COMPLETE NONORIENTABLE MINIMAL SURFACES IN $R^3$

TORU ISHIHARA

Dedicated to Professor Tadashi Nagano on his 60th birthday

Abstract. We will study complete minimal immersions of nonorientable surfaces into $R^3$. Especially, we construct a nonorientable surface $P_2$ which is homeomorphic to a Klein bottle and show that for any integer $m \geq 4$, there are complete minimal immersion of $M = P_2 - \{q\}, q \in P_2$ in $R^3$ with one end and total curvature $C(M) = -4mn$.

1. Introduction

At first, we will construct nonorientable surfaces of genus $\nu$ ($\nu \geq 1$). Let $T_{n-1}$ be a hyperelliptic Riemann surface given by

$$T_{n-1} = \left\{ (z, w) \in (\mathbb{C} \cup \{\infty\})^2, \ w^2 = \prod_{i=1}^{n} (a_i - z)(\bar{a}_i + z) \right\},$$

where, $a_i \neq a_j$ for any $i \neq j$, and $a_i \neq -\bar{a}_j$ for any $i, j$. Define a map $I: T_{n-1} \rightarrow T_{n-1}$ by $I(z, w) = (-\bar{z}, -\bar{w})$. Then, the map $I$ is an anti-holomorphic involution. Moreover it has no fixed point. In fact, let $(z, w)$ be a fixed point of $I$, that is, $z = ib$ and $w = ic$, where $b$ and $c$ are real numbers. Then, since $|a_i| > \Im(a_i)$, we have a contradiction:

$$-c^2 = \prod_{i=1}^{n}(|a_i|^2 + b^2 - 2\Im(a_i)b) > 0.$$ 

Let $P_n$ be the quotient space of $T_{n-1}$ by the equivalence relation defined by $(z_1, w_1) \sim (z_2, w_2) \iff (z_2, w_2) = I(z_1, w_1)$. Then the canonical projection $\pi: T_{n-1} \rightarrow P_n$ is the two-sheeted covering and $P_n$ is a nonorientable surface of genus $\nu$. We may consider $P_1$ as the projective plane and $P_2$ as the Klein bottle.

In the present paper, we will study minimal immersions of $M = P_n - \{q_1, \ldots, q_r\}$ into $R^3$, where $q_1, \ldots, q_r \in P_n$. Then we have the orientable double covering $\tilde{\pi}: \tilde{M} \rightarrow M$, where $\tilde{M} = T_{n-1} - \{p_1, I(p_1), \ldots, p_r, I(p_r)\}$ and $\tilde{q}_1 = \pi(p_1), \ldots, \tilde{q}_r = \pi(p_r)$. It is shown in [7] that for a complete minimal immersion $\tilde{x}: \tilde{M} \rightarrow R^3$ if and only if $\tilde{x}(I(p)) = \tilde{x}(p)$ for all $p \in \tilde{M}$. In this case, $\tilde{x}: \tilde{M} \rightarrow R^3$ is called the double surface of $x: M \rightarrow R^3$. The points $q_i$'s and also $p_i$'s, $I(p_i)$'s

Received by the editors February 13, 1990 and, in revised form, August 1, 1990.

1980 Mathematics Subject Classification (1985 Revision). Primary 53A10; Secondary 53C42.

This work was partially supported by Grant-in-Aid for Scientific Research (No. 65440040).
are called the endpoints of the immersions. We need the following representation theorem of Meeks [7] and Oliveira [8] which is followed from the classical Enneper-Weierstrass representation theorem.

**Proposition 1.1** (Meeks [7] and Oliveira [8]). Let \( g: \tilde{M} \to C \cup \{ \infty \} \) be a meromorphic function and \( \eta \) be a holomorphic 1-form on \( \tilde{M} \). Take the vector-valued 1-form

\[
\Phi = (\phi_1, \phi_2, \phi_3) = ((1 - g^2)\eta, i(1 + g^2)\eta, 2g\eta).
\]

Assume that no component of \( \Phi \) has a real period and that the poles of \( g \) coincide with zeros of \( \eta \) and the order of a pole of \( g \) is precisely half the order of a zero of \( \eta \). Then \( \tilde{x}(p) = \text{Re} \int_{p_0}^{p} \Phi \) is a conformal regular minimal immersion. Moreover \( \tilde{x}(I(p)) = \tilde{x}(p) \) for all \( p \in \tilde{M} \) if and only if \( I^*(\Phi) = \Phi \), in other words

\[
g(I(p)) = -1/g(p), \quad I^*(\eta) = -g^2 \eta, \quad \text{for } p \in M.
\]

In this case, there is a regular minimal immersion \( x: M \to \mathbb{R}^3 \) such that \( \tilde{x} = x \cdot n \).

Conversely, every regular minimal immersion \( x: M \to \mathbb{R}^3 \) can be given in this way.

The meromorphic function \( g \) in the above is the stereographic projection of the Gauss map of \( \tilde{x} \). In the case of finite total curvature, it is known from the Osserman's result [9], \( g \) is extended to a meromorphic function on \( T_{n-1} \). In the present paper, we will study only minimal immersions with finite total curvature. We will call \((g, \eta)\) in the above the Weierstrass pair of \( x \).

