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ON MAPPING CLASS GROUPS OF CONTRACTIBLE OPEN 3-MANIFOLDS

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ABSTRACT. Let W be an irreducible, eventually end-irreducible contractible open 3-manifold other than $\mathbf{R}^3$, and let V be a "good" exhaustion of W. Let $\mathscr{H}(W; V)$ be the subgroup of the mapping class group $\mathscr{H}(W)$ which is "eventually carried by V." This paper shows how to compute $\mathscr{H}(W; V)$ in terms of the mapping class groups of certain compact 3-manifolds associated to V. The computation is carried out for a genus two example and for the classical genus one example of Whitehead. For these examples $\mathscr{H}(W) = \mathscr{H}(W; V)$.

1. INTRODUCTION

The mapping class group $\mathscr{H}(W)$ of a smooth, orientable manifold W is the group of orientation preserving diffeomorphisms Diff(W) of W modulo isotopy. This paper is concerned with the case in which W is an irreducible, contractible open 3-manifold (a *Whitehead manifold*) which is eventually end-irreducible and is not homeomorphic to $\mathbf{R}^3$. Such a manifold possesses exhaustions by compact submanifolds having particularly nice properties. Given such an exhaustion V, there is a subgroup $\mathscr{H}(W; V)$ of $\mathscr{H}(W)$ whose elements are represented by diffeomorphisms which are "eventually carried by V." (See the next section for precise definitions and statements of theorems.) It is proven that the computation of $\mathscr{H}(W; V)$ can be reduced to the computation of the mapping class groups of certain compact 3-manifolds associated to the exhaustion. This reduction takes two forms, depending on whether or not V has genus one.

If V does not have genus one, then either $\mathscr{H}(W; V)$ is isomorphic to the direct limit $\overline{\mathscr{F}}(W; V)$ of a sequence $\overline{\mathscr{F}}_N(W; V)$ of groups, each of which is a subgroup of the direct product of a sequence of mapping class groups of compact 3-manifolds, or $\mathscr{H}(W; V)$ contains $\overline{\mathscr{F}}(W; V)$ as a normal subgroup with infinite cyclic quotient. The latter case occurs when V is "periodic," and $\mathscr{H}(W; V)$ is in fact the semidirect product of $\overline{\mathscr{F}}(W; V)$ and $\mathbf{Z}$. $\mathscr{H}(W; V)$ is computed explicitly for a certain genus two example similar to an example

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given by McMillan [Mc], and it is shown that for this example $\mathscr{H}(W; V)$ is equal to the entire mapping class group $\mathscr{H}(W)$.

If V does have genus one, then $\mathscr{H}(W; V)$ has a normal subgroup $\mathscr{D}(W; V)$ which is represented by Dehn twists about infinitely many of the boundary tori of elements of the exhaustion. This subgroup is most usefully described as the direct product modulo the direct sum of countably many copies of $\mathbf{Z}^2$. The quotient $\mathscr{H}(W; V)/\mathscr{D}(W; V)$ then has the structure described in the previous paragraph. This extra complication in the structure of $\mathscr{H}(W; V)$ is in part compensated for by the fact that for a genus one Whitehead manifold V can be chosen so that $\mathscr{H}(W; V) = \mathscr{H}(W)$. Thus, in principle, one can always compute the mapping class group of such a manifold. The computation is carried out explicitly for the classical example of Whitehead [Wh].

One reason for investigating mapping class groups of contractible open 3-manifolds is their application to the study of proper group actions. A group G acts properly on W if each compact subset of W meets only finitely many of its translates by elements of G. Two important classes of such actions are the finite group actions and the actions of fundamental groups of 3-manifolds having W as universal covering space.

With regard to finite group actions, it is proven in [My3] that for W an eventually end-irreducible Whitehead manifold not homeomorphic to $\mathbf{R}^3$ and G any finite subgroup of $\mathrm{Diff}(W)$ the restriction of the natural homomorphism $\mathrm{Diff}(W) \to \mathscr{H}(W)$ to G is one-to-one. Thus information about $\mathscr{H}(W)$ can place limits on the finite groups which can act (preserving orientation) on W.

For the genus two example the only finite groups which can act are $\mathbf{Z}_2$ and $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, and in fact these groups do act on the manifold. For the classical Whitehead example the only possibility is $\mathbf{Z}_2$, but there are uncountably many $\mathbf{Z}_2$ subgroups of the mapping class group. Each of them is represented by an involution.

With regard to the case in which W is the universal covering space of an orientable 3-manifold M, it is known that $\pi_1(M)$ acts properly on W and is torsion free. It is conjectured that if M is closed, then W must be homeomorphic to $\mathbf{R}^3$. Geoghegan and Mihalik have shown [Ge-Mi] that if W is a Whitehead manifold which is not homeomorphic to $\mathbf{R}^3$, then the restriction of $\mathrm{Diff}(W) \to \mathscr{H}(W)$ to any torsion-free subgroup whose action on W is proper must again be one-to-one. Thus one approach to the conjecture would be to try to show that if W is not homeomorphic to $\mathbf{R}^3$, then $\mathscr{H}(W)$ contains no subgroup isomorphic to a closed, aspherical 3-manifold group.

For the genus two example this is indeed the case, and so it cannot cover a closed 3-manifold. On the other hand the Whitehead example (and every other periodic genus one Whitehead manifold) contains subgroups isomorphic to the fundamental groups of every torus bundle over the circle. Note however that a closed, irreducible 3-manifold having such a fundamental group must by [Wa] be homeomorphic to a torus bundle and so have universal cover $\mathbf{R}^3$. More generally, by [Ha-Ru-Sc] a closed, irreducible 3-manifold whose fundamental group contains the fundamental group of a closed, orientable surface other than S^2 must be covered by $\mathbf{R}^3$. Thus one must at least modify the above approach by trying to show that every closed, aspherical 3-manifold subgroup of $\mathscr{H}(W)$ contains such a surface group.

It should also be pointed out that Whitehead's example is after all a genus one Whitehead manifold and so by [My2] admits no torsion-free proper group actions and thus cannot cover another 3-manifold. More generally, Wright has recently shown [Wr] that the same is true for any eventually end-irreducible Whitehead manifold other than $\mathbf{R}^3$. In particular this is the case for all Whitehead manifolds of positive finite genus.

The paper is organized as follows. Section 2 gives definitions, sets up notation, and gives precise statements of the theorems. Section 3 uses results of Laudenbach, Cerf, and Palais to give conditions under which an isotopically trivial diffeomorphism of a 3-manifold which leaves a surface invariant can be isotoped to the identity by an isotopy which leaves the surface invariant. In §4 this is applied to show that isotopically trivial diffeomorphisms of W which are eventually carried by V are isotopic to the identity by isotopies which are eventually carried by V. This is the main result needed in §5 to analyze the structure of $\mathscr{H}(W; V)$. In §6 certain types of incompressible surfaces in compact 3-manifolds arising in the examples are classified. The mapping class groups of these compact manifolds are studied in §7. The mapping class groups of the genus two and genus one examples are computed in §8 and §9, respectively. Section 10 shows how to embed torus bundle groups in the mapping class groups of periodic genus one Whitehead manifolds.

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2. Definitions, notation, and statements of theorems

We shall work throughout in the C^∞ category. The reader is referred to [Hi] for basic differential topology and to [He and Ja] for basic 3-manifold topology.

If X is an orientable manifold and A is a submanifold of X, then $\mathrm{Diff}(X, A)$ denotes the group of orientation preserving diffeomorphisms h of X such that $h(A) = A$. It is given the weak C^∞ topology and has basepoint the identity map of X. A path h_t in $\mathrm{Diff}(X, A)$ is called an (*ambient*) *isotopy* and its endpoints are said to be *isotopic*. It is not assumed that one endpoint is the identity. If B is a submanifold of X, then an isotopy h_t is *constant* on B if $h_t(x)$ is constant in t for each $x \in B$ and is *fixed* on B or *rel* B if $h_t(x) = x$ for each $x \in B$ and all t. Two submanifolds F and G of X are *isotopic* if there is an isotopy h_t with h_0 the identity and $h_1(F) = G$.

The group $\pi_0(\mathrm{Diff}(X, A))$ is called the *mapping class group* of (X, A) and is denoted $\mathscr{H}(X, A)$. The isotopy class of a diffeomorphism h is denoted $[h]$. Square brackets will also be used at times to denote homotopy classes, but the meaning will be clear from the context.

Let W be an irreducible, contractible open 3-manifold. W will be called a *Whitehead manifold.* An *exhaustion* for W is a sequence $V = \{V_n\}_{n\geq 0}$ of compact, codimension-zero submanifolds of W such that $W = \bigcup_{n\geq 0} V_n$, ∂V_n is connected, and $V_n \subseteq \mathrm{int}(V_{n+1})$ for all $n \geq 0$. Every W has an exhaustion. The *genus* of V is $\max\{\mathrm{genus}(\partial V_n)\}$; it is either a nonnegative integer or ∞. The *genus* of W is $\min\{\mathrm{genus}(V)\}$, taken over all exhaustions of W. The unique W of genus zero is $\mathbf{R}^3$. Let $S_n = \partial V_n$ for $n \geq 0$. Let $X_n = V_n - \mathrm{int}(V_{n-1})$, $\partial_+ X_n = S_n$, and $\partial_- X_n = S_{n-1}$ for $n \geq 1$.

Consider the following conditions on an exhaustion V:

(1) S_n is incompressible in $W - \mathrm{int}(V_0)$ for all $n \geq 0$.

(2) No S_n is a 2-sphere.

(3) No X_n is a product I-bundle or a Seifert fibered space.

(4) Either $\operatorname{genus}(S_n) = 1$ for all $n \geq 0$ or $\operatorname{genus}(S_n) > 1$ for all $n \geq 0$.

If W has an exhaustion satisfying (1), then W is *eventually end-irreducible.* Every Whitehead manifold of finite genus has this property [Br]. Every eventually end-irreducible Whitehead manifold other than $\mathbf{R}^3$ has an exhaustion which also satisfies (2). There is a subsequence of such an exhaustion which in addition satisfies (3) and (4). An exhaustion is called *good* if it satisfies all these conditions.

