$H^p$- AND $L^p$-VARIANTS OF MULTIPARAMETER
CALDERÓN-ZYGMUND THEORY

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ABSTRACT. We consider Calderón-Zygmund operators on product domains. Under certain weak conditions on the kernel a singular integral operator can be proved to be bounded on $H^p(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R})$, $0 < p \leq 1$, if its behaviour on $L^2$ and on certain scalar-valued and vector-valued rectangle atoms is known. Another result concerns an extension of the authors’ results on $L^p$-variants of 
Calderón-Zygmund theory [1, 23] to the product-domain-setting. As an application, one obtains estimates for Fourier multipliers and pseudo-differential operators.

1. Introduction

The first purpose of this paper is to extend to the multiparameter setting the results obtained by the authors in [1, 2, and 23] concerning the classical (one-parameter) Calderón-Zygmund theory of $L^p$ spaces. One of the results of [1 and 23] may be stated as follows. Let $\Phi$ be a smooth bump function on $\mathbb{R}$ supported in $[1, 4]$ such that $\sum_{m \in \mathbb{Z}} \Phi(|t|/2^m) \equiv 1$ on $\mathbb{R} \setminus \{0\}$.

Theorem A. Suppose $m$ is a (bounded) function on $\mathbb{R}^n$ such that if $\hat{m}_i(\xi) = m(2^i \xi)\Phi(|\xi|)$, we have

$$\sup_{i \in \mathbb{Z}} (\|\hat{m}_i\|_{p-p} + \|\hat{m}_i\|_{\Lambda_\varepsilon}) < \infty$$

for some $1 \leq p < 2$ and some $\varepsilon > 0$. Then $m$ is a Fourier multiplier of $L^r(\mathbb{R}^n)$ for $p < r < p'$.

(Here, $\| \cdot \|_{p-p}$ is the $L^p$ multiplier norm and $\Lambda_\varepsilon$ is the space of Lipschitz continuous functions of order $\varepsilon$.) Theorem A strengthens the classical Hörmander multiplier theorem and gives a good “almost orthogonality” criterion for Fourier multipliers in the sense that $L^p$ boundedness of each “dyadic piece” $\hat{m}_i$ implies $L^r$ boundedness of the original multiplier $m$ provided we have a tiny amount of smoothness. (This smoothness cannot be entirely dispensed with because of a classical counterexample of Littman, McCarthy and Rivière [20].) Thus one of our aims is to prove the analogue of Theorem A when the one-parameter family of dilations $\xi \mapsto 2^i \xi$ is replaced by the $n$-parameter family $(\xi_1, \ldots, \xi_n) \mapsto (2^{i_1} \xi_1, \ldots, 2^{i_n} \xi_n)$. To give a taste of what is to come, we state (for simplicity only) a result in $\mathbb{R}^2$.

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Theorem B. Suppose $m$ is a function on $\mathbb{R}^2$ such that if
\[
\hat{m}_i(\xi) = m(2^i \xi_1, \xi_2) \Phi(|\xi_1|), \quad \hat{m}_j(\xi) = m(\xi_1, 2^j \xi_2) \Phi(|\xi_2|),
\]
and
\[
\hat{m}_{ij}(\xi) = m(2^i \xi_1, 2^j \xi_2) \Phi(|\xi_1|) \Phi(|\xi_2|),
\]
we have
\[
\sup_{i \in \mathbb{Z}} \|\hat{m}_i\|_{p-p} < \infty, \quad \sup_{j \in \mathbb{Z}} \|\hat{m}_j\|_{p-p} < \infty,
\]
and
\[
\sup_{i,j \in \mathbb{Z}} \|\hat{m}_{ij}\|_{A'_{2}} < \infty.
\]
Then $m$ is a Fourier multiplier of $L^r(\mathbb{R}^2)$ for $p < r < p'$.

(Here again, $\|\cdot\|_{p-p}$ is the $L^p(\mathbb{R}^2)$ multiplier norm and $A'_{2}$ is the 2-parameter Lipschitz space with differences taken in both variables.)

The second purpose of the paper is to obtain a strengthened $H^p$-theory ($0 < p \leq 1$) of product domain Calderón-Zygmund singular integrals. Much work on this topic has been done recently by Journé [15, 16, 17], R. Fefferman [6, 7], Soria [24], and Pipher [22], with Journé’s work being especially important. The moral of R. Fefferman’s point of view [7] is that despite the fact that $H^p$ cannot be characterized by “rectangle atoms,” nevertheless a linear operator which “behaves well” with respect to rectangle atoms will be bounded from $H^p$ to $L^p$, at least in the 2-parameter setting. Journé [17] has shown that in three parameters, this philosophy breaks down, but, however, remains valid (see also H. Lin [19]) for convolution operators. Perhaps the principal achievement of this paper is to show that with a different interpretation of rectangle atoms—the Fefferman philosophy remains valid with any number of parameters. We stress at this point that the analysis of this paper still relies heavily on Journé’s geometric ideas contained in [15].

Let us make things a little more precise. The $H^p$-space on the product domain $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \mathbb{R}^n$ consists of those tempered distributions $f$ whose multiparameter area integral $S(f)$ is in $L^p$. Here,
\[
Sf(x) = \left( \int_\Gamma |\psi \ast f(y)|^2 \, dy \frac{dt}{t_1 \cdots t_n} \right)^{1/2},
\]
where $\Gamma(x)$ is the cone $\{(y,t) \in \mathbb{R}^n \times \mathbb{R}_+^n \mid |y_i - x_i| < t_i, \ i = 1, \ldots, n\}$, and
\[
\psi_t(x) = \prod_{i=1}^n \frac{1}{t_i} \psi \left( \frac{x_i}{t_i} \right)
\]
where $\psi$ is a $C^\infty$ function of compact support in $[-\frac{1}{4}, \frac{1}{4}]$ satisfying
\[
\int_\mathbb{R} \psi(t)p(t) \, dt = 0
\]
for all polynomials $p$ of degree $\leq M$ for sufficiently large $M$. (If $0 < p \leq 1$, we require $M > n/p$.) Of course a similar definition could have been made for the product space $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}$, but for simplicity only we assume each $d_i = 1$. Also, this definition of $S(f)$ and $H^p$ makes sense for $f$ taking values in a Hilbert space $H$, and all statements we shall make remain valid in this setting.

We now introduce vector-valued rectangle atoms.
Definition. A \((p, R)\) rectangle atom on \(\mathbb{R}^m\) with values in a Hilbert space \(H\) is a function \(a : \mathbb{R}^m \to H\) which is supported on some rectangle \(R\) in \(\mathbb{R}^m\) (here and always a rectangle means a rectangle with sides parallel to the coordinate axes) which satisfies

(i) \(\int_R |a(x)|^2_H \, dx \leq |R|^{1-2/p}\);
(ii) \(\int_R a(x_1, \ldots, x_j, \ldots, x_m)p(x_j) \, dx_j = 0\) for almost all \((x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m)\), \(j = 1, \ldots, m\), and for all polynomials \(p\) of degree \(\leq M\), where \(M\) is sufficiently large (we require \(M > n/p\)).

It is easy to see that every \((p, R)\) rectangle atom is in \(L^p(\mathbb{R}^m, H)\); see [4]. We shall refer to \(R\) as the rectangle associated to \(a\). We use the letter \(m\) instead of \(n\) in the above definition because we shall be considering rectangle atoms living on a factor subspace \(\mathbb{R}^m\) of \(\mathbb{R}^n\) with values in \(L^2(\mathbb{R}^{n-m})\).

To be more systematic, given a subset \(\alpha \subseteq \{1, 2, \ldots, n\}\), we denote by \(\mathbb{R}^\alpha\) the subspace of \(\mathbb{R}^n\) spanned by the unit coordinate vectors \(e_s\), \(s \in \alpha\). Similarly we define \(\mathbb{N}^\alpha\), \(\mathbb{Z}^\alpha\). We shall denote by \(\alpha'\) the complement of \(\alpha\) in \(\{1, \ldots, n\}\) so that \(\mathbb{R}^n = \mathbb{R}^\alpha \oplus \mathbb{R}^{\alpha'}\) for all \(\alpha\). \(\Pi_\alpha : \mathbb{R}^n \to \mathbb{R}^\alpha\) is the natural projection; when \(\alpha\) is a singleton, say \(\alpha = \{s\}\), we shall often write \(\Pi_s\) instead of \(\Pi_{\{s\}}\). If \(R\) is a dyadic rectangle in \(\mathbb{R}^\alpha\), we denote by \(2L_\alpha(R)\) the sidelength of the projection \(\Pi_\alpha R\) and by \(L_\alpha(R)\) the vector in \(\mathbb{Z}^\alpha\) with components \(L_s(R)\), \(s \in \alpha\).

Now consider a singular integral operator with kernel \(K(x, y)\) so that

\[ Tf(x) = \int K(x, y)f(y) \, dy. \]

For \(\alpha \subseteq \{1, \ldots, n\}\) and \(m \in \mathbb{Z}^\alpha\), let \(T_m^\alpha\) be the operator with kernel

\[ K(x, y) \prod_{s \in \alpha} \Phi \left( \frac{|x_s - y_s|}{2m_{s+2}} \right) = K(x, y)\Phi_m^\alpha(\bar{x} - \bar{y}) \quad (\text{with } \bar{x}, \bar{y} \in \mathbb{R}^\alpha). \]

When \(\alpha = \{1, \ldots, n\}\), \(T_m^\alpha\) will be denoted by \(T_m\), and \(L_\alpha(R)\) by \(L(R)\).

We are now ready to state our first result.

Theorem 1. Let \(T\) be a singular integral operator which is bounded on \(L^2(\mathbb{R}^n)\). Let \(0 < p \leq 1\) and \(\epsilon > 0\). Suppose that for all \(\alpha \subseteq \{1, \ldots, n\}\), \(1 \leq |\alpha| \leq n-1\), for all \((p, R)\) \(L^2(\mathbb{R}^\alpha)\)-valued rectangle atoms \(a\) associated to some rectangle \(R\) in \(\mathbb{R}^\alpha\), for all \(l_{\alpha} \in \mathbb{N}^\alpha\) we have

\[ \| T_{L_{\alpha}(R)+l_{\alpha}} a \|_{L^p(\mathbb{R}^n)} \leq c \prod_{s \in \alpha} 2^{-\epsilon l_s}. \]

Suppose furthermore that for all \(C\)-valued \((p, S)\) rectangle atoms \(b\) associated to some rectangle \(S\) in \(\mathbb{R}^\alpha\), for all \(l \in \mathbb{N}^\alpha\) and for all \(r = 1, \ldots, n\), we have

\[ \left\| \sum_{l_r > 0} T_{L(S)+l_r} b \right\|_p \leq c \prod_{s \neq r} 2^{-\epsilon l_s}. \]

Then \(T\) is bounded from \(H^p\) to \(L^p\).

Some remarks concerning Theorem 1 are in order.

Remark 1. Results for Calderón-Zygmund operators on \(H^p\)- and BMO-spaces in the multiparameter \((n \geq 3)\) case have been obtained by Journé [15] and
Pipher [22]. They considered classes of Calderón-Zygmund operators which are defined by an iteration based on an induction over the dimension. Those iterated conditions are not appropriate in the Hilbert-space valued setting, and do not contain, for example, the standard Hörmander-Marcinkiewicz multiplier theorem (Corollary 5.2). In contrast, Theorem 1 immediately extends to the vector-valued setting, with rectangle atoms in $H^p(\mathbb{R}^n, H_0)$ and $H^p(\mathbb{R}^m, L^2(H_0))$ and operator-valued kernels with values in $\mathcal{L}(H_0, H_1)$.

We should like to remark that, in the special case of BMO-estimates, a result like Theorem 1 could already have been obtained by the technique that Journé used to prove $L^p$-boundedness ($p \geq 2$) for the Littlewood-Paley-function associated to arbitrary intervals.

**Remark 2.** Theorem 1, as stated, is weaker in the case $n = 2$ than Fefferman’s result [7] because we require the notion of vector-valued atoms. However there is a sharper version (see Theorem 3.1 below) which is very close to Fefferman’s theorem. In fact, it requires the notion of $L^2(\mathbb{R}^a)$-valued atoms in $\mathbb{R}^a$ only for those $\alpha$ with $|\alpha| \leq n - 2$. This illustrates the difference between the geometrically simple case $n = 2$ and the higher-parameter case.

Complementary to the $H^p$ theory is an $L^p$ theory, $1 < p < 2$, which eventually yields results such as Theorem B above. With this in mind, we introduce a multiparameter Littlewood-Paley decomposition. Let $\phi$ be a smooth function such that $\phi^2 = \Phi$. For $\alpha \subseteq \{1, \ldots, n\}$ and $j \in \mathbb{Z}^a$ we define

$$(Q_j^\alpha f)^\sim(\xi) = \prod_{s \in \alpha} \phi(2^j|\xi_s|) \hat{f}(\xi)$$

and

$$(P_j^\alpha f)^\sim(\xi) = \prod_{s \in \alpha} \left[1 - \sum_{k_i \leq j_s} \phi^2(2^{k_i}|\xi_s|)\right] \hat{f}(\xi).$$

Then we have

**Theorem 2.** Suppose that $T$ is an $L^2$ bounded singular integral operator. Suppose that for all $\beta, \gamma \subseteq \{1, \ldots, n\}$ with $\beta \cap \gamma = \phi$, $\beta \cup \gamma \neq \phi$, for all $l \in \mathbb{N}^{\beta \cup \gamma}$, all $m \in \mathbb{N}^\beta$, and that for some $\epsilon > 0$ and some large $N$ we have

$$\sum_{k \in \mathbb{Z}^{\beta \cup \gamma}} \sup_{j \in \mathbb{Z}^{\beta \cup \gamma}} \|Q_{\beta \cup \gamma}^\beta T_{\beta \cup \gamma}^{\beta \cup \gamma} Q_{\pi \beta \beta}^{\beta} P_{\pi \gamma \gamma}^{\gamma} \|_{r-r} \leq c \prod_{s \in \beta} 2^{-\epsilon l_s} \prod_{s \in \beta} 2^{N m_s},$$

for $r = p$ and for $r = 2$ (with the obvious interpretation if $\beta$ or $\gamma = \varnothing$). Then $T$ is bounded on $L^p$.