Particularly, we are concerned with elliptic Riemann surfaces and corresponding nonorientable surfaces of genus 2 with one end. Put

\[
T_1 = \{(z, w) \in (C \cup \{ \infty \})^2; w^2 = (1 - z^2)(1 - k^2 z^2)\}, \quad 0 < k < 1.
\]

Then \( T_1 \) is homeomorphic to a torus. The quotient space \( P_2 \) is homeomorphic to a Klein bottle. As for any two points \( p_1, p_2 \) of \( T_1 \), we have a conformal transformation which maps \( p_1 \) to \( p_2 \), we may assume that the endpoint is \( \pi(\infty, \infty) \), where \( \pi: T_1 \to P_2 \) is the natural projection. We will study complete minimal immersions of \( M = P_2 - \{ \pi(\infty, \infty) \} \). Our main result is the following

**Theorem.** Let \( M \) be a nonorientable surface \( P_2 - \{ q \} \) with one end \( q \). Then

1. there is no complete minimal immersion of \( M \) into \( \mathbb{R}^3 \) whose total curvature \( C(M) \) is greater than \(-14\pi\),
2. for any \( 0 < k < 1 \) and any integer \( m \geq 4 \), there are complete minimal immersions of \( M \) into \( \mathbb{R}^3 \) with \( C(M) = -4m\pi \).

### 2. A Weierstrass pair for a nonorientable surface

In this section, we will determine the forms of \( g \) and \( \eta \) introduced in the previous section. Since \( g \) is a meromorphic function on the hyperelliptic surface \( T_{n-1} \), it is represented as

\[
g(z, w) = (Rw + P)/Qw, \quad (z, w) \in T_{n-1},
\]

where \( R, P \) and \( Q \) are polynomials of \( z \). Put \( \eta = f(z, w)dz \). Then from Proposition 1.1 we have

\[
\overline{f(I(z, w))} = g(z, w)^2 f(z, w).
\]
Taking account of the conditions for a Weierstrass pair, we can put

\[ f(z, w) = \left( Q^2 w^2 / S \right) \times \left( 1 / w \right), \]

where \( S \) is a polynomial of \( z \) which vanishes only at the endpoints. We need the last factor \( 1 / w \), because the corresponding immersion is regular at points \((a_i, w(a_i))\) and \( I(a_i, w(a_i)), 1 \leq i \leq n \). Hence if these all points are endpoints, we may drop the factor \( 1 / w \). Substituting (2.1) and (2.3) into (2.2), we have

\[ S \tilde{Q}^2 w^2 + \tilde{S} Q^2 (R^2 w^2 + P^2) = -2 \tilde{S} \tilde{Q}^2 P R w, \]

where \( \tilde{S} = S(-z) \) and \( \tilde{Q} = Q(-z) \). The left-hand side of the above equation depends on \( z \) only. On the other hand, its right-hand side depends on \( z \) and \( w \). Hence they must vanish. As \( S \neq 0 \) and \( \tilde{Q} \neq 0 \), we have \( P = 0 \) or \( R = 0 \).

At first, we assume \( P = 0 \). Then we can put

\[ g = A \prod_{i=1}^{m} \left( z - d_i / \prod_{j=1}^{s} (z - c_j), \quad (c_i \neq d_j \text{ for all } i, j). \right. \]

Since \( g \) satisfies \( g(I(p)) = -1 / g(p) \), it follows

\[ |A|^2 (-1)^{s-m} \prod_{i=1}^{m} (z - d_i)(z + \bar{d}_i) = -\prod_{j=1}^{s} (z - c_j)(z + \bar{c}_j). \]

Hence it must be \( s = m \). But then we have \( |A|^2 = -1 \). Thus, we get \( R = 0 \) and \( P \neq 0 \). Now we can set

\[ g(z, w) = A \prod_{i=1}^{m} (z - b_i) / \left( w \prod_{j=1}^{m} (z - c_j) \right), \quad (b_i \neq c_j \text{ for all } i, j). \]

From the condition about \( g \), it follows

\[ (-1)^{t-m-n} |A|^2 \prod_{j=1}^{t} (z - b_j)(z + \bar{b}_j) = \prod_{k=1}^{m} (z - c_k)(z + \bar{c}_k) \prod_{i=1}^{n} (z - a_i)(z + \bar{a}_i). \]

Hence we get \( t = m + n \) and \( |A|^2 = 1 \). Since \( c_i \neq b_i \), if it is necessary, permuting the order of \( b_i \)’s and replacing \( a_i \) by \(-\bar{a}_i\), we may put \( c_i = -\bar{b}_i \), for \( 1 \leq i \leq m \), \( b_{m+j} = a_j \) for \( 1 \leq j \leq n \). Thus we obtain

\[ g(z, w) = A \prod_{j=1}^{n} (z - b_j) \prod_{i=1}^{m} (z - a_i) / w \prod_{j=1}^{m} (z + \bar{b}_j), \quad |A| = 1. \]

