3}{4} N^* r_k \right],$$

for each $k \geq 1 + k_0$, and monotone in each of the remaining intervals. Define for $\rho = |y| < 1$, 

$$u(y) \equiv u(\rho) \equiv \int_{\rho}^1 \exp \int_t^1 \frac{f(s)}{s} ds dt,$$

and claim that 

$$Mu \leq 0 \quad \text{in } D\{0\}.$$ 

The idea of defining a radial $M$-supersolution in the form (2.11) comes from Gilbarg and Serrin [5] and Bauman [4]. It follows from (2.7) and the fact that $f(r) \leq B(r)$ that $Mu \leq 0$ on $\{ \frac{3}{4} r_k < |y| < 1 \} \cup \bigcup_k T(R'_{k,1})$. On $T(S_{k,1})$, we note that 

$$Mu(y) = \frac{u'(\rho)}{\rho} \left[ - \sum_{i,j} b_{ij} \frac{y_i y_j}{\rho^2} f(\rho) + \sum_i b_{ii} - \sum_{i,j} b_{ij} \frac{y_i y_j}{\rho^2} + \sum_j b_j y_j \right].$$

Eigenvalues of $(a_{ij}(x))$ are in the form $1, \Lambda(x), \Lambda(x), \ldots, \Lambda(x)$, with 

$$1 \leq \Lambda(x) \leq \frac{B_k}{n-1} \quad \text{on } S_{k,1}.$$ 

We obtain from (2.6) that 

$$\left| \sum b_j y_j \right| = \left| \sum_{l,m} a_{lm} \sum_j \frac{\partial^2 y_j}{\partial x_l \partial x_m} y_j \right| \leq 272 \delta_k B_k \frac{n}{n-1}$$

on $T(S_{k,1})$. For $x \in S_{k,1}$, $y = Tx$, and $\rho = |y|$, we obtain from (2.8), (2.9), (2.10), (2.13) and the assumptions $B(r) \leq B^*$ and $\delta_k < (2400 B^*)^{-1}$ that 

$$\frac{\sum b_{ii} + \sum b_j y_j}{\sum b_{ij} \frac{y_i y_j}{\rho^2} - 1} \geq \frac{\sum a_{ii} - \frac{128}{n-1} B_k \delta_k - 272 \frac{n}{n-1} B_k \delta_k}{\sum a_{ij} \frac{y_i y_j}{\rho^2} + \frac{128}{n-1} B_k \delta_k} - 1 \geq 1 + (n-1) \Lambda(x) - \left( \frac{128}{n-1} + 272 \frac{n}{n-1} \right) \frac{B_k \delta_k}{\Lambda(x) + \frac{128}{n-1} B_k \delta_k} - 1$$

$$\geq \left( n - 2 + \frac{n-1}{B_k} \right) \left( 1 - 1200 B^* \delta_k \right) \geq f(|y|).$$

Hence $Mu \leq 0$ on $T(S_{k,1})$, and (2.12) is proved.
Let \( H_{x_0}(x_0) = +\infty \) and
\[
H_{x_0}(x) = u(Tx) \quad \text{on } D \setminus \{x_0\}.
\]
Since \( LH_{x_0} \leq 0 \) on \( D \setminus \{x_0\} \) and the coefficients of \( L \) are smooth off \( S \), \( H_{x_0} \) is an \( L \)-supersolution in \( D \setminus S \). We shall estimate the growth of \( H_{x_0}(x) \) near \( x_0 \).

In the rest of the paper, \( C \) denotes positive constants depending at most on \( n \), \( \alpha \) and \( a \) in the theorems; its value may vary from line to line.

3. Proof of Theorem 1

Let
\[
B = \frac{2(n-1) + 2n(\alpha + 2-n)}{n-\alpha},
\]
and note that \( B > \alpha + 1 > n - 1 \), \( B \to n - 1 \) as \( \alpha \to n - 2 \), and
\[
\alpha \frac{B - (1 + \alpha)}{n} < \frac{B - (1 + \alpha')}{B - (n-2) - (n-1)/B} < 1.
\]
Choose \( \alpha' \), \( \alpha < \alpha' < n \), so that
\[
\alpha \frac{B - (1 + \alpha')}{n} < \frac{B - (1 + \alpha')}{B - (n-2) - (n-1)/B} < 1,
\]
and denote by
\[
A = \frac{B - (1 + \alpha')}{B - (n-2) - (n-1)/B} \quad \text{and} \quad E = A - \frac{\alpha}{n}.
\]