Section 2 of this paper contains the preliminaries from the Chang-Fefferman Hardy space theory which we need and also the relevant geometric lemma (a variant of Journé’s). Sharper versions of Theorems 1 and 2 are proved in §§3 and 4 respectively. Section 5 contains the applications to Fourier multipliers and pseudodifferential operators. Finally, in §6, an example is presented to show that one cannot replace “strip-hypotheses” in multiparameter multiplier theorems by purely local hypotheses. That is, we cannot replace the first two hypotheses of Theorem B by $\sup_{i,j \in \mathbb{Z}} \|\hat{m}_{ij}\|_{p-p} < \infty$ and still have a true theorem. In fact,
if we are to insist on local hypotheses, the standard Hörmander-Marcinkiewicz theorem cannot be essentially improved upon.

2. Preliminaries from Hardy-space theory and geometry

In this section we recall some definitions and facts about the Chang-Fefferman atomic decomposition of $H^p$ functions and prove a variant of Journé's covering lemma.

We first need some notation. Given a dyadic rectangle $R$ in $\mathbb{R}^n$ and $\alpha \subseteq \{1, \ldots, n\}$, we associate a rectangle $R^\alpha_+ \subseteq \mathbb{R}^n \times \mathbb{R}^n_+$ by defining $R^\alpha_+ = \{(x, t) | x \in R, \; t \sim 2L_{\alpha}(R)\}$, where $t \sim 2L_{\alpha}(R)$ means $2L_{\alpha}(R) \leq t_s \leq 2L_{\alpha}(R)+1$ for all $s \in \alpha$. We also set $\chi_{R^\alpha_+}(x, t) = \chi_{R^\alpha_+}(x, t)$ and drop the $\alpha$ when $\alpha = \{1, \ldots, n\}$.

The following kind of estimate occurs in [3] and is proved by a duality argument and Plancherel's theorem. Let $\Psi$ be a $C^\infty$ even function of compact support in $[-\frac{1}{2}, \frac{1}{2}]$ such that $\int |\Psi(t)|^2 \frac{dt}{t} = 1$, and let $\tilde{\Psi}_\alpha(t) = \prod_{s \in \alpha} \Psi(t_s t_s)$. Let $M$ be a collection of distinct dyadic rectangles in $\mathbb{R}^n$.

\[ \sum_{R \in M} \int_{R^\alpha_+} \Psi_t (e_{R, t}) \frac{dt}{t_1 \cdots t_n} \leq c \sum_{R \in M} \|e_{R, t}\|_{L^2}^2 \frac{dt}{t_1 \cdots t_n}. \]

Moreover, if $\alpha \subseteq \{1, \ldots, n\}$ and $\mathcal{R}$ is a collection of distinct dyadic rectangles in $\mathbb{R}^n$ with common $L_{\alpha'}$, and for each $R \in \mathcal{R}, \; t \in R^\alpha_+$, we have a function $e_{R, t} : \mathbb{R}^n \to \mathbb{C}$, then

\[ \sum_{R \in \mathcal{R}} \int_{R^\alpha_+} \Psi_t^\alpha (e_{R, t} \chi_{R^\alpha_+}) \frac{dt}{t_1 \cdots t_n} \leq c \prod_{R \in \mathcal{R}} \|e_{R, t}\|_{L^2}^2 \frac{dt}{t_1 \cdots t_n} \]

(where $*_{\alpha}$ denotes convolution only in the $\alpha$-variables). \hfill \square

For $\mu \in \mathbb{Z}$, let $\Omega_\mu = \{x| Sf(x) > 2^{\mu}\}$ and let $\mathcal{R}_\mu$ be the collection of all dyadic rectangles $R$ in $\mathbb{R}^n$ satisfying $|R \cap \Omega_\mu| > \frac{1}{2} |R|$, but $|R \cap \Omega_{\mu+1}| \leq \frac{1}{2} |R|$. Then we have

Lemma 2.2 [3].

\[ \sum_{R \in \mathcal{R}_\mu} \int \|((\Psi_t * f) \chi_{R^\alpha_+})\|^2 \frac{dt_1 \cdots dt_n}{t_1 \cdots t_n} \leq c 2^{2\mu} |\Omega_\mu|. \hfill \square \]

These lemmas were used to derive an atomic decomposition for $H^1$ functions. We give a definition of an atom associated to an open set similar to the one which may be found in [3].

Definition. Let $\Omega \subseteq \mathbb{R}^n$ be an open set of finite measure. A function $b_\Omega$ is called a $(p, \Omega)$ atom (associated to $\Omega$) if $b_\Omega$ can be decomposed as $b_\Omega = \sum_R \Psi_t * e_{R, t} \frac{dt}{\prod_{s = 1}^n t_s}$ where $(x, t) \mapsto e_{R, t}(x, t)$ is supported in $R^+_{\alpha}$ and $R$ runs over a collection of dyadic rectangles supported in $\Omega$, and where

\[ \left( \sum_R \int \|e_{R, t}\|_{L^2}^2 \frac{dt}{t_1 \cdots t_n} \right)^{1/2} \leq |\Omega|^{1/2-1/p}. \]
By Lemma 2.1, a \((p, \Omega)\) atom \(b_\Omega\) satisfies \(\|b_\Omega\|_2 \leq c|\Omega|^{1/2-1/p}\); hence any \((p-R)\) atom for \(R\) a rectangle is also a \((p, R)\) rectangle atom.

Now any reasonable function has an atomic decomposition via the Calderón reproducing formula. That is, \(f = \sum_{\mu \in \mathbb{Z}} a_{\Omega_\mu} \) where

\[
a_{\Omega_\mu} = \sum_{R \in \mathcal{R}_{\Omega_\mu}} \int_{l_1 \cdots l_n} \psi_\mu \ast \left[ (\psi_\mu \ast f) \chi_{R,t} \right] \frac{dt}{l_1 \cdots l_n}
\]

(for suitably chosen \(\psi\)) and by Lemma 2.2 the functions \([c|\Omega_\mu|^{1/p} 2^{-\mu}]^{-1} a_{\Omega_\mu}\) are \((p, \tilde{\Omega}_\mu)\) atoms associated to the open set \(\tilde{\Omega}_\mu = \{ x | M \chi_{\Omega_\mu}(x) > \frac{1}{2} \} \), \(M\) denoting the strong maximal operator. (Note that by the strong maximal theorem, \(|\tilde{\Omega}_\mu| \leq c|\Omega_\mu|\).

If an operator \(T\) satisfies \(\|Tb_\Omega\|_p \leq c\) for all atoms \(b_\Omega\), then, for \(0 < p \leq 1\), it follows that \(\|Tf\|_p \leq c \sum \|T a_{\Omega_\mu}\|_p \leq c \sum |\Omega_\mu|^{2/p} \leq c \|Sf\|_p = c\|f\|_p\), and so \(T\) is bounded from \(H^p\) into \(L^p\).

For future use, we set, in accordance with the above discussion,

\[
e_{R,t} = (\psi_\mu \ast f) \chi_{R,t}, \quad e_R = \int \psi_\mu \ast e_{R,t} \frac{dt}{l_1 \cdots l_n}
\]

so that the atom \(b_\Omega\) can be decomposed as the sum of elementary particles \(\sum e_R\).

We now turn to geometry and a variant of Journé's lemma.

Let \(\Omega \subseteq \mathbb{R}^n\) be a bounded open set, \(\mathcal{R}\) a collection of dyadic rectangles supported in \(\Omega\). We define \(\tilde{\Omega} = \{ x | M \chi_{\Omega}(x) > 10^{-n} \}\) and inductively, \(\tilde{\Omega}^{(j)} = (\tilde{\Omega}^{(j-1)})^{-1}\), where \(\tilde{\Omega}^{(0)} = \Omega\). For \(R = I_1 \times I_2 \times \cdots \times I_n \in \mathcal{R}\), let \(\tilde{I}_1 = \tilde{I}_1(R, \Omega)\) be the largest dyadic interval containing \(I_1\) such that \(|(\tilde{I}_1 \times I_2 \times \cdots \times I_n) \cap \Omega| > \frac{1}{2} |\tilde{I}_1 \times I_2 \times \cdots \times I_n|\).

Notice that \(\tilde{I}_1 \times I_2 \times \cdots \times I_n \subseteq \Omega\). Inductively let \(\tilde{I}_s = \tilde{I}_s(R, \Omega)\) be the largest dyadic interval containing \(I_s\) such that \(|(\tilde{I}_1 \times \cdots \times \tilde{I}_s \times I_{s+1} \times \cdots \times I_n) \cap \tilde{\Omega}^{(s-1)}| > \frac{1}{2} |\tilde{I}_1 \times \cdots \times \tilde{I}_s \times I_{s+1} \times \cdots \times I_n|\), and notice that \(\tilde{I}_1 \times \cdots \times \tilde{I}_s \times I_{s+1} \times \cdots \times I_n \subseteq \tilde{\Omega}^{(s)}\) (which allows \(\tilde{I}_{s+1}\) to be defined). Further, let \(\kappa_s(R, \Omega) = L_s(\tilde{I}_s) - L_s(I_s)\).

Given the subset \(\{1, \ldots, k\}\) of \(\{1, \ldots, n\}\), we write \(\mathbb{R}^{1, \ldots, k}\) as \(\mathbb{R}^{1-k}\), \(\mathbb{R}^{1, \ldots, k'}\) as \(\mathbb{R}^{k+1-n}\), and similarly \(\mathbb{Z}^{1-k}\), \(\mathbb{N}^{1-k}\). If now \(m \in \mathbb{N}^{1-k}\) and \(Q\) is dyadic in \(\mathbb{R}^{1-k}\), we let \(\mathcal{D}_{Q,m}^{1-k}\) be the union of all dyadic rectangles \(S\) in \(\mathbb{R}^{k+1-n}\) such that \(Q \times S \in \mathcal{R}\) and such that \(\kappa_i(Q \times S, \Omega) = m_i\), \(i = 1, \ldots, k\). Further, for \(l \in \mathbb{N}^{1-k}\), let \(\mathcal{D}_{Q,l}^{1-k} = \bigcup_{m<l} \mathcal{D}_{Q,m}^{1-k}\) (where \(m < l\) means \(m_i < l_i\), \(i = 1, \ldots, k\)). (Both \(\mathcal{D}_{Q,m}^{1-k}\) and \(\mathcal{D}_{Q,l}^{1-k}\) depend of course on \(\Omega\) and \(\mathcal{R}\).)

**Lemma 2.3.** For \(k = 1, 2, \ldots, n - 1\) and \(l \in \mathbb{N}^{1-k}\),

\[
\sum |Q| \mathcal{D}_{Q,l}^{1-k} \leq c \prod_{s=1}^{k} l_{s}^{k-s+1} |\Omega|,
\]

where the sum is extended over the dyadic \(Q\) in \(\mathbb{R}^{1-k}\).
Proof. First let $k = 1$, $l \in \mathbb{N}$. For a dyadic interval $I$ of $\mathbb{R}^1$, let $E_I(\Omega)$ be the union of all dyadic $S$ in $\mathbb{R}^{[2^{\cdots n}]}$ such that $I \times S \subseteq \Omega$. Let $I(l)$ be the unique dyadic interval containing $I$ such that $L(I(l)) = L(I) + l$. We first claim that

$$B_{I,l}^{[1]} \subseteq [E_I(\Omega) \setminus E_{I(l)}(\Omega)]^\sim. \quad (2.1)$$

To see this, let $y \in B_{I,l}^{[1]}$ and let $S$ be a dyadic rectangle in $\mathbb{R}^{[2^{\cdots n}]}$, $y \in S$, with $I \times S \in \mathcal{B}$ and $\kappa(I \times S) < l$. Then, since $I(l) \times E_{I(l)}(\Omega) \subseteq \Omega$,

$$|(I(l) \times S) \cap (I(l) \times E_{I(l)}(\Omega))| \leq |(I(l) \times S) \cap \Omega| \leq \frac{1}{2} |I(l) \times S|,$$

(the second inequality holding since $I(l)$ is too big to be $\hat{I}(I \times S, \Omega)$). Hence, $|S \cap E_{I(l)}(\Omega)| \leq \frac{1}{2} |S|$ and consequently (since $S \cup E_{I(l)} \subseteq E_I$), $|(E_I \setminus E_{I(l)}) \cap S| \geq \frac{1}{2} |S|$, which shows that $M(E_I \setminus E_{I(l)})(y) \geq \frac{1}{2} > \frac{1}{10^3}$, establishing (2.1). ($M$ here and $\sim$ of (2.1) refer to the strong maximal function in $\mathbb{R}^{[2^{\cdots n}]}$.) We observe that the sets $I \times (E_I \setminus E_{I(l)})$ are disjoint if $I$ runs over all dyadic intervals with sidelength $L(I) = ml + i$, for $i$ fixed, for variable $m \in \mathbb{Z}$ (since if $I_1 \subsetneq I_2$ and $I_1$ and $I_2$ have lengths of the form $2^m l^1$, $m \in \mathbb{Z}$, then $I_1(l) \subseteq I_2$, and so $(E_I \setminus E_{I(l)}) \times (E_I \setminus E_{I(l)}) \subseteq E_I \setminus E_{I(l)} = \emptyset$). By the strong maximal theorem we have

$$\sum \left| I \right| |B_{I,I,l}^{[1]}| \leq c \sum_{l} \left| I \right| |E_I \setminus E_{I(l)}|$$

$$\leq c \sum_{i=0}^{l-1} \sum_{m \in \mathbb{Z}} \sum_{L(I)=ml+i} \left| I \times (E_I \setminus E_{I(l)}) \right|$$

$$\leq cl|\Omega|,$$

thus finishing the proof in the case $k = 1$.

