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# QED DOMAINS AND NED SETS IN $\overline{\mathbb{R}}^n$

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*Dedicated to Professor F. W. Gehring on his 65th birthday*

ABSTRACT. This paper contributes to the theory of quasiextremal distance (or QED) domains. We associate with every QED domain $D$ two QED constants $M(D)$ and $M^*(D)$ and exhibit how these constants reflect the geometry of $D$. For example, we give a geometric characterization for QED domains $D$ with $M^*(D) = 2$ and obtain some sharp estimates of QED constants $M(D)$ and $M^*(D)$ for different kinds of domains.

## INTRODUCTION

In this paper, we use the following notation for Euclidean $n$-space $\mathbb{R}^n$. Unit vectors in the directions of the rectangular coordinate axes in $\mathbb{R}^n$ are denoted by $e_1, \dots, e_n$. For $x \in \mathbb{R}^n$ and $0 < r < \infty$ we let $B^n(x, r)$ denote the open $n$-ball with center $x$ and radius $r$ and $S^{n-1}(x, r)$ its boundary.

1.1. **Capacities, moduli, and condensers.** Given an open set $D \subset \overline{\mathbb{R}}^n$ and a pair of disjoint compact sets $F_0$ and $F_1$ in $\overline{D}$, we let $\Delta(F_0, F_1; D)$ and $\Delta(F_0, F_1; \overline{\mathbb{R}}^n)$ denote the families of curves which join $F_0$ and $F_1$ in $D$ and in $\overline{\mathbb{R}}^n$, respectively, and $\operatorname{cap}(F_0, F_1; D)$ the conformal capacity of $F_0$ and $F_1$ relative to $D$. For a curve family $\Gamma$ we let $\operatorname{mod}(\Gamma)$ denote its modulus. By [Y1, Remark 1.6],

$$\operatorname{cap}(F_0, F_1; D) = \inf_{u \in W'} \int_D |\nabla u|^n dm, \tag{1.2}$$

where $W' = W'(F_0, F_1; D)$ is the set of all functions $u$ such that

(1) $u$ is continuous in $D \cup F_0 \cup F_1$,
(2) $0 \le u(x) \le 1$ for $x \in D$ and $u(x) = i$ for $x \in F_i$, $i = 0, 1$,
(3) $u$ is ACL in $D$.

Next, a condenser is a domain in $\overline{\mathbb{R}}^n$ whose complement consists of two disjoint compact sets $F_0$ and $F_1$. It is usually denoted by $R(F_0, F_1)$ or $R$. The

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conformal capacity and modulus of a condenser $R$ are denoted by $\operatorname{cap}(R)$ and $\operatorname{mod}(R)$, respectively. Then according to [G1]

$$\operatorname{mod}(R) = \left(\frac{\omega_{n-1}}{\operatorname{cap}(R)}\right)^{1/(n-1)} = \left(\frac{\omega_{n-1}}{\operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n))}\right)^{1/(n-1)} \tag{1.3}$$

where $\omega_{n-1}$ is the $n-1$ dimensional measure of $S^{n-1}(0, 1)$. The following result will be needed in what follows. For more details we refer the reader to [Y1, 3.4 and 3.25].

1.4. **Lemma.** *Suppose that $R = R(F_0, F_1)$ is a condenser in $\overline{\mathbb{R}}^n$. Then there exist a sequence of functions $\{u_j\} \subset W'(F_0, F_1; \overline{\mathbb{R}}^n)$ and an ACL-function $u$ in $R$ such that*

(a) $\int_R |\nabla u_j|^n dm < \infty$, $j = 1, 2, \dots$,

(b) $\nabla u_j \to \nabla u$ *in* $L^n(R)$ *and* $u_j \to u$ *uniformly on each compact subset of* $R$,

(c) $\operatorname{cap}(R) = \lim_{j\to\infty} \int_R |\nabla u_j|^n dm = \int_R |\nabla u|^n dm$,

(d) $\lim_{x\to x_0} u(x) = i$ *for each* $x_0$ *contained in a nondegenerate component of* $\partial R \subset F_i$, $i = 0, 1$.

(e) $u$ *is n-harmonic in* $R$, *i.e.,* $u$ *is a weak solution in* $R$ *to the differential equation*

$$\operatorname{div}(|\nabla u|^{n-2}\nabla u) = 0.$$

Finally for fixed $t > 0$, we let $R_T(t)$ denote the Teichmüller ring bounded by the line segment $x = se_1$, $-1 \le s \le 0$, and ray $x = se_1$, $t \le s \le \infty$. The modulus of $R_T(t)$ is denoted by

$$\operatorname{mod}(R_T(t)) = \log \Psi_n(t). \tag{1.5}$$

For further information about this topic, we refer to the reader to [G1, G2, G3, V1 and Y1].

1.6. **QED domains.** Following [GM], a closed set $E$ in $\overline{\mathbb{R}}^n$ is said to be an *M-QED exceptional set*, with $1 \le M < \infty$, if for each pair of disjoint continua $F_0$ and $F_1$ in $D = \overline{\mathbb{R}}^n \backslash E$,

$$\operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n)) \le M \cdot \operatorname{mod}(\Delta(F_0, F_1; D)). \tag{1.7}$$

In this case, $D$ is open, connected and hence a domain. We say that $D$ is an *M-QED domain.* A closed set $E$ is called an *NED set* if (1.7) holds with $M = 1$ for all choices of $F_0$ and $F_1$. This class of sets in the plane was introduced by Ahlfors and Beurling [AB].

QED domains are closely related to some other classes of domains such as uniform domains, quasisphere domains and linearly locally connected (or LLC) domains (see [GM] for definitions). For example, it was proved in [GM] that a finitely connected domain $D$ in $\overline{\mathbb{R}}^2$ is a QED domain if and only if it is a quasicircle domain. We formulate some other results related to QED domains which are needed in this paper in the following lemmas.

1.8. **Lemma** [GM, 2.11]. *If $D$ is an M-QED domain, then $D$ is c-linearly locally connected, where $c$ is a constant depending only on $M$ and $n$.*

1.9. **Lemma.** *If $D$ is a domain in $\overline{\mathbb{R}}^n$, then*

$$\operatorname{cap}(F_0, F_1; D) \ge \operatorname{mod}(\Delta(F_0, F_1; D)) \tag{1.10}$$

*for each pair of disjoint compact sets* $F_0, F_1$ *in* $\overline{D}$. *Furthermore,* (1.10) *holds with equality if* $F_0$ *and* $F_1$ *lie in* $D$ *or if* $D$ *is QED.*

Inequality (1.10) and the fact that it holds with equality when $F_0$ and $F_1$ lie in $D$ were proved by J. Hesse [H, Lemma 5.2 and Theorem 5.5]. The fact that (1.10) holds with equality when $D$ is a QED domain was proved by D. A. Herron and P. Koskela [HK, Theorem 2.6].

1.11. **Lemma** [HK, 2.8, 2.10]. *Inequality* (1.7) *holds for each pair of disjoint continua* $F_0$ *and* $F_1$ *in* $\overline{D}$ *if and only if it holds for all such pairs in* $D$. *The same conclusion holds for compact sets.*

We associate with a domain $D$ the following QED constants

$$M(D) = \sup\left(\frac{\operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n))}{\operatorname{mod}(\Delta(F_0, F_1; D))}\right), \tag{1.12}$$

where the supremum is taken over all pairs of disjoint continua $F_0$ and $F_1$ in $\overline{D}$ such that $\operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n))$ and $\operatorname{mod}(\Delta(F_0, F_1; D))$ are not simultaneously 0 or $\infty$, and

$$M^*(D) = \sup\left(\frac{\operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n))}{\operatorname{mod}(\Delta(F_0, F_1; D))}\right), \tag{1.13}$$

where the supremum is taken over all pairs of disjoint compact sets $F_0$ and $F_1$ in $\overline{D}$ such that $\operatorname{mod}(\Delta\, F_0, F_1; \overline{\mathbb{R}}^n))$ and $\operatorname{mod}(\Delta(F_0, F_1; D))$ are not simultaneously 0 or $\infty$. If $M(D) < \infty$ or $M^*(D) < \infty$, we say that $D$ is an $M(D)$-QED domain with respect to continua or compact sets, respectively. It is easy to see that both $M(D)$ and $M^*(D)$ are Möbius invariant and that

$$1 \le M(D) \le M^*(D) \le \infty. \tag{1.14}$$

Thus any domain which is QED with respect to compact sets is also QED with respect to continua. It would be interesting to decide whether or not these two classes of domains coincide. It follows that the symmetry principle for moduli of curve families (see [G1, V1]) that $M(D) = M^*(D) = 2$ if $D$ is a ball or a half space.

This paper is concerned with how these constants reflect the geometry of $D$. Section 2 is devoted to the study of NED sets and 1-QED domains. In §3 we characterize all 2-QED domains in $\overline{\mathbb{R}}^n$ geometrically. In §4 we derive some sharp lower and upper bounds of $M(D)$ and $M^*(D)$ for different kinds of domains in $\overline{\mathbb{R}}^n$. Applying some results from §4, we determine in §5 the values of $M(D)$ and $M^*(D)$ for some special planar domains. In §5 we also give a proof for the result that every QED domain in $\overline{\mathbb{R}}^2$ is a quasicircle domain. This argument yields an explicit and much sharper estimate for the quasicircle constant $K$ in terms of the QED constant $M(D)$. This estimate yields particularly simple proofs for several properties of QED domains in $\overline{\mathbb{R}}^2$, some of which are obtained in previous sections by other means.

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## 2. NED sets and 1-QED domains

In this section we will study some properties of NED sets, a class of sets which has been studied by many authors [AB, AS, GM, V2]. It is obvious that this class is Möbius invariant. Now, we generalize the concept of QED exceptional sets as follows.

2.1. **Definition.** A closed set $E$ in $\overline{\mathbb{R}}^n$ is said to be an $M$-QED *exceptional set relative to a domain* $D$, $1 \le M < \infty$, if $E \subset D$ and if for each pair of disjoint continua $F_0$ and $F_1$ in $D \backslash E$,

$$\operatorname{mod}(\Delta(F_0, F_1; D)) \le M \cdot \operatorname{mod}(\Delta(F_0, F_1; D \backslash E)). \tag{2.2}$$

The following results will be needed in what follows. For more details we refer the reader to [Y1].

