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COMPLEMENTED IDEALS IN THE FOURIER ALGEBRA AND THE RADON NIKODYM PROPERTY

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ABSTRACT. Necessary and sufficient conditions are given for an ideal $I(H)$ of the Fourier algebra to be complemented when H is a closed subgroup of G. Using the Radon Nikodym property, an example of a group G with a normal abelian subgroup H for which $I(H)$ is not complemented is presented.

1. INTRODUCTION

The problem of characterizing complemented ideals in the group algebra $L^1(G)$ of a locally compact abelian group G has received considerable attention. D. J. Newman showed that if Π is the circle group and $H^1 = \{f \in L^1(\Pi) | \hat{f}(n) = 0 \text{ for every } n < 0\}$, then H^1 is not complemented in $L^1(\Pi)$ [20]. Later in [23], Rudin proved that I is complemented in $L^1(\Pi)$ if and only if $I = I(A) = \{f \in L^1(\Pi) \mid \hat{f}(n) = 0 \text{ for every } n \in A\}$ with $A = \bigcup_{i=1}^n a_i\mathbb{Z} + b_i$.

In his memoir [22], Rosenthal showed that for an arbitrary locally compact abelian group a necessary condition for I to be complemented is that $I = I(A)$, where A is a closed element of the coset ring of $\widehat{G}$. Alspach and Matheson completed the characterization for $G = \mathbb{R}$ by proving that I is complemented if and only if $I = I(A)$, where $A = \bigcup_{i=1}^n (\tau_i\mathbb{Z} + \beta_i)\backslash F$, the τ_i's are pairwise rationally dependent and F is finite [2].

Alspach, Matheson, and Rosenblatt examined the problem for arbitrary locally compact abelian groups and were successful in giving a necessary and sufficient condition for an ideal with a discrete hull to be complemented [3]. (Their proof was incorrect but was subsequently corrected in [4].) They also developed an inductive procedure which Alspach exploited to characterize the complemented ideals in $L^1(\mathbb{R}^2)$ [1].

By identifying $L^1(G)$ with $A(\widehat{G})$, we can view these studies as a series of investigations into the ideal structure of the Fourier algebra of a locally compact abelian group. In this paper we will examine the complementation problem for the Fourier algebra of an arbitrary locally compact group G. In particular, we will focus on the question of identifying those closed subgroups H of G which are such that $I(H)$ is complemented.

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In §3, we verify a conjecture of Herz [12, p. 161] by presenting an example of a locally compact group G and of a closed subgroup H for which $I(H)$ is not complemented in $A(G)$. Our proof is geometric in nature. It shows that the question of spectral synthesis which Herz believed to be at the heart of the matter is not the true barrier to complementation.

2. Preliminaries

Throughout this paper G will denote a locally compact group with a fixed left Haar measure dx. Σ_G will denote the class of equivalence classes of strongly continuous unitary representations of G. $\widehat{G}$ is the set of irreducible classes in Σ_G.

For each $\pi \in \Sigma_G$, $\xi, \eta \in \mathscr{H}_\pi$, the Hilbert space associated with π, define $u_{\pi,\xi,\eta}(x) = \langle \pi(x)\xi, \eta\rangle$. Let $B(G)$ be the collection of all such coefficient functions. It is well known that $B(G)$ is the linear span of continuous positive definite functions on G. Furthermore $B(G)$ is an algebra with respect to pointwise addition and multiplication.

Define a norm $\|\cdot\|_{B(G)}$ on $B(G)$ by

$$\|u\|_{B(G)} = \inf_{\substack{\pi\in\Sigma,\\ \xi_i,\eta_i\in\mathscr{H}_\pi}} \left\{ \sum_{i=1}^{\infty} \|\xi_i\|\,\|\eta_i\| \mid u(x) = \sum_{i=1}^{\infty} \langle \pi(x)\xi_i, \eta_i\rangle \right\}.$$

With the norm $\|\cdot\|_{B(G)}$, $B(G)$ becomes a commutative Banach algebra called the Fourier-Stieltjes algebra. $B(G)$ was first introduced and studied by Eymard in [9].

Define $\lambda(x)\colon L^2(G) \to L^2(G)$ by $\lambda(x)f(y) = f(x^{-1}y)$ for each $f \in L^2(G)$, $x, y \in G$. Then $\lambda \in \Sigma_G$ is called the left regular representation of G. Let $A(G) = \{u(x) | u(x) = \langle \lambda(x)f, g\rangle,\ f, g \in L^2(G)\}$. Then $A(G)$ is a closed ideal of $B(G)$ [9, p. 208]. With respect to $\|\cdot\|_{A(G)} = \|\cdot\|_{B(G)|_{A(G)}}$, $A(G)$ is a commutative Banach algebra with maximal ideal space $\Delta(A(G)) = G$.

Let $u \in A(G)$. Define $(L_x u)(y) = u(x^{-1}y)$. Then $L_x u \in A(G)$ and $\|L_x u\|_{A(G)} = \|u\|_{A(G)}$. Let $A, B \subset G$ be closed. Define $\rho(A, B) = \{u \in B(G) | u(A) = 1,\ u(B) = 0\}$. Let

$$s(A, B) = \begin{cases} \inf\{\|u\|_{B(G)} | u \in \rho(A, B)\} & \text{if } \rho(A, B) \neq \varnothing, \\ \infty & \text{if } \rho(A, B) = \varnothing. \end{cases}$$

Let I be an ideal of $A(G)$. Let $Z(I) = \{x \in G | u(x) = 0 \text{ for every } u \in I\}$. Then $Z(I)$ is a closed subset of G. Given $A \subset G$ closed, $I(A) = \{u \in A(G) | u(x) = 0 \text{ for every } x \in A\}$. A closed set $A \subset G$ is called a set of spectral synthesis (or simply an s-set) if $Z(I) = A$ implies that I is dense in $I(A)$.