Suppose W has a good genus one exhaustion V. Let F_n be the canonical 2-manifold of X_n. (See the Splitting Lemma of [Ja-Sh] or Lemma 2.5 of [My2].) V is *very good* if it satisfies the following conditions:

(i) No component of F_n bounds a solid torus V_n' in W with $V_{n-1} \to V_n'$ null-homotopic.

(ii) The component of the manifold obtained by splitting X_n along F_n which contains S_n is anannular and atoroidal.

(iii) $V_n \to V_{n+1}$ is null-homotopic for all $n \geq 0$.

(In the definition of very good given in [My2] condition (i) is incorrectly stated. With the corrected definition given here all the results of [My2] are valid.)

By Lemma 2.7 of [My2] every genus one Whitehead manifold admits a very good genus one exhaustion. Note that a genus one exhaustion V which satisfies (iii) and has the property that each X_n is anannular and atoroidal is very good.

Returning now to the general case of a good exhaustion V of a Whitehead manifold W, let $h \in \operatorname{Diff}(W)$. h is *eventually carried by* V if there exist $N \geq 0$ and $s \geq -N$ such that $h(V_n) = V_{n+s}$ for all $n \geq N$. If $s = 0$, then h *eventually preserves* V. If $s \neq 0$, then h *eventually shifts* V and h is called a *shift* of V with *shift constant* s and *initial index* N. If V admits a shift let $\sigma = \min\{s\}$, taken over all shifts with positive shift constant s. Any shift with $s = \sigma$ is called a *minimal shift* of V, and V is said to be *periodic* with *period* σ.

This paper is about the subgroup $\mathscr{H}(W; V)$ of $\mathscr{H}(W)$ consisting of those isotopy classes represented by diffeomorphisms which are eventually carried by V. In general it is a proper subgroup, but there is an important special case in which it is not.

Theorem 2.1. *If V is a very good genus one exhaustion of the Whitehead manifold W, then $\mathscr{H}(W) = \mathscr{H}(W; V)$.*

Proof. This is an immediate consequence of Lemma 3.3 (the Shift Lemma) of [My2]. □

Returning again to the general case, the study of $\mathscr{H}(W; V)$ proceeds by first determining the structure of the subgroup $\mathscr{G}(W; V)$ of $\mathscr{H}(W; V)$ consisting of those isotopy classes having representatives which eventually preserve V and then examining the effect of shifts. $\mathscr{G}(W; V)$ is the nested union of the sequence of groups $\mathscr{G}_N(W; V)$, where $\mathscr{G}_N(W; V)$ consists of those classes having representatives h such that $h(V_n) = V_n$ for all $n \geq N$. For $P > N$, let $g_{N,P}: \mathscr{G}_N(W; V) \to \mathscr{G}_P(W; V)$ be the inclusion homomorphism. Note that although the diffeomorphisms used to define these groups in some sense respect the exhaustion, the isotopies between them need in no way do so.

If one requires the isotopies to respect the exhaustion, then one gets a new collection of groups, as follows. Let $\mathcal{F}_N(W;V) = \mathcal{H}(W, \bigcup_{n\geq N} S_n)$. This is the group of orientation preserving diffeomorphisms of W which preserve each V_n modulo isotopies which preserve each V_n, for $n \geq N$. There is a homomorphism $q_N: \mathcal{F}_N(W;V) \to \mathcal{H}(W)$ which allows isotopies that need not respect the exhaustion; the image of q_N is $\mathcal{G}_N(W;V)$. For $P > N$ there is a restriction induced homomorphism $f_{N,P}: \mathcal{F}_N(W;V) \to \mathcal{F}_P(W;V)$. Clearly $q_P \circ f_{N,P} = f_{N,P} \circ q_N$. Let $\mathcal{F}(W;V)$ be the direct limit of the sequence $\{\mathcal{F}_N(W;V), f_{N,P}\}$. This group can be interpreted as the group of orientation preserving diffeomorphisms of W which for some N preserve each V_n for $n \geq N$ modulo isotopies which for some $P > N$ preserve each V_n for $n \geq P$. Let q be the homomorphism of direct limits induced by the q_N.

Theorem 5.1. *$q: \mathcal{F}(W;V) \to \mathcal{G}(W;V)$ is an isomorphism.*

This theorem essentially says that diffeomorphisms which eventually preserve the exhaustion and are isotopic are isotopic by isotopies which eventually (but usually later) preserve the exhaustion.

For the practical computation of these groups it is necessary to consider yet another sequence of groups. Let $\overline{\mathcal{F}}_N(W;V)$ be the subgroup of $\mathcal{H}(V_N) \times \prod_{n=N+1}^{\infty} \mathcal{H}(X_n, S_n)$ consisting of those sequences $([h_n])$ such that the restrictions of h_n and h_{n+1} to S_n are isotopic, for all $n \geq N$. There is an obvious epimorphism $r_N: \mathcal{F}_N(W;V) \to \overline{\mathcal{F}}_N(W;V)$. For $P > N$ one can define $\overline{f}_{N,P}: \overline{\mathcal{F}}_N(W;V) \to \overline{\mathcal{F}}_P(W;V)$ by piecing together diffeomorphisms representing the first $P-N+1$ terms of a sequence in $\overline{\mathcal{F}}_N(W;V)$ to obtain a diffeomorphism of V_P and then taking its isotopy class as the first term of a sequence in $\overline{\mathcal{F}}_P(W;V)$. It turns out that $\overline{f}_{N,P}$ is well defined and $r_P \circ f_{N,P} = \overline{f}_{N,P} \circ r_N$. Let $\overline{\mathcal{F}}(W;V)$ be the direct limit of the sequence $\{\overline{\mathcal{F}}_N(W;V), \overline{f}_{N,P}\}$, and let r be the homomorphism of direct limits induced by the r_N.

Theorem 5.7.

(1) *If* genus$(V) > 1$, *then $r: \mathcal{F}(W;V) \to \overline{\mathcal{F}}(W;V)$ is an isomorphism.*

(2) *If* genus$(V) = 1$, *then there is an exact sequence*

$$0 \to \mathcal{D}(W;V) \to \mathcal{F}(W;V) \xrightarrow{r} \overline{\mathcal{F}}(W;V) \to 1,$$

where $\mathcal{D}(W;V) \cong \prod_{n=0}^{\infty} \mathbf{Z}^2 / \bigoplus_{n=0}^{\infty} \mathbf{Z}^2$, and the nth coordinate of $\{(a_n, b_n)\}$ corresponds to a Dehn twist about S_n with trace (a_n, b_n).

If G is a group and ψ is an automorphism of G let $G \times_\psi \mathbf{Z}$ denote the semidirect product of G and $\mathbf{Z}$ with respect to ψ, i.e., the elements of $G \times_\psi \mathbf{Z}$ are those of $G \times \mathbf{Z}$, and the multiplication is given by $(g_1, n_1) \cdot (g_2, n_2) = (g_1 \cdot \psi^{n_1}(g_2), n_1 + n_2)$. A homomorphism $\hat{\alpha}: G \times_\psi \mathbf{Z} \to G' \times_{\psi'} \mathbf{Z}$ is said to *preserve the semidirect product structure* if $\hat{\alpha}$ restricts to the identity $\mathbf{Z} \to \mathbf{Z}$ and to a homomorphism $\alpha: G \to G'$ such that $\psi' \circ \alpha = \alpha \circ \psi$. Any homomorphism $\alpha: G \to G'$ having this property induces a homomorphism $\hat{\alpha}: G \times_\psi \mathbf{Z} \to G' \times_{\psi'} \mathbf{Z}$ which preserves the semidirect product structure.

Theorem 5.13. *Suppose V is periodic of period σ with minimal shift h. Then conjugation by h induces automorphisms ψ, ξ, and $\overline{\xi}$ of $\mathcal{G}(W;V)$, $\mathcal{F}(W;V)$, and $\overline{\mathcal{F}}(W;V)$, respectively, having the following properties.*

(1) *$\mathcal{H}(W;V) = \mathcal{G}(W;V) \times_\psi \mathbf{Z}$, with $\mathbf{Z}$ generated by $[h]$.*

(2) *$q\colon \mathscr{F}(W; V) \to \mathscr{G}(W; V)$ induces an isomorphism*

$$\hat{q}\colon \mathscr{F}(W; V) \times_\xi \mathbf{Z} \to \mathscr{G}(W; V) \times_\psi \mathbf{Z}$$

which preserves the semidirect product structure.

(3) *$r\colon \mathscr{F}(W; V) \to \overline{\mathscr{F}}(W; V)$ induces an epimorphism*

$$\hat{r}\colon \mathscr{F}(W; V) \times_\xi \mathbf{Z} \to \overline{\mathscr{F}}(W; V) \times_{\overline{\xi}} \mathbf{Z}$$

which preserves the semidirect product structure.

(i) *If genus $(V) > 1$, then $\hat{r}$ is an isomorphism.*

(ii) *If genus $(V) = 1$, then $\ker \hat{r} = \mathscr{D}(W; V)$ and ξ restricts to an automorphism of $\mathscr{D}(W; V)$ given by*

$$\xi(\{(a_0, b_0), (a_1, b_1), \dots\}) = \{\overbrace{(0, 0), \dots, (0, 0)}^{\sigma}, (a_0, b_0), (a_1, b_1), \dots\}.$$

For one of the examples all of this becomes very simple.

Theorem 8.1. *There is a genus two Whitehead manifold W with a good genus two exhaustion V of period $\sigma = 1$ such that $\mathscr{H}(W) = \mathscr{H}(W; V) \cong (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \times_\xi \mathbf{Z}$, where ξ interchanges the summands of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.* □

For the other example things are a bit more complicated.