Let \( p_1 = (c_1, w_1), I(p_1), \ldots, p_r = (c_r, w_r), I(p_r) \) be the endpoints. If they contain the infinite point \((\infty, \infty)\), we may assume \( p_r = (\infty, \infty) \). We put \( r^* = r - 1 \) or \( r \) according to \( p_r = (\infty, \infty) \) or not. Now we will determine the form of \( \eta = f d z \). At first, we assume \( c_i \neq a_j \), \( c_i \neq -\bar{a}_j \) for all \( i, j \). As \( g \) is given by (2.5), we can put

\[ f = B \prod_{j=1}^{m} (z + \bar{b}_j)^2 \prod_{i=1}^{n} (z + \bar{a}_i) / w \prod_{k=1}^{r^*} (z - c_k)^{\alpha_k} (z + \bar{c}_k)^{\beta_k}. \]
Using (2.2), we get
\[
(\alpha_1 + \beta_1 + \cdots + \alpha_r + \beta_r + 1)B / \prod_{k=1}^{r^*} (z + c_k)^{\alpha_k} (z - \bar{c}_k)^{\beta_k}
\]
\[
= A^2 B / \prod_{k=1}^{r^*} (z - c_k)^{\alpha_k} (z + \bar{c}_k)^{\beta_k}.
\]

Hence we have \( \alpha_k = \beta_k \) when \( c_k \neq -\bar{c}_k \). We may assume that \( c_k \neq -\bar{c}_k \) for \( 1 \leq k \leq p \) and \( c_k = -\bar{c}_k \) for \( p + 1 \leq k \leq r^* \). Now we can modify \( f \) as
\[
(2.6) \quad f = B \prod_{j=1}^{m} (z + \bar{b}_j)^2 \prod_{i=1}^{n} (z + \bar{a}_i) / \left( \prod_{k=1}^{p} (z - c_k)^{\alpha_k} (z + \bar{c}_k)^{\alpha_k} \prod_{j=1}^{q} (z - d_j)^{\beta_j} \right),
\]
where \( p + q = r^* \), the \( d_j \)'s are zero or pure imaginary numbers and
\[
(2.7) \quad \overline{A B} = -(-1)^\beta A B, \quad \beta = \beta_1 + \cdots + \beta_q, \quad \alpha_k \geq 2, \quad \beta_j \geq 2 \quad \text{for} \quad 1 \leq k \leq p, \quad 1 \leq j \leq q.
\]
The above inequalities follow from the fact that the orders of the poles are greater than 1 (see [2, 8]).

When some of \( c_i \)'s coincide with some of \( a_i \)'s, we can show after all that \( f \) has the same form as (2.6). In order to study the behavior of \( \eta \) at \((\infty, \infty)\), we put \( z = 1/\xi \). Then we have, by putting \( \alpha = \alpha_1 + \cdots + \alpha_p \),
\[
\eta = (\xi)^{2\alpha + \beta - 2(m + 1)} B \prod_{j=1}^{m} (1 + \bar{b}_j \xi)^2 \prod_{i=1}^{n} (1 + \bar{a}_i \xi) / \left( \prod_{k=1}^{p} (1 - c_k \xi)^{\alpha_k} (1 + \bar{c}_k \xi)^{\alpha_k} \prod_{j=1}^{q} (1 - d_j \xi)^{\beta_j} d\xi \right).
\]
Thus, if \( p_r = (\infty, \infty) \), it is a pole of order \( \geq 2 \). Hence we have \( 2\alpha + \beta \leq 2m \). On the other hand, if all endpoints are finite, as an infinite point \((\infty, \infty)\) is neither a pole nor a zero, we get \( 2\alpha + \beta = 2(m + 1) \). Now we obtain

**Lemma 2.1.** A Weierstrass pair \((g, \eta = f dz)\) on \( \tilde{M} \) which constructs a nonorientable complete minimal immersion has the form (2.5) and (2.6). If one of the endpoints is an infinite point \((\infty, \infty)\), we have \( 2\alpha + \beta \leq 2m \). If all endpoints are finite points, we have \( 2\alpha + \beta = 2(m + 1) \).

### 3. The conditions under which \( \Phi \) has no real period

The vector-valued 1-form \( \Phi \) for a Weierstrass pair given by (2.5) and (2.6) is given by
\[
(3.1) \quad \Phi = (B \psi_1 + \overline{B} \psi_2, i(B \psi_1 - \overline{B} \psi_2), (-1)^n 2AB \psi_3), \quad \overline{A B} = (-1)^{\beta+1} A B,
\]
where

\[
\psi_1 = \left( \prod_{j=1}^{m} (z + b_j)^2 \prod_{j=1}^{n} (z + a_j)/(Rw) \right) dz,
\]

\[
\psi_2 = (-1)^{n+\beta} \left( \prod_{j=1}^{m} (z - b_j)^2 \prod_{j=1}^{n} (z - a_j)/(Rw) \right) dz,
\]

\[
\psi_3 = \left( \prod_{j=1}^{m} (z - b_j)(z + \bar{b}_j)/R \right) dz,
\]

\[
R = \prod_{i=1}^{p} ((z - c_i)(z + \bar{c}_i)) \prod_{j=1}^{q} (z - d_j)^{\beta_j}.
\]