2.3. **Lemma** [Y1, 2.5]. *Suppose that $D$ is a domain in $\overline{\mathbb{R}}^n$, that $F_0$ and $F_1$ are disjoint compact sets in $D$, and that $u \in W'(F_0, F_1; D)$ with*

$$\int_D |\nabla u|^n dm < \infty.$$

*Then there exists a function $v \in W'(F_0, F_1; D)$ such that*

$$\int_D |\nabla v|^n dm \le \int_D |\nabla u|^n dm$$

*and $v$ is monotone in $D$ relative to $F = F_0 \cup F_1$, i.e., for any $\varepsilon > 0$ each point $x_0 \in D$ can be joined to the set $F$ by curves $\gamma_1$ and $\gamma_2$ in $D$ such that*

$$v(x) \le v(x_0) + \varepsilon \quad \text{for all } x \in \gamma_1$$

*and*

$$v(x) \ge v(x_0) - \varepsilon \quad \text{for all } x \in \gamma_2.$$

2.4. **Lemma** [Y1, 2.24]. *Suppose that $E$ is an $M$-QED exceptional set relative to a domain $D$ in $\overline{\mathbb{R}}^n$ with $m(E) = 0$, that $G \subset \overline{\mathbb{R}}^n$ is an open set, and that $F_0$, $F_1$ are disjoint compact sets in $G \backslash E$. Suppose also that a function $v$ in $W'(F_0, F_1; G \backslash E)$ is monotone in $G \backslash E$ relative to $F = F_0 \cup F_1$ and*

$$\int_{G \backslash E} |\nabla v|^n dm < \infty.$$

*Then $v$ can be extended to be a function $v^* \in W'(F_0, F_1; G)$.*

The main result in this section is the following series of equivalent assertions for closed sets.

2.5. **Theorem.** *Let $E$ be a closed set in $\overline{\mathbb{R}}^n$ and $G = \overline{\mathbb{R}}^n \backslash E$. Then the following statements are equivalent.*

(a) *$E$ is NED;*

(b) $\operatorname{int}(E) = \varnothing$ *and $E$ is an $M$-QED exceptional set relative to some domain $D$ with $1 \le M < \infty$;*

(c) $M^*(G) = 1$;

(d) $M(G) = 1$.

To prove Theorem 2.5, we need a result similar to [GM, 2.16].

2.6. **Lemma.** *If a closed set $E$ satisfies condition* (b) *in Theorem* 2.5, *then $D\backslash E$ is a domain and $m(E) = 0$.*

*Proof.* Since $E$ is closed, $D' = D\backslash E$ is open. Suppose $D'$ is not connected and let $D_0, D_1$ be two disjoint components of $D'$. Choose nondegenerate continua $F_i \subset D_i \cap \mathbb{R}^n$ and points $x_i \in F_i$, $i = 0, 1$. By [M1, Theorem 3.11], there is a $K$-quasiconformal map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $x_i \in f(B^n) \subset D$, $i = 0, 1$, where $B^n = B^n(0, 1)$. Let $F_i'$ be the component of $F_i \cap f(\overline{B}^n)$ containing $x_i$, $i = 0, 1$. Since $B^n$ is 2-QED, $f(B^n)$ is $M$-QED where $M = M(K)$ by [GM, 2.5] and hence

$$\begin{aligned}\operatorname{mod}(\Delta(F_0, F_1; D)) &\geq \operatorname{mod}(\Delta(F_0', F_1'; f(B^n)))\\ &\geq \frac{1}{M}\operatorname{mod}(\Delta(F_0', F_1'; \overline{\mathbb{R}}^n)) > 0\end{aligned}$$

by [GM, 2.6]. On the other hand, since $\Delta(F_0, F_1; D') = \varnothing$,

$$\operatorname{mod}(\Delta(F_0, F_1; D')) = 0.$$

These two conclusions contradict (2.2) and hence $D' = D\backslash E$ is connected.

We next show that for each $x_0 \in E \cap \overline{\mathbb{R}}^n$ and $0 < r < d(E, \partial D)$,

$$\frac{m(D' \cap B^n(x_0, r))}{m(B^n(x_0, r))} \geq \frac{c}{M} \tag{2.7}$$

where $c$ is a positive number depending only on $n$.

Fix $x_0 \in E \cap \mathbb{R}^n$, $0 < r < d(E, \partial D)$ and let $s = \frac{r}{4}$. By performing a preliminary translation, we may assume that $x_0 = 0$. Since $\operatorname{int}(E) = \varnothing$ and $D' = D\backslash E$, $D' \cap G \neq \varnothing$ for any open set $G \subset D$. Thus there exist points $z_0 \in D' \cap B^n(0, s)$, $z_1 \in D'\backslash\overline{B}^n(0, r)$. Then since we can join $z_0$ to $z_1$ by an arc $F$ in $D'$, there exist subarcs $F_0$ in $D' \cap \{s \leq |x| \leq 2s\}$ and $F_1$ in $D' \cap \{3s \leq |x| \leq 4s\}$ with end points $x_1, x_2$ and $y_1, y_2$, respectively, where $|x_1| = s$, $|x_2| = 2s$, $|y_1| = 3s$ and $|y_2| = 4s$. Let $\Gamma, \Gamma_D, \Gamma_D'$ and $\Gamma_B$ be the families of curves joining $F_0$ and $F_1$ in $\overline{\mathbb{R}}^n$, $D$, $D'$ and $B^n(0, r)$, respectively, and set

$$\rho(x) = \begin{cases} \frac{1}{s}, & \text{for } x \in D' \cap B^n(0, r),\\ 0, & \text{elsewhere}. \end{cases}$$

Since each curve in $\Gamma_D'$ contains a subarc joining $S^{n-1}(0, 2s)$ and $S^{n-1}(0, 3s)$, $\rho(x)$ is admissible for the curve family $\Gamma_D'$ and

$$\operatorname{mod}(\Gamma_D') \leq \int_{\mathbb{R}^n} \rho(x)^n dm = \frac{m(D' \cap B^n(0, r))}{s^n} = c_n \frac{m(D' \cap B^n(0, r))}{m(B^n(0, r))},$$

where $c_n$ is a constant depending only on $n$.

On the other hand, since $x_1, x_2 \in F_0$ and $y_1, y_2 \in F_1$,

$$\min_{j=0,1} \operatorname{dia}(F_j) \geq s \geq \tfrac{1}{5}|x_2 - y_1| \geq \tfrac{1}{5}d(F_0, F_1)$$

and Lemma 2.6 in [GM] implies that

$$\operatorname{mod}(\Gamma) \geq c_0 > 0,$$

where $c_0$ is a constant depending only on $n$. Thus, the hypothesis that $E$ is an $M$-QED exceptional set relative to $D$ and the fact that $B^n(0, r)$ is 2-QED

imply that

$$c_n \frac{m(D' \cap B^n(0, r))}{m(B^n(0, r))} \geq \operatorname{mod}(\Gamma'_D) \geq \frac{1}{M} \operatorname{mod}(\Gamma_D)$$
$$\geq \frac{1}{M} \operatorname{mod}(\Gamma_B) \geq \frac{1}{2M} \operatorname{mod}(\Gamma) \geq \frac{c_0}{2M}.$$

Therefore, (2.7) holds with $c = c_0/(2c_n)$.

Finally, (2.7) implies that no point $x_0 \in E \cap \mathbb{R}^n$ can be a point of density for $E$ and hence that $m(E) = 0$ as desired. □

2.8. **Proof of Theorem 2.5.** If $E$ is NED, then $m(E) = 0$ by [V2, Theorem 1]. Thus (a) $\Rightarrow$ (b) with $D = \overline{\mathbb{R}}^n$ and $M = 1$. It follows immediately from the definitions that (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

It remains to show that (b) $\Rightarrow$ (c). For this we first show that $G = \overline{\mathbb{R}}^n \backslash E$ is connected. By Lemma 2.6, $D \backslash E \subset G$ is a domain. Suppose $G$ is not connected and let $G_1, G_2$ be two different components of $G$ with $D \backslash E \subset G_1$. Since $\operatorname{int}(E) = \varnothing$ and $E \subset D$, then $\partial G_1 = E$ and $\partial G_1 \cap \partial G_2 \neq \varnothing$. Let $P \in \partial G_1 \cap \partial G_2 \subset E$ and choose $r > 0$ so that $B^n(P, r) \subset D$. Then $B^n(P, r) \backslash E \subset G_1$ and $G_2 \cap B^n(P, r) \backslash E \neq \varnothing$, hence $G_1 \cap G_2 \neq \varnothing$ contradicting the assumption that $G_1$ and $G_2$ are two different components of $G$. Therefore $G$ is connected.

Next we show that for any two disjoint compact sets $F_0, F_1$ in $G$,

$$\operatorname{cap}(F_0, F_1; \overline{\mathbb{R}}^n) \leq \operatorname{cap}(F_0, F_1; G). \tag{2.9}$$

Let $u$ be a function in $W'(F_0, F_1; G)$ with

$$\int_G |\nabla u|^n dm < \infty.$$

By Lemma 2.3, there exists a function in $u'$ in $W'(F_0, F_1; G)$ such that

$$\int_G |\nabla u'|^n dm \leq \int_G |\nabla u|^n dm$$

and such that $u'$ is monotone in $G$ relative to $F_0 \cup F_1$. Then since $G = \overline{\mathbb{R}}^n \backslash E$, Lemmas 2.4 and 2.6 and condition (b) imply that $u'$ has an extension $u^*$ which is in $W'(F_0, F_1; \overline{\mathbb{R}}^n)$. Thus

$$\operatorname{cap}(F_0, F_1; \overline{\mathbb{R}}^n) \leq \int_{\mathbb{R}^n} |\nabla u^*|^n dm = \int_G |\nabla u'|^n dm \leq \int_G |\nabla u|^n dm.$$

Taking the infimum over all such functions $u$ in $W'(F_0, F_1; G)$ yields (2.9).

Finally, Lemmas 1.9 and 1.11 and inequality (2.9) imply that $M^*(G) \leq 1$, and hence that $M^*(G) = 1$ as desired.

## 3. 2-QED domains in $\overline{\mathbb{R}}^n$

In this section, we characterize 2-QED domains in $\overline{\mathbb{R}}^n$. For this we must extend the concept of NED sets to those which are not necessarily compact.

3.1. **Definition.** A set $A$ in $\overline{\mathbb{R}}^n$ is said to be an *NED set* if each compact subset $E$ of $A$ is an NED set in the usual sense.

Our main result in this section is the following.

3.2. **Theorem.** *If $D$ is a domain in $\overline{\mathbb{R}}^n$, then $M^*(D) = 2$ if and only if $D$ is Möbius equivalent to a ball minus an NED set.*

In order to prove Theorem 3.2, we establish two lemmas first.