Theorem 9.1. *Let W be the classical Whitehead manifold* [Wh]. *W has a very good genus one exhaustion V of period $\sigma = 1$ with the following properties:*

(1) *$\mathscr{H}(W) = \mathscr{H}(W; V) \cong \mathscr{F}(W; V) \times_\xi \mathbf{Z}$.*

(2) *There is an exact sequence*

$$0 \to \mathscr{D}(W; V) \to \mathscr{F}(W; V) \times_\xi \mathbf{Z} \xrightarrow{\hat{r}} \overline{\mathscr{F}}(W; V) \times_{\overline{\xi}} \mathbf{Z} \to 1,$$

where $\ker \hat{r} = \mathscr{D}(W; V) \cong \prod_{n=0}^\infty \mathbf{Z}^2 / \bigoplus_{n=0}^\infty \mathbf{Z}^2$, $\overline{\mathscr{F}}(W; V) \cong \prod_{n=0}^\infty \mathbf{Z}_2 / \bigoplus_{n=0}^\infty \mathbf{Z}_2$, and $\hat{r}$ preserves the semidirect product structure.

(3) *ξ restricts to the automorphism of $\mathscr{D}(W; V)$ given by*

$$\xi(\{(a_0, b_0), (a_1, b_1), \dots\}) = \{(0, 0), (a_0, b_0), (a_1, b_1), \dots\}.$$

(4) *For $\overline{c} = \{c_n\} \in \overline{\mathscr{F}}(W; V)$, $\overline{\xi}(\{c_0, c_1, \dots\}) = \{0, c_0, c_1, \dots\}$.*

(5) *For every $c \in \mathscr{F}(W; V)$ such that $r(c) = \overline{c}$, and for each $\{(a_n, b_n)\} \in \mathscr{D}(W; V)$,*

$$c\{(a_n, b_n)\}c^{-1} = \{(-1)^{c_n}(a_n, b_n)\}.$$

(6) *For every $\overline{c} \in \overline{\mathscr{F}}(W; V)$ there exists $c \in \mathscr{F}(W; V)$ such that $r(c) = \overline{c}$ and*

$$c^2 = \left\{ \frac{1 + (-1)^{c_n}}{2} (c_{n-1}, c_{n+1}) \right\}.$$

The element c' of $\mathscr{F}(W; V)$ satisfies $r(c') = r(c)$ and $(c')^2 = c^2$ if and only if c and c' differ by an element of $\mathscr{D}(W; V)$ of the form

$$\left\{ \frac{1 - (-1)^{c_n}}{2} (a_n, b_n) \right\}.$$

(7) *There is an involution γ of W such that $r([\gamma]) = \{(1, 1, 1, \dots)\}$. The finite subgroups of $\mathscr{H}(W)$ are precisely the $\mathbf{Z}_2$ subgroups generated by elements of the form $[\gamma]\{(a_n, b_n)\}$. Each of these elements is represented by an involution of W.*

Thus the classical Whitehead manifold admits uncountably many nonisotopic involutions.

Finally, there is the fact that mapping class groups of Whitehead manifolds can contain fundamental groups of closed, aspherical 3-manifolds.

Theorem 10.1. *Let W be a periodic genus one Whitehead manifold. Then for every torus bundle M over the circle there is a subgroup of $\mathscr{H}(W)$ which is isomorphic to $\pi_1(M)$.*

3. Surface preserving isotopies

Let Y be an irreducible, orientable 3-manifold and let S be a closed, connected, orientable surface in Y. S is *not* assumed to be incompressible in Y. Let h be a diffeomorphism of Y such that $h(S) = S$.

Theorem 3.1. *Suppose h is isotopic to the identity and that there is an irreducible 3-dimensional submanifold M of Y which contains the track of S under the isotopy and in which S is incompressible. If S is not a fiber in a fibration of M over the circle and S does not bound a submanifold of M diffeomorphic to a twisted I-bundle over a closed surface, then the given isotopy is path homotopic in* $\mathrm{Diff}(Y)$ *to an isotopy h_t such that $h_t(S) = S$ for all $t \in [0, 1]$.*

In the course of the proof h may be changed by an isotopy which preserves S and one isotopy may be replaced by another. To avoid excessive notation the new maps will often be given the same names as the old maps.

The main tool in the proof is a theorem of Laudenbach, stated below, about paths of surfaces in a 3-manifold. To apply this theorem one needs to use some results of Cerf and Palais about spaces of embeddings. Let S be a smooth, closed, connected submanifold of a smooth manifold X. (In the applications S will be the surface above and X will be M or Y.) Give the set of smooth embeddings $\mathrm{Emb}(S, X)$ the weak C^∞ topology [Hi]. By Théorème 1 on page 114 of [Ce] or Theorem C on page 310 of [Pa] the restriction map $r\colon \mathrm{Diff}(X) \to \mathrm{Emb}(S, X)$ is a fibration. $\mathrm{Diff}(S)$ acts on $\mathrm{Emb}(S, X)$ by precomposition. Let $\mathrm{Im}(S, X)$ be the quotient space. Then by Théorème 3 on page 114 of [Ce] the quotient map $q\colon \mathrm{Emb}(S, X) \to \mathrm{Im}(S, X)$ is also a fibration.

Now let S be a closed, connected, orientable, incompressible surface in the irreducible, orientable 3-manifold M. Let T be another such surface in M which is disjoint from S. Suppose S_t is a path in $\mathrm{Im}(S, M)$ with $S_0 = S$ and $S_1 \cap T = \varnothing$ and s_t is a path in M with $s_t \in S_t$ for all $t \in [0, 1]$.

Theorem 3.2 (Laudenbach). *If $[s_t]$ is trivial in $\pi_1(M, M - T, s_0)$, then $[S_t]$ is trivial in $\pi_1(\mathrm{Im}(S, M), \mathrm{Im}(S, M - T))$.*

Proof. This is Théorème 7.3 on page 50 of [La]. □

Now suppose that f_t is a path in $\mathrm{Emb}(S, M)$ with f_0 the inclusion map, $f_1(S) = S$, and $f_1(s_0) = s_0$. Let $\alpha(t) = f_t(s_0)$.

Corollary 3.3. *If $[\alpha] \in (f_0)_*(\pi_1(S, s_0))$, then f_t is path-homotopic to f'_t such that $f'_t(S) \cap T = \varnothing$.*

Proof. Theorem 3.2 applied to $S_t = f_t(S)$ and $s_t = f_t(s_0)$ gives a homotopy $S_{t,u}$ with $S_{t,0} = S_t$ and with $S_{0,u}$, $S_{t,1}$, and $S_{1,u}$ all disjoint from T. Let $f_{t,u}$ be the lifting of $S_{t,u}$ to $\mathrm{Emb}(S, M)$ with $f_{t,0} = f_t$. The product of the paths $f_{0,t}$, $f_{t,1}$, and $\overline{f}_{1,t}$ is the required f'_t. (For a path β, $\overline{\beta}(t) = \beta(1-t)$.) $\square$

Now assume that T is a disjoint parallel copy of S in M and that f_t is a path in $\mathrm{Emb}(S, M)$ with f_0 the inclusion map and $f_1(S) = S$.

Lemma 3.4. *If $f_t(S) \cap T = \varnothing$, then f_t is path-homotopic to f'_t such that $f'_t(S) = S$.*

Proof. Let $C = S \times [0, 1]$ be embedded in M so that $S \times \{0\} = S$ and $S \times \{1\} = T$. By the isotopy extension theorem there is a path g_t in $\mathrm{Diff}(M)$ such that g_0 is the identity, the restriction of g_t to S is f_t, and the restriction of g_t to T is the identity. Then $g_1(C) = C$. It follows from Lemma 3.5 of [Wa] that the restriction of g_1 to C is isotopic rel ∂C to a level preserving diffeomorphism. Thus one may assume that $g_1(x, u) = (l_u(x), u)$ for some path l_u in $\mathrm{Diff}(S)$.

Now let k_u be a path in $\mathrm{Diff}(M)$ which pushes S across C to T through levels, i.e., k_0 is the identity and $k_u(x, 0) = (x, u)$ for $x \in S$ and $u \in [0, 1]$.

Let $h_{t,u} = k_u^{-1} \circ g_t \circ k_u$. This is a free homotopy of paths in $\mathrm{Diff}(M)$ with $h_{t,0} = g_t$ and $h_{0,u}$ the identity. One computes that

$$h_{t,1}(S) = (k_1^{-1} \circ g_t \circ k_1)(S) = (k_1^{-1} \circ g_t)(T) = k_1^{-1}(T) = S,$$

and

$$h_{1,u}(S) = (k_u^{-1} \circ g_1 \circ k_u)(S) = (k_u^{-1} \circ g_1)(S \times \{u\}) = k_u^{-1}(S \times \{u\}) = S.$$

Let $r_{t,u}$ be the restriction of $h_{t,u}$ to S. Then the product of the paths $r_{0,t}$, $r_{t,1}$, and $\overline{r}_{1,t}$ is the required f'_t. $\square$

In order to apply the previous two results in the proof of Theorem 3.1 it must be checked that the hypothesis of Corollary 3.3 holds.

Lemma 3.5. *Suppose S is not a fiber in a fibration of M over the circle and does not bound a submanifold of M diffeomorphic to a twisted I-bundle over a closed surface. If f_t is a path in* $\mathrm{Emb}(S, M)$ *such that f_0 is the inclusion map and $f_1(s_0) = s_0$, then $[\alpha] \in (f_0)_*(\pi_1(S, s_0))$.*

Proof. By the isotopy extension theorem there is a path g_t in $\mathrm{Diff}(M)$ such that g_0 is the identity and the restriction of g_t to S is f_t. Let $p\colon \widetilde{M} \to M$ be the covering space with $p_*(\pi_1(\widetilde{M}, \tilde{s}_0)) = (f_0)_*(\pi_1(S, s_0))$. Lift g_t to a path $\tilde{g}_t$ in $\mathrm{Diff}(\widetilde{M})$ with $\tilde{g}_0$ the identity. α lifts to a path $\tilde{\alpha}$ with $\tilde{\alpha}(t) = \tilde{g}_t(s_0)$.