We must determine \( b_j \) such that \( \Phi \) has no real period. The existence of real periods must be searched among the cycles that generate the fundamental group of \( \tilde{M} \). These are the ones that generate the fundamental group of \( T_{n-1} \) and the ones around the endpoints, \( p_1, \ldots, p_r \). Let \( \tilde{\beta}_1(t), 0 \leq t \leq 1 \), be a curve connecting \(-a_1\) to \( a_1 \) and \( \tilde{\beta}_i(t), 0 \leq t \leq 1 \), be curves connecting \( a_{i-1} \) to \( a_i \) for \( 2 \leq i \leq n \). Let

\[
\gamma_i(t) = \begin{cases} 
\beta_i(t) = (\tilde{\beta}_i(2t), w(\tilde{\beta}_i(2t))), & \text{for } 0 \leq t \leq 1/2, \\
\beta_i^*(t) = (\tilde{\beta}_i(2 - 2t), -w(\tilde{\beta}_i(2 - 2t))), & \text{for } 1/2 \leq t \leq 1.
\end{cases}
\]

Then, \( \gamma_1, \ldots, \gamma_n, I(\gamma_2), \ldots, I(\gamma_{n-1}) \) generate the fundamental group of \( T_{n-1} \).

The condition (3.1) implies \( \int_{I(\gamma_i)} \phi_i = \int_{\gamma_i} I^* (\phi_i) = \int_{\gamma_i} \overline{\phi}_i \), for \( 1 \leq i \leq 3 \). As \( \phi_i \) contain the factor \( 1/w \) for \( i = 1, 2 \), we have

\[
\int_{\gamma_i} \phi_i = \int_{\beta_i} \phi_i + \int_{\beta_i^*} \phi_i = 2 \int_{\beta_i} \phi_i, \quad \text{for } i = 1, 2.
\]

Similarly we have \( \int_{\gamma_i} \phi_3 = \int_{\beta_i} \phi_3 + \int_{\beta_i^*} \phi_3 = \int_{\beta_i} \phi_3 - \int_{\beta_i} \phi_3 = 0 \). Hence \( \Phi \) has no real period on the fundamental cycles if and only if

\[
\text{Re} \left( \int_{\beta_j} \phi_i \right) = 0 \quad \text{for } i = 1, 2 \text{ and } 1 \leq j \leq n,
\]

which are expressed by

\[
\text{Re} \left( \int_{\beta_j} B\psi_1 \right) - \text{Re} \left( \int_{\beta_j} \overline{B}\psi_2 \right) = 0,
\]

\[
\text{Im} \left( \int_{\beta_j} B\psi_1 \right) - \text{Im} \left( \int_{\beta_j} \overline{B}\psi_2 \right) = 0.
\]

Combining them together, we obtain, as we have \( \overline{\psi}_2 = I^*(\psi_1) \),

\[
\int_{\beta_j} \psi_1 + \int_{\beta_j} I^*(\psi_1) = 0.
\]
Put
\[ \psi_1 = \left( \sum_{j=0}^{l} A_j z^j + \sum_{i=1}^{p} \sum_{j=1}^{\alpha_i} \left( B_{ij}/(z-c_i)^j + C_{ij}/(z+c_i)^j \right) \right. \]
\[ \left. + \sum_{i=1}^{q} \sum_{j=1}^{\beta_i} D_{ij}/(z-d_i)^j \right) \frac{dz}{w}, \]
(3.3)
where \( l = 2m + n - 2\alpha - \beta \). Now, from (3.2), we get

**Lemma 3.1.** Put
\[ \int_{\beta_i} (z^j/w) dz = I^j_i, \quad \int_{\beta_i} (1/(z-c_j)^j w) dz = J^j_j(c). \]

Then the vector-valued 1-form \( \Phi \) has no real period along the fundamental cycles of \( T_{n-1} \) if and only if, for \( 1 \leq i \leq n \),
\[ \sum_{j=0}^{l} (I^j_i + (-1)^j I^j_i) A_j + \sum_{k=1}^{p} \sum_{j=1}^{\alpha_k} (J^j_i(c_k) + (-1)^j J^j_i(-c_k)) B_{kj} \]
\[ + (J^j_i(-c_k) + (-1)^j J^j_i(c_k)) C_{kj} \]
\[ + \sum_{k=1}^{q} \sum_{j=1}^{\beta_k} (J^j_i(d_k) + (-1)^j J^j_i(d_k)) D_{kj} = 0. \]

Next, we will study periods around the endpoints. As \( I^*(\psi_3) = (-1)^{\beta+1} \overline{\psi_3} \),
we can put
\[ \prod_{j=1}^{m} (z-b_j)/(z+b_j)/R \]
\[ = \sum_{j=0}^{l_1} E_j z^j + \sum_{k=1}^{p} \sum_{j=1}^{\alpha_k} (F_{kj}/(z-c_k)^j + (-1)^{\beta+j} F_{kj}/(z+c_k)^j) \]
\[ + \sum_{k=1}^{q} \sum_{j=1}^{\beta_k} G_{kj}/(z-d_k)^j, \]
(3.4)
where \( G_{kj} = (-1)^{\beta+j} G_{kj} \), \( E_j = (-1)^{\beta+j} E_j \), \( l_1 = 2m - 2\alpha - \beta \) if \( 2m - 2\alpha - \beta \geq 0 \)
and the first term of the right-hand side appears in this case only. Put \( c_{p+k} = d_k \)
for \( 1 \leq k \leq q \). Let \( \delta_j \) be a curve making one turn around \( c_k \) for \( 1 \leq k \leq p+q \).
Then, for \( 1 \leq k \leq p \), \( I(\delta_k) \) makes one turn around \( -c_k \) in the opposite
direction. Hence it holds
\[ \text{Re} \left( \int_{I(\delta_k)} \phi_i \right) = \text{Re} \left( \int_{\delta_k} \phi_i \right) = \text{Re} \left( \int_{\delta_k} \phi_i \right). \]