3.3. **Lemma.** *Suppose that $R = R(F_0, F_1)$ is a condenser in $\overline{\mathbb{R}}^n$ such that $F_0$ and $F_1$ contain nondegenerate continua $F_0'$ and $F_1'$, respectively, that $u(x)$ is the extremal function for the capacity of $R$ defined in Lemma 1.4, and that $G$ is a domain in $R$. Then*

$$\int_G |\nabla u|^n dm > 0 \tag{3.4}$$

*if either $n = 2$ or $\partial G \cap (F_0' \cup F_1') \neq \varnothing$.*

*Proof.* Suppose (3.4) does not hold. Then $\nabla u = 0$ a.e. in $G$ and hence $u$ is constant in $G$.

If $n = 2$, then $u$ is harmonic in $R$. By the uniqueness theorem for harmonic functions, $u$ is identically constant in $R$.

If $\partial G \cap (F_0' \cup F_1') \neq \varnothing$, then we may assume that $\partial G \cap F_0' \neq \varnothing$. By Lemma 1.4(d) and the continuity of $u$, $u \equiv 0$ in $G$. Thus by using the Harnack inequality [GLM, Theorem 4.15], one can show that $u \equiv 0$ in $R$.

In both cases we have shown that $u$ is identically constant in $R$. Therefore

$$\operatorname{cap}(R) = \int_R |\nabla u|^n dm = 0$$

which contradicts the fact that $\operatorname{cap}(R) > 0$. Hence (3.4) holds. □

*Remarks.* As one can see from the proof, Lemma 3.3 holds under weaker assumptions. For example, it suffices to assume that $\partial G$ meet $F_0$ or $F_1$ at a Wiener regular point and that $\operatorname{cap}(R) > 0$. However, this has no effect in what follows.

Here and in what follows we let $H$ denote the upper half space $\{x \in \overline{\mathbb{R}}^n : x_n > 0\}$. Next, for any set $F$ and point $x$ in $\overline{\mathbb{R}}^n$, we let $F^*$ and $x^*$ denote the images of $F$ and $x$ under the reflection in $\partial H$, respectively.

3.5. **Lemma.** *If $D$ is a domain in $\mathbb{R}^n$ which is symmetric with respect to the plane $\partial H$ and if $F$ is a compact set in $D \cap H$, then*

$$\operatorname{mod}(\Delta(F, F^*; D)) = 2^{1-n}\operatorname{mod}(\Delta(F, D \cap \partial H; D \cap H)). \tag{3.6}$$

*Proof.* Let $\Gamma = \Delta(F, F^*; D)$ and $\Gamma_1 = \Delta(F, D \cap \partial H; D \cap H)$, fix $\rho$ in $\operatorname{adm}(\Gamma_1)$ and set

$$\rho^*(x) = \begin{cases} \frac{1}{2}\rho(x), & \text{if } x \in \overline{H}, \\ \frac{1}{2}\rho(x^*), & \text{if } x \in H^*. \end{cases}$$

It is not difficult to see that $\rho^*$ is in $\operatorname{adm}(\Gamma)$. Therefore,

$$\operatorname{mod}(\Gamma) \leq \int_H 2^{-n}\rho^n\, dm + \int_{H^*} 2^{-n}\rho(x^*)^n\, dm \leq 2^{1-n}\int_{\mathbb{R}^n} \rho^n\, dm,$$

and taking the infimum over all $\rho$ in $\operatorname{adm}(\Gamma_1)$ yields

$$\operatorname{mod}(\Gamma) \leq 2^{1-n}\operatorname{mod}(\Gamma_1). \tag{3.7}$$

Next, for the reverse inequality, by Lemma 4.6 in [H] we need only to consider functions $\rho$ in $\operatorname{adm}(\Gamma)$ which are continuous in $D\backslash(F\cup F^*)$. Fix such a function $\rho$ and let

$$\rho^*(x)=\begin{cases}\rho(x)+\rho(x^*), & \text{if } x\in H,\\ 0, & \text{if } x\in H^*.\end{cases}$$

Suppose that $\gamma_1\in\Gamma_1$ is locally rectifiable. Choose $x_0\in\overline{\gamma}_1\cap\partial H\cap D$ and $r>0$ so that $\overline{B}^n(x_0,r)\subset D\backslash(F\cup F^*)$, and let

$$\max_{x\in\overline{B}^n(x_0,r)}\rho(x)=M<\infty.$$

Let $x_1$ be the first point where $\gamma_1$ meets $S^{n-1}(x_0,t)$ with $0<t<r$ and denote the subarc of $\gamma_1$ from $F$ to $x_1$ by $\gamma_1'$. Then

$$\gamma=\gamma_1'\cup[x_1,x_1^*]\cup(\gamma_1')^*$$

is a locally rectifiable curve in $\Gamma$, where $[x_1,x_1^*]$ is the line segment joining $x_1$ and $x_1^*$. Therefore,

$$\begin{aligned}1&\le\int_\gamma\rho\,ds\le\int_{\gamma_1'}\rho\,ds+\int_{(\gamma_1')^*}\rho ds+2tM\\&=\int_{\gamma_1'}\rho^*ds+2tM\le\int_{\gamma_1}\rho^*ds+2tM.\end{aligned}$$

Letting $t\to0$ yields $\int_{\gamma_1}\rho^*ds\ge1$. Thus $\rho^*\in\operatorname{adm}(\Gamma_1)$ and

$$\operatorname{mod}(\Gamma_1)\le\int_H(\rho^*)^ndm=\frac12\int_{\mathbb{R}^n}(\rho(x)+\rho(x^*))^ndm\le2^{n-1}\int_{\mathbb{R}^n}\rho(x)^ndm.$$

Taking the infimum over all such functions $\rho$, we obtain

$$\operatorname{mod}(\Gamma_1)\le2^{n-1}\operatorname{mod}(\Gamma). \tag{3.8}$$

Finally, (3.6) follows from (3.7) and (3.8) as desired. □

3.9. **Proof of the necessity in Theorem 3.2.** Let $M^*(D)=2$. We will divide the proof into three propositions.

**Proposition 1.** *Each component of $\overline{\mathbb{R}}^n\backslash\overline{D}$ is Möbius equivalent to the unit ball.*

*Proof*. Suppose $G$ is a nonempty component of $\overline{\mathbb{R}}^n\backslash\overline{D}$ which is not Möbius equivalent to the unit ball. Then by performing a preliminary Möbius transformation, we may assume that $D$ lies in the upper half space $H$, that $\infty\in G$, and that $\partial G\cap\partial H$ contains at least two points $P_0$ and $P_1$. For $i=0,1$ and $k=1,2,\dots$, let $C_i^k$ denote the components of $\overline{D}\cap\{x\colon x_n\le1/k\}$ which contain $P_i$. We consider two cases.

*Case* 1. Suppose that $C_0^k\ne C_1^k$ for some $k$. Then $\overline{D}\cap\{x\colon x_n\le1/k\}$ contains at least two disjoint nondegenerate components $C_0^k$ and $C_1^k$. By [HY, Theorem 2.9, p. 44] there exists a decomposition

$$\overline{D}\cap\{x\colon x_n\le1/k\}=E_0\cup E_1$$

such that $C_i^k\subset E_i$ for $i=0,1$ and that $E_0$ and $E_1$ are disjoint and compact. This decomposition induces the following decomposition

$$\overline{D}\cap\{x\colon x_n=1/k\}=F_0\cup F_1$$

such that $F_0$ and $F_1$ are disjoint compact sets with $F_i = E_i \cap \{x_n = 1/k\}$ for $i = 0, 1$. Obviously, $F_0$, $F_1$ contain interior points of $D$ and hence contain nondegenerate components. Set $H' = \{x : x_n > 1/k\}$ and $D' = D \cap H'$ and let $u(x)$ be the extremal function for the conformal capacity of the condenser $R = R(F_0, F_1)$ defined in Lemma 1.4. We claim that

$$\int_{H' \setminus \overline{D}} |\nabla u|^n dm \geq \delta > 0 \tag{3.10}$$

for some $\delta > 0$.

If $n = 2$, (3.10) follows immediately from Lemma 3.3.

If $n \geq 3$, we let $G_0$ be the unbounded component of $G \cap H'$ and $A = \partial G_0 \cap \partial H'$. Then $A$ is a closed proper subset of $\partial H' = \overline{\mathbb{R}}^{n-1}$. Let $A'$ be the boundary of $A$ in $\overline{\mathbb{R}}^{n-1}$. Since $A$ contains interior points in $\overline{\mathbb{R}}^{n-1}$, $A'$ contains a nondegenerate conponent $E$ by [N, Theorem V.14.3]. We claim that

$$E \subset A' \subset \partial G \cap \partial H' \subset \partial D \cap \partial H' = F_0 \cup F_1. \tag{3.11}$$

Since $\partial G \subset \partial D$, it suffices to show that $A' \subset \partial G$. Suppose $x_0 \in A' \subset A$. Then $x_0 \in \overline{G}$. If $x_0 \in G$, then there exists $r > 0$ such that $B^n(x_0, r) \subset G$. This implies that $B^n(x_0, r) \cap H' \subset G_0$ and that $x_0$ is an interior point of $A$ in $\overline{\mathbb{R}}^{n-1}$ contradicting the assumption that $x_0 \in A'$. Therefore (3.11) follows and

$$E \subset (F_0 \cup F_1) \cap \partial G_0.$$

Thus (3.10) follows from Lemma 3.3 since $G_0 \subset H' \setminus \overline{D}$.