Let $\mathscr{Z}(G)$ denote the center of G. If H is a subgroup of G, $C_G(H)$ will denote the centralizer of H. The following classes of locally compact groups will be considered:

[Z]—Central topological groups—$G/Z(G)$ is compact.

[SIN]—Small Invariant Neighborhood groups—every neighborhood of the identity contains a compact neighborhood which is invariant under all inner automorphisms.

[IN]—Invariant Neighborhood groups—there exists a compact neighborhood of the identity which is invariant under all inner automorphisms.

$[FC]^-$—Topologically Finite Conjugacy Class groups—the closure of each conjugacy class is compact.

For properties of these classes see [21].

3. The complemented ideal property for closed subgroups

Definition 3.1. A closed ideal I in $A(G)$ is said to be complemented if there exists a continuous projection P of $A(G)$ onto I. Let H be a closed subgroup of G. H is said to have the complemented ideal property (CIP)) in G if $I(H)$ is complemented in $A(G)$. Let $\mathrm{CIP}(G) = \{H | H \text{ has CIP in } G\}$.

Proposition 3.2. *Let H be an open subgroup of G. Then $H \in \mathrm{CIP}(G)$. In particular, if G is a discrete group, then every subgroup of G belongs to* $\mathrm{CIP}(G)$.

Proof. Since H is open, $1_H \in B(H)$ [9, p. 205] and thus $1_{G\setminus H} \in B(G)$. Therefore $A(G) = 1_H A(G) \oplus 1_{G\setminus H} A(G)$ and $1_{G\setminus H} A(G) = I(H)$. □

It should be noted that if $A \subset G$, then $1_A \in B(G)$ if and only if A belongs to $\mathscr{R}(G)$, the ring of subsets of G generated by open cosets of G. Hence if $A \in \mathscr{R}(G)$, then as in Proposition 3.2, we see that $I(A)$ is complemented in $A(G)$. For abelian groups, Rosenthal has shown for an ideal $I \subset A(G)$ to be complemented, $Z(I)$ must belong to $\mathscr{R}(G_d)$, the coset ring of G with the discrete topology [22]. Though it is very likely that Rosenthal's result holds for all locally compact groups we are unable to prove this. This does however motivate us to consider the question of identifying those subgroups with the CIP. The next proposition is our key tool in this regard.

Proposition 3.3. *Let H be a closed subgroup. Then $H \in \mathrm{CIP}(G)$ if and only if there exists a continuous linear map $\Gamma\colon A(H) \to A(G)$ such that $\Gamma u_{|H} = u$ for every $u \in A(H)$.*

Proof. Assume that $H \in \mathrm{CIP}(G)$. Let Q be a continuous projection of $A(G)$ onto $I(H)$. Let $u \in A(H)$. Let $v \in A(G)$ be such that $v_{|H} = u$ and $\|v\|_{A(G)} = \|u\|_{A(H)}$ [13, Theorem 1].

Define $\Gamma u = v - Qv$. Then $\|\Gamma u\|_{A(G)} \le (1 + \|Q\|)\|u\|_{A(H)}$. Γ is well defined since $(v_1 - Qv_1) - (v - Qv) = (v_1 - v) - Q(v_1 - v) = 0$ for any other extension v_1 of u.

Let $u_1, u_2 \in A(H)$, and $\alpha_1, \alpha_2 \in \mathbb{C}$. Choose any two extensions v_1 and v_2 of u_1 and u_2 respectively. Then

$$\Gamma(\alpha_1 u_1 + \alpha_2 u_2) = (\alpha_1 v_1 + \alpha_2 v_2) - Q(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \Gamma(u_1) + \alpha_2 \Gamma(u_2).$$

Hence Γ is linear.

Conversely, assume that $\Gamma\colon A(H) \to A(G)$ is a continuous linear extension map. For $v \in A(G)$ define $Q(v) = v - \Gamma(v_{|H})$. Then $\|Q\| \le 1 + \|\Gamma\|$, $Q(v) \in I(H)$ and since $v \in I(H)$ implies $v_{|H} = 0$, $Q(v) = v$ for every $v \in I(H)$. □

The following proposition is the starting point of [3]. It is not stated as such, but the essential ideas of the proof are included as a remark [3, p. 257].

Proposition 3.4. *Let G be an abelian locally compact group. Let H be a closed subgroup of G. Then there exists a linear isometry $\Gamma\colon A(H) \to A(G)$ such that $\Gamma u_{|H} = u$ for every $u \in A(H)$. In particular, $H \in \mathrm{CIP}(G)$.*

Definition 3.5. A representation $\pi \in \Sigma_G$ is said to be completely reducible if $\pi = \sum_{\alpha \in \mathscr{U}} \bigoplus \sigma_\alpha$ where $\sigma_\alpha \in \widehat{G}$.

It is well known that if G is abelian, then the left regular representation λ_G is completely reducible if and only if G is compact; cf. [25].

Theorem 3.6. *Let G be a locally compact group. Let H be a closed subgroup of G such that the left regular representation of H is completely reducible. Then there exists a linear isometry $\Gamma\colon A(H) \to A(G)$ such that $\Gamma u_{|_H} = u$ for every $u \in A(H)$.*

Proof. Since $A(G)_{|_H} = A(H)$, $\lambda_{G|_H}$ is quasiequivalent to λ_H [5, p. 27]. Therefore there exists cardinal numbers η_G and η_H such that $\eta_G(\lambda_{G|_H})$ is unitarily equivalent to $\eta_H\lambda_H$. Let $T\colon \eta_G\mathscr{H}_{\lambda_G} \to \eta_H\mathscr{H}_{\lambda_H}$ be an intertwining operator.