If $\tilde{\alpha}$ is a loop, then one is done, so assume $\tilde{\alpha}(0) \neq \tilde{\alpha}(1)$. Then $\tilde{\alpha}(0)$ and $\tilde{\alpha}(1)$ lie in distinct components $\widetilde{S}_0$ and $\widetilde{S}_1$ of $p^{-1}(S)$. $\tilde{g}_1(\widetilde{S}_0) = \widetilde{S}_1$, so $\widetilde{S}_0$ and $\widetilde{S}_1$ are ambient isotopic and are therefore parallel (Corollary 5.5 of [Wa]); let $\widetilde{S}_0 \times [0, 1]$ be embedded in $\widetilde{M}$ with $\widetilde{S}_0 \times \{0\} = \widetilde{S}_0$ and $\widetilde{S}_0 \times \{1\} = \widetilde{S}_1$. By Corollary 3.2 of [Wa] all the components of $p^{-1}(S)$ meeting this product are isotopic to horizontal surfaces. Thus there is a component $\widetilde{S}$ such that $\widetilde{S}_0 \cup \widetilde{S}$ bounds a product I-bundle $\widetilde{C}$ whose interior misses $p^{-1}(S)$. $\widetilde{C}$ covers a component C of the manifold obtained by splitting M along S. C is either a product I-bundle over S or a twisted I-bundle over a closed surface double covered by S. (See e.g. Theorem 10.5 of [He].) It follows that either M is an S-bundle over the circle or S bounds a twisted I-bundle over a closed surface, a contradiction. $\square$

Proof of Theorem 3.1. Let h_t be a path in $\operatorname{Diff}(Y)$ with h_0 the identity, $h_1 = h$, and $h_t(S) \subseteq M$. Choose a basepoint $s_0 \in S$. By changing h by an isotopy which preserves S one may assume that $h(s_0) = s_0$.

Let f_t be the restriction of h_t to S, regarded as a path in $\operatorname{Emb}(S, M)$. By Lemma 3.5, $[\alpha] \in (f_0)_*(\pi_1(S, s_0))$. If T is any closed incompressible surface in M disjoint from S, then by Corollary 3.3 one may assume that $f_t(S) \cap T = \varnothing$. In particular, taking T to be a disjoint parallel copy of S in M, one may then assume by Lemma 3.4 that $f_t(S) = S$.

Now regard the path homotopies which make these changes in f_t as taking place in $\operatorname{Emb}(S, Y)$. They lift to path homotopies in $\operatorname{Diff}(Y)$ which change h_t so that $h_t(S) = S$. □

4. Exhaustion preserving isotopies

Theorem 4.1. *Let V be a good exhaustion for W. Let h be a diffeomorphism of W such that for some $N \geq 0$, $h(V_n) = V_n$ for all $n \geq N$. Assume h is isotopic to the identity. Then there exists $P > N$ and an isotopy h_t with h_0 the identity and $h_1 = h$ such that $h_t(V_n) = V_n$ for all $n \geq P$.*

In the course of the proof a slightly stronger result will be established which will be needed later. The statement of this result requires some more notation.

For $n \geq 0$ let C_n^+ be a collar on S_n in X_{n+1}. For $n \geq 1$ let C_n^- be a collar on S_n in X_n. Let C_0^- be a collar on S_0 in V_0. Let $C_n = C_n^+ \cup C_n^-$ be parametrized as $S_n \times [-1, 1]$, with $S_n \times \{0\} = S_n$, $S_n \times [0, 1] = C_n^+$, and $S_n \times [-1, 0] = C_n^-$. Let $S_n^\pm = S_n \times \{\pm 1\}$. Let $X_n^0 = \overline{X_n - (C_n^- \cup C_{n-1}^+)}$ for $n \geq 1$, and let $V_0^0 = \overline{V_0 - C_0^-}$. Let $\Sigma_m = \bigcup_{n \geq m}(S_n \cup S_n^+) \cup \bigcup_{n > m} S_n^-$ and $\Gamma_m = C_m^+ \cup \bigcup_{n > m} C_n$.

By the isotopy uniqueness of collars h can be isotoped $\operatorname{rel} \bigcup_{n \geq N} S_n$ so that $h(C_n) = C_n$ for $n > N$, $h(C_N^+) = C_N^+$, and on each of these collars $h(x, u) = (h(x), u)$ for $x \in S_n$, $n \geq N$. If h has this property it is called *standard on collars.*

Theorem 4.1 is then a consequence of the following.

Theorem 4.2. *Suppose h is a diffeomorphism of W such that $h(\Sigma_N) = \Sigma_N$, h is standard on collars, and h is isotopic to the identity. Then there is a $P > N$ and an isotopy h_t with h_0 the identity, $h_1 = h$, and $h_t(\Sigma_P) = \Sigma_P$.*

We shall begin with an arbitrary isotopy h_t of h to the identity and, after modifying it, rename it h_t.

Lemma 4.3. *There is a $P > N$ and an isotopy h_t' path-homotopic to h_t such that $h_t'(S_P) = S_P$.*

Proof. There is a $P > N$ such that $h_t^{-1}(V_N) \subseteq \operatorname{int}(V_P)$. Let $M = W - \operatorname{int}(V_N)$, $Y = W$, and $S = S_P$. Then $h_t(S_P) \subseteq \operatorname{int}(M)$. Since M is not a closed surface bundle over a circle and no component of $M - S_P$ has as closure a twisted I-bundle over a closed surface, the result follows from Theorem 3.1. □

It will now be assumed that $h_t(S_P) = S_P$.

Lemma 4.4. *Let T_0 and T_1 be distinct components of Σ_P such that T_0 is contained in the compact submanifold of W bounded by T_1. Let Y_i be the* *noncompact submanifold of W bounded by T_i. Let X be the compact submanifold of W bounded by $T_0 \cup T_1$. Suppose k_t is a path in* $\mathrm{Diff}(Y_0)$ *with k_0 the identity and $k_1(T_1) = T_1$. Then k_t is path-homotopic to k'_t such that $k'_t(T_1) = T_1$. In particular the restrictions of k_1 to X and to Y_1 are isotopic to the identity.*

Proof. Apply Theorem 3.1 with $M = Y = Y_0$, $S = T_1$, and k_t in place of h_t. □

It is an immediate consequence of Lemmas 4.3 and 4.4 that the restrictions of h to V_P, C_P^+, C_n^+, C_n^-, X_n^0, and X_n, $n > P$, are isotopic to the identity. Note, however, that since the restrictions of k_t and k'_t to T_0 in Lemma 4.4 need not agree, these isotopies need not fit together to give the isotopy promised in the theorem. Some care is required to do this.

Lemma 4.5. *There is an isotopy d_t such that $d_t(\Sigma_P) = \Sigma_P$, $d = d_0$ has support in Γ_P and is level preserving in Γ_P, and $d_1 = h$.*

Proof. By Lemma 4.4 the restriction of h to $W - \mathrm{int}(\Gamma_P)$ is isotopic to the identity. Define d_t on $W - \mathrm{int}(\Gamma_P)$ to be an arbitrary isotopy which accomplishes this.

Given some C_n^+, $n > P$, let r_t be the restriction of d_t to S_n^+. Using the product structure r_t determines a product isotopy on C_n^+ and so can be regarded as a path in $\mathrm{Diff}(S_n)$. Since h is standard on collars, r_1 is the restriction of h to S_n. Let $\overline{r}_t = r_{1-t}$. The contractibility of the loop $r_t \cdot \overline{r}_t$ gives a homotopy $c_{t,u}$ in $\mathrm{Diff}(S_n)$ such that $c_{t,0}$ and $c_{1,u}$ are each the restriction of h to S_n, $c_{t,1} = r_t$, and $c_{0,u} = \overline{r}_u$. Define d_t on C_n^+ by $d_t(x, u) = (c_{t,u}(x), u)$. Then $d_0(x, u) = (c_{0,u}(x), u) = (\overline{r}_u(x), u)$ and so is level preserving. $d_1(x, u) = (c_{1,u}(x), u) = (h(x), u) = h(x, u)$. $d_t(x, 1) = (c_{t,1}(x), 1) = (r_t(x), 1)$ and so agrees with d_t as previously defined on S_n^+.

Since $d_t(x, 0) = (c_{t,0}(x), 0) = (h(x), 0)$, one can give a similar construction of d_t on C_n^- which is compatible on ∂C_n^-.

For C_P^+ one lets $S_P \times \{\frac{1}{2}\}$ play the role of S_n and proceeds as above. □

To prove Theorem 4.2 it is now sufficient to show that the restriction of d to Γ_P is isotopic to the identity $\mathrm{rel}\, \partial\Gamma_P$. The proof divides into two cases according to whether V has genus greater than or equal to one.

Lemma 4.6. *If* $\mathrm{genus}(V) > 1$, *then the restriction of d to Γ_P is isotopic to the identity via a level preserving isotopy which is fixed on $\partial\Gamma_P$.*

Proof. Consider d on C_n, $n > P$. Since d is level preserving $d(x, u) = (l_u(x), u)$ for some path l_u in $\mathrm{Diff}(S_n)$ parametrized by $[-1, 1]$. Since d is the identity on ∂C_n, l_u is a loop in $\mathrm{Diff}(S_n)$ based at the identity. $\mathrm{genus}(S_n) > 1$ implies that $\pi_1(\mathrm{Diff}(S_n))$ is trivial [Ea-Ee] and so there is a homotopy $l_{u,t}$ with $l_{u,1} = l_u$, and $l_{u,0}$, $l_{0,t}$, and $l_{1,t}$ the identity of S_n. Define d'_t on C_n by $d'_t(x, u) = (l_{u,t}(x), u)$.

A similar argument can be used for C_P^+. □

In the genus one case d consists of *Dehn twists* about the tori $S_n, n \geq P$. Recall that if T is a torus and $C = T \times [-1, 1]$, where $T = T \times \{0\}$, then a *Dehn twist in C* is a level preserving diffeomorphism f of C which is the identity on ∂C. f is thus determined by an isotopy of the identity of T to itself. The motion of the basepoint under this isotopy gives a loop whose homotopy class in $\pi_1(T)$ is called the *trace* of f. Using the analogue of Laudenbach's results in one lower dimension one can show that f is classified up to level preserving isotopy rel ∂P by its trace. More generally it can be shown that every diffeomorphism of C which is the identity on ∂C is isotopic rel ∂C to a Dehn twist, and that such diffeomorphisms are classified up to isotopy rel ∂C by their traces. Thus the mapping class group of C rel ∂C is isomorphic to $\pi_1(T) \cong \mathbf{Z}^2$.

