ON CONJUGACY SEPARABILITY OF FUNDAMENTAL GROUPS
OF GRAPHS OF GROUPS

M. SHIRVANI

Abstract. A complete determination of when the elements of a fundamental
group of a (countable) graph of profinite groups are conjugacy distinguished is
given. By embedding an arbitrary fundamental group $G$ into one with profinite
vertex groups and making use of the above result, questions on conjugacy sepa-
rability of $G$ can be reduced to the solution of equations in the vertex groups
of $G$.

1. Introduction

An element $g$ of a group $G$ is said to be conjugacy distinguished (or con-
jugacy separable) in $G$ if for every element $h$ of $G$ not conjugate to $g$ there
exists $N < G$ such that $gN$ and $hN$ are nonconjugate in $G/N$. A group is
conjugacy separable if all its elements are conjugacy distinguished. The best-
known classes of conjugacy separable groups are polycyclic-by-finite groups [10],
profinite groups [11], free-by-finite groups [5], and certain Fuchsian groups [19].
Results are also known on conjugacy separability of certain amalgamated free
products (e.g., of free groups [18]), and some one-relator groups with torsion
[1]. In the last mentioned paper the authors ask whether every one-relator group
with torsion is conjugacy separable. More generally, one can ask when the fun-
damental group of a graph of groups is conjugacy separable.

In order to investigate these problems one might adopt the following strategy:
let $G$ be a residually finite group, with $\hat{G}$ its profinite completion (i.e., $\hat{G}$ is the
inverse limit of the system of finite images $G/N$ of $G$). It is easily shown that
if $g, h \in G$ are conjugate in every finite image of $G$, then they are conjugate
in $\hat{G}$, and so $G$ is conjugacy separable if and only if whenever $g, h \in G$ are
conjugate in $\hat{G}$, then they are conjugate in $G$. Unfortunately, if $G$ is a non-
trivial fundamental group of a graph of groups, then $\hat{G}$ is still not sufficiently
well understood (cf. [7, 20]) to allow for the successful completion of the last
step. A complete answer can be given when the vertex groups are themselves
profinite groups. This, in turn, allows necessary and sufficient conditions to be
derived for the general case.

Let $X$ be a connected graph, let $G = \{A_x \mid x \in VX, H_e \mid e \in EX\}$ be a
graph of groups over $X$, and let $G = \pi_1(G, X)$ be the fundamental group of

Received by the editors September 21, 1990.
1980 Mathematics Subject Classification (1985 Revision). Primary 20E26; Secondary 20E06,
20E18.
this graph of groups (see [2, 4], or [14] for the relevant definitions and properties). We assume that \( G \) is residually finite, and denote by \( \hat{A}_x \) the profinite completion of the vertex group \( A_x \) with respect to the induced topology \( \{ A_x \cap M : M \triangleleft_f G \} \). Let \( \overline{H}_e \) denote the topological closure of \( H_e \) in \( \hat{A}_x \). The edge isomorphisms can be extended to these closures (cf. Lemma 3.1). The result is a graph of groups over \( X \) with fundamental group \( G^+ \) (for details see §3). The obvious maps \( A_x \to \hat{A}_x \) extend to an embedding of \( G \) into \( G^+ \). If we denote the conjugacy class of \( g \) in \( G \) by \( gG \) and \( \cap_{M\triangleleft_f G} g^G M \) by \( cl_G(g^G) \), then we have:

**Theorem 3.4.** For every \( g \in G \) we have \( cl_G(g^G) = cl_{G^+}(g^{G^+}) \cap G \).

The point of this result is that most elements of \( G^+ \) turn out to be conjugacy distinguished (i.e., \( cl_{G^+}(g^{G^+}) = g^{G^+} \)). If \( g \in G \) is such an element, then \( g \) is conjugacy distinguished in \( G \) if and only if \( g^{G^+} \cap G = g^G \). This condition is quite tractable, and amounts to whether certain equations hold in the vertex groups \( A_x \) and \( \hat{A}_x \). Assuming that we have enough information about the vertex groups and their completions, the conjugacy separability of \( G \) can be decided.

To state the results for \( G^+ \), let \( Y \) be a maximal subtree of \( X \), and write \( A^+ = \langle \hat{A}_x : x \in V_X \rangle \leq G^+ \). Then \( A^+ \) is the fundamental group of the graph of groups \( \mathcal{G} \) restricted to \( Y \), and \( G^+ \) is an HNN-extension with base group \( A^+ \). As a consequence of 4.4, 4.6, and 4.9 below we get

**Theorem A.** Let \( a \in A^+ \).

(i) If neither a conjugate of \( a \) belongs to a vertex group, or \( X \) is locally finite, \( a \in \hat{A}_x \), and no \( \hat{A}_x \)-conjugate of \( a \) belongs to an edge subgroup, then \( a \) is conjugacy distinguished in \( A^+ \).

(ii) Assume that \( X \) is locally finite, let \( D = \cup_{e \in EX} \overline{H}_e \), and suppose \( a \in D \) belongs to only a finite number of edge subgroups. Then \( a \) is conjugacy distinguished in \( A^+ \) if and only if there exist vertices \( x_1, \ldots, x_m \) of \( Y \) such that \( D \cap a^{A^+} = D \cap \{ a^{i_1 \cdots i_m} : a_i \in \hat{A}_{x_i} \} \).

For elements of \( G^+ \) we obtain from 5.1, 5.3, 5.6, and 5.8

**Theorem B.** (i) All elements of \( G^+ \setminus A^+ \), and all elements of \( A^+ \) satisfying condition (i) of Theorem A, are conjugacy distinguished in \( G^+ \).

(ii) Assume that \( X \) is locally finite, and let \( D' = \cup_{e \in EX \setminus E_Y} \overline{H}_e \) and \( C = \cup_{x \in V_X} \hat{A}_x \).

Then \( h \in D' \) is conjugacy distinguished in \( G^+ \) if and only if \( h^{G^+} \cap C \) is a union of a finite number of sets of the form \( C \cap \{ h^{e_1 \cdots e_n} : e_i \in EX \setminus E_Y, \ e_i = \pm 1, \ \text{and} \ x_i \in V_X \} \).

This in particular settles the problem for the case when the vertex groups are profinite, since such a \( G \) can be residually finite if and only if \( G \cong G^+ \), by 3.3 below.

2. Notation

Let \( X \) be a directed connected graph, \( \mathcal{G} \) a graph of groups \( \{ A_x : x \in V_X ; H_e : e \in EX \} \) with isomorphisms \( \theta_e \) from \( H_e \leq A_{l(e)} \) to \( H_e \leq A_{l(e)} \) over
CONJUGACY SEPARABILITY OF GRAPHS OF GROUPS

231

X, and G = \pi_1(\mathcal{G}, X) the fundamental group of \((\mathcal{G}, X)\) relative to the choice of a fixed maximal subtree \(Y\) (see [2, 4], or [14] for details). It is well known that if \(A = \pi_1(\mathcal{G}\vert_Y, Y)\) then the obvious map of \(A\) to \((A_x: x \in V_Y) \subseteq G\) is an isomorphism, and \(G\) is an HNN-extension with base group \(A\) and stable letters \(t_e\) corresponding to the edges \(e \in E_X\setminus E_Y\):

\[
G = \langle A, t_e: t_e^{-1}ht_e = h\theta_e \text{ for all } h \in H_e, e \in E_X\setminus E_Y \rangle.
\]