Now assume that $2 \leq k \leq n - 1$ and that the lemma has been proved for $1, 2, \ldots, k - 1$. Take $l = (l, l_k) = (l_1, \ldots, l_k) \in \mathbb{N}^k$. We write dyadic intervals in $\mathbb{R}^{[1^{\cdots k}]}$ as $Q \times I_k$ with $Q \subseteq \mathbb{R}^{[1^{\cdots k-1}]}$, $I_k \subseteq \mathbb{R}^{[k]}$.

We fix $m \in \mathbb{N}^{[1^{\cdots k}]}$ with $m_s < l_s$, $s = 1, \ldots, k - 1$, and consider a dyadic rectangle $R = \hat{T}_s(Q \times S) = I_1 \times \cdots \times I_n$ where $R \in \mathcal{B}$ and $\kappa_s(R, \Omega) = m_s$, $s = 1, \ldots, k - 1$, and we write $\hat{Q} = \hat{T}_s(Q \times S, \Omega) \times \cdots \times \hat{T}_{k-1}(Q \times S, \Omega)$. In other words, $\hat{Q} = I_s(m_s)$ and so $\hat{Q}$ is independent of the choice of $S$ used. By the definition of $\hat{I}_s$, we have

$$\hat{Q} \times B_{\hat{Q},m}^{[1^{\cdots k-1}]} \subseteq \hat{\Omega}^{(k-1)}.$$ 

For each $\bar{m}$, let $I_{k,m} = I_{k,m}(R)$ be the largest dyadic interval in $\mathbb{R}^{[k]}$ containing $I_k$ such that

$$|(I_{k,m} \times I_{k+1} \times \cdots \times I_n) \cap B_{\hat{Q},m}^{[1^{\cdots k-1}]}| > \frac{1}{2} |I_{k,m} \times I_{k+1} \times \cdots \times I_n|. $$

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We have
\[
||\tilde{Q} \times I_{k,m} \times I_{k+1} \times \cdots \times I_n|| \cap \tilde{Q}^{(k-1)}| \\
\geq ||\tilde{Q} \times (I_{k,m} \times \cdots \times I_n) \cap D_{Q,m}^{(1 \cdots k-1)}|| \\
> \frac{1}{2}||\tilde{Q}||I_{k,m} \times \cdots \times I_n| \\
= \frac{1}{2}||\tilde{Q} \times I_{k,m} \times \cdots \times I_n||,
\]
so that by the maximality condition in the definition of \( \tilde{I}_k \) we can conclude that \( I'_{k,m} \subseteq \tilde{I}_k \).

In analogy with (2.1) we now claim that for all \( Q = \tilde{Q} \times I_k \) dyadic in \( R^{[1 \cdots k]} \) and all \( m \in \mathbb{N}^{[1 \cdots k]} \), we have
\[
D_{Q,m}^{[1 \cdots k]} \subseteq [E_{I_k}^{(k)}(D_{Q,m}^{[1 \cdots k-1]}) \backslash E_{I_k(m_k+1)}^{(k)}(D_{Q,m}^{[1 \cdots k-1]})],
\]
where, for \( J \) an interval in \( \mathbb{R}^k \) and \( \Sigma \subseteq \mathbb{R}^{[k \cdots n]} \), \( E_J^{[k]}(\Sigma) \) is the union of all dyadic \( S \) in \( \mathbb{R}^{[k+1 \cdots n]} \) such that \( J \times S \subseteq \Sigma \).

To establish (2.2), we should like to show that if \( T = I_{k+1} \times \cdots \times I_n \subseteq D_{Q,m}^{[1 \cdots k]} \), then \( T \subseteq E_{I_k}^{[k]}(D_{Q,m}^{[1 \cdots k-1]}) \) and that
\[
|T \cap E_{I_k(m_k+1)}^{[k]}(D_{Q,m}^{[1 \cdots k-1]})| \leq \frac{1}{2}|T|.
\]
The first of these two assertions is clear from the definitions, and the above arguments show that \( I_k \subseteq I'_{k,m}(Q \times T) \subseteq \tilde{I}_k = I_k(m_k) \subseteq I_k(m_k+1) \), and hence that
\[
|(I_k(m_k+1) \times I_{k+1} \times \cdots \times I_n) \cap D_{Q,m}^{[1 \cdots k-1]}| \leq \frac{1}{2}|I_k(m_k+1) \times I_{k+1} \times \cdots \times I_n|,
\]
or \( |(I_k(m_k+1) \times T) \cap D_{Q,m}^{[1 \cdots k-1]}| \leq \frac{1}{2}|(I_k(m_k+1) \times T)|. \) Now
\[
I_k(m_k+1) \times E_{I_k}^{[k]}(D_{Q,m}^{[1 \cdots k-1]}) \subseteq D_{Q,m}^{[1 \cdots k-1]},
\]
and so
\[
|(I_k(m_k+1) \times T) \cap (I_k(m_k+1) \times E_{I_k}^{[k]}(D_{Q,m}^{[1 \cdots k-1]}))|
\leq \frac{1}{2}|I_k(m_k+1) \times T|.
\]
Dividing both sides of this inequality by \( |I_k(m_k+1)| \) establishes (2.2).

An immediate consequence of (2.2) is
\[
D_{Q,m}^{[1 \cdots k]} \subseteq [E_{I_k}^{[k]}(D_{Q,m}^{[1 \cdots k-1]}) \backslash E_{I_k}^{[k]}(D_{Q,m}^{[1 \cdots k-1]})],
\]
whenever \( m_k < l_k \). Arguing as in the first step,
and, since \( B_{Q, m}^{[1\cdots k-1]} \subseteq B_{Q, \tilde{m}}^{[1\cdots k-1]} \) \((\tilde{m} = m + 1)\) we may apply the inductive hypothesis to obtain

\[
\sum_Q |Q||B_{Q,l}^{[1\cdots k]}| \leq cl_k \sum_{m<l} \sum_{Q} |Q||B_{Q,m}^{[1\cdots k-1]}|
\]

\[
\leq c l_k \sum_{m<l} \prod_{r=1}^{k-1} (m_r + 1)^{k-1-r+1}|\Omega|
\]

\[
\leq cl_k \sum_{m<l} \prod_{r=1}^{k-1} l_r^{k-r}|\Omega|
\]

\[
= cl_k \prod_{r=1}^{k-1} l_r^{k-r} \#\{\tilde{m} < \tilde{l}\}|\Omega|
\]

\[
= cl_k \prod_{r=1}^{k-1} l_r^{k-r} l_1 \cdots l_{k-1}|\Omega|
\]

\[
= c \prod_{r=1}^{k} l_r^{k-r+1}|\Omega|
\]

which is what we had to prove. \(\Box\)

Now let \( R \) be some collection of dyadic rectangles supported in \( \Omega \) such that any two rectangles with the same projections on \( \mathbb{R}^{[1\cdots n-1]} \) are disjoint. Define \( I_s(R, \Omega) \) for \( R \in R \), \( 1 \leq s \leq n-1 \) as before. Let \( \mathcal{M}(l, R) \) be the collection of all those rectangles in \( R \) satisfying \( \kappa_s(R, \Omega) < l_s \), \( s = 1, \ldots, n-1 \). Then we have the following variant of Journé's lemma [16]:

Lemma 2.4.

\[
\sum_{R \in \mathcal{M}(l, R)} |R| \leq c \prod_{i=1}^{n-1} l_i^{n-i}|\Omega|.
\]

Proof. Two different dyadic rectangles in \( \mathcal{M}(l, R) \) with the same projection \( Q \) in \( \mathbb{R}^{[1\cdots n-1]} \) are disjoint and will be contained in \( Q \times B_{Q, l}^{[1\cdots n-1]} \). The assertion now follows from the case \( k = n-1 \) of Lemma 2.3. \(\Box\)

In the above two lemmas, we have made geometrical constructions (dilations of rectangles in some directions) depending on an ordering of these directions. Of course, we could have chosen any permutation of the standard ordering used here, and still have obtained an analogous result; we shall need this observation in the next section.

3. \( H^p \)-estimates

We first formulate a version of Theorem 1, with, among other things, improved decay assumptions. For \( \alpha \subseteq \{1, \ldots, n\} \) with \( \alpha = \{\alpha_1 < \cdots < \alpha_\nu\} \) we set \( d(l, \alpha) = l_{\alpha_1}^{\nu_1} l_{\alpha_2}^{\nu_2} \cdots l_{\alpha_\nu} \). Then we have

Theorem 3.1. Let \( T \) be a singular integral operator which is bounded on \( L^2 \) with norm at most \( A \). Suppose that for all \( \alpha, \ 1 \leq |\alpha| \leq n-2, \) all \( l \in \mathbb{N}^\alpha, \) all
\( L^2(\mathbb{R}^n) \)-valued \((p, R)\) rectangle atoms \( a \) we have

\[
\|T_{L_n(R)}^a + l b\|_{L^p(L^2)} \leq A^p.
\]

Suppose furthermore that for all \( \alpha, \ |\alpha| = n - 1 \), and all \( C\)-valued \((p, S)\) rectangle atoms \( b \) we have

\[
\|T_{L_n(S)}^a + l b\|_p \leq B^a_i
\]

and that for all \( 1 \leq r \leq n \), for all \( \tilde{I} \in \mathbb{N}^{(r)'} \),

\[
\left\| \sum_{l \geq 0} T_{L(S)} + l b \right\|_p \leq C^{(r)'}\tilde{I}.
\]

If

\[
\sup_{|\alpha| = n - 1} \sum_{l \in \mathbb{N}^o} [A^o_i d(l, \alpha)^{1/p - 1/2}] \leq A^p,
\]

\[
\sup_{|\alpha| = n - 1} \sum_{l \in \mathbb{N}^o} [B^o_i d(l, \alpha)^{1/p - 1/2}] \leq A^p,
\]

and

\[
\sup_{1 \leq r \leq n} \sum_{l \in \mathbb{N}^{(r)'}} [C^{(r)'} d(l, \{r\}')^{1/p - 1/2}] \leq A^p
\]

then \( T \) is bounded from \( H^p \) to \( L^p \) with norm at most \( CA \).

We first show how Theorem 1 can be obtained from Theorem 3.1. First of all, \( d(l, \alpha) \leq c_\varepsilon \prod_{s \in \alpha} 2^{l_s} \) for every \( \varepsilon > 0 \). Hence we only have to verify that under the assumptions of Theorem 1 we have

\[
\|T_{L_n(S)}^a + l b\|_p \leq c \prod_{s \in \alpha} 2^{-l_s} \quad \text{for } |\alpha| = n - 1.
\]

Suppose without loss of generality that \( \alpha = \{2, \ldots, n\} \) and let \( b \) be a \( C\)-valued rectangle atom associated to \( S = I \times J \) with \( I \subseteq \mathbb{R}^{(1)} \) and \( J \subseteq \mathbb{R}^{(1)'} \). Let \( \tilde{I} \) be the interval with the same centre as \( I \) but with twenty times the sidelength. Then by Hölder's inequality,

\[
\|T_{L_n(S)}^a + l b\|_p^p \leq c \int_I \left[ \int_{\mathbb{R}^n} \left| T_{L_n(S)}^a + l b \right|^2 \right]^{p/2} \, dx \left| I \right|^{1-p/2}
\]

\[
= A + B.
\]

Now \( |I|^{-1/2 + 1/p} \|b\|_2^{-1} b \) may be considered as an \( L^2(\mathbb{R}^n) \)-valued rectangle atom associated to \( I \) and so \( A \leq c \prod_{s=2}^n 2^{-l_s} \) by the hypothesis of Theorem 1. Moreover,

\[
B \leq \left\| \sum_{l \geq 0} T_{L(S)} + l b \right\|_p \leq c \prod_{s=2}^n 2^{-l_s},
\]

again by the hypothesis of Theorem 1.

In order to prove Theorem 3.1 we shall use an induction argument which reduces estimates for \((p, I \times A)\) atoms (with \( A \) living in an \( m\)-dimensional
space with \( m \geq 2 \) to estimates for \( L^2 \)-valued \((p, I)\) rectangle atoms and for \( \mathbb{C}\)-valued rectangle atoms, and, if \( m \geq 3 \), to estimates for \((p, I \times J \times B)\) atoms for \( B \) living in a space of dimension \( m - \dim J \).