If C is incompressibly embedded in a 3-manifold M, then a diffeomorphism, also denoted f, of M with support in C can also be considered a *Dehn twist about* T *in* M. It may happen that f is isotopically trivial although its restriction to C is isotopically nontrivial in C rel ∂C. However, this occurs only if T cuts off a Seifert fibered space from M. This fact is well known, but a reference is difficult to find in the literature. Therefore we shall give a proof in the following special case which arises in the present context.

Lemma 4.7. *Let M be a compact, connected, orientable, irreducible 3-manifold having two incompressible boundary components T_0 and T_1. Suppose T is an incompressible torus in M which separates T_0 from T_1. Let C be a regular neighborhood of T supporting a Dehn twist f about T. Let Q_i be the component of $\overline{M - C}$ containing T_i. Suppose neither Q_i is a Seifert fibered space. If f is isotopic to the identity of M, then its restriction to C is isotopic* rel ∂C *to the identity of C.*

Proof. Let f_t be an isotopy with f_0 the identity of M and $f_1 = f$. Let $\gamma = \{p\} \times [-1, 1]$ for some $p \in T$. Let $\delta = f(\gamma)$. Then $[\delta\overline{\gamma}] \in \pi_1(C) \cong \pi_1(T)$ is the trace of f, so it suffices to show that it is trivial.

Choose basepoints $x_i \in T_i$ and paths ε_i in Q_i from x_i to $\gamma \cap Q_i$. Parametrize $\varepsilon_0\gamma\overline{\varepsilon}_1$ by u. Let $\alpha_i(t) = f_t(x_i)$. The application of f_t to $\varepsilon_0\gamma\overline{\varepsilon}_1$ gives a homotopy $\theta_{t,u}$ with $\theta_{0,u} = \varepsilon_0\gamma\overline{\varepsilon}_1$, $\theta_{1,u} = \varepsilon_0\delta\overline{\varepsilon}_1$, $\theta_{t,0} = \alpha_0$, and $\theta_{t,1} = \alpha_1$. Thus if each $[\alpha_i]$ is trivial in $\pi_1(T_i, x_i)$ it follows from the incompressibility of T that $[\delta\overline{\gamma}]$ is trivial in $\pi_1(T)$.

Suppose some $[\alpha_i]$ is nontrivial. Then it has infinite order in $\pi_1(Q_i)$. Let β be a loop in Q_i based at x_i and parametrized by s. The application of f_t to β gives a homotopy $\omega_{t,s}$ with $\omega_{t,0} = \omega_{t,1} = \alpha_i(t)$ and $\omega_{0,s} = \omega_{1,s} = \beta(s)$. Since T is incompressible this homotopy can be deformed into Q_i. Thus $[\alpha_i]$ and $[\beta_i]$ commute in $\pi_1(Q_i)$ and so this group has an infinite cyclic central subgroup and therefore Q_i is Seifert fibered (see Corollary 12.8 of [He]), a contradiction. $\square$

We now return to the genus one version of Lemma 4.6.

Lemma 4.8. *If* genus$(V) = 1$, *then the restriction of d to each C_n, $n > P$, is isotopic to the identity via a level preserving isotopy which is fixed on ∂C_n, and the restriction of d to $V_P \cup C_P^+$ is isotopic to the identity via an isotopy which is level preserving on C_P^+ and is fixed on S_P^+.*

Proof. Let $n > P$. Recall that d is isotopic to h via an isotopy which preserves Σ_P. By Lemma 4.3 h is isotopic to the identity by an isotopy which preserves S_P. Therefore d is isotopic to the identity by an isotopy which preserves S_P. By Lemma 4.4 the restriction of d to the noncompact manifold bounded by S_P is isotopic to the identity by an isotopy which preserves S_{n-1}^+. By a second application of Lemma 4.4 the restriction of d to the compact manifold X

bounded by $S^+_{n-1} \cup S^-_{n+1}$ is isotopic to the identity. Since $X = X^0_n \cup C_n \cup X^0_{n+1}$, X^0_n and X^0_{n+1} are not Seifert fibered, and the restriction of d to $X^0_n \cup X^0_{n+1}$ is the identity, it follows from Lemma 4.7 that the restriction of d to C_n is isotopic rel ∂C_n to the identity.

$V_P \cup C^+_P$ is a solid torus with a collar attached to its boundary. If the restriction of d to C^+_P is not isotopic to the identity rel ∂C^+_P, then one can perform an isotopy of the identity of V_P to itself which can be extended by a level preserving isotopy of the restriction of d to C^+_P to the identity rel S^+_P. □

Lemmas 4.6 and 4.8 complete the proof of the theorem.

For later reference we point out the following consequence of Lemma 4.8 and its method of proof.

Lemma 4.9. *Let V be a good genus one exhaustion of W. Suppose g is a diffeomorphism of W which consists of Dehn twists about the S_n.*

(1) *Let $P \geq 0$. Then g is isotopic to the identity via an isotopy which preserves each V_n, $n \geq P$, if and only if each of the Dehn twists about S_n, $n > P$, is isotopically trivial in $C_n \operatorname{rel} \partial C_n$.*

(2) *If g is isotopic to the identity, then there exists a $P \geq 0$ such that g is isotopic to the identity via an isotopy which preserves each V_n, $n \geq P$.* □

5. The structure of $\mathscr{H}(W; V)$

Recall that $\mathscr{F}_N(W; V) = \mathscr{H}(W; \bigcup_{n\geq N} S_n)$ and that $\mathscr{F}(W; V)$ is the direct limit of the system of restriction induced homomorphisms $f_{N,P}\colon \mathscr{F}_N(W; V) \to \mathscr{F}_P(W; V)$. The homomorphism $q_N\colon \mathscr{F}_N(W; V) \to \mathscr{H}(W)$ is obtained by allowing isotopies which need not respect V and has image $\mathscr{G}_N(W; V)$. The group $\mathscr{G}(W; V)$ is the nested union of the $\mathscr{G}_N(W; V)$ under the inclusion maps $g_{N,P}$. Let $f_N\colon \mathscr{F}_N(W; V) \to \mathscr{F}(W; V)$ and $g_N\colon \mathscr{G}_N(W; V) \to \mathscr{G}(W; V)$ be the maps into the direct limits. It is clear that $q_P \circ f_{N,P} = g_{N,P} \circ q_N$ and so there is a homomorphism $q\colon \mathscr{F}(W; V) \to \mathscr{G}(W; V)$ such that $q \circ f_N = g_N \circ q$.

Theorem 5.1. *$q\colon \mathscr{F}(W; V) \to \mathscr{G}(W; V)$ is an isomorphism.*

Proof. Let $\mathscr{K}_N(W; V) = \ker q_N$. For $P > N$ one has the following commutative diagram, where $k_{N,P}$ is the restriction of $f_{N,P}$.

$$\begin{array}{ccccccccc}
1 & \to & \mathscr{K}_N(W; V) & \to & \mathscr{F}_N(W; V) & \xrightarrow{q_N} & \mathscr{G}_N(W; V) & \to & 1 \\
 & & \downarrow k_{N,P} & & \downarrow f_{N,P} & & \downarrow g_{N,P} & & \\
1 & \to & \mathscr{K}_P(W; V) & \to & \mathscr{F}_P(W; V) & \xrightarrow{q_P} & \mathscr{G}_P(W; V) & \to & 1
\end{array}$$

Since the rows are exact, passing to the direct limit gives the following exact sequence.

$$1 \to \mathscr{K}(W; V) \to \mathscr{F}(W; V) \xrightarrow{q} \mathscr{G}(W; V) \to 1.$$

The next lemma implies that $\mathscr{K}(W; V)$ is trivial and thus that q is an isomorphism. □

Lemma 5.2. $\mathscr{K}_N(W; V) = \bigcup_{P>N} \ker f_{N,P}$.

Proof. Since $q_P \circ f_{N,P} = q_N$ it is clear that the first group contains the second. If $q_N([h])$ is trivial, then by Theorem 4.1 there is a $P > N$ such that $f_{N,P}([h])$ is trivial, and thus the second group contains the first. □

Let $\mathscr{D}'_N(W;V)$ be the subgroup of $\mathscr{F}_N(W;V)$ consisting of isotopy classes having representatives which are the identity outside a regular neighborhood of $\bigcup_{n>N} S_n$, i.e., those with support in $\bigcup_{n>N} C_n$. This group will be written additively.

Lemma 5.3. $f_{N,P}(\mathscr{D}'_N(W;V)) = \mathscr{D}'_P(W;V)$.

Proof. If genus$(V) > 1$, then by Lemma 4.6 $\mathscr{D}'_N(W;V) = 0$. If genus$(V) = 1$, then as in the proof of Lemma 4.8 the restriction of a representative diffeomorphism of $\mathscr{D}'_N(W;V)$ to $V_P \cup C_P^+$ is isotopic to the identity rel S_P^+ and thus the first group is contained in the second. The reverse inclusion is obvious. □

Let $\mathscr{D}'(W;V)$ be the direct limit of the $\mathscr{D}'_N(W;V)$ under the restrictions of the $f_{N,P}$.

Lemma 5.4.

(1) *If* genus$(V) > 1$, *then* $\mathscr{D}'(W;V) = 0$.

(2) *If* genus$(V) = 1$, *then* $\mathscr{D}'(W;V) \cong \prod_{n=0}^{\infty} \mathbf{Z}^2 / \bigoplus_{n=0}^{\infty} \mathbf{Z}^2$, *where the nth coordinate of* $\{(a_n, b_n)\}$ *corresponds to a Dehn twist about* S_n *with trace* (a_n, b_n).

Proof. (1) This follows from $\mathscr{D}'_N(W;V) = 0$.