As λ_H is completely reducible, we can write

$$\lambda_H = \sum_{\alpha\in\mathscr{I}} \bigoplus m_\alpha\sigma_\alpha \quad \text{and} \quad L^2(H) = \mathscr{H}_{\lambda_H} = \sum_{\alpha\in\mathscr{I}} \bigoplus m_\alpha\mathscr{H}_{\sigma_\alpha},$$

where $\sigma_\alpha \in \widehat{H}$. It follows that $T^{-1}\mathscr{H}_{\sigma_\alpha}$ is a closed subspace of $\eta_G\mathscr{H}_{\lambda_G}$.

Let A_{σ_α} denote the closure in $B(H)$ of the space F_{σ_α}, the linear span of the coefficient functions of σ_α. Since σ_α is irreducible, A_{σ_α} is linearly isomorphic isometric to $\mathscr{H}_{\sigma_\alpha}\hat{\otimes}\mathscr{H}_{\sigma_\alpha}$ [5, 2éme partie] where $\hat{\otimes}$ denotes the projective tensor product. For $\xi, \eta \in \mathscr{H}_{\sigma_\alpha}$ define

$$\Gamma_{\sigma_\alpha}(\langle\sigma_\alpha(\cdot)\xi, \eta\rangle) = \langle\eta_G\lambda_G(\cdot)T^{-i}\xi, T^{-1}\eta\rangle.$$

Then Γ_{σ_α} extends to a linear map of $F_{\sigma_\alpha} = \mathscr{H}_{\sigma_\alpha} \otimes \mathscr{H}_{\sigma_\alpha}$ into $A_{\eta_G\lambda_G}$ with

$$\begin{aligned}\left\|\Gamma_{\sigma_\alpha}\left(\sum_{i=1}^n\langle\sigma_\alpha(\cdot)\xi_i, \eta_i\rangle\right)\right\|_{B(G)} &\le \sum_{i=1}^n \|T^{-1}\xi_i\|_{\mathscr{H}_{\eta_G\lambda_G}}\|T^{-1}\eta_i\|_{\mathscr{H}_{\eta_G\lambda_G}} \\ &= \sum_{i=1}^n \|\xi_i\|_{\mathscr{H}_{\sigma_\alpha}}\|\eta_i\|_{\mathscr{H}_{\sigma_\alpha}}.\end{aligned}$$

Hence

$$\left\|\Gamma_{\sigma_\alpha}\left(\sum_{i=1}^n\langle\sigma_\alpha(\cdot)\xi_i, \eta_i\rangle\right)\right\|_{B(G)} \le \left\|\sum_{i=1}^n\langle\sigma_\alpha(\cdot)\xi_i, \eta_i\rangle\right\|_{A(H)}.$$

Therefore, Γ_{σ_α} extends to a norm nonincreasing linear map of A_{σ_α} into $A_{\eta_G\lambda_G}$.

Since $T_{\eta_G\lambda_G|_H} = \eta_H\lambda_H T$,

$$\langle\sigma_\alpha(h)\xi, \eta\rangle = \langle\eta_G\lambda_G(h)T^{-1}\xi, T^{-1}\eta\rangle = \Gamma_{\sigma_\alpha}(\langle\sigma_\alpha(\cdot)\xi, \eta\rangle)(h)$$

for every $\xi, \eta \in \mathscr{H}_{\sigma_\alpha}$. As convergence in $B(H)$ and $B(G)$ implies uniform convergence [9, p. 182], $\Gamma_{\sigma_\alpha}u_{|_H} = u$ for every $u \in A_{\sigma_\alpha}$. Furthermore, Γ_{σ_α} is an isometry.

Finally $A(H) = l_1 - \bigoplus_{\alpha\in\mathscr{I}} A_{\sigma_\alpha}$ [5, p. 39]. Define $\Gamma\colon A(H) \to B(G)$ by

$$\Gamma u = \sum_{\alpha\in\mathscr{I}} \Gamma_{\sigma_\alpha}u_{\sigma_\alpha},$$

where $u = \sum_{\alpha\in\mathscr{I}} u_{\sigma_\alpha}$ is the unique decomposition of u.

Note that

$$\begin{aligned}\|\Gamma u\|_{B(G)} &= \left\|\sum_{\alpha\in\mathscr{I}}\Gamma_{\sigma_\alpha}u_{\sigma_\alpha}\right\|_{B(G)} = \sum_{\alpha\in\mathscr{I}}\|u_{\sigma_\alpha}\|_{B(H)} \\ &= \sum_{\alpha\in\mathscr{I}}\|u_{\sigma_\alpha}\|_{A(H)} = \|u\|_{A(H)}.\end{aligned}$$

Also since $\Gamma_{\sigma_\alpha} u_{\sigma_\alpha|_H} = u_{\sigma_\alpha|_H}$ for every $\alpha \in \mathscr{S}$, $\Gamma u_{|_H} = u$ for every $u \in A(H)$. Therefore Γ is a linear isometry of $A(H)$ into $B(G)$. But $\Gamma u \in A_{\eta_G \lambda_G} = A(G)$ [5, p. 27]. □

Corollary 3.7. *Let H be a compact subgroup of G. Then there exists a linear isometry $\Gamma\colon A(H) \to A(G)$ such that $\Gamma u_{|_H} = u$ for every $u \in A(H)$. Furthermore, $H \in \mathrm{CIP}(G)$.*

Proof. It is well known that if H is compact, then λ_H is completely reducible. □

Corollary 3.7 is due to Herz [12, Theorem 4]. His proof involves techniques from the theory of induced representations, an approach which we shall exploit later. However, since there are many noncompact groups for which λ_G is completely reducible, Theorem 3.6 is a proper extension of Herz's result. Moreover, the geometric nature of the above proof sheds new light on the complementation problem. We will develop this further.