Let
\[
\delta_k = \frac{16\sqrt{n}}{k}, \quad N_k = k^{A/E} \quad \text{and} \quad r_k = (k!)^{-1/E}
\]
and note that \( N_k \leq r_{k-1}/r_k \) for \( k \geq 1 \). To specify \( B(r) \) in (2.1), we let
\[
B(r) \equiv B \quad \text{on } \bigcup_{k \geq k_0} [N_{k+1}r_{k+1}, r_k];
\]
and note that \( n - 1 \leq B(r) \leq B \). Choose and fix integer \( k_0 \geq 105B\sqrt{n} \).

It is clear that
\[
\lim_{k \to \infty} (k!)^{\alpha/E}(\delta_k N_k r_k)^\gamma = 0 \quad \text{if } \gamma > \alpha;
\]
and that
\[
\lim_{k \to \infty} (2^{-k}(k-1)!)^{-\alpha/E}/(\delta_k N_k r_k)^\eta = 0, \quad \text{if } \eta < \alpha.
\]
Note that there are \( I_k = \prod_{j=k_0}^{k} [j^{A/E-1}] \) balls in \( \{D_{k,l}\}_l \) and that
\[
\left( \frac{k!}{2^k k_0!} \right)^{\alpha/E} \leq I_k \leq k!^{\alpha/E}
\]
when \( k \) is large. And recall that \( \mu \) is the continuous measure on \( S \) defined by
\[
\mu(D_{k,l}) = I_k^{-1} \quad \text{for each } l \text{ and } k \geq k_0.
\]
The smallest balls that carry a $\mu$-measure $I_{k-1}^{-1}$ have radii proportional to $\delta_k N_k r_k$. Using (3.5) and (3.6) one may check that for each $\eta < \alpha$, $\mu(D(x, r)) \leq C_r^n$ for all $x \in \mathbb{R}^n$ and $r > 0$. Hence $S$ has positive $\eta$-dimensional measure for all $\eta < \alpha$. In view of (3.4), $S$ has Hausdorff dimension $\alpha$.

For $0 < t < r_k$, let $K = K(t)$ be the largest integer so that $r_K \geq t$. We deduce from (2.10), (2.11), and (3.3) that, for $0 < \rho < r_k$, 

$$u(\rho) \geq \int_\rho^{r_k} \exp \left\{ - \int_t^{r_k} \frac{B}{S} ds \right\} dt.$$ 

And note from the choices of $A$ and $\delta_k$ that 

$$\sum_{k=k_0}^{K} \int_{r_k}^{r_{k+1}} B - \left( n - 2 + \frac{n-1}{B} \right) (1 - 1200B\delta_k)^{dS}$$ 

$$\leq \sum_{k=k_0}^{K} \left[ \left( B - n + 2 - \frac{n-1}{B} \right) + 1200B\delta_k(n-1) \right] \frac{A}{E} \log k$$ 

$$\leq \sum_{k=k_0}^{K} \frac{(B - \alpha' - 1)}{E} \log k + C \frac{\log k}{k}$$ 

$$= C + (B - \alpha' - 1) \log \frac{1}{t} + C \left( \log \log \frac{1}{t} \right)^2.$$ 

Therefore, for $0 < \rho < r_k$, 

$$u(\rho) \geq \int_\rho^{r_k} \exp \left\{ -C + (\alpha' + 1) \log \frac{1}{t} - C \left( \log \log \frac{1}{t} \right)^2 \right\} dt \geq C \rho^{-\gamma}$$ 

for some $\gamma$ satisfying $\alpha < \gamma < \alpha'$. From the property (2.5) of the transformation $T$, it follows that 

$$H_{x_0}(x) \geq C|x - x_0|^{-\gamma} \text{ when } |x| < r_k.$$ 

Let $v(x) = \int_S H_z(x) d\mu(z)$, where $\mu$ is the measure defined in (3.7). Clearly $v$ is an $L$-supersolution on $D \setminus S$. In view of (3.4), (3.6), and (3.8), $v$ approaches $+\infty$ as $x \to x_0$ for every $x_0 \in S$. Since $f(r)$ is bounded, $v < +\infty$ on $D \setminus S$.