(2) Lemma 4.9 implies that

$$\mathscr{D}'_N(W;V) \cong \prod_{n=N+1}^{\infty} \mathbf{Z}^2 = \prod_{n=N+1}^{P} \mathbf{Z}^2 \times \prod_{n=P+1}^{\infty} \mathbf{Z}^2$$

and that $f_{N,P}$ is projection onto the second of these factors. It is then easily checked that the direct limit of this system is isomorphic to $\prod_{n=0}^{\infty} \mathbf{Z}^2 / \bigoplus_{n=0}^{\infty} \mathbf{Z}^2$ and that the coordinates have the stated interpretation. □

Recall that $\overline{\mathscr{F}}_N(W;V)$ is the subgroup of $\mathscr{H}(V_N) \times \prod_{n=N+1}^{\infty} \mathscr{H}(X_n, S_n)$ consisting of those sequences $([h_n])$ such that the restrictions of h_n and h_{n+1} to S_n are isotopic for $n \geq N$. Restriction induces a homomorphism $r_N: \mathscr{F}_N(W;V) \to \overline{\mathscr{F}}_N(W;V)$.

For $P > N$ there is a homomorphism $\overline{f}_{N,P}: \overline{\mathscr{F}}_N(W;V) \to \overline{\mathscr{F}}_P(W;V)$ defined as follows. For $n > P$, $[h_n]$ remains the same. Choose any representatives $h_N, \ldots, h_P$ for $[h_N], \ldots, [h_P]$. These determine diffeomorphisms $h_N^0, \ldots, h_P^0$ of $V_N^0, \ldots, X_P^0$. The fact that the restrictions of h_n and h_{n+1} to S_n are isotopic implies that one can extend the h_n^0 to the regular neighborhoods $C_N, \ldots, C_{P-1}$, as well as to C_P^-, thus giving a diffeomorphism h'_P of V_P. Let $\overline{f}_{N,P}([h_N], [h_{N+1}], \ldots, [h_P], [h_{P+1}], \ldots) = ([h'_P], [h_{P+1}], \ldots)$.

Lemma 5.5. $\overline{f}_{N,P}$ *is well defined and* $r_P \circ f_{N,P} = \overline{f}_{N,P} \circ r_N$.

Proof. If genus$(V) > 1$, then a different choice of representatives would yield a diffeomorphism h''_P of V_P such that $h''_P \circ (h'_P)^{-1}$ is isotopic to a diffeomorphism which is the identity on $V_N^0 \cup X_{N+1}^0 \cup \cdots \cup X_P^0$. As in the proof of Lemma 4.6 the fact that $\pi_1(\mathrm{Diff}(S_n))$ is trivial enables one to continue to isotop this diffeomorphism to the identity, so that $[h'_P]$ is well defined.

If genus$(V) = 1$, then since V_P is a solid torus the isotopy class of h'_P is determined by that of its restriction to S_P, and so $[h'_P]$ is well defined.

The remainder of the lemma is easily checked. □

Thus there is a homomorphism $r: \mathscr{F}(W; V) \to \overline{\mathscr{F}}(W; V)$, where $\overline{\mathscr{F}}(W; V)$ is the direct limit of the system $\{\overline{\mathscr{F}}_N(W; V), \bar{f}_{N,P}\}$.

Lemma 5.6. $0 \to \mathscr{D}'_N(W; V) \to \mathscr{F}_N(W; V) \xrightarrow{r_N} \overline{\mathscr{F}}_N(W; V) \to 1$ *is exact.*

Proof. The definition of $\overline{\mathscr{F}}_N(W; V)$ implies that r_N is onto. Any representative of an element of $\mathscr{D}'_N(W; V)$ restricts to diffeomorphisms of V_N and X_n, $n > N$, which are supported in collars on the boundaries and are therefore isotopically trivial. If $r_N([h])$ is trivial, then the restrictions of h to V_N and to X_n, $n > N$, are isotopically trivial. h can be isotoped so as to be standard on collars and the identity on $V_N \cup C_N^+$. It can then be further isotoped as in the proof of Lemma 4.5 so that it has support in $\bigcup_{n>N} C_n$, and so $[h] \in \mathscr{D}'_N(W; V)$. □

Theorem 5.7.

(1) *If* genus$(V) > 1$, *then* $r: \mathscr{F}(W; V) \to \overline{\mathscr{F}}(W; V)$ *is an isomorphism.*
(2) *If* genus$(V) = 1$, *then there is an exact sequence*

$$0 \to \mathscr{D}(W; V) \to \mathscr{F}(W; V) \xrightarrow{r} \overline{\mathscr{F}}(W; V) \to 1,$$

where $\mathscr{D}(W; V) \cong \prod_{n=0}^{\infty} \mathbf{Z}^2 / \bigoplus_{n=0}^{\infty} \mathbf{Z}^2$, *and the nth coordinate of* $\{(a_n, b_n)\}$ *corresponds to a Dehn twist about* S_n *with trace* (a_n, b_n).

Proof. Passing to the direct limit gives an exact sequence

$$0 \to \mathscr{D}'(W; V) \to \mathscr{F}(W; V) \xrightarrow{r} \overline{\mathscr{F}}(W; V) \to 1.$$

Lemma 5.4 then completes the proof, with $\mathscr{D}(W; V) = \mathscr{D}'(W; V)$ in the genus one case. □

We now consider manifolds with periodic exhaustions.

Lemma 5.8. *Every shift is isotopically nontrivial.*

Proof. Suppose h is an isotopically trivial shift with shift constant s. By replacing h by h^{-1}, if necessary, we may assume $s > 0$. Let h_t be an isotopy with h_0 the identity and $h_1 = h$. Choose n such that $h_t^{-1}(V_0) \subseteq \operatorname{int}(V_n)$. Then $h_t(S_n) \subseteq W - \operatorname{int}(V_0)$. Restricting h_t to S_n gives a homotopy between the inclusion map of S_n into $W - \operatorname{int}(V_0)$ and an embedding of S_n into $W - \operatorname{int}(V_0)$ whose image is S_{n+s}. Disjoint, incompressible, homotopic closed surfaces in an irreducible 3-manifold are parallel [Wa]. It then follows from the fact that closed incompressible surfaces in a product I-bundle are isotopic to horizontal surfaces [Wa] that X_m is a product I-bundle for $n + 1 \leq m \leq s$, contradicting the fact that V is a good exhaustion. □

Now suppose that h is a minimal shift of V with shift constant σ and initial index N_0. Let $N \geq N_0$. Then it is easily checked that $h \cdot \operatorname{Diff}(W, \bigcup_{n \geq N} S_n) \cdot h^{-1} = \operatorname{Diff}(W, \bigcup_{n \geq N+\sigma} S_n)$ in $\operatorname{Diff}(W)$. Thus the map $g \mapsto h \circ g \circ h^{-1}$ induces homomorphisms $\psi_N: \mathscr{G}_N(W; V) \to \mathscr{G}_{N+\sigma}(W; V)$ and $\xi_N: \mathscr{F}_N(W; V) \to \mathscr{F}_{N+\sigma}(W; V)$. The proofs of the statements in the next lemma are straightforward.

Lemma 5.9.

(1) *ψ_N and ξ_N are isomorphisms.*
(2) $\psi_P \circ g_{N,P} = g_{N+\sigma, P+\sigma} \circ \psi_N$ *and* $\xi_P \circ f_{N,P} = f_{N+\sigma, P+\sigma} \circ \xi_N$.
(3) $\psi_N \circ q_N = q_{N+\sigma} \circ \xi_N$.
(4) *The $\overline{\psi}_N$ and ξ_N induce automorphisms ψ of $\mathscr{G}(W; V)$ and ξ of $\mathscr{F}(W; V)$ such that $q \circ \xi = \psi \circ q$.*
(5) *q induces an isomorphism*

$$\hat{q}: \mathscr{F}(W; V) \times_\xi \mathbf{Z} \to \mathscr{G}(W; V) \times_\psi \mathbf{Z},$$

which respects the semidirect product structure. □

h also induces a homomorphism $\overline{\xi}_N: \overline{\mathscr{F}}_N(W; V) \to \overline{F}_{N+\sigma}(W; V)$, given by

$$\begin{aligned}\overline{\xi}_N&([g_N], [g_{N+1}], \dots, [g_n], \dots) \\ &= ([h \circ g_N \circ h^{-1}], [h \circ g_{N+1} \circ h^{-1}], \dots, [h \circ g_n \circ h^{-1}], \dots).\end{aligned}$$

The following properties are easily verified.

Lemma 5.10. (1) *$\overline{\xi}_N$ is an isomorphism.*

(2) $\overline{\xi}_P \circ \overline{f}_{N,P} = \overline{f}_{N+\sigma, P+\sigma} \circ \overline{\xi}_N$.
(3) $\overline{\xi}_N \circ r_N = r_{N+\sigma} \circ \xi_N$.
(4) *The $\overline{\xi}_N$ induce an automorphism $\overline{\xi}$ of $\overline{\mathscr{F}}(W; V)$ such that $r \circ \xi = \overline{\xi} \circ r$.*
(5) *r induces an epimorphism*

$$\hat{r}: \mathscr{F}(W; V) \times_\xi \mathbf{Z} \to \overline{\mathscr{F}}(W; V) \times_{\overline{\xi}} \mathbf{Z},$$

which preserves the semidirect product structure.

Lemma 5.11. $\ker \hat{q} = \ker q = \mathscr{D}'(W; V)$.

Proof. This follows from Theorem 5.7 and Lemma 5.9(5). □

Lemma 5.12. *Suppose* $\text{genus}(V) = 1$. *Then ξ restricts to an automorphism of $\mathscr{D}(W; V)$ given by*

$$\xi(\{(a_0, b_0), (a_1, b_1), \dots\}) = \{\overbrace{(0,0), \dots, (0,0)}^{\sigma}, (a_0, b_0), (a_1, b_1), \dots\}.$$

Proof. Orient W. Then orient each S_n by an outward pointing normal. There is then an ordered, oriented meridian-longitude pair (m_n, l_n) on each S_n which is unique up to isotopy. (l_n bounds in $W - \text{int}(V_n)$.) This determines a choice of coordinates for the group of Dehn twists about S_n. Since, up to isotopy, h carries (m_n, l_n) to $(m_{n+\sigma}, l_{n+\sigma})$, ξ carries a Dehn twist about S_n to a Dehn twist about $S_{n+\sigma}$ having the same coordinates. Thus the effect of ξ on $\mathscr{D}(W; V)$ is to shift a representative sequence σ places to the right. Since we are working modulo direct sums the exact values of the first $N_0 + \sigma$ terms do not matter and so may be assigned as stated. □

In summary, these lemmas establish the following result.