In the notation of [16] let \(I = I(\mathcal{G}, X)\) denote the set of all sequences \((P_x)_{x \in V_X}\) satisfying the following conditions: (a) \(P_x \triangleleft A_x\) and there exists an integer \(m = m(P)\) such that \(|A_x : P_x| \leq m\) for all \(x \in V_X\), (b) \((P_{i(e)} \cap H_e)\theta_e = P_{i(e)} \cap H_e\) for all \(e \in E_X\). For \(P, Q \in I\) we write \(P \leq Q\) if \(P_x \subseteq Q_x\) for all \(x\). For \(P \in I\) let \(\mathcal{G}_P\) denote the graph of groups \((A_x/P_x, H_eP_{i(e)}/P_{i(e)}), \text{ induced isomorphisms } \theta_e, P\), and let \(G_P = \pi_1(\mathcal{G}_P, X)\). The projections \(A_x \rightarrow A_x/P_x\) clearly extend to an epimorphism \(\pi_P: G \rightarrow G_P\). It is well known that \(G_P\) is free-by-finite (e.g., [14, Exercise 2 on p. 123]). Also note that if \(M \triangleleft_f G\) then \(P = (M \cap A_x)_{x \in V_X} \in I\), and \(G/M\) is a homomorphic image of \(G_P\). In particular, this implies that if \(G\) is residually finite then \(\bigcap_{P \in I} P_x = \{1\}\) for all \(x \in V_X\). We assume this from now on. For \(g \in G\) write \(g^G\) for the conjugacy class of \(g\) in \(G\).

**Proposition 2.1.** Let \(g \in G\). Then \(\bigcap_{P \in I} g^G M = \bigcap_{P \in I} g^G \ker \pi_P\). In particular, \(\bigcap_{P \in I} M = \bigcap_{P \in I} \ker \pi_P\).

**Proof.** If \(M \triangleleft_f G\) induces \(P \in I\) then \(M \supseteq \ker \pi_P\), so \(g^G M \supseteq \bigcap_{P \in I} g^P \ker \pi_P\). Conversely, given \(P \in I\) let \(P \uparrow = \{M \triangleleft_f G: M \cap A_x = P_x \text{ for all } x \in V_X\}\). Now by a theorem of Dyer [5], \(G_P\) is conjugacy separable, which implies that

\[
(g \pi_P)^G_P = \bigcap_{P \uparrow} (g \pi_P)^G_P M \pi_P = \bigcap_{P \uparrow} (g^G M) \pi_P,
\]

whence \(g^G \ker \pi_P = \bigcap_{P \in I} g^G M\). The result follows. (For the final part take \(g = 1\).) \(\square\)

3. PROFinitE Closures

It is evident that if \(P, Q \in I\), then \(P \cap Q = (P_x \cap Q_x)_{x \in V_X}\) also belongs to \(I\). We refer to the topology on \(A_x\) with \(\{P_x: P \in I\}\) as a fundamental system of open neighbourhoods of the identity as the \(I\)-topology. For each \(x\) let \(\hat{A}_x = \lim_{P \in I} (A_x/P_x)\), the inverse limit being formed relative to the partial order \(\leq\) of \(I\) introduced above. Let \(\overline{H}_e\) denote the topological closure of \(H_e\) in \(\hat{A}_{i(e)}\). We have

**Lemma 3.1.** For every \(e \in E_X\), the isomorphism \(\theta_e: H_e \rightarrow H_e\) extends to an isomorphism \(\overline{\theta}_e: \overline{H}_e \rightarrow \overline{H}_e\) such that \((\overline{H}_e \cap \overline{P}_{i(e)})\overline{\theta}_e = \overline{H}_e \cap \overline{P}_{i(e)}\).

**Proof.** For each \(P \in I\) there are canonical isomorphisms

\[
\overline{H}_e/(\overline{H}_e \cap \overline{P}_{i(e)}) \cong \overline{H}_e \overline{P}_{i(e)}/\overline{P}_{i(e)} = H_e \overline{P}_{i(e)}/\overline{P}_{i(e)}
\]

\[
\cong H_e/(H_e \cap \overline{P}_{i(e)}) = H_e/H_e \cap P_{i(e)},
\]

and \(\overline{H}_e\) is naturally isomorphic to the inverse limit \(\lim_{P \in I} (H_e/H_e \cap P_{i(e)})\) [12]. The isomorphisms induced by \(\theta_e\) on the quotients \(H_e/H_e \cap P_{i(e)}\) are compatible.
with the inverse limit structure, so the existence of \( \overline{\vartheta}_e \) follows. (If one thinks of the inverse limit \( \hat{A}_x \) as a subgroup of the cartesian product \( \prod_{x \in V \times X} (A_x/P_x) \), then the image of the element \( h = (h_P \pi_P) \in \overline{H}_e \) is \( h \overline{\vartheta}_e = (h_P \theta_e \pi_P) \in \overline{H}_e \).) The intersection property of \( \overline{\vartheta}_e \) is also trivial from the construction. \( \square \)

Thus we have a graph of groups \( \mathcal{G}^+ = \{ \hat{A}_x, \overline{H}_e, \overline{\vartheta}_e \} \) over \( X \). Let \( G^+ = \pi_1(\mathcal{G}^+, X) \). The natural embeddings \( A_x \to \hat{A}_x \) extend to a homomorphism \( \mu : G \to G^+ \). The next few results exploit the relationship between \( G^+ \) and the residual properties of \( G \).

Lemma 3.2. Let \( G, G^+ \), and \( \mu \) be as above. Then \( G^+ \) is residually finite and \( \ker \mu = \bigcap_{M \in G} M \).

Proof. The edge subgroups of \( G^+ \) are closed in their vertex groups by definition, and 3.1 implies that if \( P \in I(\mathcal{G}) \) then \( \overline{P} = (\overline{P}_x) \in I(\mathcal{G}^+) \). The residual finiteness of \( G^+ \) is therefore a consequence of the theorem of [16]. For the second part let \( P \in I \), and consider the following diagram with exact top row:

\[
\begin{array}{ccc}
\ker \mu & \to & G \\
\pi_P \downarrow & & \downarrow \pi_P^+
\end{array}
\]

The diagram is easily seen to commute, and \( A_x/P_x \cong \hat{A}_x/\overline{P}_x \) implies that \( \mu_P \) is an isomorphism. Then \( 1 = (\ker \mu) \mu \pi_P = (\ker \mu) \pi_P \mu_P \) implies that \( \ker \mu \subseteq \ker \pi_P \) for all \( P \), so \( \ker \mu \subseteq \bigcap_P \ker \pi_P = \bigcap_{M \in G} M \) by 2.1. On the other hand, \( G/\ker \mu \) is a subgroup of \( G^+ \); being residually finite, the reverse inclusion follows. \( \square \)

Before proceeding further we mention that, in general, profinite closedness of the edge subgroups in their vertex groups is not necessary for the residual finiteness of \( G \) (cf. [15, 17]). In the case of profinite vertex groups, however, we have the following result.

Corollary 3.3. Let \( G = \pi_1(\mathcal{G}, X) \), and assume that \( A_x = \lim_{P \in I} (A_x/P_x) \) for all \( x \in V \times X \). Then \( G \) is residually finite if and only if every edge subgroup is profinitely closed in its vertex group.

Proof. Suppose there exists an element \( a \in \overline{H}_e \backslash H_e \) for some \( e \in E \). Then \( G \) is not residually finite by 3.2. The converse follows from 3.2 and the fact that when the edge subgroups are closed we have \( G = G^+ \). \( \square \)

If \( S \) is a subset of a group \( H \), write \( \text{cl}_H(S) \) for \( \bigcap_{M \in H} SM \).

Theorem 3.4. Let \( G, G^+, \) and \( \mu \) be as above. If \( g \in G \) then \( \text{cl}_G(g^G \mu) = \text{cl}_{G^+}((g \mu)^{G^+}) \cap G \mu \).