Next, let $\{u_j\} \subset W'(F_0, F_1; \overline{\mathbb{R}}^n)$ be the sequence of functions defined in Lemma 1.4. Since $\nabla u_j \to \nabla u$ in $L^n(R)$ as $j \to \infty$, for any $\varepsilon > 0$ we can choose $j$ such that

$$\left( \int_R |\nabla u_j - \nabla u|^n dm \right)^{1/n} < \varepsilon. \tag{3.12}$$

Next since $R(F_0, F_1)$ is symmetric with respect to $\partial H'$, it is not difficult to see that

$$\operatorname{cap}(F_0, F_1; \overline{\mathbb{R}}^n) = \int_R |\nabla u|^n dm = 2 \int_{H'} |\nabla u|^n dm.$$

Thus by (3.10), (3.12) and the fact that $u_j \in W'(F_0, F_1; D')$,

$$\begin{aligned} \operatorname{cap}(F_0, F_1; \overline{\mathbb{R}}^n) &= 2 \int_{H' \setminus \overline{D}} |\nabla u|^n dm + 2 \int_{D'} |\nabla u|^n dm \\ &\geq 2\delta + 2 \left\{ \left( \int_{D'} |\nabla u_j|^n dm \right)^{1/n} - \varepsilon \right\}^n \\ &\geq 2\delta + 2 \left( \operatorname{cap}(F_0, F_1; D')^{1/n} - \varepsilon \right)^n. \end{aligned}$$

Letting $\varepsilon \to 0$, we conclude that

$$\operatorname{cap}(F_0, F_1; \overline{\mathbb{R}}^n) \geq 2\delta + 2\operatorname{cap}(F_0, F_1; D'). \tag{3.13}$$

On the other hand, since each curve $\gamma$ in $\Delta(F_0, F_1; D)$ contains a subcurve in $\Delta(F_0, F_1; D')$,

$$\operatorname{mod}(\Delta(F_0, F_1; D)) \leq \operatorname{mod}(\Delta(F_0, F_1; D')). \tag{3.14}$$

Then (3.13), (3.14) and Lemma 1.9 imply that

$$\operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n)) \geq 2\delta + 2\operatorname{mod}(\Delta(F_0, F_1; D)) \tag{3.15}$$

which contradicts the assumption that $M^*(D) = 2$. Hence $G$ is Möbius equivalent to the unit ball and this completes the proof of Proposition 1 for Case 1.

*Case* 2. Suppose next that $C_0^k = C_1^k$ for all $k$. Then by [N, Theorem IV.5.3],

$$C = \bigcap_{k=1}^{\infty} C_1^k$$

is a continuum in $\partial D \cap \partial H$ containing $P_0$ and $P_1$. We claim that there exist two disjoint nondegenerate continua $F_0$ and $F_1$ in $\partial D \cap \partial H$ such that

$$\int_{H \setminus \overline{D}} |\nabla u|^n dm \geq \delta > 0 \tag{3.16}$$

for some $\delta > 0$, where $u$ is the extremal function for the capacity of the condenser $R(F_0, F_1)$.

Let $G_0$ be the unbounded component of $G \cap H$ and let $A = \partial G_0 \cap \partial H$. If $n = 2$, then we choose two disjoint nondegenerate continua $F_0$ and $F_1$ in $C \subset \partial D \cap \partial H$. Since $G_0 \subset H \setminus \overline{D}$, (3.16) holds for some $\delta > 0$ by Lemma 3.3.

If $A = \partial H$, then we also choose two disjoint nondegenerate continua $F_0$ and $F_1$ in $C \subset \partial D \cap \partial H$. Since $G_0 \subset H \setminus \overline{D}$ and

$$F_0 \cup F_1 = (F_0 \cup F_1) \cap \partial G_0 \neq \varnothing,$$

(3.16) holds for some $\delta > 0$ by Lemma 3.3.

If $A \neq \partial H = \overline{\mathbb{R}}^{n-1}$ and $n \geq 3$, as in Case 1 we can choose a nondegenerate continuum $E$ in $\partial G_0 \cap \partial D \cap \partial H$. Then we choose two disjoint nondegenerate continua $F_0$ and $F_1$ in $E$. By Lemma 3.3 again, (3.16) holds for some $\delta > 0$. This proves our claim.

Next, since $F_0$ and $F_1$ are nondegenerate continua, the extremal function $u$ is in $W'(F_0, F_1; R)$ and hence $W'(F_0, F_1; D)$ by Lemma 1.4. Thus by the symmetry of $R(F_0, F_1)$ and (3.16),

$$\begin{aligned} \operatorname{cap}(F_0, F_1; \overline{\mathbb{R}}^n) &= 2\int_H |\nabla u|^n dm \geq 2\delta + 2\int_D |\nabla u|^n dm \\ &\geq 2\delta + 2\operatorname{cap}(F_0, F_1; D). \end{aligned} \tag{3.17}$$

This together with Lemma 1.9 contradicts the hypothesis that $M^*(D) = 2$ and completes the proof of Proposition 1 for Case 2. □

**Proposition 2.** *$\overline{\mathbb{R}}^n \setminus \overline{D}$ has one and only one nonempty component.*

*Proof*. Since $M^*(D) = 2$, Theorem 2.5 with $E = \overline{\mathbb{R}}^n \setminus D$ implies that

$$\operatorname{int}(\overline{\mathbb{R}}^n \setminus D) \neq \varnothing.$$

Thus $\overline{\mathbb{R}}^n \setminus \overline{D}$ has at least one nonempty component. Suppose $\overline{\mathbb{R}}^n \setminus \overline{D}$ has two different nonempty components $G_0$ and $G_1$. By performing a preliminary Möbius transformation and appealing to Proposition 1, we may assume that $G_0 = B^n(0, 1)$ and $G_1 = \overline{\mathbb{R}}^n \setminus \overline{B}$ where $B$ is a ball containing $B^n(0, 1)$. Choose

a nondegenerate continuum $F_0$ in $S^{n-1}(0, 1) \cap H$ and denote the symmetric images of $F_0$ and $H$ in $\partial H$ by $F_0^* = F_1$ and $H^*$, respectively. Let $D_1$ denote the domain bounded by spheres $S^{n-1}(0, 1)$ and $\partial B$ and $D_1'$ the symmetric image of $D_1$ in $S^{n-1}(0, 1)$. Then since $D \subset D_1$,

$$\text{(3.18)} \qquad \operatorname{mod}(\Delta(F_0, F_1; D)) \leq \tfrac{1}{2}\operatorname{mod}(\Delta(F_0, F_1; D_1 \cup D_1' \cup S(0, 1))).$$

Next choose $r > 0$ so that $D_1 \subset B^n(0, r)$ and let $G = H \cap B^n(0, r)$, $E = \partial H \cap B^n(0, r)$ and $F = \partial H \cap \overline{B}^n(0, r)$. By Lemma 3.5,

$$\text{(3.19)} \qquad \operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n)) = 2^{1-n}\operatorname{mod}(\Delta(F_0, \partial H; H)),$$

and

$$\text{(3.20)} \qquad \operatorname{mod}(\Delta(F_0, F_1; B^n(0, r))) = 2^{1-n}\operatorname{mod}(\Delta(F_0, E; G)).$$

Now we claim that

$$\text{(3.21)} \qquad \operatorname{mod}(\Delta(F_0, \partial H; H)) > \operatorname{mod}(\Delta(F_0, E; G)).$$

Let $u$ be the extremal function for the condenser $R = R(F_0, \overline{H}^*)$. By Lemma 3.3, we see that

$$\int_{H \setminus \overline{G}} |\nabla u(x)|^n dm \geq \delta > 0.$$

On the other hand, by Lemma 1.4, $u$ is in $W'(F_0, \overline{H}^*; \overline{\mathbb{R}}^n)$, hence in

$$W'(F_0, F; G).$$

Therefore, since $E \subset F$, it follows from Lemma 1.9 that

$$\begin{aligned}\operatorname{mod}(\Delta(F_0, \partial H; H)) = \operatorname{cap}(R(F_0, \overline{H}^*)) &\geq \delta + \int_G |\nabla u(x)|^n dm \\ &\geq \delta + \operatorname{cap}(F_0, F; G) \geq \delta + \operatorname{mod}(\Delta(F_0, E; G)).\end{aligned}$$

This proves (3.21). Finally (3.18), (3.19), (3.20) and (3.21) imply that

$$\begin{aligned}\operatorname{mod}(\Delta(F_0, F_1; D)) &\leq \tfrac{1}{2}\operatorname{mod}(\Delta(F_0, F_1; B^n(0, r))) \\ &< \tfrac{1}{2}\operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n)).\end{aligned}$$

This contradicts the assumption that $M^*(D) = 2$ and hence completes the proof of Proposition 2. □

**Proposition 3.** *$D$ is Möbius equivalent to the unit ball minus an NED set.*

*Proof.* By Propositions 1 and 2, we may assume that $D \subset B^n(0, 1) = B$ and $\overline{D} = \overline{B}$. We must show that $A = B \setminus D$ is an NED set. Let $E$ be any compact subset of $A$ and choose $0 < r < 1$ such that $E \subset B^n(0, r)$. We claim that $E$ is $M$-QED exceptional relative to $B$. In fact, for any pair of disjoint continua $F_0$ and $F_1$ in $B \setminus E$, we have

$$\begin{aligned}\operatorname{mod}(\Delta(F_0, F_1; B)) &\leq \operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n)) \leq 2\operatorname{mod}(\Delta(F_0, F_1; D)) \\ &\leq 2\operatorname{mod}(\Delta(F_0, F_1; B \setminus E)).\end{aligned}$$

This shows that $E$ is 2-QED exceptional relative to $B$. Obviously, $\operatorname{int}(E) = \varnothing$. Therefore, Theorem 2.5 implies that $E$ is NED. By Definition 3.1, $A$ is an

NED set. This completes the proof of Proposition 3 and hence the proof of the necessity in Theorem 3.2. □

3.22. **Proof of the sufficiency in Theorem 3.2.** Suppose that $D$ is a domain in the unit ball $B$ and that $A = B\backslash D$ is an NED set. Then $\partial B \subset \partial D$ and it is easy to see that $M^*(D) \geq 2$. It remain to show that

$$\operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n)) \leq 2 \operatorname{mod}(\Delta(F_0, F_1; D)) \tag{3.23}$$

for each pair of disjoint compact sets $F_0$ and $F_1$ in $D$. For this we first show that

$$\operatorname{cap}(F_0, F_1; B) \leq \operatorname{cap}(F_0, F_1; D) \tag{3.24}$$

Let $u$ be any function in $W'(F_0, F_1; D)$. with

$$\int_D |\nabla u|^n dm < \infty.$$

Then by Lemma 2.3, there exists a function $v$ in $W'(F_0, F_1; D)$ such that

$$\int_D |\nabla v|^n dm \leq \int_D |\nabla u|^n dm$$

and $v$ is monotone in $D$ relative to $F_0 \cup F_1$.