Now, \( \text{Re}(\int_{\delta_k} \phi_i) = 0, 1 \leq k \leq p+q \), for \( i = 1, 2 \) if and only if
\[ \int_{\delta_k} \psi_1 + \int_{\delta_k} I^*(\psi_1) = 0, \quad 1 \leq k \leq p+q. \]
On the other hand, \( \Re(\int_{\delta_k} \phi_3) = 0 \), \( 1 \leq k \leq p + q \), if and only if

\[
(3.6) \quad \Im \text{Res}_{z=c_k} \left( AB \prod_{j=1}^{m} (z - b_j)(z + \overline{b}_j)/R \right) = 0, \quad 1 \leq k \leq p + q.
\]

If \( c_k \neq a_i \) for all \( 1 \leq i \leq n \), then we put

\[
(3.7) \quad R_j(c_k) = (1/w)^{(j-1)}(c_k)/(j - 1)!. \quad \text{On the other hand, if } c_j = a_i \text{ for some } i, \text{ we put}
\]

\[
(3.8) \quad R_k(c_j) = (1/w)_{2k}(0)/2k!,
\]

where

\[
\omega_i = \sqrt{(a_1 - a_i + t^2)(a_1 + a_i - t^2) \cdots (a_{i-1} + a_i - t^2)(a_i + a_i - t^2) \cdots (a_{i+1} - a_i + t^2)(a_n - a_i + t^2)(a_n + a_i - t^2)}. \]

Now, we can state the conditions under which the 1-form \( \Phi \) has no real period around the endpoints.

**Lemma 3.2.** The vector-valued 1-form \( \Phi \) has no real period around the endpoints which are different from \((\infty, \infty)\) if and only if

\[
(3.9) \quad \sum_{j=1}^{\alpha_k} (B_{kj} R_j(c_k) - (-1)^j C_{kj} \overline{R_j(c_k)}) = 0, \quad 1 \leq k \leq p,
\]

\[
\sum_{j=1}^{\beta_k} D_{kj} (R_j(d_k) - (-1)^j R_j(d_k)) = 0, \quad 1 \leq k \leq q,
\]

\[
(3.10) \quad \Re(F_{k1}) = 0, \quad 1 \leq k \leq p \text{ if } \beta \text{ is even},
\]

\[
\Im(F_{k1}) = 0, \quad 1 \leq k \leq p \text{ if } \beta \text{ is odd.}
\]

If the endpoint \( p_r \) is \((\infty, \infty)\), then \( p + q = r - 1 \) and \( 2m \geq 2p + q \). The 1-form \( \Phi \) has no real period around the endpoint \( p_r = (\infty, \infty) \) if and only if

\[
(3.11) \quad \sum_{j=n-1}^{l} A_{j} (\overline{R}_{2+j-n}(0) - (-1)^j \overline{R}_{2+j-n}(0)) = 0,
\]

where

\[
\overline{R}_{j}(0) = (1/\bar{\omega})^{(j-1)}(0)/(j - 1)! \quad \text{with } \bar{\omega} = \sqrt[n]{\prod_{i=1}^{n}(a_i \xi - 1)(a_i \xi + 1)}.
\]

**Proof.** Using (3.3), we get from (3.5), for \( 1 \leq k \leq p \),

\[
\sum_{j=1}^{\alpha_k} (B_{kj} \text{Res}_{z=c_k} (1/(z - c_k)^j w) - (-1)^j C_{kj} \overline{\text{Res}_{z=c_k} (1/(z - c_k)^j w)}) = 0
\]

and for \( 1 \leq k \leq q \),

\[
\sum_{j=1}^{\beta_k} D_{kj} (\text{Res}_{z=d_k} (1/(z - d_k)^j w) - (-1)^j \overline{\text{Res}_{z=d_k} (1/(z - d_k)^j w)}) = 0.
\]
If \( c_j \neq a_i \) for all \( 1 \leq i \leq n \), we obtain (3.9). If \( c_j = a_i \) for some \( i \), we may take a local coordinate \( t \) near \( c_j \) such that \( a_i - z = t^2 \). Using (3.8), we also have (3.9) in this case. The conditions (3.6) are reduced to \( \text{Im}((ABF_k)^0) = 0 \), \( 1 \leq k \leq p \), \( \text{Im}((ABG_k)^1) = 0 \), \( 1 \leq k \leq q \). If \( \beta \) is even, \( AB \) is an imaginary number and \( G_k \) are also imaginary numbers. Hence we get from the above \( \text{Re}(F_k) = 0 \). Similarly, if \( \beta \) is odd, we have \( \text{Im}(F_k) = 0 \). Now assume \( p_r = (\infty, \infty) \). Set \( z = 1/\xi \). Then we have

\[
\psi_1 = -\sum_{j=n-1}^{l} A_j/(\xi^{2+j-n}w) + (\text{the holomorphic part near } p_r),
\]

\[
\psi_2 = -\sum_{j=n-1}^{l} (-1)^j A_j/(\xi^{2+j-n}w) + (\text{the holomorphic part near } p_r).
\]

Hence we obtain (3.11). From the conditions (3.6), we get

\[
\text{Im}(AB(F_k - (-1)^{\beta} F_k)) = 0, \quad \text{Im}(ABG_k) = 0.
\]

But these hold without any assumption.