We are not assuming that \( \mu \) is injective.
Proof. Let $S = \text{cl}_{G}(g^G) = \bigcap_{P \in I} S^G \ker \pi_P$ (by 2.1). Then using $\ker \mu \subseteq \ker \pi_P$ (cf. the proof of 3.2) we have

$$S \mu = \bigcap_{P \in I} (g^G \ker \pi_P) \mu = \bigcap_{P \in I} (g^G \mu)(\ker \pi_P) \mu.$$

Now it is easy to see from the commutative diagram in the proof of 3.2 and the fact that $\mu_P$ is an isomorphism, that $(\ker \pi_P) \mu = \ker \pi_P^+ \cap G \mu$. The proof of 2.1 applies without change to show that $\text{cl}_{G^*}(y^{G^*}) = \bigcap_{P \in I} y^{G^*} \ker \pi_P^+$ for all $y \in G^+$. Thus

$$S \mu = \bigcap_{P \in I} (g^G \mu)(\ker \pi_P^+ \cap G \mu) = \bigcap_{P \in I} (g^G \mu) \ker \pi_P^+ \cap G \mu,$$

which is certainly contained in $\bigcap_{P \in I} (g^G \mu) \ker \pi_P^+ \cap G \mu$. Conversely, suppose we have an element $z \mu \in \bigcap_{P \in I} (g^G \mu) \ker \pi_P^+$, where $z \in G$. Then for each $P$ there exists $w_P \in G^+$ such that $z \pi_P^+ = (w_P^{-1}(g^G \mu)w_P)\pi_P^+$. Since $\mu_P$ is an isomorphism, there exists $u_P \in G$ such that $u_P \pi_P \mu_P = w_P \pi_P^+$. The above equation now becomes $z \pi_P \mu_P = (u_P^{-1}(g^G \mu)\mu_P \pi_P^+$, and since $\mu_P$ is injective we obtain $z \in \bigcap_{P \in I} g^G \ker \pi_P = S$. We have now shown that

$$S \mu = \bigcap_{P \in I} (g^G \mu) \ker \pi_P^+ \cap G \mu \subseteq \bigcap_{P \in I} (g^G \mu) \ker \pi_P^+ \cap G \mu \subseteq S \mu,$$

as required. □

Since conjugacy separable groups are residually finite, we henceforth assume that $\ker \mu = \langle 1 \rangle$, and identify $G$ with $G \mu$. The statement of 3.4 can then be written more simply as $\text{cl}_{G}(g^G) = \text{cl}_{G^*}(g^{G^*}) \cap G$. What is the point of this result? Let $g$ and $g'$ be elements of $G$ which are conjugate in every finite image of $G$. By 2.1 this means that for every $P \in I$ there exists $w_P \in G$ such that $g' \pi_P = (w_P^{-1}(g^G \mu)w_P)\pi_P$, and at first glance it is conceivable that the "length" of the elements $w_P$ (in whatever sense) might be unbounded. Now if we have $\text{cl}_{G^*}(g^{G^*}) = g^{G^*}$, then 3.4 implies that $g' \in \text{cl}_{G^*}(g^G) = g^{G^*} \cap G$, so $g' = w^{-1}g w$, where $w \in G^+$. This, of course, means that the above $w_P$ can be chosen to have bounded length. It turns out that conjugacy separability is commonplace for elements of $G^+$. Before we proceed with this, we state the following, which does not require $\ker \mu = \langle 1 \rangle$.

Corollary 3.5. If $g$ is conjugacy distinguished in $G$, then

$$(g^G \mu)^+ \cap G \mu = g^G \mu.$$

Proof. For if $(g^G \mu)^+ \cap G \mu = S \mu$, where $S \neq g^G$, then

$$g^G \mu = \text{cl}_{G}(g^G \mu) = \text{cl}_{G^*}((g^G \mu)^+) \cap G \mu \supseteq (g^G \mu)^+ \cap G \mu = S \mu,$$

and so $\text{cl}_{G}(g^G) \supset S \supset g^G$. □

It is necessary to study conjugacy separability of the base group of $G^+$ first. The next section is devoted to the study of $A^* = \pi_1(\mathcal{G}|_Y, Y)$, where $Y$ is a maximal subtree of $X$.
4. Conjunctivity separability of $A^+$

We begin with the following general fact.

**Proposition 4.1.** Let $\mathcal{G}$ be a graph of groups over $X$, and assume that

\[
\bigcap_{e \in E} H_e P_{i(e)} = H_e \quad \text{for all } e \in E X,
\]

with $G = \pi_1(\mathcal{G}, X)$. Let $Y$ be a connected subgraph of $X$, and put $G(Y) = \langle A_x : x \in V Y \rangle \leq G$. Then $G(Y) = \bigcap_{e \in I} G(Y) \ker \pi_p$. If $Y$ is finite then $G(Y) = \bigcap_{M \in G} G(Y) M$.

**Proof.** Fix a vertex $x_0 \in V Y$, and for each edge $e \in E X$ let $T_e \equiv 1$ be a left transversal of $H_e$ in $A_{i(e)}$. Then every element of $G$ is uniquely represented by a normal word $t_1 \cdots t_n a$, where $a \in A_{x_0}$, $t_i \in T_{e_i}$ for $1 \leq i \leq n$, $(e_1, \ldots, e_n)$ is a closed path at $x_0$, and if $e_i = e_{i-1}$ then $t_i \neq 1$ [8, Corollary 1]. Moreover, the proof shows that the elements of $G(Y)$ are represented by paths that entirely belong to $Y$. Now condition (1) means that, given any element $g = t_1 \cdots t_n a$ in normal form in $G$, we can find $P \in I$ such that $(t_1 \pi_P) \cdots (t_n \pi_P)(a \pi_P)$ is the normal form of $g \pi_P$ in $G_P$ (relative to suitable transversals of the $H_e \pi_p$ in the $A_{i(e)} \pi_p$). In particular, if $g \in G \setminus G(Y)$ then $g$ is represented by a path that goes outside $Y$, and since $g \pi_P$ is represented by the same path, we have $g \pi_P \notin G_P(Y) = \langle A_x \pi_P : x \in V Y \rangle = G(Y) \pi_P$. The first part follows.

Now suppose that $Y$ is finite. Then for any $P \in I$ we have the finitely generated subgroup $G(Y) \pi_p$ of the free-by-finite group $G_P$. Such subgroups are profinitely closed (e.g., [2, p. 229]). This evidently means that $G(Y) \ker \pi_p = \bigcap_{M \in P \uparrow} G(Y) M$, and the second part follows (cf. the proof of 2.1). \hfill $\Box$

To simplify the notation let $Y$ be a tree, $\mathcal{G}$ a graph of groups over $Y$, and $A = \pi_1(\mathcal{G}, Y)$. Also write $C = \bigcup_{x \in V Y} A_x$ and $D = \bigcup_{e \in E Y} H_e$, the subsets of vertex elements and edge elements respectively in $A$.

**Proposition 4.2.** Let $g$ be an element of $A$. Then the following assertions are true:

(i) If $g \in D$, then there exists a subtree $Y_g$ of $Y$ such that $g \in H_e$ if and only if $e$ is an edge of $Y_g$.

(ii) If $g \notin D$, then there exists a FINITE subtree $Y_g$ of $Y$ such that $g \in A(Y_g)$, and if $Z$ is any subtree of $Y$ with $g \in A(Z)$, then $Y_g \subseteq Z$.

**Proof.** (i) Let $Y_g$ consist of all vertices $x$ of $Y$ such that $g \in A_x$, and all geodesics between them. If $g \in A_x \cap A_y$, and $p$ is the $Y$-geodesic from $x$ to $y$, then $g$ belongs to every vertex group of $p$ (since adjacent vertex groups in $A$ generate their amalgamated free product). In other words, $g$ belongs to every vertex group of $Y_g$. Moreover, $H_e = A_{i(e)} \cap A_{i(e)}$ for any edge $e$, so $g \in H_e$ if and only if $i(e)$ and $t(e)$ belong to $Y_g$.