Proposition 3.8. *Let G be a locally compact group. Let H be a closed subgroup of G and let π be a completely reducible representation of H. If every $u \in A_\pi$ extends to some $v \in B(G)$, then there exists a linear isometry $\Gamma\colon A_\pi \to B(G)$ for which $\Gamma u_{|_H} = u$.*

Proof. Let ω denote the universal representation of G. Then $A_{\omega_{|_H}}$ contains any element of $B(H)$ which extends to $B(G)$. By assumption $A_\pi \subseteq A_{\omega_{|_H}}$. Hence π is quasiequivalent to a subrepresentation of $\omega_{|_H}$ [5, p. 40]. The remainder of the proof is similar to the proof of Theorem 3.6. □

Example 3.9. (i) Let G be a unimodular group. Let

$$\widehat{G}_d = \{\pi \in \widehat{G} | \pi \text{ is square integrable}\}.$$

Let $A_d(G)$ denote the closed subspace of $A(G)$ generated by the extreme points of the unit ball of $A(G)$, Mauceri [19] has shown that

$$A_d(G) = l_1 - \bigoplus_{\pi \in \widehat{G}_d} A_\pi$$

and that there exists a closed subspace $A_c(G)$ such that $A(G) = A_d(G) \oplus A_c(G)$.

Let H be a closed subgroup of G. Then Proposition 3.8 establishes a linear isometry $\Gamma\colon A_d(H) \to A(G)$ for which $\Gamma_{u_{|_H}} = u$ for every $u \in A_d(H)$. Therefore if $H \notin \mathrm{CIP}(G)$, the difficulty lies within $A_c(H)$.

(ii) Let $\mathrm{AP}(G)$ denote the space of continuous almost periodic functions on G. Let G^{ap} denote the almost periodic compactification of G. If $\varphi\colon G \to G^{\mathrm{ap}}$ is the canonical homomorphism, then $\ker\varphi$ is such that every $f \in \mathrm{AP}(G)$ is constant on cosets of $\ker\varphi$.

Assume that H is a compact subgroup of G. Then $\varphi(H) \cong H/(\ker\varphi \cap H)$ is a compact subgroup of G^{ap}. Moreover $A(\varphi(H))$ is isometrically isomorphic to the closed subalgebra of $A(H)$ which consists of functions which are constant on cosets of $\ker\varphi \cap H$. We will denote this algebra by $A_\varphi(H)$.

It follows from Theorem 3.6 that there exists a linear isometry $\Gamma_1\colon A(\varphi(H)) \to A(G^{\mathrm{ap}})$ for which $\Gamma_1\colon u_{|_{\varphi(H)}} = u$. This provides us with the means of constructing a linear isometry $\Gamma\colon A_\varphi(H) \to \mathrm{AP}(G) \cap B(G)$ for which $\Gamma_{u_{|_H}} = u$.

This follows since Eymard proved that $\mathrm{AP}(G) \cap B(G)$ can be identified in a natural way with $A(G^{\mathrm{ap}})$ [9, p. 203].

In case G is noncompact, $(\mathrm{AP}(G) \cap B(G)) \cap A(G) = \{0\}$. Hence the extending map is entirely distinct from the map constructed in the proof of Theorem 3.6. Finally, we note that if G is abelian, then $\ker \varphi \cap H = \{e\}$. Hence all of $A(H)$ extends to $\mathrm{AP}(G) \cap B(G)$.

(iii) Let H be the "$ax + b$" group. We recall that H can be realized as the matrix group

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{R}^+, \ b \in \mathbb{R} \right\}.$$

Then λ_H is completely reducible, cf. [16]. For any $n \geq 2$, $G = \mathrm{GL}(n, \mathbb{R})$ contains subgroups isomorphic to H. Let H_1 be such a group. Then there is a linear isometry $\Gamma\colon A(H_1) \to A(G)$ for which $\Gamma_{u_{|H_1}} = u$. Consequently, $H_1 \in \mathrm{CIP}(G)$.

Definition 3.10. A Banach space X is said to possess the Radon-Nikodym property if for any finite measure space (Ω, Σ, μ) and any μ-continuous vector measure $L\colon \Sigma \to X$ of bounded total variation, there exists a Bochner integrable function $g\colon \Sigma \to X$ such that

$$L(E) = \int_E g \, d\mu \quad \text{for every } E \in \Sigma \quad \text{(cf. [8])}.$$

In [7, p. 535] Chu showed that the pre-dual of a von Neumann algebra has the RNP if and only if every bounded norm closed convex subset is the closed convex hull of its extreme points.

Granirer and Leinert proved that if $K \subset G$ is compact, $A_K(G) = \{u \in A(G) | \operatorname{supp} u \subseteq K\}$ has the RNP [11, p. 464]. K. Taylor showed that $A(G)$ has the RNP if and only if λ_G is completely reducible [25]. As it is easy to see that the RNP is preserved by linear isometries the previous example shows that if $n \geq 2$, then $G = \mathrm{GL}(n, \mathbb{R})$ has closed subspaces with the RNP which are not of the form $A_K(G)$ for some compact set K even though $A(G)$ does not have the RNP, cf. [25]. However, we have:

Proposition 3.11. *Let G be a locally compact group. Let H be a closed subgroup of G such that λ_H is completely reducible. Then $I(H)$ has the RNP if and only if $A(G)$ has the RNP.*

Proof. If $A(G)$ has the RNP, then so does every closed subspace.