4. Preliminaries for the proof of the theorem

In this section, we will apply Lemmas 3.1 and 3.2 to elliptic Riemann surface and corresponding nonorientable surfaces \( M = P_2 - \{ \pi(\infty, \infty) \} \) of genus 2 with one end. In this case, a Weierstrass pair in Lemma 2.1 is given by

\[
g = kA \prod_{j=1}^{m} (z - b_j)^2 \prod_{i=1}^{2} (z - a_i)/w \prod_{j=1}^{m} (z + b_j),
\]

\[
f = B \prod_{j=1}^{m} (z + b_j)^2 \prod_{i=1}^{2} (z + a_j)/w,
\]

where \( w = \sqrt{(1 - z^2)(1 - k^2 z^2)} \), \( |A| = 1 \), \( AB \) is a pure imaginary number, \( a_1 = 1 \), \( a_2 = a = 1/k \) or \(-1/k \). Notice \( \deg g = 2m+2 \). In the present case, we can put

\[
\psi_1 = k \left( \prod_{j=1}^{m} (z + b_j)^2 (z + 1)(z + a)/w \right) dz = k \left( \sum_{j=1}^{2m+2} A_j z^j \right) dz/w.
\]

To apply Lemma 3.1, we may take the following curves. \( \beta_1(t) = (t, w(t)) \), \(-1 \leq t \leq 1 \), \( \beta_2(t) = (t, w(t)) \), \( 1 \leq t \leq 1/k \). Then we have

\[
I_1' = \int_{\beta_1} (z^j/w) dz = \int_{1}^{1} (t^j/w) dt,
\]

\[
I_2' = \int_{\beta_2} (z^j/w) dz = \int_{1}^{1/k} (t^j/w) dt.
\]

Remark that

\[
\int_{-1}^{0} (t^{2j}/w) dt = \int_{0}^{1} (t^{2j}/w) dt \quad \text{and} \quad \int_{-1}^{0} (t^{2j+1}/w) dt = -\int_{0}^{1} (t^{2j+1}/w) dt.
\]
Moreover all $I^{2j}_i$ are real numbers. Now, as $l = 2m+2$, the relations in Lemma 3.1 are reduced to

$$\sum_{j=0}^{m+1} I^{2j}_i A_{2j} = 0.$$  \hspace{1cm} (4.1)

On the other hand, $\Phi$ has no real period around the endpoint if and only if it holds (3.11), that is,

$$\sum_{h=0}^{m} A_{2h+1} \tilde{R}_{2h+1} = 0,$$  \hspace{1cm} (4.2)

where we use the fact that $\tilde{R}_k = \tilde{R}_k(0)$ are real numbers.

Let $B_j$ be the elementary symmetric polynomials of $b_j$. Moreover we set

$$S_h = \sum_{i+j=h} B_i B_j, \quad 0 \leq h \leq 2m.$$  

Then we have

$$A_{2m+2} = S_0 = 1, \quad A_{2m+1} = \overline{S}_1 + (1 + a),$$

$$A_h = \overline{S}_{2m+2-h} + (1 + a)\overline{S}_{2m+1-h} + a\overline{S}_{2m-h}, \quad 2 \leq h \leq 2m,$$

$$A_1 = (1 + a)\overline{S}_{2m} + a\overline{S}_{2m-1}, \quad A_0 = a\overline{S}_{2m},$$

where $a = 1/k$ or $-1/k$ and $k$ is the modulus of the elliptic Riemann surface. Thus, we have

**Lemma 4.1.** The vector-valued 1-form $\Phi$ has no real period if and only if for $i = 1, 2$,

$$\sum_{h=0}^{m} (I_i^{2m+2-2h} + aI_i^{2m-2h}) S_{2h} + (1 + a) \sum_{h=0}^{m-1} I_i^{2m-2h} S_{2h+1} = 0,$$  \hspace{1cm} (4.3)

$$\sum_{h=0}^{m-1} (a\tilde{R}_{2m-2h-1} + \tilde{R}_{2m-2h+1}) S_{2h+1} + (1 + a) \sum_{h=0}^{m} \tilde{R}_{2m-2h+1} S_{2h} = 0,$$  \hspace{1cm} (4.4)

where

$$I_1^i = \int_0^1 (t^j / w) dt, \quad I_2^i = \int_1^{1/k} (t^j / w) dt,$$

$$\tilde{R}_j = (1/\tilde{w})(j^{-1})(0)/(j-1)!, \quad \tilde{w} = \sqrt{(\xi^2 - 1)(\xi - k^2)}.$$  

The following formula concerning elliptic integrals is well known (see, for example, the formulas 17.1.4 and 17.1.5 on p. 589 in [1]).

$$2j + 3)I_i^{2j+4} - (2j + 2)(a^2 + 1)I_i^{2j+2} + (2j + 1)a^2 I_i^{2j} = 0, \quad i = 1, 2.$$  \hspace{1cm} (4.5)

5. A PROOF OF THE THEOREM

At first, we will show the part (1) of the theorem, that is, there is no complete minimal immersion of $M$ into $R^3$ with total curvature $-4\pi(m + 1), \ m =$
At some calculations in the present sections, we use the computer algebra system REDUCE 3.2.