Next let $B_j = B^n(0, j/(j+1))$, $E_j = A \cap \overline{B}_j$ and $D_j = \operatorname{int}(D \cup E_j)$ for $j = 1, 2, \ldots$. Then $D = D_j \backslash E_j$. Since $A$ is an NED set and $E_j$ is a compact subset of $A$, $E_j$ is NED. By Lemma 2.4, $v$ has an extension $u_j$ which is in $W'(F_0, F_1; D_j)$ such that

$$\int_{D_j} |\nabla u_j|^n dm = \int_D |\nabla v|^n dm \leq \int_D |\nabla u|^n dm. \tag{3.25}$$

In this way we obtain sequences of functions $\{u_j\}$ and sets $\{D_j\}$ such that $D \subset D_j \subset D_{j+1}$, $\overline{D}_j = \overline{D}_{j+1} = \overline{D}$, $u_j$ is an extension of $v$, $u_j = u_{j+1}$ in $D_j$ and $\bigcup_j D_j = B$. For $x \in B$ we let

$$u'(x) = u_j(x) \quad \text{if } x \in D_j.$$

Then $u'(x)$ is well defined and admissible for $\operatorname{cap}(F_0, F_1; B)$ and it follows from (3.25) that

$$\begin{aligned}\operatorname{cap}(F_0, F_1; B) &\leq \int_B |\nabla u'|^n dm = \lim_{j\to\infty} \int_{B_j} |\nabla u'|^n dm \\ &= \lim_{j\to\infty} \int_{B_j} |\Delta u_j|^n dm \leq \int_D |\nabla u|^n dm.\end{aligned}$$

Taking the infimum over all such functions $u$ in $W'(F_0, F_1; D)$ yields (3.24). Thus (3.23) follows from (3.24), Lemma 1.9 and the fact that $B$ is 2-QED.

Finally (3.23) and Lemma 1.11 imply that (3.23) holds for each pair of disjoint compact sets $F_0$ and $F_1$ in $\overline{D}$. Hence $M^*(D) \leq 2$. This completes the proof of Theorem 3.2. □

3.26. *Remark.* The description in Theorem 3.2 for the set $A = B\backslash D$ is sharp in the sense that there exists a 2-QED domain $D$ such that $A = B\backslash D$ is NED but $\overline{A}$ is not NED. Consider the set

$$A = \{z = (1 + im)2^{-n} : m = 0, 1, 2, \ldots, 2^n, n = 0, 1, \ldots\}$$

in the right half plane $H$ of the complex plane $\mathbb{C}$. It is easy to see that each compact subset $E$ of $A$ is finite and hence is NED. Then by Theorem 3.2, $D = H\backslash A$ is QED with $M^*(D) = 2$. But, obviously, $\overline{A}$ contains the vertical line segment joining the points 0 to $i$. Thus $\overline{\mathbb{C}}\backslash\overline{A}$ is not linearly locally connected and hence not QED by Lemma 1.8. Therefore $\overline{A}$ is not an NED set.

## 4. Estimates for QED constants

In this section we first derive a sharp lower bound of the QED constants for general QED domains. Then we estimate the values of QED constants for some special domains in $\mathbb{R}^n$. The following results about the function $\Psi_n(t)$ defined by the modulus of the Teichmüller ring in $\mathbb{R}^n$ will be needed in what follows.

4.1. **Lemma.** *Fix $-\infty \le a \le c \le b \le +\infty$ and let $f(t)$, $g(t)$ be positive functions in $(a, b)$ such that $f(t)$ and $g(t)$ tend to $\infty$ when $t$ tends to $c$. Then*

$$\lim_{t\to c}\frac{\log \Psi_n(f(t))}{\log \Psi_n(g(t))} = \lim_{t\to c}\frac{\log \Psi_n(f(t))}{\log g(t)} = \lim_{t\to c}\frac{\log f(t)}{\log g(t)}, \tag{4.2}$$

*provided that the last limit in* (4.2) *exists.*

*Proof.* Since

$$\frac{\log \Psi_n(f(t))}{\log g(t)} = \frac{\log\{\Psi_n(f(t))/f(t)\} + \log(f(t))}{\log g(t)},$$

the second equality in (4.2) follows from the fact that [G1, p. 225]

$$\lim_{r\to\infty}\frac{\Psi_n(r)}{r} = \lambda_n, \tag{4.3}$$

where $\lambda_n$ is a positive constant depending only on $n$. The first equality in (4.2) then follows from the second. □

4.4. **Lemma** [G1, Corollary 1, p. 226]. *If $R = R(F_0, F_1)$ is a ring with $a, b \in F_0\backslash\{\infty\}$ and $c, d \in F_1\backslash\{\infty\}$, then*

$$\left(\frac{\operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n))}{\omega_{n-1}}\right)^{1/(1-n)} = \operatorname{mod}(R) \le \log \Psi_n\left(\frac{|a-c|\,|b-d|}{|a-b|\,|c-d|}\right). \tag{4.5}$$

4.6. **Theorem.** *If $D \subset \overline{\mathbb{R}}^n$ is a QED domain, then either $M(D) = 1$ or $M(D) \ge 2$.*

*Proof.* Suppose $D$ is a QED domain with $M(D) > 1$. By Theorem 2.5 with $E = \overline{\mathbb{R}}^n\backslash D$, $\operatorname{int}(\overline{\mathbb{R}}^n\backslash D) \ne \varnothing$. Hence by performing preliminary Möbius transformations, we may assume that $D$ lies in the upper half space $H$ and $\partial D \cap \partial H$ contains the two points 0 and $e_1$.

Fix $\varepsilon > 0$ and let $t > 0$ be any number with $t + t^\varepsilon < 1$, $r = t/2$ and $s = t^\varepsilon/2$.

Then there exist points $z_0 \in D \cap B^n(0, r)$, $z_1 \in D \cap B^n(e_1, s)$. Since we can join $z_0$ to $z_1$ by an arc $F$ in $D$, there exist subarcs $F_0$ in $D \cap \{r \le |x| \le 2r\}$ and $F_1$ in $D \cap \{s \le |x - e_1| \le 2s\}$ with end points $x_1, x_2$ and $y_1, y_2$, respectively, where $|x_1| = r$, $|x_2| = 2r$, $|y_1 - e_1| = s$ and $|y_2 - e_1| = 2s$. Let

$$\Gamma = \Delta(F_0, F_1; \overline{\mathbb{R}}^n), \quad \Gamma_D = \Delta(F_0, F_1; D),$$

$$\Gamma^* = \Delta(F_0 \cup F_0^*, F_1 \cup F_1^*; \overline{\mathbb{R}}^n), \quad \Gamma_H = \Delta(F_0, F_1; H),$$

and let $\Gamma'$ be the curve family joining $S^{n-1}(0, t)$ and $S^{n-1}(0, 1 - t^\varepsilon)$. Here $F_0^*$ and $F_1^*$ are the reflections of $F_0$ and $F_1$ in $\partial H$, respectively. Then by the basic properties of moduli of curve families, see [GV, Lemma 3.3 and G1, pp. 214–216],

$$\operatorname{mod}(\Gamma^*) = 2 \operatorname{mod}(\Gamma_H) \geq 2 \operatorname{mod}(\Gamma_D) \tag{4.7}$$

and

$$\omega_{n-1}\left(\log\left(\frac{1-t^\varepsilon}{t}\right)\right)^{1-n} = \operatorname{mod}(\Gamma') \geq \operatorname{mod}(\Gamma^*). \tag{4.8}$$

Next since $|x_1 - y_1| \leq 2$, $|x_2 - y_2| \leq 2$, $|x_1 - x_2| \geq |x_2| - |x_1|$, and

$$|y_1 - y_2| \geq |y_2 - e_1| - |y_1 - e_1|,$$

we have

$$\frac{|x_1 - y_1|\,|x_2 - y_2|}{|x_1 - x_2|\,|y_1 - y_2|} \leq \frac{16}{t^{1+\varepsilon}}. \tag{4.9}$$

Then by Lemma 4.4 and (4.9),

$$\left(\frac{\operatorname{mod}(\Gamma)}{\omega_{n-1}}\right)^{1/(1-n)} \leq \log \Psi_n(f(t)), \tag{4.10}$$

where $f(t) = 16/t^{1+\varepsilon}$. Therefore by (4.8), (4.10) and Lemma 4.1,

$$\limsup_{t\to 0}\left(\frac{\operatorname{mod}(\Gamma)}{\operatorname{mod}(\Gamma^*)}\right)^{1/(n-1)} \geq \lim_{t\to 0}\frac{\log((1-t^\varepsilon)/t)}{\log \Psi_n(f(t))} = \frac{1}{1+\varepsilon}. \tag{4.11}$$

Thus it follows from (4.7) and (4.11) that

$$\limsup_{t\to 0}\frac{\operatorname{mod}(\Gamma)}{\operatorname{mod}(\Gamma_D)} \geq 2\limsup_{t\to 0}\frac{\operatorname{mod}(\Gamma)}{\operatorname{mod}(\Gamma^*)} \geq 2\left(\frac{1}{1+\varepsilon}\right)^{n-1}$$

which yields

$$M(D) \geq 2\left(\frac{1}{1+\varepsilon}\right)^{n-1}. \tag{4.12}$$

Since (4.12) holds for any $\varepsilon > 0$, it follows that $M(D) \geq 2$ as desired. □

4.13. **Corollary.** *If $D$ is a QED domain in $\overline{\mathbb{R}}^n$, then either $M^*(D) = 1$ or $M^*(D) \geq 2$.*

*Proof.* If $M^*(D) > 1$, then $M(D) > 1$ by Theorem 2.5. Thus $M^*(D) \geq M(D) \geq 2$ by Theorem 4.6. □

4.14. *Remark.* The lower bounds for $M(D)$ and $M^*(D)$ are sharp since $M^*(D) = M(D) = 2$ whenever $D$ is an open ball or half space.

Now we estimate the QED constants for some special domains in $\overline{\mathbb{R}}^n$. First we introduce some notation. We say that a domain $G$ in $\overline{\mathbb{R}}^n$ is *raylike at a point* $Q$ if for each point $P$, $P \in G$ if and only if $Q + t(P - Q) \in G$ for

$0 < t < \infty$. That is, $G$ is raylike at $Q$ if and only if each open ray from $Q$ lies either in $G$ or in $\mathbb{R}^n \backslash G$. Next let

$$x = (r, \theta, x_3, \dots, x_n) = (r, \theta, x')$$

be the cylindrical coordinates in $\mathbb{R}^n$, where $x' = (x_3, \dots, x_n) \in \mathbb{R}^{n-2}$ and $\mathbb{R}^{n-2} = \varnothing$ when $n = 2$. We say that a domain $D$ is a *dihedral wedge of angle* $\alpha\pi$, $0 < \alpha \leq 2$, if it can be mapped by a Möbius transformation onto the domain

$$A_\alpha = \{x = (r, \theta, x') : 0 < \theta < \alpha\pi, 0 < |x| < \infty\}. \tag{4.15}$$

The following results give sharp lower bounds for the QED constants of some special domains.