Conversely, assume that $I(H)$ has the RNP. Since λ_H is completely reducible, $A(H)$ has the RNP. It follows from [10, Lemma 3.9] that $A(G)/I(H)$ has the RNP. Hence $A(G)$ has the RNP; cf. [8, p. 211]. □

Proposition 3.12. *Let G be a locally compact group for which $A(G)$ has the RNP. Let H be a closed subgroup of G. if $H \in \mathrm{CIP}(G)$, then $A(H)$ has the RNP.*

Proof. Let P be a projection of $A(G)$ onto $I(H)$. Then $(1 - P)(A(G))$ has the RNP. Define $\Gamma_0\colon (1 - P)(A(G)) \to A(H)$ by $\Gamma_0 u = u_{|H}$ for every $u \in (1 - P)(A(G))$. Γ_0 is linear and $\|\Gamma_0\| \leq 1$. If we let $\Gamma\colon A(H) \to A(G)$ be the map constructed as in the proof of Proposition 3.3, then $\Gamma = \Gamma_0^{-1}$ and $\|\Gamma\| < \infty$. Therefore, $A(H)$ and $(1 - P)(A(G))$ are linearly isomorphic and

homeomorphic. It follows that every norm closed convex subset of $A(H)$ is the closed convex hull of its extreme points. Therefore, by Chu's result [7], $A(H)$ has the RNP. □

Example 3.13. Let $G = \mathbb{R} \odot \mathbb{R}^+$ be the "$ax + b$" group. Then $A(G)$ has the RNP. Also G has a closed normal subgroup H which is isomorphic to $\mathbb{R}$. But $A(\mathbb{R})$ does not have the RNP [cf. 25]. Hence by Proposition 3.12, $I(H)$ is not complemented in $A(G)$.

Observe that $\mathbb{R}$ is an s-set in G [24] hence the problem of spectral synthesis hinted at in [12, p. 161] is avoided. In addition, the above example shows that the assumptions that a subgroup is normal or abelian or that the group G is amenable will not insure complementation. This seems to indicate that the complementation problem is much more difficult in the noncommutative setting than in the abelian case which is in its own right far from trivial.

4. Product groups and central subgroups

Theorem 4.1. *Let H be a closed subgroup of a separable locally compact σ-compact group G. Let C be a subgroup of $C_G(H)$ and $K = CH$. If K is closed in G, then there exists a linear isometry $\Gamma\colon A(H) \to A(K)$ such that $\Gamma u_{|_H} = u$ for every $u \in A(H)$. In particular, $H \in \mathrm{CIP}(K)$.*

Proof. Observe that $K/H \cong C/C \cap H$. Let $\varphi\colon K/H \to C/C \cap H$ be the canonical isomorphism. Let $\psi_1\colon C/C \cap H \to C$ be a Borel cross section [cf. 18, p. 102]. Define $\psi\colon K/H \to K$ by $\psi = \psi_1 \circ \xi$. Then ψ is a Borel cross section and $\psi(\xi) \in C$ for every $\xi \in K/H$.

Let Ω be a compact subset of K/H with $\mu_{K/H}(\Omega) > 0$. Let $x \in K$ and $\xi \in K/H$. Define $\xi^x = \pi(x^{-1}y)$, where $\pi\colon K \to K/H$ is the canonical homomorphism and $y \in K$ is such that $\pi(y) = \xi$.

Define $\tau\colon K \times K/H \to H$ by

$$\tau(x, \xi) = \psi(\xi)^{-1} x \psi(\xi^x).$$

For each $u \in A(H) \cap C_{00}(H)$, let

$$\Gamma_0 u(x) = \frac{1}{\mu_{K/H}(\Omega)} \int_{K/H} 1_\Omega(\xi^x) 1_\Omega(\xi) \mu(\tau(x, \xi))\, d\mu_{K/H}(\xi).$$

Then $\mathrm{supp}(\Gamma_0 u)$ is compact. Furthermore $\Gamma_0 u \in A(K)$ with $\|\Gamma_0 u\|_{A(K)} \leq \|u\|_{A(H)}$; cf. [13, p. 114].

Let $h \in H$. Then $\xi^h = \xi$ and since $\psi(\xi) \in C$,

$$\tau(h, \xi) = \psi(\xi)^{-1} h \psi(\xi) = h.$$

Therefore $\Gamma_0 u(h) = u(h)$. Hence $\Gamma_0 u_{|_H} = u$ and $\|\Gamma_0 u\|_{A(K)} = \|u\|_{A(H)}$. As $A(H) \cap C_{00}(H)$ is dense in $A(H)$, Γ_0 can be extended to an isometry $\Gamma\colon A(H) \to A(K)$.

Corollary 4.2. *Let H be a closed central subgroup of a locally compact group G. Then there exists an isometry $\Gamma\colon A(H) \to A(G)$ for which $\Gamma u_{|_H} = u$ for every $u \in A(H)$.*

Proof. (i) Assume that G is separable and σ-compact. Then the result follows immediately from the proof of Theorem 4.2.