In the case $m = 0$, the equation (4.3) is reduced to

\[(5.1) \quad I_i^0 + aI_i^0 = 0, \quad i = 1, 2.\]

From the fundamental formulas of elliptic integrals (see, for example, §3.1 of [6]), we have

\[(5.2) \quad I_1^0 = K(k), \quad I_2^0 = -\sqrt{-1}K'(k) = -\sqrt{-1}K'(k),\]

\[(5.3) \quad I_i^2 = a^2(I_i^0 - E_i(k)),\]

where $k' = \sqrt{1 - k^2},$

\[(5.4) \quad E_1 = E(k) = \int_0^1 \frac{\sqrt{(1 - k^2 z^2)/(1 - z^2)}}{z} dz,\]

\[(5.5) \quad E_2 = E(k) = \int_0^{1/k} \frac{\sqrt{(1 - k^2 z^2)/(1 - z^2)}}{z} dz = -\sqrt{-1}(K'(k) - E'(k)).\]

By substituting (5.2), (5.3), (5.4) and (5.5) into (5.1), we obtain $KE' + K'E - KK' = 0$. This contradicts the formula

\[(5.6) \quad KE' + K'E - KK' = \pi/2.\]

In the case $m = 1$, the equation (4.3) is reduced to

\[(5.7) \quad I_i^4 + aI_i^2 + S_2(I_i^2 + aI_i^0) + (1 + a)S_1 I_i^2 = 0.\]

Put $b_1 = b$. Then $S_1 = 2b$, $S_2 = b^2$. Using (5.2), (5.3), (5.4), (5.5) and (4.5), we get

\[(5.8) \quad 3((a + 1)K - aE)b^2 + 6a(a + 1)(K - E)b + a(2a^2 + 3a + 1)K - a(2a^2 + 3a + 2)E = 0,\]

\[(5.9) \quad 3(aE' + K')b^2 + 6a(a + 1)E'b + a(2a^2 + 3a + 2)E' - aK' = 0.\]

If these two equations have a common solution, it must be $b = -(a + 1)/3$. Substituting this into (5.8) and (5.9), we obtain

\[(a^2 - a + 1)(aE - aK - K) = 0, \quad (a^2 - a + 1)(aE' + K') = 0.\]

Here we use the computer algebra system. These equations contradict to (5.6).

In this case $m = 2$, (4.3) is reduced to

\[(5.10) \quad A_1 w^2 + 2A_2 zw + A_3(z^2 + 2w) + 2A_4 z + A_5 = 0,\]

\[(5.11) \quad B_1 w^2 + 2B_2 zw + B_3(z^2 + 2w) + 2B_4 z + B_5 = 0,\]

where $z = B_1 = b_1 + b_2$, $w = B_2 = b_1 b_2$ and $A_1 = I_1^2 + aI_1^0$, $A_2 = (1 + a)I_1^2$, $A_3 = (I_1^4 + aI_1^0)$, $A_4 = (1 + a)I_1^4$, $A_5 = I_1^4 + aI_1^4$, $B_1 = I_2^2 + aI_2^0$, $B_2 = (1 + a)I_2^2$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
COMPLETE NONORIENTABLE MINIMAL SURFACES

\( B_3 = (I_2^4 + aI_2^2), \ B_4 = (1 + a)I_2^4, \ B_5 = I_2^6 + aI_2^4. \) From the equations (5.10) and (5.11), we get

\[
(5.12) \quad 2F_1zw + F_2(z^2 + 2w) + 2F_3z + F_4 = 0,
\]

where \( F_i = A_{i+1}B_i - A_iB_{i+1} \). Hence we obtain, when \( z \neq -2(a + 1)/3 \),

\[
(5.13) \quad w = -(5(a+1)z^2 + 52a^2 + a + 2)z + (4a^2 - a + 4)(a + 1))/(5(3z + 2(a + 1))).
\]

Substituting this into (5.10) and (5.11), we obtain the following equation coming from symbolic algebra manipulation.

\[
(5.14) \quad 25G_4z^4 + 50G_3z^3 + 10G_2z^2 + 20G_1z + G_0 = 0,
\]

where \( G_4 = a^2 - a + 1, \ G_3 = 2a^3 - a^2 - a + 2, \ G_2 = 14a^4 + a^3 - 14a^2 + a + 14, \ G_1 = 4a^5 + 3a^4 - 4a^3 - 4a^2 + 3a + 4, \ G_0 = 16a^6 + 24a^5 - 30a^4 + 17a^3 + 24a + 16 \). If \( z = -2(a + 1)/3 \), substituting this into (5.12), we have \((K'E + KE' - KK')a^4(a - 4)(a + 1)^2 = 0\). But, we have no real solution for this equation. Hence we get \( z \neq -2(a + 1)/3 \).