**Proposition 3.2.** (a) Let \( b_\Omega \) be a \((p, \Omega)\) atom in \( \mathbb{R}^n \). Suppose that \( T \) is bounded on \( L^2(\mathbb{R}^n) \) with norm at most \( A \). Suppose furthermore that for all \( \alpha \subseteq \{1, \ldots, n\}, 1 \leq |\alpha| \leq n - 2 \), for all \( l \in \mathbb{N}^\alpha \), for all dyadic rectangles \( Q \) in \( \mathbb{R}^\alpha \), for all open sets \( A \) in \( \mathbb{R}^{\alpha'} \) and all \((p, Q \times A)\) atoms, we have

\[
(3.1) \quad \|T_{\Omega}(Q)+l^a\|^p \leq \Gamma^\alpha_l, \quad l \in \mathbb{N}^\alpha.
\]

Moreover, suppose that for all \( \alpha, |\alpha| = n - 1 \), for all \( l \in \mathbb{N}^\alpha \), for all dyadic \( S \) in \( \mathbb{R}^n \) and all \( \mathbb{C}\)-valued \((p, S)\) rectangle atoms, we have

\[
(3.2) \quad \|T^\alpha_{\Omega}(S)+l^a\|^p \leq D^p_l
\]

and also, writing \( l = (\tilde{l}, l_n), \tilde{l} \in \mathbb{N}^{\{n\}'} \),

\[
(3.3) \quad \left\| \sum_{l_n \geq 0} T_{\Omega}(S)+l^a \right\|^p \leq E_l.
\]

Then

\[
\|T b_\Omega\|^p \leq c \left[ A^p + \sup_{1 \leq |\alpha| \leq n - 2} \sum_{l \in \mathbb{N}^\alpha} (\Gamma^\alpha_l)^p d(l, \alpha)^{-p/2}
\]

\[
+ \sup_{|\alpha| = n - 1} \sum_{l \in \mathbb{N}^\alpha} (D^p_l)^p d(l, \alpha)^{-p/2}
\]

\[
+ \sum_{l \in \mathbb{N}^{\{n\}'}} E^p_l d(\tilde{l}, \{n\}')^{1-p/2} \right].
\]

(b) Suppose \( b_{l \times A} \) is a \((p, I \times A)\) atom where \( I \subseteq \mathbb{R}^\beta, A \subseteq \mathbb{R}^\gamma, \beta \cap \gamma = \varnothing, \beta \cup \gamma = \{1, \ldots, n\}, \gamma = \{\gamma_1 < \cdots < \gamma_\nu\} \) with \( \nu \geq 2 \). Suppose that for all \( L^2(\mathbb{R}^\gamma)\)-valued \((p, I)\) rectangle atoms, we have

\[
(3.4) \quad \|T a\|_{L^p(\mathbb{R}^\beta, L^2(\mathbb{R}^\gamma))} \leq A,
\]

and that for all \( \alpha \subseteq \gamma, |\alpha| \leq |\gamma| - 2, \) for all \( l \in \mathbb{N}^\alpha \), all \( Q \) dyadic in \( \mathbb{R}^\alpha \), for all open sets \( B \) in \( \mathbb{R}^{\gamma - \alpha} \) and for all \((p, I \times Q \times B)\) atoms \( a \) we have

\[
(3.5) \quad \|T^\alpha_{\Omega}(Q)+l^a\|^p \leq \Gamma^\gamma_l.
\]

Suppose furthermore that for all \( \alpha \subseteq \gamma, |\alpha| = |\gamma| - 1, \) for all \( l \in \mathbb{N}^\alpha \), all dyadic \( S \) in \( \mathbb{R}^\gamma \) and all \((p, I \times S)\) rectangle atoms, we have

\[
(3.6) \quad \|T^\alpha_{\Omega}(S)+l^a\|^p \leq D^p_l,
\]

and if \( l = (\tilde{l}, l_n), \tilde{l} \in \mathbb{N}^{\gamma - \{\gamma_\nu\}} \),

\[
(3.7) \quad \left\| \sum_{l_n > 0} T^\gamma_{\Omega}(S)+l^a \right\|^p \leq E^p_l \{\gamma_\nu\}.
\]
Then
\[
\|Tb_{I \times A}\|_p \leq c \left[ A^p + \sup_{|\alpha| \leq |\gamma| - 2} \sum_{I \in \mathcal{N}_0} (\Gamma_0^p) d(I, \alpha)^{1-p/2} + \sup_{|\alpha|=|\gamma| - 1} \sum_{I \in \mathcal{N}_0} (D_0^p) d(I, \alpha)^{1-p/2} + \sum_{I \in \mathcal{N}_{|\gamma| - 1}} (E_0^p) d(I, \gamma - \{\gamma\})^{1-p/2} \right].
\]

Proof of Theorem 3.1. By the discussion of §2, it suffices to show \(\|Tb_{\Omega}\|_p \leq C\) for all \((p, \Omega)\) atoms. Proposition 3.2(a) reduces this to the \(L^2\)-boundedness of \(T\), appropriate estimates of the form (3.2), and (3.3)—all of which are hypotheses of Theorem 3.1—and estimates of the form (3.1). To check (3.1) we apply Proposition 3.2(b) to the operator \(T_{\Omega}^{\alpha} \) to the operator \(T_{\Omega}^{\alpha} \), reducing matters to (3.4), (3.6), and (3.7)—all of which follow directly from the hypotheses of Theorem 3.1—and (3.5). Hence Proposition 3.2 gives an iterative scheme for proving Theorem 3.1. We leave the verification of the constants to the reader. □

In order to prove Proposition 3.2, we carry out a further geometric construction which is essentially the same as in [22].

Suppose \(\gamma \subseteq \{1, \ldots, n\}, \gamma = \{\gamma_1 < \cdots < \gamma_m\}\). Let \(<\) be any total ordering on the subset of \(\gamma\) such that \(\varnothing < \cdots < \tilde{\gamma} < \gamma\), where if \(m \geq 2\), \(\tilde{\gamma}\) contains \(\gamma_m\). (This last requirement is merely a technical convenience.) For \(\alpha \subseteq \gamma\), let \(\tilde{\alpha}\) denote its predecessor with respect to the above ordering and define \(N(\alpha) = N(\tilde{\alpha}) + |\alpha|\) (so that \(N(\gamma) = \sum_{k=0}^{m} k^{|k|} = m2^{m-1}\)).

Let \(\Omega \subseteq \mathbb{R}^q\) be an open set and let \(\mathcal{R}\) be a family of dyadic rectangles supported in \(\Omega\). For each \(R \in \mathcal{R}\) and each \(\alpha \subseteq \gamma\), we will define several "enlargements" \(u_{\alpha}^\Omega(R, \Omega)\) \((r = 0, \ldots, |\alpha|)\) and \(w_{\alpha}^\Omega(R, \Omega)\), such that for every \(\alpha\) we will be able to apply Lemma 2.3 to the family of rectangles \(\{u_{\alpha}^\Omega(R, \Omega)\}_{R \in \mathcal{R}}\).

We set \(w_{\alpha}(R) = w_{\alpha}(\Omega) \subseteq \Omega^{(N(\alpha))}\). Given \(\alpha = \{\alpha_1 < \cdots < \alpha_m\}\), we set \(u_{\alpha}^\Omega(R, \Omega) = w_{\alpha}(R)\) and if \(u_{\alpha}^\Omega(R, \Omega) = J_{\gamma_1} \times \cdots \times J_{\gamma_m}\), we proceed to define \(\tilde{I}_{\alpha_1}, \tilde{I}_{\alpha_2}, \ldots, \tilde{I}_{\alpha_m}\) as in §2. That is, \(\tilde{I}_{\alpha_1}\) is the largest dyadic interval containing \(I_{\alpha_1}\) such that
\[
|J_{\gamma_1} \times \cdots \times \tilde{I}_{\alpha_1} \times \cdots \times I_{\gamma_m} \cap \tilde{\Omega}^{(N(\alpha))}| > \frac{1}{2} |J_{\gamma_1} \times \cdots \times I_{\gamma_m}|
\]
and we define \(u_{\alpha}^\Omega(R, \Omega) = J_{\gamma_1} \times \cdots \times \tilde{I}_{\alpha_1} \times \cdots \times I_{\gamma_m} := J_{\gamma_1} \times \cdots \times J_{\gamma_m}\), which is clearly supported in \(\tilde{\Omega}^{(N(\alpha))}\). If \(\nu > 1\) we let \(\tilde{J}_{\alpha_2}\) be the largest dyadic interval containing \(J_{\alpha_2}\) such that
\[
|J_{\gamma_1} \times \cdots \times \tilde{J}_{\alpha_2} \times \cdots \times J_{\gamma_m} \cap \tilde{\Omega}^{(N(\alpha))}| > \frac{1}{2} |J_{\gamma_1} \times \cdots \times J_{\gamma_m}|
\]
and set \(u_{\alpha}^\Omega(R, \Omega) = J_{\gamma_1} \times \cdots \times J_{\gamma_m} \subseteq \tilde{\Omega}^{(N(\alpha))}\). Similarly, we proceed to define \(u_{\alpha}^\Omega(R, \Omega)\) supported in \(\tilde{\Omega}^{(N(\alpha)+r)}\) for \(r = 3, \ldots, \nu\). We set
\[
w_{\alpha}(R, \Omega) = \begin{cases} u_{\alpha}^\Omega(R, \Omega) & \text{if } \alpha \neq \gamma, \\ u_{\alpha_{|\alpha|-1}}(R, \Omega) & \text{if } \alpha = \gamma. \end{cases}
\]
We also set \(\kappa_{\alpha}(R, \Omega) = L_{\alpha} w_{\alpha}(R, \Omega) - L_{\alpha} u_{\alpha}^\Omega(R, \Omega), \) and \(\nu_{\alpha}(R, \Omega) = L_{\alpha} w_{\alpha}(R) - L_{\alpha} u_{\alpha}^\Omega(R, \Omega)\) for \(s \in \gamma\), so that we clearly have
\[
(3.8) \quad \kappa_{\alpha}(R, \Omega) \leq \nu_{\alpha}(R, \Omega).
\]
Notice also that $\kappa^2_l(R, \Omega) = \nu^2_l(R, \Omega)$ and that $\kappa^2_{\gamma_m}(R, \Omega) = 0$.

**Lemma 3.3.** (a) For $Q$ dyadic in $\mathbb{R}^\alpha$, $l \in \mathbb{N}^\alpha$ ($\alpha \subseteq \gamma$), $\Omega$ open in $\mathbb{R}^\gamma$, let $\mathcal{P}^\alpha_{Q, l}(\Omega)$ be the union of all projections $\Pi_{y_\alpha(A_0)(R, \Omega)}$ where $\nu^\alpha_s(R, \Omega) < l_s$, $s \in \alpha$, and $\Pi_{A_0}(R, \Omega) = Q$. Then

$$\sum_{Q} |Q||A^\alpha_{Q, l}(\Omega)| \leq cd(l, \alpha)|\Omega|.$$ 

(b) For $\tilde{l} \in \mathbb{N}^{\gamma \setminus \{\gamma_m\}}$, let $\mathcal{P}^\gamma_l(\Omega)$ be the family of all dyadic $S$ in $\mathbb{R}^\gamma$ such that $S = u^l_0(R, \Omega)$ for some $R \in \mathcal{R}$ and such that $\nu^\gamma_s(R, \Omega) = \kappa^2_s(R, \Omega) < l_s$ for $s \neq \gamma_m$. Then

$$\sum_{S \in \mathcal{P}^\gamma_l(\Omega)} |S| \leq cd(\tilde{l}, \gamma \setminus \{\gamma_m\})|\Omega|.$$ 

**Proof.** Part (a) follows immediately from Lemma 2.3 applied to the rectangles $\{u^l_0(R)\}_{R \in \mathcal{R}}$ which are contained in the set $\mathcal{Q}^{(N(\alpha))}$. To see part (b), observe that since $\gamma$ contains $\gamma_m$, the maximality condition in the definition of $u^\gamma_{m-1}(R, \Omega)$ ensures that two different $S$ in $\mathcal{P}^\gamma_l(\Omega)$ with the same projections in $\mathbb{R}^\gamma \setminus \{\gamma_m\}$ are disjoint. Now apply Lemma 2.4. □

**Proof of Proposition 3.2.** We prove only part (a); the proof of part (b) is exactly similar and will be omitted. Let $b_\Omega$ be a $(p, \Omega)$ atom; we consider $Tb_\Omega$ separately in $\mathcal{F}^{(2^{-1})}$ and in $\mathcal{R} \setminus \mathcal{Q}^{(2^{-1})}$. As in [6], the estimate in $\mathcal{F}^{(2^{-1})}$ is an easy consequence of the $L^2$-boundedness of $T$ and the strong maximal theorem:

$$\|Tb_\Omega\|_{L^p(\mathcal{Q}^{(2^{-1})})} \leq \|\hat{\Omega}^{(2^{-1})}\|^{1-p/2}\|Tb_\Omega\|_2 \leq CA^p|\Omega|^{1-p/2}\|b_\Omega\|_2 \leq CA^p.$$ 

To estimate $Tb_\Omega$ in $\mathbb{R}^n \setminus \mathcal{Q}^{(2^{-1})}$, we use the formula

$$\sum_{m \in \mathbb{Z}^n} a_m = \sum_{1 \leq |r| \leq n} (-1)^{|r|-1} \sum_{m \in \mathbb{Z}^n, \, m_i > 0} a_m + \sum_{1 \leq i \leq n} a_m (m_{s_i} = 0)$$

(which may be proved by applying the formula

$$\prod_j (1 - P_j) = 1 - \sum_j P_j + \sum_{j \neq k} P_j P_k - \sum_{j \neq k \neq l} P_j P_k P_l \cdots$$

with $P_j = \chi_{\{m_i > 0\}}$) to write, as in [15], for $x \notin \mathcal{Q}^{(2^{-1})}$,

$$Tb_\Omega(x) = \sum_{1 \leq |r| \leq n} (-1)^{|r|-1} \sum_{R \, m_i > l_s, \omega^\gamma(R, \Omega)} T^\gamma_{m}e_R(x)$$

$$= \sum_{1 \leq |r| \leq n} (-1)^{|r|-1} I_\alpha$$

with $\gamma = \{1, \ldots, n\}$ and

$$b_\Omega = \sum_{R} e_R = \sum_{R} \int \psi_l * e_R \cdot \frac{dl}{t_1 \cdots t_n},$$
since the missing term
\[ \sum_{1 \leq |a| \leq n} \sum_{R} \sum_{m \leq L_{\alpha} u_{\alpha}^\gamma(R, \Omega)} T_{m}^\alpha e_{R} \]
is supported in \( \Omega(n^{2n-1}) \). For \( 1 \leq |\alpha| \leq n-2 \) we substitute \( l_{s} = m_{s} - L_{s} u_{\alpha}^\gamma(R, \Omega) \)
and write
\[ I_{\alpha} = \sum_{R} \sum_{l \in \mathbb{N}^{a}} \sum_{l_{s} > L_{\alpha} u_{\alpha}^\gamma(R)} T_{l + L_{s} u_{\alpha}^\gamma(R)}^\alpha e_{R} \]
\[ = \sum_{l \in \mathbb{N}^{a}} \sum_{R : \nu_{\alpha}^\gamma(R, \Omega) < l_{s}} T_{l + L_{s} u_{\alpha}^\gamma(R)}^\alpha e_{R}. \]