4.16. **Theorem.** *Suppose that $D$ and $G$ are domains in $\mathbb{R}^n$ with*

$$D \cap B^n(Q, r) = G \cap B^n(Q, r)$$

*for some finite point $Q$ and $r > 0$. If $G$ is raylike at $Q$, then*

$$M(D) \geq M(G). \tag{4.17}$$

*Proof.* By performing a preliminary similarity transformation, we may assume that $Q = 0$ and that $r = 1$. We may also assume that $M(D) < \infty$.

It remains to show that

$$\operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n)) \leq M(D)\operatorname{mod}(\Delta(F_0, F_1; G)) \tag{4.18}$$

for each pair of disjoint nondegenerate continua $F_0, F_1$ in $G$. Fix such a pair $F_0, F_1$ in $G$ and $t \in (0, 1)$ and choose $s \in (1, \infty)$ such that $F_0, F_1 \subset B^n(0, s)$. Let $f(x) = (t/s)x$ for $x \in \mathbb{R}^n$ and $F_i' = f(F_i)$, $i = 0, 1$. Since $G$ is raylike at $0$, $f(G) = G$. Then since $f$ is a Möbius transformation, it follows that

$$\operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n)) = \operatorname{mod}(\Delta(F_0', F_1'; \overline{\mathbb{R}}^n)) \tag{4.19}$$

and that

$$\operatorname{mod}(\Delta(F_0, F_1; G)) = \operatorname{mod}(\Delta(F_0', F_1'; G)). \tag{4.20}$$

On the other hand, since $F_0', F_1' \subset B^n(0, t) \cap G$ and $D \cap B^n(0, 1) = G \cap B^n(0, 1)$, each curve in the family

$$\Gamma = \Delta(F_0', F_1'; D) \backslash \Delta(F_0', F_1'; G)$$

contains a subcurve which joins $S^{n-1}(0, t)$ and $S^{n-1}(0, 1)$. Thus

$$\begin{aligned} \operatorname{mod}(\Delta(F_0', F_1'; D)) &\leq \operatorname{mod}(\Delta(F_0', F_1'; G)) + \operatorname{mod}(\Gamma) \\ &\leq \operatorname{mod}(\Delta(F_0', F_1'; G)) + \omega_{n-1}\left(\log \frac{1}{t}\right)^{1-n} \end{aligned} \tag{4.21}$$

by [G1, pp. 214–216], where $\omega_{n-1}$ is a constant as in (1.3). Next since $F_0'$ and $F_1'$ are also disjoint nondegenerate continua in $D$,

$$\operatorname{mod}(\Delta(F_0', F_1'; \overline{\mathbb{R}}^n)) \leq M(D)\operatorname{mod}(\Delta(F_0', F_1'; D)). \tag{4.22}$$

Combining relations (4.19) through (4.22), we obtain

$$\operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n)) \leq M(D)(\operatorname{mod}(\Delta(F_0, F_1; G)) + \omega_{n-1}\left(\log \tfrac{1}{t}\right)^{1-n}). \tag{4.23}$$

Letting $t \to 0$ yields (4.18).

Finally, (4.18) together with Lemma 1.11 implies that (4.18) holds for each pair of disjoint nondegenerate continua $F_0$ and $F_1$ in $\overline{G}$ and hence that $M(G) \leq M(D)$. □

4.24. **Theorem.** *Let $A_\alpha \subset \mathbb{R}^n$ be the dihedral wedge of angle $\alpha\pi$ defined in* (4.15). *Then*

$$M(A_\alpha) \geq \frac{2}{\alpha}. \tag{4.25}$$

*Proof.* We first assume that $n = 2$. In this case we identify $\overline{\mathbb{R}}^2$ with $\overline{\mathbb{C}}$, the extended complex plane, and use complex notation. For any $t$ with $0 < t < \infty$, we let $F_0$ be the closed line segment joining $0$ to $e^{i\alpha\pi}$ and $F_1$ be the closed ray from $t$ to $\infty$ on the real axis. Then by using the conformal mapping $\phi(z) = z^{1/\alpha}$ from $A_\alpha$ onto the upper half plane, we obtain

$$\operatorname{mod}(\Delta(F_0, F_1; A_\alpha)) = \tfrac{1}{2}\operatorname{mod}(\Delta(\phi(F_0), \phi(F_1); \mathbb{C})) = \pi(\log \Psi_2(t^{1/\alpha}))^{-1}. \tag{4.26}$$

On the other hand, since $0, e^{i\alpha\pi} \in F_0$ and $t, \infty \in F_1$, it follows from Lemma 4.4 that

$$\operatorname{mod}(\Delta(F_0, F_1; \mathbb{C})) \geq 2\pi(\log \Psi_2(t))^{-1}. \tag{4.27}$$

Combining (4.26) and (4.27), we obtain

$$\frac{\operatorname{mod}(\Delta(F_0, F_1; \mathbb{C}))}{\operatorname{mod}(\Delta(F_0, F_1; D))} \geq \frac{2\log \Psi_2(t^{1/\alpha})}{\log \Psi_2(t)}. \tag{4.28}$$

By Lemma 4.1,

$$\lim_{t\to\infty} \frac{\log \Psi_2(t^{1/\alpha})}{\log \Psi_2(t)} = \lim_{t\to\infty} \frac{\log t^{1/\alpha}}{\log t} = \frac{1}{\alpha}. \tag{4.29}$$

Thus (4.25) follows from (4.28) and (4.29) as desired.

We next assume that $n > 2$. In this case we let $F_0$ be the segment $x_1 = \cdots = x_{n-1} = 0$, $-1 \leq x_n \leq 0$, and $F_1$ be the ray $x_1 = \cdots = x_{n-1} = 0$, $1 \leq x_n \leq \infty$. Then

$$\operatorname{mod}(\Delta(F_0, F_1; A_\alpha)) = \frac{\alpha}{2}\operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^n)) \tag{4.30}$$

by [GV, Lemma 7.1]. This implies (4.25) as desired. □

The next result allows one to obtain sharp upper bounds of $M^*(D)$ for certain domains in $\mathbb{R}^n$.

4.31. **Theorem.** *Suppose that $\Omega$ is a quasiball in $\mathbb{R}^n$ and that $G, D$ are domains in $\mathbb{R}^n$ with $D = G \cap \Omega$. Suppose also that $f$ is a $K$-quasiconformal mapping of $G$ onto itself with $f(D) = G\backslash\overline{D}$ and $f(x) = x$ for $x \in \partial D \cap G$. Then*

$$M^*(D) \leq (K+1)M^*(G). \tag{4.32}$$

*Proof.* We observe first that (see [GLM, 6.9] or [Z, 4.2]) if $f$ is a $K$-quasiconformal mapping of a domain $D \subset \mathbb{R}^n$ onto $D' \subset \mathbb{R}^n$, then for any function $u$ in the local Sobolev space $W^1_{n,\mathrm{loc}}(D')$, the function $v(x) = u \circ f(x)$ is in $W^1_{n,\mathrm{loc}}(D)$ with

$$|\nabla v(x)|^n \leq K|J(x, f)| \cdot |\nabla u|^n \circ f(x) \tag{4.33}$$

a.e. in $D$, where $J(x, f)$ is the Jacobian of $f$.

For the proof of (4.32), we may assume that $M^*(G) < \infty$ and suppose first that $\partial\Omega\backslash G \neq \varnothing$. Then by performing a preliminary Möbius transformation, we may assume that $\infty \in \partial\Omega\backslash G$. Let $H$ denote the upper half space $\{x_n > 0\}$ and $g$ denote the reflection in $\partial H$, choose a quasiconformal self-mapping $h$ of $\mathbb{R}^n$ with $h(H) = \Omega$, and let $D_1 = f(D)$,

$$D_2 = h \circ g \circ h^{-1}(D), \quad G_1 = D \cup D_2 \cup (\partial D \cap G), \quad G_2 = G \cap G_1.$$

Fix any two disjoint compact sets $F_0$, $F_1$ in $D$ and any function $u$ in

$$W'(F_0, F_1; D)$$

with

$$\int_D |\nabla u|^n dm < \infty.$$

Then fix $t \in (0, 1)$ and let

$$v(x) = \frac{1+t}{1-t}(u(x) - t).$$

Since $v$ belongs to $W^1_{n,\mathrm{loc}}(D)$, it follows that

$$u_1(x) = v \circ h(x) \in W^1_{n,\mathrm{loc}}(h^{-1}(D)).$$

Then since $h^{-1}(G_1)$ is symmetric with respect to the plane $\partial H$ and

$$h^{-1}(G_1) \cap H = h^{-1}(D),$$

it is not difficult to check that the classical derivatives of

$$W(x) = \begin{cases} u_1(x), & \text{if } x \in h^{-1}(D), \\ u_1 \circ g(x), & \text{if } x \in g(h^{-1}(D)) \end{cases}$$

are the weak derivatives of $w$ in $h^{-1}(G_1)$ (see [M2, 1.1.3]). Thus $w \in W^1_{n,\mathrm{loc}}(h^{-1}(G_1))$ and

$$u_2(x) = w \circ h^{-1}(x) \in W^1_{n,\mathrm{loc}}(G_1) \subset W^1_{n,\mathrm{loc}}(G_2).$$

Fix $\varepsilon > 0$. Since $u_2(x) = v(x) \leq -t$ for $x \in F_0$ and $u_2(x) = v(x) \geq 1 + t$ for $x \in F_1$, we can choose a smooth convolution approximation $\phi$ of $u_2$ in $G_2 \supset D$ with $\phi(x) \leq 0$ for $x \in F_0$, $\phi(x) \geq 1$ for $x \in F_1$, and

$$\int_D |\nabla \phi|^n dm \leq \int_D |\nabla u_2|^n dm + \varepsilon = \int_D |\nabla v|^n dm + \varepsilon. \tag{4.34}$$

Next let $\phi_1(x) = \phi \circ f(x)$ for $x \in f^{-1}(G_2)$ and

$$\psi(x) = \begin{cases} \phi(x), & \text{if } x \in \overline{D} \cap G, \\ \phi_1(x), & \text{if } x \in D_1. \end{cases}$$