(ii) Among all subtrees $Z$ with $g \in A(Z)$ pick one with the fewest number of vertices, and call it $Y_g$. Fix a vertex $x_0$ in $Y_g$. Then the normal form $t_1 \cdots t_n a$ of $g$ can be represented by a closed path $p$ at $x_0$, with $p$ entirely contained in $Y_g$ (since $g \in A(Y_g)$). Now let $Z$ be a subtree such that $g \in A(Z)$. Suppose first that $Y_g = \{x_0\}$. If $x_0 \notin V Z$ then $g \in A_{x_0} \cap A(Z)$ must be an edge element (consider the geodesic from $x_0$ to $A(Z)$), a contradiction. So in this case $Y_g \subseteq Z$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
There remains the case where $Y_g$ has more than one vertex. If $Z$ contains every end vertex of $Y_g$, then $Y_g \subseteq Z$, so suppose $Z$ does not contain some end vertex $x$ of $Y_g$, and let $e$ be the edge of $Y_g$ that ends at $x$. Now the path $p$ must visit $x$ at least once (since otherwise $g \in A(Y_g \setminus \{e, x\})$, and if $t_i \in A_x$ then $t_i \neq 1$ (since $p$ looks like $\ldots, e, x, \bar{e}, \ldots$ at $x$, and we have a normal form). If $Z$ and $Y_g$ were disjoint then $g \in A(Z) \cap A(Y_g)$ would be an edge element, which is not the case. Thus $Z' = Z \cup Y_g \setminus \{e, x\}$ is a subtree of $Y$ such that $g \in A(Z')$. If $x_0$ is on $Z'$ then we have the immediate contradiction that $g$ can be represented in normal form by a path $q$ in $Z'$, and $q \neq p$ since $x$ is on $p$ but not on $q$. So we must have $x_0 = x$. But since $Z$ and $Y_g$ are not disjoint we can always choose $x_0$ to be a common vertex. This final contradiction shows that $Z \supseteq Y_g$, as required.

To define the notion of a reduced form for elements of $A$ we need a definition. Let $A'$ be a nonempty subtree of a finite tree $A$. By a reduction process $R$ from $\Delta$ to $\Delta'$ we mean a sequence $\Delta_1, \ldots, \Delta_m$ of subtrees of $\Delta$ such that $\Delta_1 = \Delta$, $\Delta_m = \Delta'$, and $\Delta_{i+1}$ is obtained from $\Delta_i$ by deleting an end vertex $x_i$ (for $i = 1, \ldots, m - 1$). We let $y_i$ be the vertex of $\Delta_i$ adjacent to $x_i$, so when $\Delta \subseteq Y$ we have

$$A(\Delta_i) = (A(\Delta_{i+1}) \ast A_{x_i} : H_{y_i}x_i = H_{x_i}y_i \text{ via } \theta_{y_i}x_i), \quad i = 1, \ldots, m - 1.$$ 

Let $\Delta \supseteq \Delta'$ be finite subtrees of $Y$, and $R = \{\Delta_1, \ldots, \Delta_m\}$ a reduction process from $\Delta$ to $\Delta'$. Say $g$ is reduced (resp. cyclically reduced) relative to $R$ if either $g \in A(\Delta')$ or $g \in A(\Delta_i) \setminus A(\Delta_{i+1})$ for some $i \leq m - 1$, and then $g$ is reduced (resp. cyclically reduced) in the amalgamated free product $A(\Delta_i)$ of $A(\Delta_{i+1})$ and $A_{x_i}$. Every element of $A(\Delta)$ can be written in reduced form, and is conjugate, in $A(\Delta)$, to a cyclically reduced element of $A(\Delta)$. Note that an element $m$ may be cyclically reduced relative to one reduction process, but not another.

**Lemma 4.3.** Assume that (1) holds. Let $g$ be a nonvertex element of $A$ such that $g$ is cyclically reduced relative to some reduction process from $Y_g$ to a single vertex. Then $\cl_A(g^A) = \bigcup g_\alpha^A$, where each $g_\alpha \in A(Y_g)$.

**Proof.** There exists $P_0 \in I$ such that $Y_{g \pi P} = Y_g$ for all $P \subseteq P_0$. To begin with, we can choose $P_1$ such that $g \pi P$ is cyclically reduced in $A(Y_g) \pi P$ relative to $R$, for all $P \subseteq P_1$. This is because (1) can be used to ensure that the finitely many vertex elements in the cyclically reduced form of $g$ are excluded from the finitely many edge subgroups encountered by $R$. In particular, if $P \subseteq P_1$ then $g \pi P$ is not an edge element of $A \pi P$. For such $P$ we have $Y_{g \pi P} \subseteq Y_g$ (since $g \pi P \in A(Y_g) \pi P$), and if $Q \subseteq P$ then $Y_{g \pi P} \supseteq Y_{g \pi Q}$. Since $Y_g$ is finite, there exists a subtree $\Delta \supseteq Y_g$ and $P_0 \subseteq P_1$ such that for all $P \subseteq P_0$ we have $Y_{g \pi P} = \Delta$. Then $g \in \bigcap_{P \subseteq P_0} A(\Delta) \ker \pi_P = A(\Delta)$ by 4.1, and then $\Delta = Y_g$ by definition of $Y_g$.

Now suppose $P \subseteq P_0$. Then $g \pi P$ is a cyclically reduced nonvertex element of $A(Y_g) \pi P$, so the conjugacy theorem for amalgamated free products [9] implies that the conjugacy class of $g \pi P$ in $A(Y_g) \pi P$ contains no vertex elements. If $\Delta \supseteq Y_g$ is a finite subtree then the conjugacy class of $g \pi P$ in $A(\Delta) \pi P$ contains no vertex elements (consider a reduction process from $\Delta$ to $Y_g$). Now
let $g' \in \text{cl}_A(g^A)$, and let $\Delta \supseteq Y_g$ be a finite subtree with $g' \in A(\Delta)$. Let $R = \{\Delta_1, \ldots, \Delta_m\}$ be a reduction process from $\Delta$ to $Y_g$. Replacing $g'$ by a conjugate in $A(\Delta)$ we may assume that $g'$ is cyclically reduced relative to $R$, and we may choose $P_0$ such that $g'\pi_P$ is cyclically reduced for all $P \subseteq P_0$. If $P \subseteq P_0$ then $g\pi_P$ has minimal length in its $A(\Delta)\pi_P$-conjugacy class, and belongs to $A(\Delta_2)\pi_P\backslash H_{x_1}y_1\pi_P$. The conjugacy theorem for amalgamated free products \cite{9} implies that $g'\pi_P \in A(\Delta_2)\pi_P$. Continuing in this way we finally obtain $g'\pi_P \in A(Y_g)\pi_P$, whence $g' \in \bigcap_{P \subseteq P_0} A(Y_g)\ker\pi_P = A(Y_g)$, as required. □

We can now prove

**Theorem 4.4.** Let $g$ be a nonvertex element of $A$ such that $g$ is cyclically reduced relative to some reduction process $R$ from $Y_g$ to a vertex. Assume that (1) holds and that $H_{x_1}y_1$ is compact in the $I$-topology (where $x_1, y_1$ is the first edge deleted in the reduction process). Then $g$ is conjugacy distinguished in $A$.