(ii) Suppose that G is σ-compact. Let K be a compact normal subgroup of G such that G/K is separable. Let $\varphi: G \to G/K$ be the canonical homomorphism. By (i), there exists a linear isometry $\Gamma_{0_K}: A(\varphi(H)) \to A(G/K)$ such that $\Gamma_{0_K} u_{|_{\varphi(H)}} = u$. But $\varphi(H) \cong H/(K \cap H)$. Hence $A(\varphi(H))$ can be naturally identified with the subalgebra of $A(H)$ consisting of functions which are constant on cosets of $(K \cap H)$. Similarly $A(G/K)$ can be identified with the subalgebra of $A(G)$ consisting of functions which are constant on cosets of K. Composition of these mappings gives us a linear isometry Γ_K from $A(H/(H \cap K)) \subseteq A(H)$ into $A(G/K) \subseteq A(G)$ for which $\Gamma_K u_{|_H} = u$.

Assume that K_1 is any other compact normal subgroup of G such that G/K_1 is separable and that $K_1 \subseteq K$. Observe that $A(H/K \cap H) \subseteq A(H/K_1 \cap H)$. Moreover, because the process used to produce the isometry Γ in Theorem 4.1 is essentially an *averaging procedure*, a careful examination will show that Γ_{K_1} is an extension of Γ_K. Since every $u \in A(H)$ belongs to $A(H/K_1 \cap H)$ for some such compact normal group K_1, the desired isometry is obtained by an obvious directed limit of the Γ_K's.

(iii) Assume that G is any locally compact group and that H is any closed subgroup which is also σ-compact. Then there exists an open σ-compact group $G_0 \subset G$ such that $H \subseteq G_0$. Identify $A(G_0)$ with the obvious subalgebra of $A(G)$ and apply (ii) to obtain the desired isometry. □

Corollary 4.2 is due to Herz [12, Theorem 5]. His proof is for the case G separable, metric and σ-compact (see [12, p. 161]). For the sake of completeness we have outlined how the separability assumption can be dropped, something which Herz asserts but does not exhibit. In fact, a careful modification of the above procedure will allow us to remove the separability assumption from the statement of Theorem 4.1 which for the rest of the paper we shall assume has been done.

Since we do not know of any way to extend Corollary 4.2 to non-σ-compact groups, we are not able to obtain Proposition 3.4 as a corollary to the above result.

Let G_1 and G_2 be locally compact groups. Let $u_i \in A(G_i)$. Define $u \in A(G_1 \times G_2)$ by $u(x, y) = u_1(x)u_2(y)$. In this manner we get a linear map $\psi: A(G_1)\hat{\otimes}A(G_2) \to A(G_1 \times G_2)$. Furthermore ψ has a dense range and $\|\psi\| \leq 1$; cf. [17].

Losert proved that if the irreducible representations of either G_1 or G_2 are bounded, then ψ is surjective and $\|\psi^{-1}\| = \min(d_1, d_2)$ where $d_i = \sup_{\pi \in \widehat{G_i}} \dim \pi$. Consequently if either G_1 or G_2 is abelian, then ψ is an isometry.

Conversely, if ψ is a surjection, then one of the G_i's must have an open abelian subgroup of finite index and ψ is an isometry only if one of the G_i's is abelian.

Proposition 4.3. *Let $G = G_1 \times G_2$. Then for each $i = 1, 2$, there exists a linear isometry $\Gamma_i: A(G_i) \to A(G)$ such that $\Gamma_i u_{|_{G_i}} = u$ for each $u \in A(G_i)$. Consequently $G_i \in \mathrm{CIP}(G)$.*

Proof. Let $u_2 \in A(G_2)$ be such that $u_2(e_2) = 1$ and $\|u_2\|_{A(G_2)} = 1$. Then $\Gamma_1 v = \psi(v \otimes u_2)$ for every $v \in A(G_1)$ determines the desired map. Γ_2 is obtained in a similar fashion. □

Lemma 4.4. *Let $G = G_1 \times G_2$. Let $A = A_1 \times A_2$ where A_i is closed in G_i. Assume that A is a set of spectral synthesis in G. Then $I_G(A)$ is the closed linear span $\langle J\rangle^-$ of $J = \{\psi(I_{G_1}(A_1) \otimes (G_2)) \cup \psi(A(G_1) \otimes I_{G_2}(A_2))\}$.*

Proof. Clearly $J \subset I_G(A)$. Therefore, $\langle J\rangle^- \subseteq I_G(A)$ as well. Let $v \in A(G)$. Then there exists a net

$$\left\{\sum_{i=1}^{n_\alpha} w_{\alpha_i} \otimes z_{\alpha_i}\right\}_{\alpha\in\mathscr{U}}$$

in $A(G_1) \otimes A(G_2)$ such that

$$v = \lim_\alpha \psi\left(\sum_{i=1}^{n_\alpha} w_{\alpha_i} \otimes z_{\alpha_i}\right).$$

Assume that $u_1 \in I_{G_1}(A_1)$ and $u_2 \in A(G_2)$. Then

$$\psi(u_1 \otimes u_2)v = \lim_\alpha \psi\left[\sum_{i=1}^{n_\alpha}(u_1 w_{\alpha_i}) \otimes (u_2 z_{\alpha_i})\right].$$

Hence $\psi(u_1 \otimes u_2)v \in \langle J\rangle^-$. Similarly if $u_1 \in A(G_1)$, $u_2 \in I_{G_2}(A_2)$, then $\psi(u_1 \otimes u_2)v \in \langle J\rangle^-$. Therefore $\langle J\rangle^-$ is a closed ideal in $A(G)$.