Clearly (0.2) holds with $\Lambda_{n, \alpha} = B/(n-1)$.

The number $B$ was chosen so that among other properties, (3.1) holds. As a consequence, (0.3) and (0.4) follow.

### 4. Proof of Theorem 2 $(n > 3)$

Let 

$$\delta_k = \frac{16\sqrt{n}}{k^{3/2}}, \quad N_k = k^{2n-5/2}$$

and 

$$r_k = (k!)^{-2n+(4+(2+a)/(n-2))/k}$$

for $k \geq 1$. Choose an integer $k_0 \geq 10^5(1 + a)n$, so that $N_k \leq r_{k-1}/r_k$ for $k \geq k_0$. It is easy to check that

$$\lim_{k \to \infty} (k!)^{2n(n-2)}r_k^\gamma = 0 \quad \text{if } \gamma > n - 2,$$

$$\lim_{k \to \infty} (k - 1)!^{2n(n-2)}/(\delta_k N_k r_k)^{n-2} \left(\log \frac{1}{\delta_k N_k r_k}\right)^{-1-a} = 0,$$

$$\lim_{k \to \infty} (k - 1)!^{2n(n-2)}/(\delta_k N_k r_k)^{n-2} \left(\log \frac{1}{\delta_k N_k r_k}\right)^{-3-a} = \infty.$$

There are $(k!/k_0!)^{2n(n-2)}$ balls in $\{D_{k,i}\}$, and that

$$\mu(D_{k,i}) = (k_0!/k!)^{2n(n-2)}$$

for $k \geq k_0$. In view of (4.2) and (4.3), $S$ has Hausdorff dimension $n - 2$ and positive $h$-measure, where $h(r) = r^{n-2}(\log \frac{1}{r})^{-1-a}$.

Let

$$\beta = 12n(1 + a),$$

and $B(r)$ be the function described in §2, with

$$B(r) = n - 1 + \frac{\beta}{\log \frac{1}{r}} \quad \text{on} \quad \bigcup_{k \geq k_0} [N_k r_k, r_k].$$

Clearly $n - 1 \leq B(r) \leq n - 1 + \beta/k_0 \equiv B^*$, and $\delta_k < (2400B^*)^{-1}$ for $k \geq k_0$.

Stirling’s formula shows that

$$\left(\log \frac{1}{r_{k-1}}\right)^{-1} \leq \frac{1}{2nk \log k} \left(1 + \frac{C}{\log k}\right),$$

when $k$ is sufficiently large. Since $(\log \frac{1}{s})^{-1}$ is an increasing function of $s$ ($0 < s < 1$), and $r_{k-1} \geq N_k r_k$, we obtain from (4.7) that

$$1 - n - 1 = \frac{B_k - (n - 1)}{B_k} \leq \frac{\beta}{(n - 1) \log 1/r_{k-1}}$$

$$\leq \frac{\beta}{2n(n-1)k \log k} \left(1 + \frac{C}{\log k}\right)$$

for large $k$, here $B_k$ is the number defined in (2.3).

Thus, for $0 < \rho < r_{k_0}$,

$$u(\rho) \geq \int_{\rho}^{r_{k_0}} \exp \left\{ \int_{\rho}^{r_{k_0}} \frac{n - 1}{S} + \frac{\beta}{s \log \frac{1}{s}} ds - \sum_{k = k_0}^{K} \int_{r_k}^{N_k r_k} \frac{\beta}{s \log \frac{1}{s}} ds \right. - \left. \sum_{k = k_0}^{K} \int_{r_k}^{N_k r_k} (n - 1) - \left(n - 2 + \frac{n - 1}{B_k}\right) (1 - 1200B^* \delta_k) \frac{ds}{s} \right\} dt,$$