Next, we will investigate the condition (4.4). In the case \( m = 2 \), this is reduced to

\[
(5.15) \quad (a + 1)R_{1w}w^2 + 2(aR_1 + R_3)wz + (1 + a)R_3(z^2 + 2w)
+ 2(aR_3 + R_5)z + (1 + a)R_5 = 0.
\]

By calculation, we have \( R_{2i} = 0 \) and \( R_1 = 1, \ R_3 = (1 + a^2)/2 \),

\[
R_5 = (3a^4 + 2a^2 + 3)/2^3, \quad R_7 = (5a^63a^4 + 3a^2 + 5)/2^5, \\
R_9 = (35a^8 + 20a^6 + 18a^4 + 20a^2 + 35)/2^5.
\]

Substituting (5.13) into (5.15), we obtain

\[
(5.16) \quad 100H_4z^4 + 50H_3z^3 + 5H_2z^2 + 60H_1z + H_0 = 0,
\]

where \( H_4 = 5a^2 - 8a + 5, \ H_3 = 23a^3 - 15a^2 - 15a + 23, \ H_2 = 223a^4 + 64a^3 - 318a^2 + 64a + 223, \ H_1 = 9a^5 + 11a^4 - 12a^3 - 12a^2 + 11a + 9, \ H_0 = 108a^6 + 232a^5 - 36a^4 - 320a^3 - 36a^2 + 232a + 108 \). Thus the problem is to find a common solution for (5.14) and (5.16). Now we consider Sylvester's resultant of those equations, which is a determinant of order 8. The computation of the determinant is accomplished by the computer algebra system REDUCE 3.2. In fact the resultant of the equations is as follows

\[
R = 390625(3721a^{12} + 7442a^{11} + 8299a^{10} - 16958a^9
- 14717a^8 - 1684a^7 + 48530a^6 - 1684a^5
- 14717a^4 - 16958a^3 + 8299a^2 + 7442a + 3721)
\cdot (4a^2 - a + 4)(a + 1)^4(a - 1)^8.
\]

We can show that there is no solution \( a \) for \( R = 0 \) which satisfies \( |a| > 1 \). Thus we obtain part (1) of the theorem.

In order to prove the part (2) of the theorem, when \( m = 3 \) we put \( x = 1, \ y = b_1 + b_2 + b_3, \ z = b_1b_2 + b_2b_3 + b_3b_1, \ w = b_1b_2b_3. \) Then the equations
(4.3) for \( i = 1, 2 \) are rewritten respectively as
\[
A_1 w^2 + A_3 z^2 + A_5 y^2 + A_7 x^2 \\
+ 2(A_2 wz + A_3 wy + A_4 wz + A_5 zx + A_6 yx) = 0,
\]
\[
B_1 w^2 + B_3 z^2 + B_5 y^2 + B_7 x^2 \\
+ 2(B_2 wz + B_3 wy + B_4 wz + B_5 zx + B_6 yx) = 0,
\]
where \( A_1, A_2, A_3, A_4, A_5 \) and \( B_1, B_2, B_3, B_4, B_5 \) are the same coefficients in (5.10) and (5.11), and \( A_6 = (1 + a)I_1^6, A_7 = I_1^8 + aI_2^6, B_6 = (1 + a)I_2^6, B_7 = I_2^8 + aI_2^6 \). The equation (4.4) is also written as
\[
C_1 w^2 + C_3 z^2 + C_5 y^2 + C_7 x^2 \\
+ 2(C_2 wz + C_3 wy + C_4 wx + C_5 yz + C_5 zx + C_6 yx) = 0.
\]

Though \( x = 1 \), we now assume that \( x \) takes any complex values. Then the above three equations give three quadrics in the 3-dimensional complex projective space \( CP^3 = \{(w, z, y, x)\} \). Then it is evident that the intersection of the three quadrics is not empty. We will show that there is no point with coordinate \( (w, z, y, 0) \) in the intersection. In fact, if we have such a point in the intersection, the three equations are reduced to
\[
A_1 w^2 + A_3 z^2 + A_5 y^2 + 2(A_2 wz + A_3 wy + A_4 yz) = 0,
\]
\[
B_1 w^2 + B_3 z^2 + B_5 y^2 + 2(B_2 wz + B_3 wy + B_4 yz) = 0,
\]
\[
C_1 w^2 + C_3 z^2 + C_5 y^2 + 2(C_2 wz + C_3 wy + C_4 yz) = 0.
\]

These equations have no solution with \( y \neq 0 \), because in this case the above equations are reduced to the equations (4.3) and (4.4) for \( m = 2 \). On the other hand, if \( y = 0 \), the equations are reduced to (4.3) and (4.4) for \( m = 1 \) or \( m = 0 \). Hence there are no solutions. Thus the intersection consists of points with coordinate \( (w, x, y, 1) \).

Now it is evident that for \( m \geq 4 \), equations (4.3) and (4.4) have common solutions.

ACKNOWLEDGMENT

The author would like to express his cordial thanks to the referee for his correcting many errors in the original draft.

REFERENCES


**Department of Mathematics and Computer Sciences, The University of Tokushima, Josanjima, Tokushima 770, Japan**