Assume that $(x_1, x_2) \notin A_1 \times A_2$. We may assume that $x_1 \neq A$. We can find $u_1 \in I_{G_1}(A_1)$ such that $u_1(x_1) = 1$ and $u_2 \in A(G_2)$ with $u_2(x_2) = 1$. Since $\psi(u_1 \otimes u_2)(x_1, x_2) = 1$, $2(\langle J\rangle^-) = A_1 \times A_2 = A$ But A is an s-set, so $\langle J\rangle^- = I_G(A)$. □

Proposition 4.5. *Let $G = G_1 \times G_2$, where G_1 has an abelian subgroup of finite index. Let $A = A_1 \times A_2$, where A_i is closed in G_i. Assume that A is set of spectral synthesis in G. Suppose the $I_{G_i}(A_i)$ is complemented in $A(G_i)$ for $i = 1, 2$. Then $I_G(A)$ is complemented in G.*

Proof. Let P_i be a continuous projection of $A(G_i)$ onto $I_{G_i}(A_i)$. Define $P\colon A(G) \to A(G)$ by

$$Pu = u - \psi[((1 - P_1) \otimes (1 - P_2))(\psi^{-1}u)].$$

Note that since G_1 has an abelian subgroup of finite index ψ^{-1} exists and $\|\psi^{-1}\| < \infty$. Therefore P is continuous.

Let $u_i \in A(G_i)$ and let $(x_1, x_2) \in A_1 \times A_2$. If $u = \psi(u_1 \otimes u_2)$, then

$$\begin{aligned} P(u)(x_1, x_2) &= u_1(x_1)u_2(x_2) - [(1 - P_2)u_1](x_1)[(1 - P_2)u_2](x_2) \\ &= u_1(x_1)u_2(x_2) - u_1(x_1)u_2(x_2) = 0. \end{aligned}$$

Therefore $Pu \in I_G(A)$ for every $u \in A(G)$.

Let $u_1 \in I_{G_1}(A_1)$ and $u_2 \in A(G_2)$. Let $u = \psi(u_1 \otimes u_2)$. Then

$$\begin{aligned} \psi[(1 - P_1) \otimes (1 - P_2)(\psi^{-1}u)] &= \psi[(1 - P_1)u_1 \otimes (1 - P_2)u_2] \\ &= \psi(0 \otimes (1 - P_2)u_2) = 0. \end{aligned}$$

Therefore $Pu = u$. Similarly if $u_1 \in A(G_1)$ and $u_2 \in I_{G_2}(A_i)$, then for $u = \psi(u_1 \otimes u_2)$, $Pu = u$. By Lemma 4.4, we have $Pu = u$ for every $u \in I_G(A)$. □

Corollary 4.6. *Let G_i be a locally compact abelian group for each $i = 1, 2$. Let $A_i \subset G_i$ be such that $I_{G_i}(A_i)$ is complemented in $A(G_i)$. Then $A = A_1 \times A_2$ is such that $A_G(A)$ is complemented in $A(G_1 \times G_2)$.*

Proof. Since $I_{G_i}(A_1)$ is complemented, $A_i \in \mathscr{R}(G_{i_d})$. Therefore $A = A_1 \times A_2 \in \mathscr{R}((G_1 \times G_2)_d)$. It follows that A is an s-set [22]. The corollary is therefore implied by Proposition 4.5. □

Lemma 4.7. *Let A, B be closed subsets of the locally compact group G. Suppose that there exists continuous projections P of $A(G)$ onto $I(A)$ and Q of $A(G)$ onto $I(B)$. If $s(A/B, B/A) < \infty$, then there exists a continuous projection Γ of $A(G)$ onto $I(A \cup B)$ with $\|\Gamma\| \leq (s(A\backslash B, B\backslash A) + 2)(\|P\| + \|Q\|)$.*

Proof. Let $u \in \rho(A\backslash B, B\backslash A)$ with $\|u\|_{B(G)} \leq s(A\backslash B, B\backslash A) + 1$. Let $v \in A(G)$. Define $\Gamma v = uPv + (1-u)Qv$. It is easy to see that Γ is the desired projection. □

Corollary 4.8. *Let $A \subset G$ be closed. Let F be finite. If there exists a continuous projection of $A(G)$ onto $I(A)$, then there exists a continuous projection of $A(G)$ onto $I(A \cup F)$.*

Proposition 4.9. *Let A, B be disjoint closed subsets of G. Assume that $s(A, B) < \infty$. Then, there exists a continuous projection of $A(G)$ onto $I(A \cup B)$ if and only if there exists continuous projections of $A(G)$ onto $I(A)$ and $I(B)$ respectively.*

Proof. If both $I(A)$ and $I(B)$ are complemented, then by Lemma 4.7, $I(A \cup B)$ is complemented in $A(G)$.

Conversely, assume that P is a continuous projection of $A(G)$ onto $I(A \cup B)$. Let $u \in \rho(A, B)$. Define $Qv = P(uv) + v - uv$, $Q_1 v = P((1-u)v) - uv$. Then Q, Q_1 are the desired projections onto $I(A)$ and $I(B)$ respectively. □

Proposition 4.10. *Let G be a locally compact group and let H be an open abelian subgroup of G which is of finite index in G. If $A \subset G$ is such that $I(A)$ is complemented in $A(G)$, then $A \in \mathscr{R}(G_d)$.*

Proof. It follows from Proposition 4.9 that $I(H \cap A)$ is complemented in $A(G)$ by means of a continuous projection P. If we identify $A(H)$ with the closed subalgebra of $A(G)$ consisting of functions in $A(G)$ which are zero on $G\backslash H$, the restriction of P to $A(H)$ determines a continuous projection of $A(H)$ onto $I_H(H \cap A)$. By Rosenthal's result $H \cap A \in \mathscr{R}(H_d) \subseteq \mathscr{R}(G_d)$.