**Proof.** By 4.3 we have $\text{cl}_A(g^A) = \bigcup g_o^A$, where each $g_o \in A(Y_g)$, and we may assume that each $g_o$ is cyclically reduced relative to $R$. Fix $a$, and choose $P_0$ such that for all $P \subseteq P_0$, $g\pi_P$ and $g_a\pi_P$ are cyclically reduced in $A(Y_g)\pi_P$ relative to $R$. The conjugacy theorem for amalgamated free products can now be applied to deduce that $g_a\pi_P$ is conjugate to some cyclic permutation of $g\pi_P$, via an element of $H_{x_1}y_1\pi_P$. Replacing $g$ by a cyclic permutation if necessary we have $g_a\pi_P = (h^{-1}g\pi_P)\pi_P$ for all $P \subseteq P_0$, where $h\pi_P \in H_{x_1}y_1$. Consider the function $f: H_{x_1}y_1 \rightarrow A(Y_g)$ given by $f(h) = g_a^{-1}h^{-1}gh$. With the appropriate profinitely topologies it is easy to see that $f$ is continuous and $\ker\pi_P \cap A(Y_g)$ is closed in $A(Y_g)$. Thus for each $P \subseteq P_0$, $f^{-1}(\ker\pi_P)$ is a nonempty closed subset of $H_{x_1}y_1$, and $P \subseteq Q$ implies that $f^{-1}(\ker\pi_P) \subseteq f^{-1}(\ker\pi_Q)$, so we have the finite intersection property. If

$$h \in \bigcap_{P \subseteq P_0} f^{-1}(\ker\pi_P) = f^{-1}\left(\bigcap_{P \subseteq P_0} \ker\pi_P\right) = f^{-1}(\{1\}),$$

then $g_a^{-1}h^{-1}gh = 1$, so $g_a \in g^A$. The result follows. □

The next lemma collects the information we need on conjugacy of edge and vertex elements.

**Lemma 4.5.** Let $C = \bigcup_{x \in \text{VY}_h} A_x$ and $D = \bigcup_{e \in \text{EY}} H_e$.

(i) Let $h \in D$, and let $a \in C$ be such that $h^a \in C$. Then $a \in A_x$ for some $x \in \text{VY}_h$.

(ii) Let $a \in A_x$, $b \in A_y$, and $ab \in C$. Then there exists a vertex $z$ such that $a, b \in A_z$.

(iii) Let $h \in D$, and $w$ be a nonvertex element of $A$ such that $h^w \in C$. Let $w = a_1 \cdots a_r$ be the reduced form of $w$ (relative to some reduction process $R$ of $Y_w$), where the $a_i \in C$. Then $h^{a_1\cdots a_i} \in D$ for $1 \leq i \leq r - 1$.

(iv) Let $a \in A_x$ be such that $a^4 \cap D = \emptyset$. Then $a^4 \cap D = \emptyset$, and $a^4 \cap C = a^4$.

**Proof.** (i) Choose $x$ as close to $Y_h$ as possible subject to $a \in A_x$, and get a contradiction to $h^a \in C$ by assuming that $x \notin \text{VY}_h$ and considering the geodesic from $x$ to $Y_h$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
(ii) Similar to (i): choose $x_1$ and $y_1$ as close to each other as possible (on the geodesic joining $x$ and $y$) subject to $a \in A_{x_1}$ and $b \in A_{y_1}$.

(iii) First note that $Y_h \cap Y_w \neq \emptyset$, for otherwise a consideration of the geodesic from $Y_h$ to $Y_w$ gives the contradiction $h^w \notin C$. Thus $a = h^w \in A(Y_w)$. The proof is by induction on $r$, where $w = a_1 \cdots a_r$. Consider the first step in the reduction process: if $h^{a_1}$ is not an edge element then $h^w$ is reduced as written (in the amalgamated free product of $A(\Delta_2)$ and $A_{x_1}$) and so cannot be a vertex element. Therefore $h_1 = h^{a_1} \in D$, and $a = h^{a_2 \cdots a_r}$. The result follows by induction on $r$ (using part (i)).

(iv) Suppose $a^w = h \in D$ for some nonvertex element $w = a_1 \cdots a_r$, in reduced form. Then $h^{a_1^{-1} \cdots a^{-1}_r} = a \in C$, and by part (iii) we get $a^{a_1} = h^{a_2 \cdots a_r} \in D$. So $a^A \cap D \neq \emptyset$ implies that $a^b \in D$ for some vertex element $b$. By part (ii), $a$ and $b$ must belong to the same vertex group, which has to be $A_x$ since $a$ belongs to no other vertex group. But now we have $a^A \cap D \neq \emptyset$, a contradiction. It follows that $a^A \cap D = \emptyset$.

For the second part let $b = aw \in A_y$, and choose $y$ as close to $x$ as possible subject to $b \in A_y$. Claim that $y = x$. For if not, then $Y_w \neq \{x\}$, and we have two cases to consider:

Case 1. $x \notin V Y_w$. Then there exists a finite subtree $\Delta \supseteq Y_w$ and an edge $yx$ with $y \in V \Delta$. Now $a \in A_x \setminus H_{xy}$, $w \in A(\Delta)$, and $b = w^{-1}aw \in A(\Delta)$ (as $\Delta$ has to contain $y$ because $b = aw \in A(\Delta \cup \{x\})$ and $y \neq x$). This forces $w \in H_{yx} = H_{xy}$, and $a^w \in (A(\Delta) \cap A_x = H_{xy}$, contradicting $a^A \cap D = \emptyset$.

Case 2. $x \in V Y_w$. Let $R = \{\Delta_1, \ldots, \Delta_m\}$ be a reduction process from $Y_w$ to a point, such that the first vertex deleted is $x_1 \neq x$ (this can be done since $Y_w \neq \{x\}$). Write $w = a_1 \cdots a_r$ in reduced form relative to $R$. Then $b = a^{-1}_1 \cdots a^{-1}_r a_1 \cdots a_r \in A_y \subseteq A(\Delta_x) \cup A(x_1)$, so we must have $a_1 \in A_x$ and $a^{a_1} \in H_{xy_1}$ (since $r \geq 2$). This is a contradiction.

We have now shown that $b = aw \in A_x$. An argument similar to that in Case 2 now shows that $w \in A_x$, so $a^A \cap C \subseteq a^{A_x}$, are required. □

We can now prove

**Theorem 4.6.** Let $g \in A_x$ be such that $g^{A_x} \cap D = \emptyset$. Assume that $X$ is locally finite, that every edge subgroup of $A$ is compact in the $I$-topology, and that $\bigcap_{P \subseteq I} g^{A_P} P_x = g^{A_x}$. Then $g$ is conjugacy distinguished in $A$.

**Proof.** If $cI(A(g^A))$ contains a nonvertex cyclically reduced element $g_0$, then by 4.4 we have $cI(A(g^A)) = cI(A(g^A)) = g^A$, so $g^A = g_0^A = cI(A^A)$ and we are done. (Note that condition (1) is a consequence of the compactness assumption, so 4.4 is applicable.) We may therefore suppose that $cI(A(g^A)) = \bigcup a_i^A$, where each $a_i$ is a vertex element of $A$. Let $S = \bigcup_{i(e) = x} H_e$, and note that $S$ is compact since $X$ is locally finite. Put $T_P = g^{A_x} \pi_P \cap S \pi_P$. We claim that there exists $P_0$ such that $T_P = \emptyset$ for all $P \subseteq P_0$. Suppose not. Note that $Q \subseteq P$ implies that $T_Q$ maps into $T_P$ under the obvious map (from $A_Q$ to $A_P$), so if $T_P \neq \emptyset$ then we have an inverse system of nonempty finite sets. Let $(h^P \pi_P) \in \lim T_P$, where each $h_P \in S$. Since $S$ is compact there exists $h \in S$ such that $h^P \pi_P = h^P \pi_P$ for all $P$. This clearly means that $h \in \bigcap_{P \subseteq I} g^{A_P} P_x = g^{A_x}$, contradicting $g^{A_x} \cap D = \emptyset$. Hence there exists $P_0$ such that $g^{A_x} \pi_P \cap D \pi_P = \emptyset$ for all $P \subseteq P_0$. By 4.5(iv)
we have $g^A \pi P \cap D \pi P = \emptyset$, and $a_i \pi P \in g^A \pi P \cap C \pi P = g^A \pi P$. Thus first of all $a_i \in \bigcap_{P \subseteq P_0} A_x \ker \pi P = A_x$, and then $a_i \in \bigcap_{P \subseteq P_0} g^A \pi P_x = g^A_x$. Thus $\text{cl}_A(g^A) = g^A$, as required. □

Remark 1. For $g$ as in 4.6, the condition $\bigcap_P g^A_x \pi P_x = g^A_x$ is implied by $\text{cl}_A(g^A) = g^A$. For if $b \in \bigcap g^A_x \pi P_x$, then $b \in \text{cl}_A(g^A) = g^A$, and so $b \in g^A \cap C = g^A_x$ by 4.5(iv).