Let \( \mathcal{H}_{Q, l}^\alpha \) be the family of all rectangles \( R \) such that \( \nu_{\alpha}^\gamma(R, \Omega) < l_{s}, s \in \alpha \),
and \( \Pi u_{\alpha}^\gamma(R) = Q \). Then, if we set \( a_{Q, l}^\alpha = \sum_{R \in \mathcal{H}_{Q, l}^\alpha} e_{R} \), we have
\[ I_{\alpha} = \sum_{l \in \mathbb{N}^{a}} \sum_{Q \text{dyadic in } \mathbb{R}^{a}} T_{l + L_{l}^a Q}^\alpha a_{Q, l}^\alpha. \]

Now
\[ |Q \times \mathcal{A}_{Q, l}^{\alpha, \gamma}|^{-1/p+1/2} \left( \sum_{R \in \mathcal{H}_{Q, l}^\alpha} \int \|e_{R, l}\|^2_{1/2} \frac{dt}{t_{1} \cdots t_{n}} \right)^{-1/2} a_{Q, l}^\alpha \]
is a constant multiple (for a fixed absolute constant) of a \((p, Q \times \mathcal{A}_{Q, l}^{\alpha, \gamma})\) atom. Consequently, we have
\[ \|T_{l + L_{l}^a Q}^\alpha a_{Q, l}^\alpha\|_{p} \leq C(\Gamma_{l})^{p} |Q \times \mathcal{A}_{Q, l}^{\alpha, \gamma}|^{1-p/2} \left( \sum_{R \in \mathcal{H}_{Q, l}^\alpha} \int \|e_{R, l}\|^2_{1/2} \frac{dt}{t_{1} \cdots t_{n}} \right)^{p/2}, \]
and by applying Hölder's inequality and Lemma 3.3(a), we get
\[ \sum_{Q} \|T_{l + L_{l}^a Q}^\alpha a_{Q, l}^\alpha\|_{p}^{p} \leq C(\Gamma_{l})^{p} \left( \sum_{Q} |Q| \|\mathcal{A}_{Q, l}^{\alpha, \gamma}\| \right)^{1-p/2} \left( \sum_{Q} \sum_{R \in \mathcal{H}_{Q, l}^\alpha} \int \|e_{R, l}\|^2_{1/2} \frac{dt}{t_{1} \cdots t_{n}} \right)^{p/2} \]
\[ \leq C(\Gamma_{l})^{p} d(l, \alpha)^{1-p/2}, \]
since \( b_{Q} \) is a \((p, \Omega)\) atom. The bound for \( I_{\alpha} \) follows by summation on \( l \in \mathbb{N}^{a} \).

For \( |\alpha| = n - 1 \) we use a slightly different argument. Let \( \mathcal{M}(\mathcal{A}_{Q, l}^{\alpha, \gamma}) \) be the family of all one-dimensional dyadic intervals which are maximal in \( \mathcal{A}_{Q, l}^{\alpha, \gamma} \). Each \( R \in \mathcal{H}_{Q, l}^{\alpha} \) is contained in a unique \( Q \times I \) with \( I \in \mathcal{M}(\mathcal{A}_{Q, l}^{\alpha, \gamma}) \). For \( J \in \mathcal{M}(\mathcal{A}_{Q, l}^{\alpha, \gamma}) \) let
\[ a_{Q, J, l}^\alpha = \sum_{R \in \mathcal{H}_{Q, l}^{\alpha}, \eta, R \subseteq J} e_{R} ; \]
clearly \( a_{Q, J, l}^\alpha |Q \times J|^{-1/p+1/2} a_{Q, J, l}^\alpha \|_{2}^{-1} \) is a \((p, Q \times J)\) atom. Hence,
\[ \|T_{l + L_{l}^a Q}^\alpha a_{Q, J, l}^\alpha\|_{p} \leq (D_{l})^{p} |Q \times J|^{1-p/2} a_{Q, J, l}^\alpha \|_{2}^{p}. \]
Therefore,

\[ \|I_\alpha\|_p^p \leq \sum_{l \in \mathbb{N}_0} (D_l^p)^p \sum_{Q} |Q|^{1-p/2} \left( \sum_{J} |J|^{1-p/2} \|a^\alpha_{Q, J, l}\|_2^p \right) \]

\[ \leq \sum_{l \in \mathbb{N}_0} (D_l^p)^p \sum_{Q} |Q|^{1-p/2} |\mathcal{A}_Q^\alpha, \gamma|^{1-p/2} \left( \sum_{J \in \mathcal{M}_{\mathcal{A}_Q^\alpha, \gamma}} \|a^\alpha_{Q, J, l}\|_2^p \right)^{p/2} \]

(since different \( J \in \mathcal{M}_{\mathcal{A}_Q^\alpha, \gamma} \) are disjoint)

\[ \leq \sum_{l \in \mathbb{N}_0} (D_l^p)^p |\Omega|^{1-p/2} \left( \sum_{Q} \sum_{J \in \mathcal{M}_{\mathcal{A}_Q^\alpha, \gamma}} \|a^\alpha_{Q, J, l}\|_2^p \right)^{p/2} \]

\[ \leq c \sum_{l \in \mathbb{N}_0} (D_l^p)^p d(l, \alpha)^{1-p/2}, \]

by Lemma 3.3, Lemma 2.1, and the definition of the atom \( b_\Omega \).

Finally, we must estimate \( I_\gamma \) for \( \gamma = \{1, \ldots, n\} \). For \( S \in \mathcal{M}_I^\gamma(\Omega) \), let

\[ a_S = \sum_{l \geq 0} u_0^l(R, \Omega) = S e_R. \]

Then

\[ I_\gamma = \sum_R \sum_{m_l > L_n u_0^\gamma(R)} T_m e_R \]

\[ = \sum_S \sum_{u_0^\gamma(R) = S} \sum_{m_l > L_n u_0^\gamma(R)} \sum_{s = 1, \ldots, n-1} T_m e_R \]

\[ = \sum_{l \in \{n\}'} \sum_{S \in \mathcal{M}_I(\Omega)} \sum_{l_a \geq 0} T_{L(S) + l} a_S. \]

Since \( |S|^{1/2-1/p} \|a_S\|_2^{-1} a_S \) is a \((p, S)\) atom, we get

\[ \left\| \sum_{l_a \geq 0} T_{L(S) + l} a_S \right\|_p^p \leq c E_I |S|^{1-p/2} \|a_S\|_2^p, \]

and, as before, by Hölder’s inequality, Lemma 3.3, and Lemma 2.1,

\[ \sum_{S \in \mathcal{M}_I^\gamma(\Omega)} \left\| \sum_{l_a \geq 0} T_{L(S) + l} a_S \right\|_p^p \leq c E_I^p \left( \sum |S| \right)^{1-p/2} \left( \sum \|a_S\|_2^2 \right)^{p/2} \]

\[ \leq E_I^p d(\{\bar{l}, \{n\}'\})^{1-p/2}. \]

Thus

\[ \|I_\gamma\|_p^p \leq \sum_{l \in \{n\}'} E_I^p d(\{\bar{l}, \{n\}'\})^{1-p/2}, \]

concluding the proof of Proposition 3.2 and Theorem 3.1. □
4. $L^p$-estimates

Recall the definition of $Q^\alpha_j$ and $P^\alpha_j$ from §1, and that of $d(l, \alpha)$ from §3. We now state an improved version of Theorem 2, which is new even in the classical one-parameter setting.

**Theorem 4.1.** Suppose $T$ is a singular integral operator which is bounded on $L^2$ with norm at most $A$, and let $1 < p < 2$. Suppose that for all $\beta, \gamma \subseteq \{1, \ldots, n\}$ with $1 \leq |\beta \cup \gamma| \leq n-1$, $\beta \cap \gamma = \emptyset$ (allowing the possibility that $\beta$ or $\gamma = \emptyset$), for all $m \in \mathbb{N}^\beta$, $l \in \mathbb{N}^{\beta \cup \gamma}$, and some large $N$,

$$\sum_{k \in \mathbb{Z}^{\beta \cup \gamma}} \sup_{j \in \mathbb{Z}^{\beta \cup \gamma}} \|Q_j^{\beta \cup \gamma} T_{j+1}^{\beta \cup \gamma} O_{\pi_j^{\beta \cup \gamma} + m} \otimes P_{\pi_j^{\gamma}}\|_{r-r} \leq A_i^{\beta, \gamma}(r) \prod_{s \in \beta} 2^{m_s}$$

for $r = p$ and $r = 2$, and suppose that for all $\beta, \gamma$ with $\beta \cup \gamma = \{1, \ldots, n\}$, $\beta \cap \gamma = \emptyset$, $m \in \mathbb{N}^\beta$, $l \in \mathbb{N}^{\{n\}'}$ and some large $N$,

$$\sum_{k \in \mathbb{Z}^{\{1, \ldots, n\}}} \sup_{j \in \mathbb{Z}^{\{1, \ldots, n\}}} \left\| \left( \sum_{l \geq 0} Q_j^{\beta} T_{j+1}^{\beta} Q_{\pi_j^{\beta} + m} \otimes P_{\pi_j^{\gamma}} \right) \right\|_{r-r} \leq B_i^{\beta, \gamma}(r) \prod_{s \in \beta} 2^{m_s}$$

for $r = p$ and $r = 2$. Let

$$C(r) = \sup_{1 \leq |\beta \cup \gamma| \leq n-1} \sum_{l \in \mathbb{N}^{\beta \cup \gamma}} A_i^{\beta, \gamma}(r) d(l, \beta \cup \gamma)^{1/r - 1/2}$$

$$+ \sup_{\beta \cup \gamma = \{1, \ldots, n\}} \sum_{l \in \mathbb{N}^{\{n\}'}} B_i^{\beta, \gamma}(r) d(l, \{n\}')^{1/r - 1/2}.$$

If $A + C(2) + C(p) < \infty$, then $T$ is bounded on $L^p$.

**Remark.** In the one-parameter case, the theorem reduces to the following statement. Suppose $T$ is an $L^2$-bounded operator such that

$$\sum_{k \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} \left\| \left( \sum_{l \geq 0} Q_j^{\beta} T_{j+1}^{\beta} Q_{\pi_j^{\beta} + m} \otimes P_{\pi_j^{\gamma}} \right) \right\|_{r-r} \leq C(r)$$

for $m \in \mathbb{N}$, some large $N$, for $p \leq r \leq 2$. Then $T$ is bounded on $L^p$.

**Proof.** Let $f \in L^p$; then by the Calderón reproducing formula, we may write

$$f = \sum_{\mu \in \mathbb{Z}} \sum_{R \in \mathcal{R}_\mu} \int \psi_{t} * [(\psi_{t} * f) \chi_{R, \mu}] \frac{dt}{t_1 \cdots t_n}$$

where $\psi$, $R$, and $\Omega$ are defined as in §2, and where we additionally assume $\int \Psi(t)^i dt = 0$, $i = 0, \ldots, M$, with $M > N$. For $\mu \in \mathbb{Z}$, $t \in (\mathbb{R}_+)^n$, and $R$ running over the collection of all dyadic rectangles in $\mathbb{R}^n$, we consider vector-valued functions $F = \{e_{R, \mu}^\mu\}$ as members of one of the function spaces $Y$, with norm

$$\|F\|_{Y} = \left( \sum_{\mu} \left[ |\Omega_{\mu}|^{1/r - 1/2} \left( \sum_{R} \left\| e_{R, \mu}^\mu \right\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t_1 \cdots t_n} \right)^{1/2} \right]^{1/2} \right)^{r/2}$$
where \(|\Omega_\mu|\) are fixed weights coming from a fixed \(f \in L^p\). We shall apply estimates for general \(F\) to the particular choice \(F(f) = \{(\psi_t * f)\chi_{R,t}^\mu\}\) where \(\chi_{R,t}^\mu = \chi_{R,t}\) (see §2) if \(R \in \mathcal{R}_\mu\) and is zero otherwise. Then we may rewrite (4.1) as

(4.2) \[ f = \sum_{\mu \in \mathbb{Z}} \sum_{R} \int \psi_t \ast [F(f)]_{R,t}^\mu \frac{dt}{t_1 \cdots t_n}, \]

and, by Lemma 2.2 we have

(4.3) \[ \|F(f)\|_{L^p} \leq C\|S(f)\|_{L^p} \leq C'\|f\|_{L^p}. \]

For general \(F\) we consider the operator

(4.4) \[ \tilde{T}(F) = \sum_{\mu \in \mathbb{Z}} \sum_{R} \int \psi_t \ast \left( e_{R,t}^\mu \chi_{R,t}^\mu \right) \frac{dt}{t_1 \cdots t_n}. \]

Hence, by (4.2), \(Tf = \tilde{T}(F(f))\). As in §3 we may decompose \(\tilde{T}\) by the formula

(4.4) \[ \tilde{T}(F) = V(F) + \sum_{1 \leq |\alpha| \leq n} (-1)^{|\alpha|-1} \sum_{\mu} \sum_{R} \sum_{m_j > L_{e_{R,0}}(R, \Omega_\mu)} T_m(e_{R}^\mu(F)) \]

where \(\omega(R, \Omega_\mu)\) is the \(\omega^{(1, \ldots, n)}(R, \tilde{\Omega}_\mu)\) of §3 and where

(4.4) \[ V(F) = \sum_{\mu} \sum_{R} \sum_{m_j \leq L_{\omega(R, \Omega_\mu)}} T_m(e_{R}^\mu(F)). \]

For \(\tilde{\mu} \in \mathbb{Z}\), let

(4.4) \[ F_{\tilde{\mu}} = \begin{cases} e_{R}^\mu, & \mu = \tilde{\mu}, \\ 0, & \mu \neq \tilde{\mu}, \end{cases} \]

so that \(V(F) = \sum_{\tilde{\mu} \in \mathbb{Z}} V(F_{\tilde{\mu}})\).