But $A = \bigcup_{i=1}^n (x_i H \cap A)$ for some finite set $x_1, \ldots, x_n \in G$. By translating and repeating the above argument we get that $x_i H \cap A \in \mathscr{R}(G_d)$ and hence that $A \in \mathscr{R}(G_d)$. □

Proposition 4.10 is a modest improvement of Rosenthal's result. Modifications of the above argument will yield the following two results.

Proposition 4.11. *Let G be a locally compact group with an open abelian subgroup. Let $A \subset G$ be compact. If $I(A)$ is complemented in $A(G)$, then $A \in \mathscr{R}(G_d)$.*

Theorem 4.12. *Let G be a locally compact group with an open subgroup H which is such that every closed subgroup H_1 of H belongs to* CIP(H). *Let H_2 be any closed subgroup of G. There exists a linear map $\Gamma\colon A(H_2) \to A(G)$ such that $\Gamma u_{|H_2} = u$ for every $u \in A(G)$ and if $K \subset H_2$ is compact, then $\|\Gamma_{|A_K(H_2)}\| \leq C_K$ for some constant which depends only upon K.*

Proof. Let $H_2 = \bigcup_{\alpha \in \mathscr{A}} x_2(H \cap H_2)$. If $u \in A(H_2)$, then we can write u as the formal sum $u = \sum_\alpha u_\alpha$ where u_α is supported on $x_\alpha(H \cap H_2)$. Since H is open, if $K \subset H_2$ is compact, then each $u \in A(H_2)$ supported on K can be written as a finite sum $\sum_{i=1}^{n_K} u_{\alpha_i}$. Furthermore n_K is determined by K independent of u and $\|u_{\alpha_i}\|_{A(H_2)} \leq \|u\|_{A(H)}$.

Clearly $(H \cap H_2) \in \mathrm{CIP}(G)$. Let Γ_0 be the extending map associated with $(H \cap H_2)$ and $C_1 = \|\Gamma\|$. By repeated translation, we can extend each u_{α_i} via Γ_0 to an element $v_{\alpha_i} \in A(G)$ with $\|v_{\alpha_i}\|_{A(G)} \leq C\|u_{\alpha_i}\|_{A(H)}$. Let $\Gamma u = \sum_{i=1}^{n_K} \Gamma u_{\alpha_i}$. Note that $\|\Gamma u\| \leq n_K C_1 \|u\|_{A(H_2)}$. Thus $C_K = n_K C_1$.

This process is easily seen to determine a well-defined extending map from the space $A(H_2) \cap C_{00}(H_2)$ into $A(G)$.

To determine Γ on all of $A(H_2)$, extend a basis of $A(H_2) \cap C_{00}(H_2)$ to a basis of $A(H_2)$. For each of the new basis elements simply choose any extension in $A(G)$. Then extend the mapping linearly. □

Theorem 4.13. *Let G be a locally compact group with an open central subgroup. Let H be a closed subgroup of G. Then there exists a linear isometry $\Gamma\colon A(H) \to A(G)$ with $\Gamma u_{|_H} = u$ for every $u \in A(H)$. In particular, $H \in \mathrm{CIP}(G)$.*

Proof. Since $Z(G)$ is open, $K = Z(G)H$ is open in G and hence is also closed. We may now apply Theorem 4.1, and the remark following Corollary 4.2.

We are in a position to expand our investigation to a number of large classes of locally compact groups.

Proposition 4.14. *Let $G \in$ [SIN] be almost connected. Let H be a closed subgroup of G. If there exists a closed subgroup C with $Z(G) \subseteq C \subseteq C_G(H)$ for which $K = HC$ is closed in G, then $H \in \mathrm{CIP}(G)$.*

Proof. Since G is almost connected, there exists a compact subgroup K_1 in G and a vector subgroup V such that $V \times K_1$ is of finite index in G [21, p. 698]. If K is closed in G, then $K \cap (V \times K_1)$ is closed in $V \times K$. Furthermore $K \cap (V \times K_1)$ is of finite index in K and there exists a compact subgroup K^* of K_1 such that $K \cap (V \times K_1) = V \times K^*$. By Corollary 4.6, $V \times K^* \in \mathrm{CIP}(G)$. Hence $K \in \mathrm{CIP}(K)$ and thus $H \in \mathrm{CIP}(G)$. □

Corollary 4.15. *Let $G \in$ [SIN] be almost connected. Let H be a closed abelian subgroup of G. Then $H \in \mathrm{CIP}(G)$.*

Proposition 4.16. *Let $G \in$ [SIN] be almost connected. Let V be a closed vector subgroup of G. Then $V \in \mathrm{CIP}(G)$.*

Proof. As G is almost connected, G has a compact normal subgroup K such that G/K is an almost connected [SIN]-group [21, p. 698]. Since $V \cap K = \{e\}$, $\varphi\colon G \to G/K$ is an isomorphism of V onto $\varphi(V)$. However, $\varphi(V) \in \mathrm{CIP}(G/K)$ by Corollary 4.15 and the extending map can be lifted from $A(G/K)$ to $A(G)$. □

Proposition 4.17. *Let G be an $[\mathrm{FC}]^-$ group. Let H be a closed subgroup of G with no nontrivial compact subgroups. Then $H \in \mathrm{CIP}(G)$.*

Proof. Let G be an $[\mathrm{FC}]^-$ group. Then G has a compact normal subgroup K such that G/K is the direct product of a vector group and a discrete group. It follows that if $H \cap K = \{e\}$, then $H \in \mathrm{CIP}(G)$.

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