where $K = k(t)$ is the largest integer satisfying $r_K \geq t$. We deduce from (4.8)
that
\[
\sum_{k=k_0}^{K} \int_{r_k}^{r_{k+1}} \left( n - 1 - \frac{n - 1}{B_k} \right) \left( 1 - 1200B_k \delta_k \right) \frac{ds}{s}
\]
\[
\leq \sum_{k=k_0}^{K} \left[ \frac{\beta}{2n(n-1)k} \log k \left( 1 + \frac{C}{\log k} \right) + Ck^{-3/2} \right] \log N_k
\]
\[
\leq C + \frac{\beta(2n - \frac{5}{2})}{2n(n-1)} \log K + C \log \log K
\]
\[
\leq C + \frac{\beta(2n - \frac{5}{2})}{2n(n-1)} \log \log \frac{1}{t} + C \log \log \log \frac{1}{t}.
\]
Again, from (4.7) and monotonicity of \((\log \frac{1}{t})^{-1}\), it follows that
\[
\sum_{k=k_0}^{K} \int_{r_k}^{r_{k+1}} \frac{\beta}{s \log \frac{1}{t}} ds \leq \frac{\beta(2n - \frac{5}{2})}{2n} \sum_{k=k_0}^{K} \left( 1 + \frac{C}{\log k} \right) / k
\]
\[
\leq C + \frac{\beta(2n - \frac{5}{2})}{2n} \log \log \frac{1}{t} + C \log \log \log \frac{1}{t}.
\]
We conclude from (4.6), (4.9), (4.10) and (4.11) that
\[
u(p) \geq \int_{\rho}^{r_k_0} \exp \left\{ -C + (n-1) \log \frac{1}{t} \right\}
\]
\[
+ \beta \left[ 1 - \frac{2n - \frac{5}{2}}{2n} \left( 1 + \frac{1}{n-1} \right) \right] \log \log \frac{1}{t}
\]
\[
\geq C \int_{\rho}^{r_k_0} t^{-n+1} \left( \log \frac{1}{t} \right)^{\beta/4(n-1)} \left( \log \log \frac{1}{t} \right)^{-C} dt
\]
\[
\geq C \rho^{-n+2} \left( \log \frac{1}{\rho} \right)^{3+a}
\]
for \(0 < \rho < r_k_0\). Therefore,
\[
H_{x_0}(x) \geq C|x - x_0|^{-n+2} \left( \log \frac{1}{|x - x_0|} \right)^{3+a} \quad \text{for} \quad |x| < r_k_0.
\]
The relation
\[
\frac{2n - \frac{5}{2}}{2n} \left( 1 + \frac{1}{n-1} \right) < 1
\]
used in (4.12) is prepared in the choices of \(r_k\) and \(N_k\).

The ellipticity of \(a_{ij}, (0.5)\), follows from the choice of \(B(r)\); and the continuity of \(a_{ij}\) in \(\mathbb{R}^n\) follows from \(\lim_{r \to 0} B(r) = n - 1\). Recall that \(LH_{x_0} \leq 0\) on \(D \setminus \{x_0\}\); it follows from the maximum principle and the solvability of the Dirichlet problem for operators with continuous coefficients [5, pp. 220 and 252] and \(H_{x_0}\) is \(L\)-supersolution in \(D\).

Again, because \(a_{ij}\) are continuous, Green functions \(G(x, x_0)\) exist in \(D\) (see Bauman [3, 4]). In fact, for each \(x_0 \in D\), \(G(\cdot, x_0)\) is a positive \(L\)-solution.
in $D \setminus \{x_0\}$ with boundary value vanishing continuously on $|x| = 1$. Let $\vec{x} = (\frac{1}{4}, 0, 0, 0, \ldots, 0)$ and assume that $G$ is normalized so that $G(\vec{x}, x_0) = 1$. We claim that for each $x_0 \in S$,

$$G(x, x_0) \geq C|x - x_0|^{-n + 2} \left(\frac{1}{|x - x_0|}\right)^{3+a}$$

whenever $0 < |x - x_0| < r_{k_0}$.

Let $g(r) = \sup\{G(x, x_0) : |x - x_0| = r\}$ for $0 < r < r_{k_0}$. Applying (4.13) and the maximum principle to the region $D \setminus \{|x - x_0| \leq r\}$, we obtain

$$1 = G(\vec{x}, x_0) \leq C g(r) r^{n-2} (\log \frac{1}{r})^{-3+a} H_{x_0}(\vec{x}).$$

Because $f(r)$ is bounded and $|x_0 - \vec{x}| > \frac{1}{4}$, $H_{x_0}(\vec{x}) < C < \infty$ for all $x_0 \in S$. Hence $g(r) \geq C r^{-n+2} (\log \frac{1}{r})^{3+a}$ for $0 < r < r_{k_0}$. Thus (4.14) follows from the Harnack principle.