Remark 2. The usual proof of the conjugacy separability of the profinite group $\hat{A}_x$ in fact shows that for $g \in \hat{A}_x$ we have $\bigcap_{P \subseteq I} g^\hat{A}_x \pi P_x = g^\hat{A}_x$.

The next two results give partial information about conjugacy separability of edge elements.

Lemma 4.7. Assume that (1) holds. Let $h$ be an edge element of $A$, and assume that there exists $P_0 \in I$ such that $Y_{h \pi P_0} = Y_h$. Then $\text{cl}_A(h^A) \cap D = \text{cl}_A(A_y(h^A_y)) \cap D$.

Proof. The condition implies that $h \pi P \notin H \pi P$ for all $P \subseteq P_0$, where $e$ is an edge leading out of $Y_h$. Let $z \in \text{cl}_A(h^A) \cap D$, so $z \pi P = h^w \pi P$ for some $w \in A$. We claim that $w$ can be chosen in $A(Y_h)$. For if not, let $\Delta$ be the smallest tree containing $h \pi P \cap w \pi P$, and consider a reduction process $\{\Delta_1, \ldots, \Delta_m\}$ from $\Delta$ to $\Delta \cap Y_h$. Now the first vertex $x_1 \notin VY_h$, and $w \pi P \notin A(\Delta_2) \pi P$ by the minimal choice of $\Delta$, and yet $(h \pi P)^w \pi P \in A$ belongs to a vertex group. By the conjugacy theorem for amalgamated free product, we must have $h \pi P \in H_{Y_{x_1}} \pi P$, contradicting the choice of $P$. We have therefore shown that

$$h_0 \in \bigcap_{P \subseteq P_0} h^A \pi P \ker \pi P \subseteq \bigcap_{P \subseteq P_0} A(Y_h) \ker \pi P = A(Y_h),$$

and so

$$h_0 \in \bigcap_{P \subseteq P_0} h^A \pi P \ker \pi P \cap A(Y_h)
= \bigcap_{P \subseteq P_0} h^A \pi P \ker \pi P \cap A(Y_h) = \text{cl}_A(A_y(h^A_y)) \cap D.$$ □

Let $h$ be an edge element of $A$. For any sequence $\sigma = (x_1, \ldots, x_r)$ of vertices of $A$ put $D_\sigma(h) = D \cap \{h^{a_1 \cdots a_r} : a_i \in A_{x_i}\}$. We have

Theorem 4.8. Assume that $X$ is locally finite and $D$ is compact. Let $h$ be an edge element of $A^+$. Then $h$ is conjugacy distinguished in $A^+$ if and only if $h^{A^+} \cap D = D_\sigma(h)$ for some finite sequence $\sigma$ of vertices.

Proof. We know that $\text{cl}_{A^+}(h^{A^+}) = \bigcup h^A_{\alpha}$ with each $h_\alpha \in D$. Let $K = h^{A^+} \cap D = \bigcup_{\sigma} D_\sigma(h)$, the union being over all finite sequences $\sigma$. If $z \in \text{cl}_{A^+}(h^{A^+}) \cap D$ and $P \in I$ then there exists a sequence $\sigma(P)$ such that $z \pi P \in D_{\sigma(P)}(h) \pi P \subseteq K \pi P$, so $z \in K \ker \pi P$. Thus

$$\text{cl}_{A^+}(h^{A^+}) \cap D \subseteq \bigcap_{P} K(\ker \pi P) = \text{cl}_{A^+}(K).$$

Now $D$ is a compact subset of $A^+$, and so is closed. Therefore $\text{cl}_{A^+}(K) = \text{cl}_D(K)$. Conversely if $z \in \text{cl}_D(K)$ and $P \in I$ then $z \pi P \in K \pi P$, so $z \pi P \in \text{cl}_{A^+}(h^{A^+}) \cap D$. Therefore $\text{cl}_{A^+}(K) = \text{cl}_D(K)$. □
$D_\sigma(h)\pi_P$ for some sequence $\sigma$ (depending on $P$). In other words, $z \in \text{cl}_{A^+}(h^{A^+})$, and we have shown that $\text{cl}_{A^+}(h^{A^+}) \cap D = \text{cl}_{D}(h^{A^+} \cap D)$. We also note that for a fixed sequence $\sigma$, the set $D_\sigma(h)$ is a complete subspace of the compact set $D$ (since each vertex group $A_x$ is compact), and is therefore compact. So if $h^{A^+} \cap D = D_\sigma(h)$ then $\text{cl}_{A^+}(h^{A^+}) \cap D = \text{cl}_{D}(K) = K = h^{A^+} \cap D$, which implies that every $h_\alpha \in h^{A^+}$, and so $h$ is conjugacy distinguished in $A^+$. Conversely, suppose $\text{cl}_{A^+}(h^{A^+}) = h^{A^+}$. Then $K = h^{A^+} \cap D = \text{cl}_{D}(K)$, so the compact (since closed) set $K$ is the (countable) union of the closed sets $D_\sigma(h)$. Enumerate the finite sequences $\{\sigma_1, \sigma_2, \ldots\}$, and put $T_i = \bigcup_{1 \leq j \leq i} D_{\sigma_j}(h)$. Then $K = \bigcup_{i=1}^{\infty} T_i$, and each $T_i$ is closed. By the Baire Category Theorem (or rather its proof, using compactness), some $T_{i_0}$ has nonempty interior, and so contains a set of the form $hP_x \cap D$, where $h \in D$ and $P \in I$. Since $P_x \cap A_x$ and $K$ is compact, it is easy to see that $K$ is the union of finitely many $T_i$, whence $K = T_j$ for some $j$. Let $\sigma$ be the concatenation of the sequences $\sigma_1, \ldots, \sigma_j$. Then it is clear that $D_{\sigma_i}(h) \subseteq D_{\sigma}(h)$ for $1 \leq i \leq j$, and so $h^{A^+} \cap D = K = T_j \subseteq D_{\sigma}(h)$, as required. □

The following are immediate consequences of 4.7 and 4.8.

**Corollary 4.9.** Let $X$ be locally finite, and let $h$ be an edge element of $A^+$ such that $Y_h$ is finite. Then $h$ is conjugacy distinguished in $A^+$ if and only if for some finite sequence $\sigma$ of vertices of $Y_h$ we have $h^{A^+} \cap D = D_\sigma(h)$. □

**Corollary 4.10.** Let $A^+$ be the amalgamated free product of countably many groups. Then an element $h$ of the associated subgroup $H$ is conjugacy distinguished in $A^+$ if and only if $h^{A^+} \cap H = D_\sigma(h)$ for some finite sequence $\sigma$ of vertices. □

### 5. CONJUGACY SEPARABILITY OF $G^+$

If $X$ is not a tree then $G^+$ is a nontrivial HNN-extension over the base group $A^+ = \langle A_x : x \in V X \rangle$. By the length $|g|$ of an element $g \in G^+$ we mean the number of $t$-symbols appearing in an HNN-reduced form for $g$. Thus $g = a_1 t_{e_1} \cdots t_{e_n} a_{n+1}$, where all $a_i \in A^+$, $a_{n+1}$ is arbitrary, each $e_i = \pm 1$, each $e_i \in EX \setminus EY$ (where $Y$ is a fixed maximal subtree of $X$), and if $e_{i-1} = e_i$ and $e_{i-1} + e_i = 0$ then $a_i \notin H_{\pm e_i}$ (we write $H_{-e}$ instead of $H_e$ for ease of notation). The reduced element $g$ is cyclically reduced if $a_{n+1} = 1$ and $t_{e_i}^a a_1 t_{e_1}^b \cdots a_n$ is also reduced.