For \(j \in \mathbb{Z}^\alpha, l \in \mathbb{N}^\alpha (|\alpha| \leq n)\), let \(\mathcal{U}_{\alpha,j,l}\) be the family of all dyadic rectangles \(R\) in \(\mathbb{R}^n\) with the property that if \(\mathcal{R}_\mu\) contains \(R\), then \(\nu_s(R, \tilde{\Omega}_\mu) < l_s\), \(s \in \alpha\), and \(L_{\alpha}(u_0^\alpha(R, \tilde{\Omega}_\mu)) = j\) (with \(u_0^\alpha\) as in §3). If \(\beta \subseteq \alpha, \beta \neq \emptyset, i \in \mathbb{N}^\beta\), let \(\mathcal{U}_{t, j, l}^\alpha, \beta\) be the family of all dyadic \(R \in \mathcal{U}_{\alpha,j,l}\) such that \(L_{\beta}(R) = \Pi_{j \neq i} j - i\). Similarly for \(j \in \mathbb{Z}^\alpha, i \in \mathbb{N}^\alpha\), let \(\mathcal{U}_{j, l}^\beta\) be the family of all dyadic rectangles \(R\) in \(\mathbb{R}^n\) with the property that if \(\mathcal{R}_\mu\) contains \(R\), then \(\nu_s(R, \tilde{\Omega}_\mu) < l_s\), \(1 \leq s \leq n-1\), and \(L(u_0(R, \tilde{\Omega}_\mu)) = j\) (with \(\nu_s = \nu_s^{(1, \ldots, n)}\) and \(u_0 = u_0^{(1, \ldots, n)}\)). If \(\beta \subseteq \{1, \ldots, n\}\) and \(i \in \mathbb{N}^\beta\), let \(\mathcal{U}_{j, i, l}^\beta\) be the subfamily of \(R\) in \(\mathcal{U}_{\alpha,j,l}\) such that \(L_{\beta}(R) = \Pi_{j \neq i} j - i\).
Setting $e_R(F) = \sum_\mu e_R^\mu(F)$ (only one term in the sum is nonzero) we consider the operators

$$\sigma_{j,l}^\alpha(F) = \sum_{R \in \mathcal{U}_{j,l}^\alpha} e_R(F), \quad \sigma_{j,i,l}^{\alpha,\beta}(F) = \sum_{R \in \mathcal{U}_{j,i,l}^{\alpha,\beta}} e_R(F),$$

$$\tau_{j,i}(F) = \sum_{R \in \mathcal{W}_{j,i}} e_R(F), \quad \tau_{j,i,l}^{\beta}(F) = \sum_{R \in \mathcal{U}_{j,i,l}^{\beta}} e_R(F).$$

With this notation, we may, as in §3, rewrite (4.4) as

$$\tilde{T}(F) - V(F) = \sum_{1 \leq |\alpha| \leq n-1} (-1)^{|\alpha|-1} \sum_{l \in \mathbb{N}^n} T_{j+l}^{\alpha} \sigma_{j+l}^\alpha(F)$$

$$+ (-1)^{n-1} \sum_{l \in \mathbb{N}(n')} \sum_{j \in \mathbb{Z}^n} T_{j+l} \tau_{j,i}(F).$$

We shall need some estimates for the operators $\sigma$ and $\tau$.

**Lemma 4.2.** Let $1 \leq r \leq 2$.

(i) $(\sum_{j \in \mathbb{Z}^n} \|Q_{j,l}^{\alpha,\beta}(F)\|_r)^{1/r} \leq c d(l, \alpha)^{1/r-1/2} \|F\|_r$.

(ii) If $\emptyset \neq \beta \subseteq \alpha$, $i, m \in \mathbb{N}^\beta$,

$$\left(\sum_{j \in \mathbb{Z}^n} \|Q_{j+m,l}^{\beta}(\sigma_{j,i,l}^\alpha)(F)\|_r^r\right)^{1/r} \leq C \prod_{s \in \beta} 2^{-(m_s+i_s)_s} d(l, \alpha)^{1/r-1/2} \|F\|_r.$$

(iii) $(\sum_{j \in \mathbb{Z}^n} \|\tau_{j,i}(F)\|_r)^{1/r} \leq c d(l, \{n\})^{1/r-1/2} \|F\|_r$.

(iv) If $\beta \neq \emptyset$, $i, m \in \mathbb{N}^\beta$,

$$\left(\sum_{j \in \mathbb{Z}^n} \|Q_{j+m,l}^{\beta}(\tau_{j,i,l}(F))\|_r^r\right)^{1/r} \leq C \prod_{s \in \beta} 2^{-(m_s+i_s)_s} d(l, \{n\})^{1/r-1/2} \|F\|_r.$$

**Proof.** We prove only (ii) in the case that $\alpha - \beta \neq \emptyset$; all the other cases may be handled in the same way. Let $W_{l}^{\beta} = \psi_{l}^{\beta} *_{\beta} f$; then clearly for $t_s \sim 2^j - i_s$, $s \in \beta$, the $L^p$ operator norm of $Q_{j+m,l}^{\beta} W_{l}^{\beta}$ is dominated by $\prod_{s \in \beta} 2^{-M(m_s+i_s)_s}$. For $l = \pi_{\beta} l'$, let

$$a_{j,i,l}^{\alpha,\beta} = \sum_{R \in \mathcal{U}_{j,i,l}^{\alpha,\beta}} \int_{l' \in \mathbb{R}^n} \psi_{l'}^{\beta} *_{\beta} [e_{R',l}^{\mu} \chi_{R',l}^{\mu}] \frac{d l'}{\prod_{s \in \beta} t_s}.$$

Hence, considering the case $r = 2$ of (ii),

$$\left(\sum_{j} \|Q_{j+m,l}^{\beta}(\sigma_{j,i,l}^\alpha)(F)\|^2\right)^{1/2} \leq \sum_{j} \int_{i_s \sim 2^j - i_s} \|Q_{j+m,l}^{\beta} W_{l}^{\beta}\|^2_{2} \|\sigma_{j,i,l}^\alpha\|^2 \frac{d l'}{\prod_{s \in \beta} t_s},$$

(by the Cauchy-Schwarz inequality), and, by using Lemma 2.1 with respect to the $\beta'$ variables, we can dominate this by $\prod_{s \in \beta} 2^{-M(m_s+i_s)} \|F\|_{Y_2}$. 

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Now we consider the case $r = 1$ of (ii); once this is established the general case follows by interpolation. For a dyadic $J$ in $\mathbb{R}^\alpha$ with $L_\alpha(J) = j$ and $i \in (\mathbb{R}^+)^\beta$, we set

$$a_{J, i, l, \mu}^{\alpha, \beta, i} = \sum_{R \in \mathbb{Z}^{\alpha,l}_R, \cap S_\mu} \int \psi_{t'} * \beta' \left[ e_{R, t'}^{\mu} \chi_{R, t'}^{\mu} \right] \frac{dt'}{\prod_{s} t_s}.$$

Notice that $a_{J, i, l, \mu}^{\alpha, \beta, i}$ is supported in $J \times \mathcal{A}_J^{\alpha, \beta} (\Omega_\mu)$, where $\mathcal{A}_J^{\alpha, \beta}$ is the $\mathcal{A}_J^{\alpha, \beta, \{1, \ldots, n\}}$ of Lemma 3.3. Hence,

$$\sum_{j \in \mathbb{Z}^\alpha} \| Q^\beta_{\pi, j+l} \sigma_{j, i, l, \mu}^{\alpha, \beta} (F) \|_1$$

$$\leq C \sum_{\mu \in \mathbb{Z}, J \text{ dyadic in } \mathbb{R}^\alpha} \int_{s \sim 2^{\alpha}} \| Q^\beta_{\pi, j+l} \sigma_{j, i, l, \mu}^{\alpha, \beta} (F) \|_{1/2} \left( \int \| a_{J, i, l, \mu}^{\alpha, \beta, i} \|_2 \frac{dt}{\prod_{s} t_s} \right)^{1/2}$$

$$\leq C \sum_{\mu} \prod_{s \in \beta} 2^{-(m_s+i_s)M} \| J \times \mathcal{A}_J^{\alpha, \beta} (\Omega_\mu) \|^{1/2} \left( \sum_{J \times \mathcal{A}_J^{\alpha, \beta} (\Omega_\mu)} \| a_{J, i, l, \mu}^{\alpha, \beta, i} \|_2 \frac{dt}{\prod_{s} t_s} \right)^{1/2}$$

(by the strong maximal theorem and two applications of Cauchy-Schwarz)

$$\leq C \prod_{s \in \beta} 2^{-(m_s+i_s)M} \sum_{\mu} \| \Omega_\mu \|^{1/2} d(l, \alpha)^{1/2} \left( \sum_{R \in \mathcal{R}_\mu} \| e_{R, t'}^{\mu} \|_2^2 \frac{dt}{t_1 \cdots t_n} \right)^{1/2}$$

(by Lemmas 2.1 and 3.3)

$$= C \prod_{s \in \beta} 2^{-(m_s+i_s)M} d(l, \alpha)^{1/2} \| F \|_{Y_1}. \quad \Box$$

Next we shall prove an estimate for $\tilde{T}(F) - V(F)$ which will follow from the case $r = p$ of the following lemma.

**Lemma 4.3.** Let $1 < p \leq 2$ and $r \in \{2, p\}$.

(i) If $1 \leq |\alpha| \leq n - 1$ and $l \in \mathbb{N}^\alpha$,

$$\left\| \sum_{j \in \mathbb{Z}^\alpha} T_{j+l}^\alpha \sigma_{j, i, l}^\alpha (F) \right\|_r \leq C \sup_{\beta, \gamma, \mu = \alpha} A_{\beta, \gamma}^\mu (r) d(l, \alpha)^{1/r-1/2} \| F \|_{Y_1}.$$

(ii) If $\tilde{l} \in \mathbb{N}^{\{n\}}$,

$$\left\| \sum_{j \in \mathbb{Z}^\alpha} \sum_{l_n > 0} T_{j+l}^\alpha \sigma_{j, i, l}^\alpha (F) \right\|_r \leq C \sup_{\beta, \gamma, \mu = \{1, \ldots, n\}} B_{\beta, \gamma}^\mu (r) d(l, \{n\})^{1/r-1/2} \| F \|_{Y_1}.$$
Proof. As in [1 and 23], we combine Littlewood-Paley theory and Calderón-Zygmund theory. Thus
\[
\left\| \sum_{j \in \mathbb{Z}^n} T^{\alpha}_{j+i} \sigma_{j,i}(F) \right\| \leq \sum_{k} \left\| \sum_{j \in \mathbb{Z}^n} Q^\alpha_{j+k} Q^\alpha_{j+k} T^{\alpha}_{j+i} \sigma_{j,i}(F) \right\|_{r} \\
\leq C \sum_{k} \left( \sum_{j \in \mathbb{Z}^n} \left\| Q^\alpha_{j+k} T^{\alpha}_{j+i} \sigma_{j,i}(F) \right\|_{r} \right)^{1/r}
\]
by Littlewood-Paley theory and the embedding $l' \subseteq l^2$. Using the identity
\[
I = \sum_{\beta \subseteq \alpha} \sum_{m \in \mathbb{N}^\beta} Q^{\beta^2}_{\pi_{j},m} \otimes P^{\beta'}_{\pi_{j},m} \text{ on } \mathbb{R}^n
\]
(valid for all $j$), we see this expression can in turn be dominated by
\[
C \sum_{\beta \subseteq \alpha} \sum_{m \in \mathbb{N}^\beta} \sum_{k} \left( \sum_{j \in \mathbb{Z}^n} \left\| Q^\alpha_{j+k} T^{\alpha}_{j+i} Q^{\beta}_{\pi_{j},m} \otimes P^{\beta'}_{\pi_{j},m} \right\|_{r} \right)^{1/r}
\]
(by the hypothesis of Theorem 4.1)
\[
\leq C \sum_{\beta \subseteq \alpha} \sum_{m \in \mathbb{N}^\beta} \sum_{i \in \mathbb{N}^\beta} A^{\beta \cdot \gamma}_{i}(r) \prod_{s \in \beta} \left( \sum_{j} \left\| Q^{\beta}_{\pi_{j},i} \sigma_{j,i}(F) \right\|_{r} \right)^{1/r}
\]
(by Lemma 4.2. Since we have chosen $M$ to be greater than $N$, we may sum up and obtain part (i). Part (ii) is proved in a similar way. □

We have shown that $\| \tilde{T}(F) - V(F) \|_p \leq CC(p)\| F \|_{Y_\rho}$, and we now consider $V(F)$ and show
\[
\| V(F) \|_r \leq CA + C(2)\| F \|_{Y_\gamma}
\]
for $1 \leq r \leq 2$. Once we have done this, we are finished, since then
\[
\left\| T \psi \right\|_p = \| \tilde{T}(F(f)) \|_r \leq \| \tilde{T}(F(f)) - V(F(f)) \|_p + \| V(F(f)) \|_p
\leq C(A + C(2) + C(p))\| F(f) \|_{Y_\gamma}
\leq C(A + C(2) + C(p))\| f \|_p \text{ by (4.3)}.
\]
We establish (4.6) also by interpolation. For $r = 2$ we have
\[
\| V(F) \|_2 \leq \| \tilde{T}(F) \|_2 + \| V(T) - \tilde{T}(F) \|_2
\]
\[
\leq A \left\| \sum_{\mu} \sum_{R} \int \psi_{i} * \epsilon_{R} \chi_{R} \, dt \right\|_2 + CC(2)\| F \|_{Y_2}
\]
(by assumption and Lemma 4.3 in the case $r = 2$)
\[
\leq CA + C(2)\| F \|_{Y_2} \text{ (by Plancherel's theorem)}.
\]
Now let \( r = 1 \) and write \( V(F) = \sum_{\mu \in \mathbb{Z}} V(F_\mu) \), observing that \( V(F_\mu) \) is supported in \( \Omega_\mu^{(n2^k-1)} \). Hence,
\[
\|V(F)\|_1 \leq C \sum_{\mu} |\Omega_\mu|^{1/2} \|V(F_\mu)\|_2 \leq C \sum_{\mu} |\Omega_\mu|^{1/2} [A + C(2)] \|F_\mu\|_2 \quad \text{(by (4.7))}
\]
\[
\leq C[A + C(2)] \|F\|_{Y_1},
\]
which establishes (4.6), and hence Theorem 4.1. \( \square \)

5. Fourier multipliers and pseudodifferential operators

In this section we give some applications of Theorems 1.1 and 1.2 and prove \( H^p \)- and \( L^p \)-results for convolution and pseudodifferential operators. The computations needed to check the hypotheses of Theorems 1.1 and 1.2 are easy modifications of those carried out in [1 and 2] in the one-parameter case; so we will be very concise and omit most of the details.