In view of (4.14), the maximum principle and the solvability of the Dirichlet problem, $G(\cdot, x_0)$ is actually an $L$-supersolution on $D$. The function $v(x) = \int_S G(x, z) d\mu(z)$ approaches $+\infty$ on $S$ in view of (4.4) and (4.14), and it is the function desired.

5. Proof of Theorem 2 ($n = 2$)

Let $\delta_k \equiv \delta \equiv \frac{50000(1+a)^2}{N^4}$, $N_k \equiv N \equiv 16000000(1+a)^2$, $r_k \equiv e^{-\frac{4}{1+a}}^{1+a}/32\sqrt{2}$ for $k \geq 1$. Choose integers $k_0 \geq 1 + a$ so that

$$r_{k-1}/r_k \geq N^4 \text{ when } k \geq k_0.$$ 

Note that

$$4^{-k} \left(\frac{1}{N\delta r_k}\right)^{1+a} = 1.$$ 

We note that there are $4^{k-k_0}$ disks in $\{D_{\delta, \epsilon}\}$, and that $\mu(D_{\delta, \epsilon}) = 4^{-k+k_0}$. In view of (5.2), $S$ has positive finite $h$-measure for $h(r) = (\log \frac{1}{r})^{-1-a}$.

Choose

$$B(r) = 1 + 20(1+a)^2 \left(\frac{1}{r}\right)^{-1} \text{ on } \bigcup_{k \geq k_0} [N_{k+1}r_{k+1}, r_k].$$ 

Clearly $1 \leq B(r) \leq 6(1+a)^2$. Let $f(r)$ be the function in §2, satisfying all the properties there except (2.10); instead, let $f(r) \equiv 0$ on $\bigcup_{k \geq k_0} [\frac{1}{4} r_k, \frac{3}{4} N_k r_k]$. The fact that $Mu \leq 0$ in $D \setminus \{0\}$ is not affected by the change of $f(r)$ due to the estimate (2.14).

For $0 < \rho < r_{k_0}$,

$$u(\rho) \geq \int_\rho^{r_{k_0}} \exp \left\{ \int_t^{r_{k_0}} \frac{1}{s} + \frac{20(1+a)^2}{s \log \frac{1}{s}} ds \right\} dt,$$

$$- \sum_{k=k_0}^{K} \int_{r_k}^{N_k r_k} \frac{1}{s} + \frac{20(1+a)^2}{s \log \frac{1}{s}} ds \right\} dt,$$
where $K = K(t)$ is the largest integer so that $r_K \geq t$. We deduce from (5.1) that
\[
\sum_{k=k_0}^{K} \int_{r_k}^{N_{r_k}} \frac{1}{s \log \frac{1}{s}} \, ds \leq C + \frac{1}{4} \int_{t}^{r_{k_0}} \frac{1}{s \log \frac{1}{s}} \, ds \leq C + \frac{1}{4} \log \log \frac{1}{t},
\]
and that
\[
\sum_{k=k_0}^{K} \int_{r_k}^{N_{r_k}} \frac{1}{s} \, ds = (K - K_0) \log N \leq C + 14(1 + a)^2 \log \log \frac{1}{t}.
\]
Combining the above estimates, we obtain
\[
u(\rho) \geq \int_{\rho}^{1} \exp \left\{ C + \log \frac{1}{t} + (1 + a)^2 \log \log \frac{1}{t} \right\} \, dt \geq C \left( \log \frac{1}{\rho} \right)^{1+(1+a)^2}
\]
for $0 < \rho < r_{k_0}$.

In view of (5.3), $a_{ij}$ are continuous in $D$. Thus the normalized Green function exists on $D$ and satisfies
\[
G(x, x_0) \geq C \left( \log \frac{1}{|x-x_0|} \right)^{1+(1+a)^2} \quad \text{for } |x-x_0| < r_{k_0}.
\]
The function $v(x) = \int_{S} G(x, y) \, d\mu(y)$ has all the properties in the theorem.

6. PROOFS OF THEOREMS 3 AND 4

We follow the constructions in §2 and indicate the necessary changes.

Given $B^* = n - 1$, $k_0$, $\{\delta_k\}$, $\{r_k\}$ and $\{N_k\}$, let $S$ be the Cantor set and $\mu$ be the measure on $S$ defined in §2.