Then $\psi$ is ACL in $D$ and $D_1$ and continuous in $G = (\overline{D} \cap G) \cup D_1$. We shall show that $\psi$ is ACL in $G$. For this we let $\psi' = \psi \circ h$, $\phi' = \phi \circ h$, $\phi'_1 = \phi_1 \circ h$. Then $\phi'$ and $\phi'_1$ are ACL in $h^{-1}(G_2)$ and $h^{-1}(f^{-1}(G_2))$, respectively, with $\phi'(x) = \phi'_1(x)$ for $x \in \partial H \cap h^{-1}(G)$. We claim that $\psi'$ is ACL in $h^{-1}(G)$. Since the ACL-property is a local property, $\psi'(x) = \phi'(x)$ when $x_n \geq 0$, and $\psi'(x) = \phi'_1(x)$ when $x_n \leq 0$. It suffices to show that $\psi'$ is ACL in a

neighborhood of each point of $\partial H \cap h^{-1}(G)$. Fix $P \in \partial H \cap h^{-1}(G)$ and $r > 0$ so that

$$B = B^n(P, r) \subset h^{-1}(G_2) \cap h^{-1}(f^{-1}(G_2)).$$

Then $\phi'$ and $\phi_1'$ are ACL in $B$ with $\phi'(x) = \phi_1'(x)$ for $x \in B \cap \partial H$, and hence $\psi'$ is ACL in $B$. This proves our claim that $\psi'$ is ACL in $h^{-1}(G)$. Thus, $\psi = \psi' \circ h^{-1}$ is ACL in $G$, and hence

$$\begin{aligned}\operatorname{cap}(F_0, F_1; G) &\le \int_G |\nabla \psi|^n dm = \int_D |\nabla \phi|^n dm + \int_{D_1} |\nabla(\phi \circ f)|^n dm \\ &\le \int_D |\nabla \phi|^n dm + K \int_{D_1} |J(x, f)||\nabla \phi|^n \circ f(x) dm \\ &\le (K+1) \int_D |\nabla \phi|^n dm \le (K+1) \int_D |\nabla v|^n dm + (K+1)\varepsilon\end{aligned}$$

by (4.33), [V1, 24.5] and (4.34). Letting $\varepsilon \to 0$ yields

$$\operatorname{cap}(F_0, F_1; G) \le (K+1) \int_D |\nabla v|^n dm = (K+1) \left(\frac{1+t}{1-t}\right)^n \int_D |\nabla u|^n dm.$$

Finally letting $t \to 0$ and taking the infimum over all such functions $u$ in $W'(F_0, F_1; D)$, we obtain

$$\operatorname{cap}(F_0, F_1; \mathbb{R}^n) \le M^*(G)\operatorname{cap}(F_0, F_1; G) \le (K+1)M^*(G)\operatorname{cap}(F_0, F_1; D).$$

This together with Lemmas 1.9 and 1.11 implies (4.32) as desired.

Suppose next that $\partial\Omega \subset G$. Then the above argument with $H$ replaced by the unit ball $B^n(0, 1)$ yields the same conclusion. □

## 5. QED constants for planar domains

We begin by deriving the following sharp upper bound of $M^*(D)$ for quasidisks.

5.1. **Theorem.** *If $D$ is a Jordan domain in $\overline{\mathbb{C}}$ and $\partial D$ admits a $K$-quasiconformal reflection, then*

$$M^*(D) \le K + 1. \tag{5.2}$$

*Proof.* We may assume that $\infty \in \partial D$. Let $f$ be a $K$-quasiconformal reflection in $\partial D$. Then applying Theorem 4.31 to $\Omega = D$, $G = \mathbb{C}$ and $f$, we obtain

$$M^*(D) \le (K+1)M^*(\mathbb{C}) = K + 1$$

as desired. □

As some applications of Theorems 4.16, 4.24, 4.31 and 5.1, we calculate the QED constants for some domains in $\mathbb{C}$. Let $P_n$ denote the regular $n$-gon with vertices at the $n$th roots of unity, $T(\alpha, \beta)$ a triangle with interior angles $\alpha\pi$, $\beta\pi$ and $(1-\alpha-\beta)\pi$, and $B_\alpha = B \cap A_\alpha$, where $B = \{|z| < 1\}$ and $A_\alpha$ is as in (4.15).

5.3. **Corollary.** *Using the above notation we have*

$$M^*(A_\alpha) = M(A_\alpha) = \frac{2}{\alpha}, \quad \text{if } 0 < \alpha \le 1, \tag{5.4}$$

$$M^*(P_n) = M(P_n) = \frac{2n}{n-2}, \quad \text{if } n \ge 3, \tag{5.5}$$

$$M^*(T(\alpha, \beta)) = M(T(\alpha, \beta)) = \frac{2}{\alpha}, \quad \text{if } \alpha \le \beta \text{ and } \alpha + \beta = \frac{1}{2}, \tag{5.6}$$

$$M^*(T(\alpha, \alpha)) = M(T(\alpha, \alpha)) = \frac{2}{1-2\alpha}, \quad \text{if } \alpha \ge \frac{1}{3}, \tag{5.7}$$

$$M^*(B_\alpha) = M(B_\alpha) = \frac{2}{\alpha}, \quad \text{if } 0 < \alpha \le \frac{1}{2}. \tag{5.8}$$

*Proof*. Since the mapping $f(z) = \overline{z}^\lambda z^{1-\lambda}$, $\lambda = 1/\alpha$, is a $K$-quasiconformal reflection in $\partial A_\alpha$ with $K = 2\lambda - 1$, it follows from Theorem 5.1 that

$$M^*(A_\alpha) \le K + 1 = \frac{2}{\alpha}.$$

This together with (4.25) and (1.14) implies (5.4).

For the proof of (5.5), we first notice that each interior angle of $P_n$ is $\frac{n-2}{n}\pi$. Then since $A_\alpha$ with $\alpha = \frac{n-2}{n}$ is raylike at the origin, Theorems 4.16 and 4.24 imply that

$$M(P_n) \ge M(A_\alpha) \ge \frac{2}{\alpha} = \frac{2n}{n-2}. \tag{5.9}$$

Next we want to construct a $K$-quasiconformal reflection in $\partial P_n$ with $K = \frac{n+2}{n-2}$. Set $\theta = \frac{n-2}{2n}$, $\omega = e^{2\pi i/n}$. Next let $D$ be the disk in $A_{2/n}$ which is tangent to $\partial A_{2/n}$ at $\omega$ and 1, and let $G = P_n \cap A_{2/n}$ and $G' = A_{2/n} \backslash \overline{P}_n$. Then

$$\zeta(z) = -e^{\theta\pi i}\frac{z-\omega}{z-1}$$

maps $P_n \cap D$ and $B \cap D$ conformally onto $A_\theta$ and $A_{1/2'}$, respectively, with $\zeta(\omega) = 0$ and $\zeta(1) = \infty$. Also,

$$w(\zeta) = |\zeta|^{-2/(n-2)}\zeta^{n/(n-2)}$$

maps $A_\theta$ onto $A_{1/2}$ $K_1$-quasiconformally with $K_1 = \frac{n}{n-2}$. Thus

$$\phi(z) = \begin{cases} \zeta^{-1} \circ w \circ \zeta(z), & \text{if } z \in P_n \cap D, \\ z, & \text{if } z \in G \backslash D, \end{cases}$$

is a $K_1$-quasiconformal mapping from $G$ onto $A_{2/n} \cap B$. Similarly, one can construct a $K_2$-quasiconformal mapping $\psi$ from $A_{2/n} \backslash \overline{B}$ onto $G'$ with $K_2 = \frac{n+2}{n}$ and $\psi(z) = z$ on $G' \backslash D$. Therefore we obtain $f(z) = \psi(1/\overline{\phi(z)})$ which is a $K_1K_2$-quasiconformal reflection from $\overline{G}$ onto $\overline{G}'$ with $f(z) = z$ on $\partial P_n \cap \partial G$ and $f(z) = 1/\overline{z}$ on $\partial G \backslash \partial P_n$. By the symmetry of $P_n$, we can extend $f$ to be a $K$-quasiconformal reflection in $\partial P_n$ with $K = K_1K_2 = \frac{n+2}{n-2}$.

Finally it follows from Theorem 5.1 that

$$M^*(P_n) \le K + 1 = \frac{2n}{n-2}. \tag{5.10}$$

And hence (5.5) follows from (5.9) and (5.10).

For the proof of (5.6), we let $T = T(\alpha, \beta)$ be the triangle with vertices at 1, $-1$ and a third point which is on the upper half unit circle. Then as above, we can construct a $K$-quasiconformal mapping $f$ of the upper half plane $H$ onto itself such that $f(T) = H \backslash \overline{T}$, $f(z) = z$ for $z \in \partial T \cap H$, and

$$K = \max\left\{\frac{1-\alpha}{\alpha}, \frac{1-\beta}{\beta}\right\} = \frac{1-\alpha}{\alpha}.$$

Applying Theorem 4.31 to $G = H$, $D = T$, $\Omega = \mathbb{C} \backslash f(\overline{T})$ and $f$, we obtain

$$M^*(T) \le (K+1)M^*(H) = 2/\alpha. \tag{5.11}$$

On the other hand, Theorems 4.16 and 4.24 imply that $M(D) \ge 2/\alpha$. This together with (5.11) and (1.14) yields (5.6).

By a similar argument as in the proof of (5.5), one can prove (5.7) and (5.8). □

5.12. *Remark.* It follows from Theorem 5.1 and (5.5) that the above quasiconformal reflection in $\partial P_n$ is extremal in terms of the dilatation. R. Kühnau also had a similar idea about the construction of the extremal quasiconformal reflection in $\partial P_n$ [K].

5.13. *Remark.* There are analogues of Theorem 5.1 and Corollary 5.3 in higher dimensions. For example,

$$M^*(A_\alpha) = M(A_\alpha) = \frac{2}{\alpha}$$

for $0 < \alpha \le 1$, $n \ge 2$, and

$$M^*(Q_n) = M(Q_n) = 2^n$$

when $n = 2, 3, 4$, or $5$, where $Q_n$ is a cube in $\mathbb{R}^n$. Proofs will appear in [Y2].

Next we consider some general properties of QED domains in $\overline{\mathbb{R}}^2$. Following [GM], a domain $D$ in $\overline{\mathbb{R}}^2$ is said to be a *$K$-quasicircle domain* if each component of $\partial D$ is either a point or a $K$-quasicircle. The main tool in the rest of this section is the following result.