We need some notation: \( M_p, 1 \leq p \leq \infty \), denotes the standard space of Fourier multipliers \( m \) in \( L^p \). The norm is given by the norm of the operator \( T \) where \( (Tf)(\xi) = m(\xi) \hat{f}(\xi) \). Similarly we define for \( \alpha \subset \{1, \ldots, n\} \) the space \( M_\alpha^{1/2} \) consisting of those \( m \) such that the multiplier transformation \( T \) is bounded on the mixed norm space \( L^1(\mathbb{R}^\alpha, L^2(\mathbb{R}^{\alpha'})) \). (Note that by [13], \( M_1 \subset M_\alpha^{1/2} \)). The dyadic decomposition \( \Phi_t \) (or \( \Phi_t^\alpha \)) is used to express some Lipschitz conditions with respect to multiplier norms.

Proposition 5.1. Suppose that \( m \) is a bounded function, \( \|m\|_\infty \leq A \), and that for some \( \delta > 0 \),

\[
(5.1) \quad \sup_{t \in (\mathbb{R}_+)^n} \|\phi m(t\cdot) \ast \Phi_t\|_{M_1} \leq A \prod_{s=1}^n 2^{-s\delta}
\]

and that for each \( \alpha \subset \{1, \ldots, n\} \)

\[
(5.2) \quad \sup_{t \in (\mathbb{R}_+)^n} \|\phi^\alpha m(t\cdot) \ast \Phi_t^\alpha\|_{M_\alpha^{1/2}} \leq A \prod_{s=\alpha} 2^{-s\delta}.
\]

Then we have the inequality

\[
\|T^{-1}[m\hat{f}]\|_{H^p} \leq CA\|f\|_{H^p} \quad \text{for} \quad \frac{1}{1 + \delta} < p \leq 1.
\]

Proof. It suffices to show that \( T : H^p \to L^p \) since \( H^p \) may be characterized as a space of distributions whose iterated Hilbert transforms are in \( L^p \) (see [11, 15]).

In order to keep the notation simple, we assume \( n = 2 \) and the general case is proved in the same way. Then we have to check the four hypotheses of the theorem. We only examine the hypothesis involving mixed norms in \( L^p(L^2) \); the other inequalities are obtained in the same manner.

Let \( a \) be an \( L^2 \)-valued rectangle atom. We may consider \( a \) as a function supported in \( I \times \mathbb{R} \), where \( I = \{x_1, a \leq x_1 \leq b\} \). By translation and dilation invariance, we may assume that \( I = [-1, 1] \). Then we have to show the
inequality

(5.3) \[ \| T^1_i a \|_{L^p(L^2)} \leq c 2^{-\varepsilon l_i} \]

for some \( \varepsilon > 0 \). In this proof, \( T^1_i \) denotes the operator with kernel in

\[ m^*(x-y)\Phi_i^\ast((x_1-y_1)/2^i) \]

(we shall write \( l, k, i \) instead of \( l_1, k_1, i_1 \)). Since \( T^1_i a \) is supported in the strip \( \{(x_1, x_2), |x_1| \leq 2^{l+1}\} \), we have by Hölder's inequality,

\[ \| T^1_i a \|_{L^p(L^2)} \leq c 2^{l(1-p)} \| T^1_i a \|_{L^1(L^2)}^p. \]

Using the dyadic decomposition \( (Q^1_k) \) on the multiplier side with respect to the \( \xi_1 \)-variable, we can write

\[ T^1_i a = \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} [TQ^1_k Q^1_k] T^1_i a \]

and if \( \phi = \phi \psi \) and

\[ m_{ikl}(\xi_1, \xi_2) = \phi(2^l \xi_1) \int 2^i \Phi(2^l (\xi_1 - y_1)) m(y_1, \xi_2) \phi(2^k y_1) dy_1, \]

we get

\[ \| T^1_i a \|_{L^p(L^2)} \leq 2^{l(1-p)} \left[ \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} m_{ikl} \| M_{12} \|_{L^1(L^2)} \right]^p. \]

If \( i > 0 \), the expression \( Q^1_i a \) is small because the rectangles have cancellation in the \( x_1 \)-direction: using Taylor's formula, we obtain

(5.4) \[ \| Q^1_i a \|_{L^1(L^2)} \leq c \min(1, 2^{-i M(p)}) \]

where \( M(p) > 1 - \frac{1}{p} \). By dilation invariance, we have to examine

\[ \tilde{m}_{ikl}(\xi_1, \xi_2) = m_{ikl}(2^{-i} \xi_1, \xi_2) \]

(5.5)

\[ = \phi(\xi_1) \int 2^{l-k} \Phi(2^{l-i} (\xi_1 - 2^{l-k} y_1)) \phi(y_1) m(2^{-k} y_1, \xi_2) dy_1. \]

The essential terms occur if \( i \) is close to \( k \); we use the estimate

\[ \| \tilde{m}_{ikl} \|_{M_{12}} \leq \| \phi(2^{l-i} \cdot) \|_{M_{12}} \| \Phi(2^{-k} \cdot) \|_{M_{12}} \]

(5.6)

\[ \leq c \min(1, 2^{(k-l)\delta}) \]

(see also the Remark below). From (5.5) it is easy to check that for every \( j, N > 0 \), we have the estimates

\[ |\partial_{\xi_1}^j [\tilde{m}_{ikl}]| \leq c_{j,N} 2^{l-k} \min(1, 2^{-(l-i)N}) \quad \text{if } k \geq i + 5, \]

\[ |\partial_{\xi_1}^j [\tilde{m}_{ikl}]| \leq c_{j,N} \min(1, 2^{-(l-k)N}) \quad \text{if } k \leq i - 5. \]

Recalling that \( \tilde{m}_{ikl} \) has compact support in \( \{ \frac{1}{4} \leq |\xi_1| \leq 4 \} \), we obtain

(5.7) \[ \| m_{ikl} \|_{M_{12}} \leq c \sup_{\xi_1} \left[ \int_{\xi_2} |m_{ikl}(\xi_1, \xi_2)|^2 + |\partial_{\xi_1} m_{ikl}(\xi_1, \xi_2)|^2 d\xi_2 \right]^{1/2} \]

\[ \leq \left\{ \begin{array}{ll}
  c_N 2^{l-k} \min(1, 2^{-(l-i)N}) & \text{if } k \geq i + 5, \\
  c_N \min(1, 2^{-(l-k)N}) & \text{if } k \leq i - 5.
\end{array} \right. \]
Putting the estimates (5.4), (5.6), and (5.7) together, we get

\[ \|T^i_a\|_1 \leq c 2^{l(1-p)} \left( \sum_i \sum_{|k-i| \leq 5} 2^{(k-i)\delta} \min(1, 2^{-iM(p)}) \right)^p + c 2^{l(1-p)} \left( \sum_i \sum_{k \geq i+5} 2^{l-k} \min(1, 2^{-i(l-i)N}) \min(1, 2^{-iM(p)}) \right)^p + c 2^{l(1-p)} \left( \sum_i \sum_{k \leq i-5} \min(1, 2^{-i(l-i)N}) \min(1, 2^{-iM(p)}) \right)^p \]

and since \( \delta > 1 - \frac{1}{p} \), \( M(p) > 1 - \frac{1}{p} \), we get the bound (5.3).

Similarly one can do the estimates required for the rectangle atoms. For these estimates we use (5.1) and the restriction theorem of de Leeuw ([18]; see also Jodeit [14]). □

**Remark.** The condition (5.1) is equivalent to the condition

\[ \sup_{h \in \mathbb{R}^n, \varepsilon > 0} \|\Delta_{h\varepsilon}^l - \Delta_{h\varepsilon}^l\|_{\mathcal{M}_l} \leq c \prod_{s=1}^n |h_s|^{-\varepsilon}, \]

all \( h \in \mathbb{R}^n \), some \( \varepsilon > 0 \), where \( \Delta_{h\varepsilon}^l \) denotes the difference operator \( \Delta_{h\varepsilon}^l f(x) = f(x + h\varepsilon_x) - f(x) \) in the \( x_s \)-direction. A similar remark applies to (5.2). This observation can be used to show that the hypotheses of Proposition 5.1 are really independent of the choice of \( \phi \) (see [2, 23] for similar arguments).

Proposition 5.1 implies as a corollary an \( H^p \)-version of the Hörmander multiplier theorem in product spaces. The same result was proved by R. Fefferman and K. C. Lin [9] in the two-parameter case, using R. Fefferman's [7] result on rectangle atoms. For \( p = 1 \) the corollary already follows from the BMO-estimates for convolution operators proved by H. Lin [19].

Let us define the multiparameter Sobolev-space \( \mathcal{L}^p \) by

\[ \|g\|_{\mathcal{L}^p} = \left\| \mathcal{F}^{-1} \left( \prod_{s=1}^n (1 + |\xi_s|^2)^{\gamma/2} \hat{g} \right) \right\|_p. \]

Then we have

**Corollary 5.2** (Hörmander-Marcinkiewicz type multiplier theorem). Suppose that for \( \gamma > \frac{1}{p} - \frac{1}{2} \), \( 0 < p \leq 1 \),

\[ \sup_{t \in (\mathbb{R}_+)^n} \|\phi m(t_{1^*}, \ldots, t_{n^*})\|_{\mathcal{L}^{p^*}} \leq A. \]

Let \( (T^i f)^*_N(\xi) = m(\xi) \hat{f}(\xi) \). Then \( \|T f\|_{H^p} \leq c A\|f\|_{H^p} \).

**Proof.** Suppose again \( n = 2 \) for simplicity. For fixed \( t_1 \), let \( \tilde{K}_{t_1} = \phi m(t_{1^*}) \).
We apply the Cauchy-Schwarz inequality and Plancherel’s theorem to get

\[
\|K_{t_1} \Phi_{l_1} * g\|_{L^1(L^2)} \leq \int_{y_1,x_1} \left( \int_{y_2,x_2} \left| \int_{x_2} K_{t_1} \Phi_{l_1}(x-y) g(y_1,y_2) \, dy_2 \right|^2 \, dx_2 \right)^{1/2} \, dx_1 \, dy_1 \leq c_2^{l_1/2} \int_{y_1} \left( \int_{x_2} \left| \int_{y_2} K_{t_1} \Phi_{l_1}(x-y) g(y) \, dy_2 \right|^2 \, dx \right)^{1/2} \, dy_1 = c_2^{l_1/2} \int \left( \int \left| \hat{K}_{t_1} \Phi_{l_1}(\xi_1, \xi_2) \int g(y) e^{iy_2 \xi_2} \, dy_2 \right|^2 \, d\xi \right)^{1/2} \, dy_1 \leq c_2^{l_1/2} \sup_{\eta_2} \left( \int_{\xi_1} \left| \hat{K}_{t_1} \Phi_{l_1}(\xi_1, \eta_2) \right|^2 \, d\xi_1 \right)^{1/2} \cdot \left( \int_{y_1} \left( \int_{\xi_2} \left| \int g(y) e^{iy_2 \xi_2} \, dy_2 \right|^2 \, d\xi_2 \right)^{1/2} \, dy_1 \right) \leq c_2^{l_1/2} \sup_{t_2} \sup_{\eta_2} \left( \int_{\xi_1} \left| \Phi_m(t_1, t_2) \Phi_{l_1}(\xi_1, \eta_1) \right|^2 \, d\xi_1 \right)^{1/2} \, \|g\|_{L^1(L^2)}.
\]

Similarly one may prove

\[
\|K \Phi_l\|_1 \leq c_2^{l_1/2} \|m\|^2 \left( \int |m \Phi_l|^2 \, d\xi \right)^{1/2}.
\]

From these estimates, the corollary follows by Proposition 5.1. \[ \square \]

We now give a generalization of Proposition 5.1 for pseudodifferential operators

\[
Tf = \sigma(x, D)f = \int \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i \xi \cdot x} \, d\xi.
\]

Let us first introduce some notation. In order to keep this simple, we restrict ourselves to the two-parameter case. Let \( X_p \) be the space of symbols \( \sigma \) such that \( \sigma(\cdot, D) \) is bounded in \( L^p \). The norm in this space is the operator-norm of \( \sigma(\cdot, D) \). Let \( X_{p2} \) be the space of symbols \( \sigma \) such that \( \sigma(\cdot, D) \) is bounded on \( L^p(L^2) \), where the \( L^p \)-norm is taken with respect to the \( x_1 \)-variable. Further, set \( \|\sigma\|_{X_p} = \|\tau_{x_1}\|_{X_{p2}} \), where \( \tau_{x_1}(x_1, x_2, \xi_1, \xi_2) = \sigma(x_2, x_1, \xi_2, \xi_1) \). (Notice that \( X_{2p} \) is a space of operators in \( L^p(L^2) \), not in \( L^2(L^p) \)!) Further, we introduce the localized and dilated symbols

\[
\hat{\sigma}_{l_1}^1(x, \xi) = \phi^1(x_1) \sigma(2^{l_1} x_1, x_2, 2^{-l_1} \xi_1, \xi_2),
\]

\[
\hat{\sigma}_{l_2}^2(x, \xi) = \phi^2(x_2) \sigma(x_1, 2^{l_2} x_2, \xi_1, 2^{-l_2} \xi_2),
\]

\[
\hat{\sigma}_k(x, \xi) = \phi(\xi_1) \phi(\xi_2) \sigma(2^{k_1} x_1, 2^{k_2} x_2, 2^{-k_1} \xi_1, 2^{-k_2} \xi_2).
\]