Let $B(r)$ be a new function, smooth on $r > 0$, with values $\frac{1}{2} < B(r) \leq n - 1$, satisfying (2.1),
\[
B(r) < n - 1 \quad \text{on} \quad \bigcup_{k \geq k_0} [N_{k+1} r_{k+1}, r_k],
\]
and monotone in each of the remaining intervals. Define an operator $L$ associated with this $B(r)$ as in (2.2). Let
\[
\beta_k = \inf\{B(r): 0 < r \leq N_k r_k\},
\]
then
\[
\frac{\beta_k}{n-1} |\xi|^2 \leq \sum_{i,j} a_{ij} \xi_i \xi_j \leq |\xi|^2 \quad \text{on} \quad R_{k,l}.
\]

Fix $x_0 \in S$, let $y = Tx$ be the diffeomorphism and $M$ be the operator defined before. Clearly (2.5) ~ (2.7) are retained; and (2.8) and (2.9) can be replaced respectively by
\[
\left| \sum_{i,j} b_{ij}(Tx) \xi_i \xi_j - \sum_{i,j} a_{ij}(x) \xi_i \xi_j \right| \leq 128 \delta_k |\xi|^2 \quad \text{on} \quad S_{k,1},
\]
and
\[
\left| \sum_{i,j} b_{ij}(Tx) \xi_i \xi_j - \sum_{i,j} a_{ij}(x) \xi_i \xi_j \right| \leq 128 \delta_k \quad \text{on} \quad S_{k,1}.
\]

Suppose that $F$ is a smooth function on $r > 0$, with values $F(r) \geq B(r)$, $F(r) \equiv n - 1$ for $r \geq \frac{3}{4} r_{k_0}$, $F(r) = B(r)$ on $\bigcup_{k \geq k_0} [N_{k+1} r_{k+1}, r_k]$, $F(r) = \left( n - 2 + \frac{n-1}{\beta_k} \right) (1 + 5000 \delta_k)$ on $[\frac{3}{4} r_k, \frac{3}{4} N_k r_k]$. 

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for each \( k \geq 1 + k_0 \), and that \( F \) is monotone in each of the remaining intervals. Define for \( \rho = |y| < 1 \),

\[
U(y) = U(\rho) = \int_\rho^1 \exp \int_t^1 \frac{F(s)}{s} \, ds \, dt.
\]

Arguing as in §2, we conclude that for \( x \in S_{k,1} \), and \( y = Tx \),

\[
\sum b_{ii} + \sum b_{ij} y_j - \frac{1}{\rho^2} \leq f(|y|).
\]

From this, we may deduce that \( MU(y) \geq 0 \) on \( \{|y| < 1\}\{\{0\} \). Thus,

\[
Q \psi(x) \equiv U(Tx) \quad \text{on} \ D \setminus \{x_0\}
\]

is an \( L \)-subsolution in \( D \setminus S \).

To complete the proof of Theorem 3, we let \( \delta_k \) and \( N_k \) be the numbers defined in (4.1), let \( \tau > a/(n-2) \) and \( r_k = (k!)^{-2n-\tau^k} \). Fix an integer \( k_0 \geq 20(n^2 + \tau^2) \), so that \( N_k \leq r_{k-1}/r_k \) and \( \delta_k \leq (2400n)^{-1} \) for \( k \geq k_0 \). It is ready to check that

\[
\lim_{k \to \infty} (k!)^{2n(n-2) \rho^{n-2}} \left( \log \frac{1}{r_k} \right)^a = 0,
\]

\[
\lim_{k \to \infty} ((k-1)!)^{-2n(n-2)} (\delta_k N_k r_k)^{-\eta} = 0 \quad \text{if} \ \eta < n-2,
\]

and

\[
\sum_{k \geq k_0} ((k-1)!)^{-2n(n-2)} r_{k-1}^{n-2} \left( \log \frac{1}{r_k} \right)^{-2n(n+\tau)} < \infty.
\]

There are \( (k!/k_0!)^{2n(n-2)} \) balls in \( \{D_{k,i}\}_t \) for each \( k \geq k_0 \), and \( \mu(D_{k,i}) = (k_0!/k!)^{2n(n-2)} \). From (6.2) and (6.3) it follows that \( S \) has Hausdorff dimension \( n-2 \), and zero \( h \)-measure for \( h(r) = r^{-2}(\log 1/r)^a \).