**Theorem 5.1.** Let $g \in G^+$ be cyclically reduced and of length at least one. Then $g$ is conjugacy distinguished in $G^+$.

**Proof.** Let $z$ be a cyclically reduced element in $\text{cl}_{G^+}(g^{G^+})$. It is easy to see that the edge subgroups $\overline{H}_{\pm e}$ are closed in $A^+$, so we can find $P_0 \in I$ such that $|g| = |g_P|$, $|z| = |z_P|$, and both $g_P$ and $z_P$ are cyclically reduced in $G_P$ for all $P \subseteq P_0$. The conjugacy theorem for HNN-extensions (e.g. [3]) now implies that $|g_P| = |z_P|$, so $|g| = |z|$. Modulo the same $P$ we know that $g_{-P}$ and some cyclic permutation of $z_{-P}$ have the same sequence of $t$-symbols with the same exponents. Replacing $z$ by a suitable cyclic permutation we may assume that $g = a_1 t_{e_1} \cdots a_n t_{e_n}$ and $z = b_1 t_{e_1} \cdots b_n t_{e_n}$. For $P \subseteq P_0$ the conjugacy theorem in $G_P$ implies the existence of elements $h_0, \ldots, H_n, P$ such that, modulo $P$
\[
\begin{align*}
  h_0, p b_1 & \equiv a_1 h_1, p, & h_1, p & \in \bar{H}_{e_1 e_1}, \\
  (h_1, p \theta_{e_1}^{p}) b_2 & \equiv a_2 h_2, p, & h_2, p & \in \bar{H}_{e_2 e_2}, \\
  & \vdots \\
  (h_{n-1}, p \theta_{e_n}^{p}) b_n & \equiv a_n h_n, p, & h_n, p & \in \bar{H}_{e_n e_n}, \\
  (h_n, p \theta_{e_n}^{p}) & \equiv h_0, p, & \text{so } h_0, p & \in \bar{H}_{-e_n e_n} = \bar{H}_{e_0 e_0}.
\end{align*}
\]

Since \( \bigcup_{i=0}^{n} \bar{H}_{e_i e_i} \) is compact, there exist elements \( h_i \in \bar{H}_{e_i e_i} \) such that \( h_i p P = h_i, p \pi P \) for all \( P \). Then we have the equations \( h_0 b_1 = a_1 h_1, \ (h_1 \theta_{e_1}^{p}) b_2 = a_2 h_2, \ldots, \ h_n \theta_{e_n}^{p} = h_0, \) and so \( z = h_0^{-1} g h_0 \). The result follows. \( \square \)

To deal with conjugacy separability of elements of \( A^+ \) in \( G^+ \), we begin with the following trivial observation, which can be proved by using a length argument (as in the proof of 5.1) and the conjugacy theorem for HNN-extensions.

**Lemma 5.2.** If \( a \in A^+ \) then \( \text{cl}_{G^+}(a^{G^+}) = \bigcup a^{G^+} \), where each \( a_a \in A^+ \). In particular, \( a \) is conjugacy distinguished in \( G^+ \) if and only if \( \text{cl}_{G^+}(a^{G^+}) \cap A^+ = a^{G^+} \cap A^+ \). \( \square \)

The conjugacy theorem for elements of the base of an HNN-extension states that if \( a, b \in A^+ \) are conjugate in \( G^+ \), then there exist elements \( h_1, \ldots, h_m \in \bigcup_{e \in EX \setminus EY} \bar{H}_e \), such that \( a \sim h_1 \) in \( A^+ \) (i.e., conjugate in \( A^+ \)), \( b \sim h_m \) in \( A^+ \), and for \( 1 \leq i \leq m-1 \), either \( h_{i+1} \sim h_i \) in \( A^+ \) or \( h_{i+1} = h_i \theta_e \) for some \( e \in EX \setminus EY \). Now we have

**Proposition 5.3.** Let \( a \in A^+ \) be a nonvertex cyclically reduced element (relative to some reduction process of \( Y_a \)). Then \( \text{cl}_{G^+}(a^{G^+}) \cap A^+ = a^{A^+} \), and so \( a \) is conjugacy distinguished in \( G^+ \).

**Proof.** Choose \( P_0 \) such that for all \( P \subseteq P_0 \) we have \( Y_a \pi_P = Y_a \) and \( a \pi_P \) is cyclically reduced (cf. the proof of 4.4). Then \( (a \pi_P)^{A^+} \pi_P \cap D \pi_P = \emptyset \) by 4.5(iii) and using the fact that no nonvertex conjugate of an edge element can be cyclically reduced. Now let \( b \in \text{cl}_{G^+}(a^{G^+}) \cap A^+ \). Then \( b \pi_P \) is conjugate to \( a \pi_P \) in \( G_P \). Since no \( A^+ \pi_P\)-conjugate of \( a \pi_P \) can belong to an edge subgroup in \( G_P \), we must have \( b \pi_P \) conjugate to \( a \pi_P \) in \( A^+ \pi_P \), and so \( b \in \text{cl}_{A^+}(a^{A^+}) = a^{A^+} \), the latter by 4.4. The final part follows from 5.2. \( \square \)

For vertex elements of \( A^+ \) we have the following (recall that \( C = \bigcup_{x \in V_X} \hat{A}_x \)):

**Proposition 5.4.** Assume that \( X \) is locally finite. Let \( a \) be a vertex element of \( A^+ \), and let \( b \in \text{cl}_{G^+}(a^{G^+}) \cap C \). Then there exists a sequence \( c_0, c_1, \ldots \) of elements of \( C \) such that (i) \( c_0 = a \); (ii) for each \( i \geq 1 \), either \( c_i \sim c_{i-1} \) in \( C \), or \( c_i = c_{i-1} \theta_e \) for some \( e \in EX \setminus EY \); and (iii) \( c_i \rightarrow b \) as \( i \rightarrow \infty \).

**Proof.** For \( P \in I \) let \( S_P(a, b) \) denote the set of all sequences \( \sigma = \{a_i, p \pi_P : i \geq 0\} \), where each \( a_i, p \in C \), \( a_0, p \pi_P = a \pi_P \), \( a_i, p \pi_P = b \pi_P \) for all sufficiently large \( i \), and each \( a_i, p \pi_P \) is either \( C \pi_P\)-conjugate to, or the \( \theta_e, p\)-image of, \( a_{i-1}, p \pi_P \) (in other words, each \( \sigma \in S_P(a, b) \) represents a possible conjugacy \( a \pi_P \sim b \pi_P \) in \( G_P^+ \)). For fixed \( i \) and \( P \) write \( T_i(P) \) for the set of \( i \)th components of all \( \sigma \in S_P(a, b) \). Suppose \( a \in \hat{A}_X \), and let \( \Delta_i \) denote the (finite)
subtree of radius \( i \) with centre \( x \). A simple inductive argument shows that \( T_i(P) \subseteq \bigcup_{y \in V_{\Delta_i}} \hat{A}_y \pi y P \), so each \( T_i(P) \) is finite. If \( Q \subseteq P \) then every sequence in \( S_Q(a, b) \) maps to a sequence in \( S_P(a, b) \) under the induced map \( G_Q^+ \rightarrow G_P^+ \), and so \( T_i(Q) \) maps into \( T_i(P) \).