Theorem 5.13. *Suppose V is periodic of period σ with minimal shift h. Then conjugation by h induces automorphisms ψ, ξ, and $\bar{\xi}$ of $\mathscr{G}(W; V)$, $\mathscr{F}(W; V)$, and $\overline{\mathscr{F}}(W; V)$, respectively, having the following properties.*

(1) *$\mathscr{H}(W; V) = \mathscr{G}(W; V) \times_\psi \mathbf{Z}$, with $\mathbf{Z}$ generated by $[h]$.*

(2) *$q: \mathscr{F}(W; V) \to \mathscr{G}(W; V)$ induces an isomorphism*

$$\hat{q}: \mathscr{F}(W; V) \times_\xi \mathbf{Z} \to \mathscr{G}(W; V) \times_\psi \mathbf{Z}$$

which preserves the semidirect product structure.

(3) *$r: \mathscr{F}(W; V) \to \overline{\mathscr{F}}(W; V)$ induces an epimorphism*

$$\hat{r}: \mathscr{F}(W; V) \times_\xi \mathbf{Z} \to \overline{\mathscr{F}}(W; V) \times_{\bar{\xi}} \mathbf{Z},$$

which preserves the semidirect product structure.

(i) *If* $\operatorname{genus}(V) > 1$, *then $\hat{r}$ is an isomorphism.*

(ii) *If* $\operatorname{genus}(V) = 1$, *then* $\ker \hat{r} = \mathscr{D}(W; V)$ *and ξ restricts to an automorphism of $\mathscr{D}(W; V)$ given by*

$$\xi(\{(a_0, b_0), (a_1, b_1), \ldots\}) = \{\overbrace{(0,0), \ldots, (0,0)}^{\sigma}, (a_0, b_0), (a_1, b_1), \ldots\}.$$

6. Incompressible surfaces in certain compact 3-manifolds

In this section we consider two compact 3-manifolds X and X'. Copies of these manifolds will appear as the manifolds X_n associated to exhaustions V of certain Whitehead manifolds. X will be used in §8 to construct a genus two example; X' will be used in §9 to construct a genus one example. The present section classifies certain kinds of incompressible surfaces in X and X', the results being stated in Lemmas 6.6 and 6.7, respectively. This information will be used in §7 to compute the mapping class groups $\mathscr{H}(X_n, S_n)$ and in §§8 and 9 to show that $\mathscr{H}(W) = \mathscr{H}(W; V)$.

X and X' will be assembled from cubes with handles which are glued along incompressible planar surfaces in their boundaries. The incompressible surfaces of interest will be analyzed by examining how they intersect the cubes with handles and the planar surfaces. Two incompressible surfaces which are under consideration will generally be assumed to be in general position and to have an intersection of minimal complexity in an isotopy class, where the complexity is the lexicographically ordered pair (number of arcs, number of simple closed curves). In particular, since all the 3-manifolds will be irreducible, it will be assumed that no simple closed curves of intersection bound disks on a surface.

Before building X and X' we will need the following general fact.

Lemma 6.1. *Suppose S is a connected, incompressible, boundary-compressible surface in the 3-manifold M. Let S' be a surface obtained by boundary-compressing S. If each component of S' is boundary-parallel in M, then S is boundary-parallel in M.*

Proof. Let D be a boundary-compressing disk for S. Lete $\alpha = D \cap S$ and $\beta = D \cap \partial M$. Let Z be a regular neighborhood of D in the manifold obtained by splitting M along S. Thus $Z \cap S$ is a regular neighborhood of α in S. Let D_0 and D_1 be the components of $\overline{\partial Z - (Z \cap (S \cup \partial M))}$. Then S' is obtained by replacing $Z \cap S$ by $D_0 \cup D_1$.

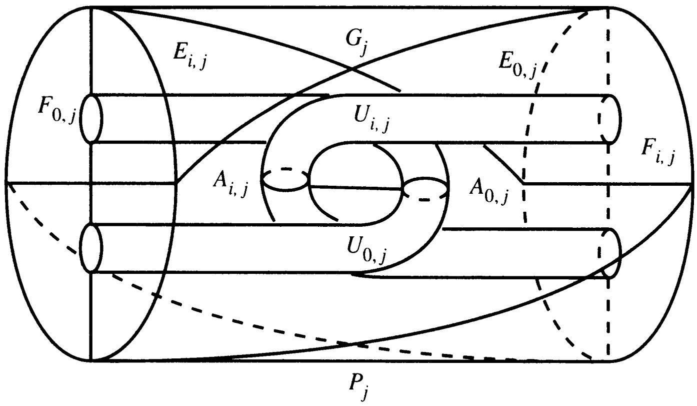

FIGURE 1

Suppose S' is connected. There is a 3-manifold Y in M homeomorphic to $S' \times [0, 1]$ with $S' = S' \times \{0\}$ and the remainder of ∂Y contained in ∂M. Z is either contained in Y or meets Y only in $D_0 \cup D_1$. The first case cannot occur since it would imply that S is compressible in M. The second case implies that S is boundary-parallel in M.

Suppose S' has two components S_0' and S_1'. Each S_i' is boundary-parallel via a product Y_i, as above. Then either one product is contained in the other, say Y_0 contained in Y_1, and Z is contained in Y_1, or the Y_i are disjoint and Z may be assumed to meet Y_i in D_i. The first case again implies that S is compressible and the second that it is boundary-parallel. □

Let P_j be the 3-manifold in Figure 1. It is a cube with two handles which admits the structure of a product I-bundle, as follows. Let S_j be a once-punctured torus. Set $P_j = S_j \times [0, 1]$ and $G_j = \partial S_j \times [0, 1]$. For $i = 0, 1$ let $S_{i,j} = S_j \times \{i\}$. Let $\zeta_{i,j}$ be a simple closed curve and $\eta_{i,j}$ a properly embedded arc in S_j, chosen so that for $i \neq k$, $\zeta_{i,j}$ meets each of $\zeta_{k,j}$ and $\eta_{k,j}$ transversely in a single point and is disjoint from $\eta_{i,j}$, while $\eta_{i,j}$ and $\eta_{k,j}$ are disjoint. Let $U_{i,j}$ be a regular neighborhood of $\zeta_{i,j}$ in $S_{i,j}$. Let $F_{i,j} = \overline{S_{i,j} - U_{i,j}}$ and $F_j = F_{0,j} \cup F_{1,j}$. Let $E_{i,j} = \eta_{i,j} \times [0, 1]$ and $A_{i,j} = \zeta_{i,j} \times [0, 1]$.

Lemma 6.2.

(1) *Every disk D in P_j which misses ∂G_j is boundary-parallel.*
(2) *Every disk D in P_j which meets F_j in a single arc is boundary-parallel.*
(3) *Every disk D in P_j which meets F_j in exactly two disjoint arcs is boundary-parallel.*
(4) *Every disk D in P_j which meets F_j in exactly three disjoint arcs is either boundary-parallel or is isotopic to some $E_{i,j}$ by an isotopy which preserves F_j.*
(5) *Every incompressible annulus A in P_j which misses ∂F_j is boundary-parallel.*
(6) *Every incompressible annulus A in P_j such that $A \cap F_j$ is a single arc is either boundary-parallel or is isotopic to some $A_{i,j}$ by an isotopy which preserves F_j.*

(7) *Every incompressible disk with two holes K in P_j which misses ∂F_j is boundary-parallel.*

(8) *Every incompressible once-punctured torus L in P_j which misses ∂G_j is boundary-parallel.*

Proof. (1) and (2) follow from Lemma 4.7 of [My1]. (3) follows from Lemma 4.9 of [My1].

(4) If ∂D misses, say, $F_{1,j}$, then D can be isotoped so that ∂D lies in $S_{0,j}$ and so D is boundary-parallel by (1). Hence we may assume that $D \cap F_{0,j}$ consists of two arcs α_0', α_0'' and $D \cap F_{1,j}$ of one arc α_1. Let β_0 be the arc in ∂D joining α_0' and α_0'' and let β_1', β_1'' be the arcs in ∂D joining α_1 to α_0', α_0'', respectively. We may assume that none of these arcs are boundary-parallel. It follows that β_0 is a spanning arc of $U_{0,j}$ and α_1 is isotopic to $\eta_{1,j} \times \{1\}$ in $F_{1,j}$. The existence of D implies that $\alpha_0' \cup \beta_0 \cup \alpha_0''$ is isotopic to $\eta_{1,j} \times \{0\}$ in $S_{0,j}$. This isotopy can be chosen so that it preserves $U_{0,j}$ and hence $F_{0,j}$. Thus one may assume that $\beta_0 = \eta_{1,j} \cap U_{0,j}$ and $\alpha_0' \cup \alpha_0'' = \eta_{1,j} \cap F_{0,j}$. One may further assume that β_1' and β_1'' are product arcs in G_j. By Lemma 3.4 of [Wa] there is an isotopy of P_j fixed on $S_{0,j} \cup G_j$ which carries D to $\eta_{1,j} \times [0,1] = E_{1,j}$. Since α_1 and $\eta_{1,j} \times \{1\}$ are isotopic in $F_{1,j}$ it follows from the proof of this lemma that the isotopy can be chosen so as to preserve $F_{1,j}$ as well as $F_{0,j}$.

(5) This follows from Lemma 4.8 of [My1].