Let

\[
\beta = 16n^2(n+\tau),
\]

and

\[
B(r) = n - 1 - \frac{\beta}{\log \frac{1}{r}} \quad \text{on} \ \bigcup_{k \geq k_0} [N_{k+1}r_{k+1}, r_k].
\]

Thus, for \( 0 < \rho < r_{k_0} \),

\[
U(\rho) \leq \int_\rho^1 \exp \left\{ \int_t^1 \frac{n-1}{s} - \frac{\beta}{s \log \frac{1}{s}} \, ds + C \sum_{k=k_0}^K \int_{r_k}^{N_k r_k} \frac{\beta}{s \log \frac{1}{s}} \, ds 
\right.
\]

\[
\left. + \sum_{k=k_0}^K \int_{r_k}^{N_k r_k} \left[ -(n-1) + \left( n-2 + \frac{n-1}{\beta_k} \right)(1+5000\delta_k) \right] ds \frac{1}{s} \right\} \, dt,
\]

where \( K = K(t) \) is the largest integer satisfying \( r_K \geq t \). Note that for large \( k \), inequality (4.7) still holds and

\[
\frac{n-1}{\beta_k} - 1 \leq \frac{9\beta}{16n(n-1)k \log k}.
\]

Thus \( U(\rho) \leq C \rho^{-n+2}(\log \frac{1}{\rho})^{-\beta/8n} \).
Let $G(\cdot, x_0)$ be the normalized Green function on $D$ with
\[ G(\left(\frac{1}{2}, 0, 0, \ldots 0\right), x_0) = 1. \]
Arguing as in §4, we obtain
\begin{equation}
G(x, x_0) \leq C |x - x_0|^{-n+2} \left( \log \frac{1}{|x - x_0|} \right)^{-\beta/8n}
\end{equation}
for all $x_0 \in S$ and $|x - x_0| < r_k$. We may also prove that
\[ G(x, x_0) \to +\infty \quad \text{as } x \to x_0 \]
for each $x_0 \in S$, by constructing a positive $L$-supersolution approaching $+\infty$ at $x_0$. Thus $G(\cdot, x_0)$ is an $L$-supersolution on $D$. In view of (6.4), (6.5), and (6.6),
\[ w(x) = \int_S G(x, y) d\mu(y) \]
has all the properties stated in Theorem 3.

To prove Theorem 4, we need only to verify that the coefficients of $L$ can be chosen so that (0.6) and (0.7) are fulfilled. Let
\[ b = \begin{cases} 
1 - 2\alpha, & \text{if } 0 < \alpha < \frac{1}{4}, \\
(n - 1) \left( 1 - \frac{n - 2 - \alpha}{n - \frac{3}{2} - \alpha} \right), & \text{if } n - \frac{9}{4} < \alpha < n - 2,
\end{cases} \]
and note that $\frac{1}{2} < b < 1 + \alpha < n - 1$ and that
\[ \frac{\alpha}{n} < \frac{1 + \alpha - b}{n - 2 + (n - 1)/b - b} < 1. \]
Choose $\alpha', 0 < \alpha' < \alpha$ so that
\[ \frac{\alpha}{n} < \frac{1 + \alpha' - b}{n - 2 + (n - 1)/b - b} < 1, \]
and denote by
\[ A = \frac{1 + \alpha' - b}{n - 2 + (n - 1)/b - b}, \quad E = A - \frac{\alpha}{n}. \]
Let $\delta_k$, $N_k$ and $r_k$ be defined according to (3.2), associated with the current choices of $A$ and $E$; and let the function $B(r)$ in (6.1) be chosen so that
\[ B(r) \equiv b \quad \text{on } \bigcup_{k \geq k_0} [N_k + r_{k+1} - r_k]. \]
It is ready to check that $S$ has dimension $\alpha$ and that the $L$-subsolution $Q_{x_0}(x)$ satisfies
\[ Q_{x_0}(x) \leq C |x - x_0|^{-\gamma}, \quad \text{when } |x - x_0| < r_k \]
for some $\gamma$ with $\alpha' < \gamma < \alpha$. The rest of the proof is routine and follows from the observation
\[ \sum_{k \geq k_0} (k - 1)!^{-\alpha/E} (\delta_k N_k r_k)^{-\gamma} < \infty \quad \text{if } \gamma < \alpha. \]
References


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