5.14. **Theorem.** *If $D$ is a QED domain in $\overline{\mathbb{R}}^2$, then $D$ is a $K$-quasicircle domain with*

$$K \le L^2 \tag{5.15}$$

*where*

$$L = \Psi_2^{-1}(\Psi_2(1)^{(M(D)-1)}). \tag{5.16}$$

5.17. *Remark.* Gehring and Martio verified that every QED domain is LLC [GM, 2.11]. This, in conjunction with earlier work of Gehring [G4, Lemma 5], established that a planar $M$-QED domain is a $K$-quasicircle domain. However,

this argument does not give an explicit estimate for $K$. Our proof is classical in spirit (see also [HK, 3.9]). We show that the boundary homeomorphism induced by two conformal mappings is quasisymmetric, and then use the extension theorem of Beurling-Ahlfors [BA]. What is new in our argument is that we explicitly compute the best possible quasisymmetry constant $L$ in terms of $M(D)$. Notice that this estimate has the important property that $L = 1$ when $M(D) = 2$. This is essential in what follows.

5.18. *Proof of Theorem* 5.14. Suppose that $D$ is a QED domain in $\overline{\mathbb{R}}^2$ and that $C$ is a nondegenerate component of $\partial D$. By Lemma 1.8, $D$ is linearly locally connected and hence $C$ is a Jordan curve by [N, Theorem 16.3, p.168]. We may assume that $C$ is bounded and that $D$ is contained in the bounded component $D_1$ of $\overline{\mathbb{R}}^2 \backslash C$. Let $D_2 = \overline{\mathbb{R}}^2 \backslash \overline{D}_1$ and $H, H^*$ denote the upper, lower half plane, respectively.

Next we choose two conformal mappings $f_1 \colon D_1 \to H$ and $f_2 \colon D_2 \to H^*$ such that the map

$$h(x) = f_1 \circ f_2^{-1}(x)$$

induced by their homeomorphic extensions is an increasing homeomorphism from $\overline{\mathbb{R}}$ onto $\overline{\mathbb{R}}$ with $h(\infty) = \infty$. We shall show that $h$ satisfies the Beurling-Ahlfors $M$-condition (see [A, Chapter 4]). For this purpose, we take any point $x \in \mathbb{R}$ and positive number $t$ and denote $F_0 = f_2^{-1}([x - t, x])$, and $F_1 = f_2^{-1}([x + t, \infty])$. Then $F_0$ and $F_1$ are two disjoint continua in $\overline{D}$ and by Lemma 1.11,

$$\begin{aligned} \operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^2)) &\le M(D)\operatorname{mod}(\Delta(F_0, F_1; D)) \\ &\le M(D)\operatorname{mod}(\Delta(F_0, F_1; D_1)). \end{aligned} \tag{5.19}$$

On the other hand, since $\Delta(F_0, F_1; D_1)$ and $\Delta(F_0, F_1; D_2)$ are subfamilies of $\Delta(F_0, F_1; \overline{\mathbb{R}}^2)$ and lie in disjoint domains $D_1$ and $D_2$, respectively it follows that

$$\operatorname{mod}(\Delta(F_0, F_1; D_1)) + \operatorname{mod}(\Delta(F_0, F_1; D_2)) \le \operatorname{mod}(\Delta(F_0, F_1; \overline{\mathbb{R}}^2)). \tag{5.20}$$

Therefore by (5.19) and (5.20),

$$\operatorname{mod}(\Delta(F_0, F_1; D_2)) \le (M(D) - 1)\operatorname{mod}(\Delta(F_0, F_1; D_1)). \tag{5.21}$$

By the conformal invariance and symmetry principle for moduli of curve families, it follows that

$$\begin{aligned} \operatorname{mod}(\Delta(F_0, F_1; D_1)) &= \frac{1}{2}\operatorname{mod}(\Delta(f_1(F_0), f_1(F_1); \overline{\mathbb{R}}^2)) \\ &= \pi\left(\log \Psi_2\left(\frac{h(x+t) - h(x)}{h(x) - h(x-t)}\right)\right)^{-1} \end{aligned} \tag{5.22}$$

and

$$\operatorname{mod}(\Delta(F_0, F_1; D_2)) = \pi(\log \Psi_2(1))^{-1}. \tag{5.23}$$

Combining (5.21), (5.22), (5.23) and using the fact that $\Psi_2(r)$ is increasing in $r$, we obtain

$$\frac{h(x+t) - h(x)}{h(x) - h(x-t)} \le L \tag{5.24}$$

where $L$ is as in (5.16).

In a similar manner, if we replace $F_0$ and $F_1$ by $f_2^{-1}([x, x+t])$ and $f_2^{-1}([-\infty, x-t])$, respectively, we obtain

$$\frac{h(x+t)-h(x)}{h(x)-h(x-t)} \geq \frac{1}{L}. \tag{5.25}$$

Hence by (5.24) and (5.25), $h$ satisfies the $M$-condition

$$\frac{1}{L} \leq \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leq L \tag{5.26}$$

for all $x \in \mathbb{R}$ and $t > 0$. According to the Beurling-Ahlfors extension theorem [BA], there exists a $K(L)$-quasiconformal mapping $f\colon \overline{\mathbb{R}}^2 \to \overline{\mathbb{R}}^2$ such that $f(x) = h(x)$ on $\mathbb{R}$ and

$$K(L) \leq L^2. \tag{5.27}$$

Finally, let

$$F(x) = \begin{cases} f_1^{-1}(f(x)), & \text{if } x \in \overline{H}, \\ f_2^{-1}(x), & \text{if } x \in \overline{H}^*. \end{cases}$$

Then $F(x)$ is a $K$-quasiconformal mapping from $\overline{\mathbb{R}}^2$ onto itself with $K$ satisfying (5.15) and $F(\mathbb{R}) = C$. This shows that $C$ is a $K$-quasicircle and completes the proof of Theorem 5.14. □

5.28. *Remarks.* Since $M(D) \leq M^*(D)$, (5.21) also holds if $M(D)$ is replaced by $M^*(D)$. Thus (5.15) also holds if we replace $M(D)$ by $M^*(D)$.

From the proof of Theorem 5.14, we see that $F$ is conformal in $D_2$, one of the components of $\overline{\mathbb{R}}^2 \backslash C$. Hence $C$ admits a $K$-quasiconformal reflection with $K$ satisfying (5.15).

5.29. **Theorem.** *If $D$ is a domain in $\overline{\mathbb{R}}^2$, then $M(D) = 2$ if and only if $D$ is Möbius equivalent to the unit disk minus an NED set.*

*Proof.* Suppose $D$ is a QED domain in $\overline{\mathbb{R}}^2$ with $M(D) = 2$. Then by Theorem 5.14, each nondegenerate component $C$ of $\partial D$ is a $K$-quasicircle with

$$K \leq (\Psi_2^{-1}(\Psi_2(1)^{(M(D)-1)}))^2 = 1.$$

Therefore $C$ is Möbius equivalent to the unit circle.

Since $M(D) = 2$, $\partial D$ has at least one nondegenerate component by Theorems 2.5 and 4.6. Suppose next that there are two nondegenerate components $C_1$ and $C_2$ of $\partial D$. We may assume that $C_1$ is the real line $\mathbb{R}$ and $C_2$ is a circle in the upper half plane $H$. Let $D_2$ denote the domain bounded by $C_2$, $D_1 = H \backslash \overline{D}_2$ and $G = \overline{\mathbb{R}}^2 \backslash (\overline{D}_2 \cup \overline{D}_2^*)$, where $D_2^*$ is the reflection image of $D_2$ with respect to $C_1 = \mathbb{R}$. Then since $D \subset D_1$,

$$\operatorname{mod}(\Delta(F_0, F_1; D)) \leq \operatorname{mod}(\Delta(F_0; F_1; D_1)) \leq \tfrac{1}{2}\operatorname{mod}(\Delta(F_0, F_1; G)), \tag{5.30}$$

for each pair of disjoint nondegenerate continua $F_0$ and $F_1$ in $C_1 \subset \overline{D}$.

On the other hand, let $u$ be the extremal function for the ring $R = R(F_0, F_1)$. By Lemma 3.3,

$$\int_{D_2} |\nabla u(x)|^2 dm \geq \delta > 0$$

for some $\delta > 0$. Since $u$ is also admissible for $\operatorname{cap}(F_0, F_1; G)$,

$$\operatorname{cap}(F_0, F_1; G) \leq \int_G |\nabla u(x)|^2 dm \leq \operatorname{cap}(F_0, F_1; \overline{\mathbb{R}}^2) - \delta.$$

Thus by Lemma 1.9 and the hypothesis that $M(D) = 2$,

$$\operatorname{mod}(\Delta(F_0, F_1; G)) \leq 2 \operatorname{mod}(\Delta(F_0, F_1; D)) - \delta$$

which contradicts inequality (5.30). Therefore, $\partial D$ has only one nondegenerate component and hence $\overline{\mathbb{R}}^2 \backslash \overline{D}$ is Möbius equivalent to the unit disk.

Finally, the same argument as in Proposition 3 of §3 shows that $D$ is Möbius equivalent to the unit disk minus an NED set.

The sufficiency follows from the sufficiency of Theorem 3.2 and inequality (1.14). □

Combining Theorems 3.2 and 5.29, we obtain the following result.

5.31. **Corollary.** *If $D$ is a domain in $\overline{\mathbb{R}}^2$, then $M(D) = 2$ if and only if $M^*(D) = 2$.*

Since the Teichmüller function $\Psi_2(r)$ is strictly increasing in $r$, we obtain the following results from inequality (5.15) and Remark 5.28.

5.32. **Corollary.** *If $D$ is a domain in $\overline{\mathbb{R}}^2$, then $M(D) \notin (1, 2)$, $M^*(D) \notin (1, 2)$, and $M(D) = 1$ if and only if $M^*(D) = 1$.*

We close this section by deriving the following estimate of QED constants for Jordan domains in $\mathbb{R}^2$.

5.33. **Theorem.** *If $D$ is a Jordan domain in $\overline{\mathbb{R}}^2$, then*

$$M^*(D) \leq (\Psi_2^{-1}(\Psi_2(1)^{(M(D)-1)}))^2 + 1. \tag{5.34}$$

*Furthermore, $D$ is QED with respect to compact sets if and only if it is QED with respect to continua.*

*Proof.* If $M(D)$ is infinite, then so are both sides of (5.34). Next, we assume that $M(D) < \infty$. By Remark 5.28, $\partial D$ admits a $K$-quasiconformal reflection and (5.34) follows from Theorem 5.1 and inequality (5.15) as desired. □

5.34. *Remark.* It is known that the upper bound for $K(L)$ in (5.27) can be replaced by some sharper bounds. Consequently, the upper bounds for $K$ and $M^*(D)$ in (5.15) and (5.34), respectively, can be replaced by corresponding sharper bounds. However, the question of whether or not $M(D) = M^*(D)$ remains open.

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