We construct the elements \( c_i \) inductively, beginning with \( c_0 = a \). Assume that \( c_1, \ldots, c_{i-1} \) have been chosen subject to (ii) and \( c_j \pi P \in T_j(P) \) for all \( P \), and \( 1 \leq j \leq i - 1 \). Let \( K_P = \{ c' \pi P \in T_i(P) : \text{there exists } \sigma \in S_P(a, b) \text{ such that } \sigma_{i-1} c_{i-1} \pi P \text{ and } c_i = c' \pi P \} \). Then each \( K_P \neq \emptyset \), and if \( Q \subseteq P \) then \( K_Q \) maps into \( K_P \) under the induced map \( G_Q \rightarrow G_P \). Let \( (d_P \pi P) \) belong to the inverse limit of the sets \( K_P \). Then each \( d_P \in \bigcup_{y \in V_{\Delta_i}} \hat{A}_y \), and the latter set is compact, so there exists \( c_i \in \bigcup_{y \in V_{\Delta_i}} \hat{A}_y \) such that \( c_i \pi P = d_P \pi P \) for all \( P \). Therefore, for every \( P \) there exists \( \sigma \in S_P(a, b) \) such that \( \sigma_{i-1} = c_{i-1} \pi P \) and \( c_i = c_P \pi P \). Also \( c_i \pi P \in T_i(P) \) for all \( P \). To verify (ii), note that for each \( P \) the element \( c_i \pi P \) is either conjugate to \( c_{i-1} \pi P \) by some element of \( \bigcup_{y \in V_{\Delta_i}} \hat{A}_y \pi P \), or is the image of \( c_{i-1} \pi P \) under some \( \theta_{e, P} \), where \( e \in E_{\Delta_i} \).

The number of possibilities is therefore finite and independent of \( P \), and so at least one must hold for all \( P \). If \( c_i \pi P = c_{i-1} \pi P \theta_{e, P} \) for all \( P \), then \( c_i \pi P = \theta_{e, P} \) and we get \( c_i \pi P = (c_{i-1} \pi P) \pi P \) for all \( P \), whence \( c_i = c_{i-1} \pi P \). If \( c_i \pi P \sim c_{i-1} \pi P \) in \( \bigcup_{y \in V_{\Delta_i}} \hat{A}_y \pi P \) for all \( P \), then \( c_i \sim c_{i-1} \) in \( \bigcup_{y \in V_{\Delta_i}} \hat{A}_y \), using compactness. This proves (ii) for \( c_i \), and so the inductive construction goes through. Finally given \( P \) it is clear that \( c_i \pi P = b \pi P \) for all sufficiently large \( i \), so \( c_i \rightarrow b \) as \( i \rightarrow \infty \). □

**Corollary 5.5.** Let \( a \in A^+ \) be a vertex element. Then \( a \) is conjugacy distinguished in \( G^+ \) if and only if \( a G^+ \cap C \) is a closed subset of \( A^+ \).

**Proof.** By 5.2 and the proof of 5.3 we see that \( \text{cl}_{G^+}(a G^+) = \bigcup a_o G^+ \), where each \( a_o \in C \). By 5.4, for each \( a_o \) there exists a sequence \( c_i \in C \) such that \( c_i \sim a \) in \( G^+ \) and \( c_i \rightarrow a_o \) as \( i \rightarrow \infty \). Thus each \( a_o \in \text{cl}_{A^+}(a G^+ \cap C) \), and so \( \text{cl}_{G^+}(a G^+) \cap C \subseteq \text{cl}_{A^+}(a G^+ \cap C) \). The reverse inclusion being trivial, we have \( \text{cl}_{G^+}(a G^+) \cap C = \text{cl}_{A^+}(a G^+ \cap C) \). The result follows. □

**Corollary 5.6.** Let \( a \) be a vertex element of \( A^+ \), and let \( D' = \bigcup e \in E_X\setminus E_Y \hat{H}_e \). If \( a A^+ \cap D' = \emptyset \), then \( a \) is conjugacy distinguished in \( G^+ \) if and only if \( a \) is conjugacy distinguished in \( A^+ \).

**Proof.** Since no \( A^+ \)-conjugate of \( a \) belongs to an associated subgroup, we have \( a G^+ \cap A^+ = a A^+ \) by the conjugacy theorem for HNN-extensions, and so \( a G^+ \cap C = a A^+ \cap C \). The result follows from 5.5 and the fact that \( \text{cl}_{A^+}(a A^+) = \bigcup a_o A^+ \), with all \( a_o \in C \) (this is a consequence of 4.4). □

To deal with elements of \( D' = \bigcup e \in E_X\setminus E_Y \hat{H}_e \) we need the following:

**Lemma 5.7.** If \( D \) is compact, then so is \( C = \bigcup x \in V_X \hat{A}_x \).

**Proof.** It is sufficient to prove that \( C \) is complete, since we can always regard \( C \) as a subspace of the profinite completion \( \hat{A}^+ \). So let \( \{ a_P : P \in I \} \) be a subset of \( C \) such that for all \( Q \subseteq P \) we have \( a_Q \pi P = a_P \pi P \). If every \( a_P \in D \), then the compactness of \( D \) implies the existence of \( h \in D \) such that \( a_P \pi P = h \pi P \) for all \( P \), and we are done. Now suppose that \( a_P \pi P_0 \notin D \pi P_0 \) for some \( P_0 \). Then \( a_P \pi P \notin D \pi P \) for all \( P \subseteq P_0 \), for if \( a_P \in A_y \neq A_x \) for some \( P \subseteq P_0 \),
then \( a_p \pi p_0 = a_p \pi p_0 \in \hat{A}_x \pi p_0 \cap \hat{A}_x \pi p_0 \subseteq D \pi p_0 \), contrary to assumption. But now the compactness of \( \hat{A}_x \) gives the result. \( \square \)

Consider the set of all finite sequences \( \sigma = (x_1, e_1^{t_1}, \ldots, e_n^{t_n}, x_{n+1}) \), where \( x_i \in VX \), \( e_i \in E \setminus EY \), and \( e_i = \pm 1 \). For such a sequence let

\[
B_{\sigma}(h) = C \cap \{h^g : g = a_1 t_1^{e_1} \cdots t_n^{e_n} a_{n+1}, \quad a_i \in \hat{A}_x \text{ for all } i\}.
\]

**Theorem 5.8.** Assume that \( X \) is locally finite and \( D \) is compact. Then \( h \in D' \) is conjugacy distinguished in \( G^+ \) if and only if there exist finite sequences \( \sigma_1, \ldots, \sigma_m \) such that \( h^{G^+} \cap C = \bigcup_{i=1}^m B_{\sigma_i}(h) \).

**Proof.** By 5.5 the element \( h \) is conjugacy distinguished in \( G^+ \) if and only if \( K = h^{G^+} \cap C \) is closed in \( A^+ \). Since \( C \) is compact (by 5.7) and therefore closed, this is equivalent to \( K \) being closed in \( C \), i.e., to \( K \) being compact. Now \( K = \bigcup_{\sigma} B_{\sigma}(h) \), the union being over the countably many sequences \( \sigma \). An application of Baire’s Category Theorem (cf. the proof of 4.8) shows that \( K \) is compact if and only if it is the union of finitely many \( B_{\sigma_i}(h) \). (One needs to show that each \( B_{\sigma}(h) \) is closed. For this, use the compactness of the \( \hat{A}_x \) to show that \( B_{\sigma}(h) \) is complete.) \( \square \)

**References**


Department of Mathematics, University of Alberta, Edmonton, Alberta, T6G 2G1 Canada