As in Proposition 5.1, Lipschitz conditions with respect to the \( x \)- and \( \xi \)-variables are expressed by certain decay conditions of the norms \( \|\hat{\sigma}_k \|_{X_p} \), \( \|\hat{\Phi}_l \|_{X_p} \), etc. as \( l_1, l_2 \to \infty \). (For a detailed discussion, see [2, §3].)
Proposition 5.3. (a) Suppose that \( \varepsilon > 0 \) and \( T = \sigma(\cdot, D) \) satisfies the following conditions:

\[
\begin{align*}
(5.8) \quad & \|\sigma\|_{X_2} \leq A, \\
(5.9) \quad & \sup_{k_1} \|\hat{\sigma}_{k_1}^1 \ast_{\xi_1} \hat{\Phi}_{k_1}^l \|_{X_{12}} \leq A 2^{-\varepsilon l_1}, \\
(5.10) \quad & \sup_{k_2} \|\hat{\sigma}_{k_2}^2 \ast_{\xi_2} \hat{\Phi}_{k_2}^l \|_{X_{12}} \leq A 2^{-\varepsilon l_2}, \\
(5.11) \quad & \sup_{k_1, k_2} \|\hat{\sigma}_{k_1} \ast_{\xi_1} \hat{\Phi}_{k_1} \|_{X_1} \leq A 2^{-\varepsilon l_1} 2^{-\varepsilon l_2}, \\
(5.12) \quad & \sup_{k_1} \sup_{x_2, \xi_2} \|\hat{\sigma}_{k_1}(\cdot, x_2, \cdot, \xi_2)\|_{X_2(\mathbb{R})} \leq A, \\
(5.13) \quad & \sup_{k_2} \sup_{x_1, \xi_2} \|\hat{\sigma}_{k_2}(x_1, \cdot, \xi_1, \cdot)\|_{X_2(\mathbb{R})} \leq A 2^{-\varepsilon l_1}, \\
(5.14) \quad & \sup_{k_2} \sup_{x_1, \xi_1} \|\hat{\sigma}_{k_2}(x_1, \cdot, \xi_1, \cdot)\|_{X_2(\mathbb{R})} \leq A, \\
(5.15) \quad & \sup_{k_2} \sup_{x_1, \xi_1} \|\hat{\sigma}_{k_2}(x_1, \cdot, \xi_1, \cdot)\|_{X_2(\mathbb{R})} \leq A 2^{-\varepsilon l_2}, \\
(5.16) \quad & \sup_{x, \xi} |\sigma(x, \xi)| \leq A.
\end{align*}
\]

Then \( T \) is bounded from \( H^p \) to \( L^p \), \( 1/(1 + \varepsilon) < p < 1 \), the operator norm being bounded by \( c_p A \).

(b) Suppose that (5.8)-(5.16) hold with \( X_1, X_{12}, X_{21} \) replaced by \( X_{\infty}, X_{\infty 2}, X_{2\infty} \) and suppose that furthermore

\[
\begin{align*}
(5.17) \quad & \sup_{k_2} \sup_{x_2, \xi_2} \|\hat{\Phi}_{k_2}^l \ast_{x_2} \hat{\sigma}_{k} \ast \hat{\Phi}_{k_1}^l(\cdot, x_2, \cdot, \xi_2)\|_{X_{\infty}(\mathbb{R})} \leq A 2^{-\varepsilon l_1} 2^{-\varepsilon l_2}, \\
(5.18) \quad & \sup_{k_1, x_1, \xi_1} \|\hat{\Phi}_{k_1}^l \ast_{x_1} \hat{\sigma}_{k} \ast \hat{\Phi}_{k_2}^l(x_1, \cdot, \xi_1, \cdot)\|_{X_{\infty}(\mathbb{R})} \leq A 2^{-\varepsilon l_1} 2^{-\varepsilon l_2}, \\
(5.19) \quad & \sup_{k_1, x, \xi} |\hat{\Phi}_{k_1}^l \ast_{x} \hat{\sigma}_{k}(x, \xi)| \leq A 2^{-\varepsilon l_1} 2^{-\varepsilon l_2}, \\
(5.20) \quad & \sup_{k_1, x_1, \xi_1} \|\hat{\Phi}_{k_1}^l \ast_{x_1} \hat{\sigma}_{k_1}(x_1, \cdot, \xi_1, \cdot)\|_{X_2(\mathbb{R})} \leq A 2^{-\varepsilon l_1}, \\
(5.21) \quad & \sup_{k_1, x, \xi} |\hat{\Phi}_{k_1}^l \ast_{x} \hat{\sigma}_{k_1}(x, \xi)| \leq A 2^{-\varepsilon l_1}, \\
(5.22) \quad & \sup_{k_2, x_2, \xi_2} \|\hat{\Phi}_{k_2}^l \ast_{x_2} \hat{\sigma}_{k_2}(\cdot, x_2, \cdot, \xi_2)\|_{X_2(\mathbb{R})} \leq A 2^{-\varepsilon l_2}, \\
(5.23) \quad & \sup_{k_2, x, \xi} |\hat{\Phi}_{k_2}^l \ast_{x} \hat{\sigma}_{k_2}(x, \xi)| \leq A 2^{-\varepsilon l_2}.
\end{align*}
\]

Then \( T^* \) is bounded from \( H^p \) to \( L^p \), \( 1/(1 + \varepsilon) < p < 1 \), the operator norm only depending on \( A \). \( \square \)

We will not give the proof of Proposition 5.3 here. The result follows by straightforward modifications of computations in [1, §5] and [2, §4]. The hypotheses are more complicated than in the multiplier result because there is no version of the de Leeuw restriction theorem on multipliers available in the context of the pseudodifferential operators. Furthermore it is usually nontrivial to verify the \( L^2 \)-boundedness (5.8). In order to do this one may use a product version of the T1-theorem of David and Journé, proved by Journé [15], or certain...
product versions of the Calderón-Vaillancourt theorems, proved in [2]. For the $H^p$-boundedness we state a corollary generalizing the product $S^0_{1\delta}$-theorem. For simplicity we shall use the abbreviation $\mathcal{D}_x^\varepsilon$ to indicate a (fractional) derivative in the $x$-variable. Since we do not prove endpoint results, the reader may replace this by a Lipschitz condition as in Proposition 5.2. The corollary extends certain previously known $L^p$-results [25, 21, 2] to the Hardy-space setting.

**Corollary 5.4.** Suppose that $0 < p \leq 1$, $\gamma > \frac{1}{p}$ and for some $\varepsilon > 0$,

\begin{equation}
\sup_k \sup_x \| \mathcal{D}_x^\varepsilon \tilde{\sigma}_k(x, \cdot) \|_{L^\gamma} < \infty,
\end{equation}

\begin{equation}
\left( \sum_k \sup_x \| \mathcal{D}_x^\varepsilon \tilde{\sigma}_k(x, \cdot) \|_{L^\gamma}^2 \right)^{1/2} < \infty,
\end{equation}

\begin{equation}
\left( \sum_k \sup_x \| \mathcal{D}_x^\varepsilon \mathcal{D}_x^\varepsilon \tilde{\sigma}_k(x, \cdot) \|_{L^\gamma}^2 \right)^{1/2} < \infty,
\end{equation}

\begin{equation}
\left( \sum_k \sup_x \| \mathcal{D}_x^\varepsilon \mathcal{D}_x^\varepsilon \mathcal{D}_x^\varepsilon \tilde{\sigma}_k(x, \cdot) \|_{L^\gamma}^2 \right)^{1/2} < \infty.
\end{equation}

Then $T = \sigma(\cdot, D)$ is bounded from $H^p$ to $L^p$. If $\mathcal{L}^1_\gamma$ is replaced by $\mathcal{L}^2_{\gamma-1/2}$ and if $\varepsilon > \frac{1}{p} - 1$ then $T^*$ is bounded from $H^p$ to $L^p$. $\square$

The proof of Corollary 5.4 consists of verifying the conditions in Proposition 5.3. This can be done by slightly modifying arguments in Coifman and Meyer [5, p. 14] and [2, §§3.2 and 3.3]. We omit the details.

The $\mathcal{L}^1_\gamma$ result for $T$ does not require Theorem 1 and has already been deduced from Journé's and Pipher's theorems [15 and 22]. However the $\mathcal{L}^2_\gamma$ result for $T^*$ does require Theorem 1. Corollary 5.4 contains product space versions of the $S^0_{1\delta}$ theorems. In fact, suppose $\sigma$ has support in $\{|\xi_1| \geq 4, \ |\xi_2| \geq 4\}$. If

\[ \left| \left( \frac{\partial}{\partial \xi} \right)^\alpha \left( \frac{\partial}{\partial x} \right)^\beta \sigma(x, \xi) \right| \leq \left| \xi_1 \right|^{\delta_1 - \alpha_1} \left| \xi_2 \right|^{\delta_2 - \alpha_2} \]

for some $0 \leq \delta \leq 1$, we say then that $\sigma$ is in the product $S^0_{1\delta}$ class. It is easy to see that such a $\sigma$ satisfies

\[ \sup_x \| \mathcal{D}_x^\varepsilon \mathcal{D}_x^\varepsilon \tilde{\sigma}_k(x, \cdot) \|_{L^\gamma} \leq C 2^{-k_1(1-\delta)\varepsilon} 2^{-k_2(1-\delta)\varepsilon}, \]

etc. and hence, in the case $0 \leq \delta < 1$, satisfies the hypotheses of Corollary 5.4 for all $\gamma > 0$.

We remark that every Lipschitz type condition in Propositions 5.1 and 5.3 may be replaced by a Dini-type condition (see [25] for certain results of this nature for $L^p$, $1 < p < \infty$).

We now turn to $L^p$-results, $1 < p < 2$. We restrict ourselves to Fourier multipliers, although similar results may be clearly obtained for pseudodifferential operators in analogy with those in [2].
Proposition 5.5. Let \( 1 < p < 2 \) and suppose that for all \( \alpha \), \( 1 \leq |\alpha| \leq n \), and some \( \delta > 0 \), we have

\[
\sup_{r^\alpha \in (\mathbb{R}^n)^n} \| \phi^\alpha m(r^\alpha) \ast \tilde{\Phi}^\alpha_r \|_{M_p} \leq A \prod_{s \in \alpha} 2^{-l_s \delta}.
\]

If \( (Tf)(\xi) = m(\xi) \hat{f}(\xi) \), then \( \| Tf \|_p \leq CA \| f \|_p \). \(\square\)

The proof consists of verifying the hypotheses of Theorem 2 or Theorem 4.1. This is done exactly as in [1, §§3 and 5], to which we refer the interested reader for details. Finally, we observe that Theorem B of the Introduction is an immediate consequence of Proposition 5.5, since the hypotheses of Theorem B for a given \( p \) imply, by interpolation, those of Proposition 5.5 for each \( r \) with \( p < r < 2 \).

6. A COUNTEREXAMPLE

We now wish to show that the "semilocal" assumptions in Propositions 5.1 and 5.2 cannot be replaced by purely "local" ones unless much more smoothness is assumed.

Proposition 6.1. For each \( 1 < p < 2 \) there is a bounded function \( m \) such that

\begin{enumerate}
\item \( \sup_{t_1, t_2 > 0} \| \phi m(t_1, t_2) \|_{M_1} < \infty \),
\item \( \sup_{t_1, t_2 > 0} \| \phi m(t_1, t_2) \|_{A_{1/p-1/2}} < \infty \),
\item \( m \notin M_p \).
\end{enumerate}

Proof. We define

\[
m(\xi_1, \xi_2) = \sum_{j=1}^{\infty} \hat{\theta}(\xi_1) \phi(2^{-j} \xi_2) e^{i j \xi_1} j^{1/2 - 1/p}
\]

where \( \theta \) is supported in \( \{|x_1| \leq \frac{1}{10}\} \) and \( \phi \) is supported in \( \{|\xi_2| \leq \frac{1}{10}\} \). It is easy to see that \( m \) satisfies (i) and (ii). By a result of Herz and Rivière [13], it suffices to show that \( T \) associated to \( m \) is not a bounded operator on the mixed norm space \( L^p(L^2) \). Choose a positive sequence \( a_j \) with \( (\sum |a_j|^{1/2} < 1 \) but \( \sum a_j^{2p-2} = \infty \), and define \( \hat{f}(\xi_1, \xi_2) = \hat{\theta}(\xi_1) \sum_j \phi(2^{-j} \xi_2) a_j 2^{-j/2} \). Then \( f \in L^p(L^2) \) with norm \( c(\sum |a_j|^{1/2}) \leq C \). Further, by Plancherel's theorem with respect to \( x_2 \) and the fact that

\[
\{\theta \ast \theta(\cdot - j)\}_{j \in \mathbb{Z}}
\]

have disjoint supports, we obtain

\[
\| Tf \|_{L^p(L^2)}^p = \int_{x_1} \left( \int_{\xi_2} \left| \sum_j \theta \ast \theta(x_1 - j) \phi(2^{-j} \xi_2) 2^{-j/2} a_j j^{1/2 - 1/p} \right|^2 d\xi_2 \right)^{p/2} dx_1
\]

\[
\geq \sum_j \int_{x_1} |\theta \ast \theta(x_1 - j)|^p dx_1 a_j^{p/2} = \infty. \quad \square
\]

We conclude with two remarks. Firstly, condition (ii) of Proposition 6.1 is sharp in the sense that \( \sup_{t_1, t_2 > 0} \| \phi m(t_1, t_2) \|_{L^1} < \infty \) for \( \frac{1}{q} = \frac{1}{p} - \frac{1}{2} \), \( \gamma > \frac{1}{p} - \frac{1}{2} \) implies \( m \in M_p \), and of course we have \( \| \cdot \|_{L^1} \leq c \| \cdot \|_{A_{1/p-1/2}} \), for
functions with compact support. Secondly, Proposition 5.1 and the proof of Proposition 6.1 suggest the question of whether one can relax the \( M_p \) hypotheses on dyadic strips in Proposition 5.5 to \( M_p \) conditions on dyadic rectangles and \( M_{p2} \) conditions on the strips. This seems to be an open problem.

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