(6) Let $\alpha = A \cap F_j$. We may assume that α is a non-boundary-parallel arc in $F_{0,j}$ and that the arc component β of $\overline{\partial A - \alpha}$ is a spanning arc of $U_{0,j}$. For homological reasons the simple closed curve component of $\overline{\partial A - \alpha}$ cannot lie on $U_{0,j}$ or G_j and so must lie on $U_{1,j}$. γ must therefore be isotopic to $\zeta_{1,j} \times \{1\}$ in $U_{1,j}$. It follows that $\alpha \cup \beta$ must be isotopic to $\zeta_{1,j} \times \{0\}$ in $S_{0,j}$. This isotopy can be chosen so as to preserve $U_{0,j}$ and hence $F_{0,j}$. Thus one may assume that $\alpha = (\zeta_{1,j} \times \{0\}) \cap F_{0,j}$ and $\beta = (\zeta_{1,j} \times \{0\}) \cap U_{0,j}$. By Lemma 3.4 of [Wa] there is an isotopy of P_j fixed on $S_{0,j} \cup G_j$ which carries A to $\zeta_{1,j} \times [0,1] = A_{1,j}$. Since γ is isotopic to $\zeta_{1,j} \times \{1\}$ in $U_{1,j}$ it follows from the proof of this lemma that the isotopy can be chosen so as to preserve $U_{1,j}$ and hence $F_{1,j}$.

(7) If some component of ∂K is isotopic in ∂P_j to some $\zeta_{i,j} \times \{i\}$, then for homological reasons a second component is isotopic to $\zeta_{i,j} \times \{i\}$ and the third component is isotopic to $\partial S_{i,j}$. It follows that K can be isotoped so that ∂K lies in $F_{i,j}$. If none of the components of ∂K are isotopic in ∂P_j to some $\zeta_{i,j} \times \{i\}$, then again for homological reasons they are all isotopic to $\partial S_{i,j}$ for some i and so K can be isotoped so that ∂K lies in $F_{i,j}$. It then follows from Corollary 3.2 of [Wa] that K is boundary-parallel.

(8) This follows from Corollary 3.2 of [Wa]. □

Lemma 6.3. *Suppose A is an incompressible annulus in P_j which meets F_j in exactly two arcs α_0 and α_1. Assume that they lie in the same component γ of ∂A and let β_0 and β_1 be the components of $\overline{\gamma - (\alpha_0 \cup \alpha_1)}$. Then some α_k or some β_k is boundary-parallel.*

Proof. Suppose α_0 and α_1 lie in, say, $F_{0,j}$. Then if some β_k meets G_j it must be boundary-parallel in G_j. So assume γ misses G_j. Then if no α_k is boundary-parallel in $F_{0,j}$ and no β_k is boundary-parallel in $U_{0,j}$, γ is homol-

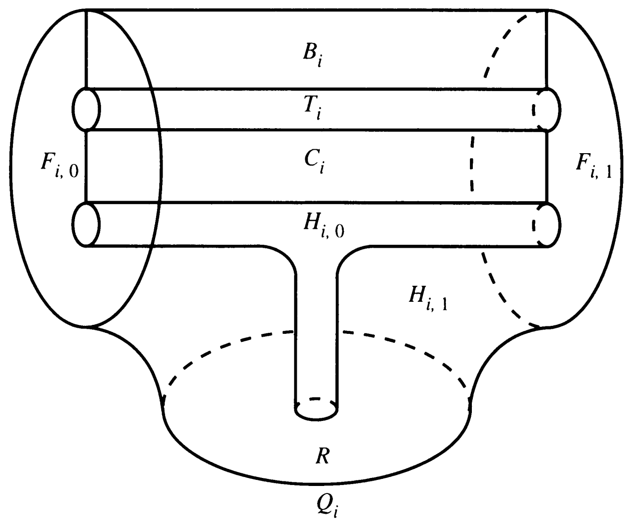

FIGURE 2

ogous in P_j or $\pm 2\zeta_{1,j}$. But the other component of ∂A must be homologous to $\pm\zeta_{0,j}$ to $\pm\zeta_{1,j}$ or be homologically trivial, so this cannot occur.

Suppose α_i lies in $F_{i,j}$. Then the β_k must be spanning arcs in G_j, and α_i must miss $U_{i,j}$. Assuming the α_i are not boundary-parallel, they must separate the components of $\partial U_{i,j}$. It follows that γ must be homologous in P_j to $\zeta_{0,j}+\zeta_{1,j}$ (properly oriented). But since the other component of ∂A is homologous to $\pm\zeta_{0,j}$ or $\pm\zeta_{1,j}$, or is homologically trivial, this is impossible. □

Let Q_i be the 3-manifold in Figure 2. It is a cube with three handles. ∂Q_i consists of two annuli T_i and R and four disks with two holes $F_{i,0}$, $F_{i,1}$, $H_{i,0}$, and $H_{i,1}$. The union of the boundary components of these surfaces is denoted J_i. B_i and C_i are disks embedded in Q_i as shown.

Lemma 6.4.

(1) *Each component of $\partial Q_i - J_i$ is incompressible in Q_i. There is no incompressible once-punctured torus L in Q_i which misses J_i.*
(2) *Every disk D in Q_i which meets J_i in at most two points is boundary-parallel.*
(3) *Every disk D in Q_i which meets R in at most three arcs and is otherwise disjoint from J_i is boundary-parallel.*
(4) *Every incompressible annulus A in Q_i such that $A\cap R$ is either empty or a single arc and A is otherwise disjoint from J_i is boundary-parallel.*
(5) *Every incompressible disk with two holes K in Q_i which misses J_i is boundary-parallel.*
(6) *Every incompressible disk with three holes M in Q_i which misses J_i is boundary-parallel.*

Proof. (1) The inclusion induced homomorphisms from the first homology groups of each component of $\partial Q_i - J_i$ into Q_i are one-to-one. This implies that each of these components is incompressible in Q_i (since they are all planar surfaces) and that there are no incompressible once-punctured tori in Q_i whose boundaries miss J_i.

(2) If D misses J_i, then the result follows from (1) and the irreducibility of Q_i. If $D \cap J_i \neq \varnothing$ it must consist of two points lying on the same component of J_i. Let α and β be the two arcs into which the intersection divides ∂D. If one of these lies in T_i or R, then it is boundary-parallel in this surface and so D can be isotoped in Q_i to a disk which misses J_i, and the result follows. If neither α nor β lie in T_i or R, then α lies in, say, $F_{i,j}$ and β in $H_{i,k}$. If either of these is boundary-parallel then the result follows as above. If neither is boundary-parallel then D can be isotoped, preserving J, so that D is transverse to $B_i \cup C_i$ and meets $(B_i \cup C_i) \cap F_{i,j}$ in a single point, this point lying in B_i if β lies in $H_{i,0}$ and in C_i if β lies in $H_{i,1}$. But this is impossible since $B_i \cap H_{i,0}$ and $C_i \cap H_{i,1}$ are empty.

(3) By (1) we may assume that $D \cap R \neq \varnothing$. We may further assume that this intersection consists of spanning arcs of R. Since R separates ∂Q_i this implies that it consists of exactly two arcs. We may also assume that no component of $D \cap H_{i,k}$ is boundary-parallel. It follows that D can be isotoped, preserving J_i, so that $D \cap \partial B_i$ is a single point, but this is impossible since B_i does not meet $R \cup H_{i,0}$.

(4) First assume A misses R. Isotop A so that it misses F_j and J_i.

Suppose $A \cap B_i$ contains an arc α which is boundary-parallel in A. Assume α is outermost on A. Since A misses J_i the endpoints of α must lie in the same component of $\partial Q_i - J_i$. There are then disks D_0 in A and D_1 in B_i whose intersection is α and whose union is a disk D with ∂D in $\partial Q_i - J_i$. The incompressibility of $\partial Q_i - J_i$ and irreducibility of Q_i then imply that D is boundary-parallel and so the intersection can be simplified by an isotopy. We may thus assume that the intersection consists of at most spanning arcs of A.

If α is an intersection arc whose endpoints lie in the same component of $\partial Q_i - J_i$, then there is a boundary-compressing disk D for A such that ∂D lies in $A \cup (\partial Q_i - J)$. It then follows from incompressibility and irreducibility that A is boundary-parallel.

If α is an intersection arc whose endpoints lie on different components of $\partial Q_i - J_i$, then A has one boundary component on T_i and the other on $H_{i,1}$, but this is homologically impossible.

One may thus assume that A misses B_i. Let Q_i' be the manifold obtained by splitting Q_i along B_i. It is homeomorphic to $H_{i,0} \times [0, 1]$ with $H_{i,0}$ identified with $H_{i,0} \times \{0\}$. One may assume that ∂A lies in $H_{i,0}$. It then follows from Corollary 3.2 of [Wa] that A is parallel to an annulus in $H_{i,0}$.

Now suppose A meets R in a single arc. Then it must be boundary-parallel in R and so one can reduce to the case above.

(5) Isotop K so that it misses $F_{i,j}$ and R.

If $K \cap B_i$ contains an arc which is boundary-parallel in K, then since K misses J_i the endpoints of the arc lie in the same component of $\partial Q_i - J_i$ and so by the usual arguments the arc can be removed by an isotopy which preserves J_i. So assume there are no such arcs.

Suppose α is an intersection arc which does not separate K.

If the endpoints of α lie in the same component of $\partial Q_i - J_i$, then there is a boundary-compressing disk D for K such that $D \cap \partial Q_i$ misses J_i. The boundary-compression yields an incompressible annulus A in Q_i which misses J_i. By (4) A is boundary-parallel and so by Lemma 6.1 so is K.

We may therefore assume that no component of $K \cap B_i$ has its endpoints in the same component of $\partial Q_i - J_i$. Thus one component τ of ∂K must lie on T_i and a second σ on $H_{i,1}$. For homological reasons the third θ must lie on $H_{i,0}$. Moreover, for some j, σ is parallel in $H_{i,1}$ to $H_{i,1} \cap F_{i,j}$ and θ is parallel in $H_{i,0}$ to $H_{i,0} \cap F_{i,j}$. K thus cuts off a submanifold N of Q_i such that ∂N is the union of $F_{i,j}$, K, and annuli in T_i, $H_{i,0}$, and $H_{i,1}$. K can be isotoped so that $N \cap B_i$ is a disk B' and $N \cap C_i$ is a disk C'. Splitting N along $B' \cup C'$ gives a manifold N' with $\partial N'$ a 2-sphere. Thus N' is a 3-cell and N is homeomorphic to $K \times [0, 1]$ with $K \times \{0\} = K$ and $K \times \{1\} = F_{i,j}$; so K is boundary-parallel.

Suppose every intersection arc separates K and so has its endpoints in the same component of ∂K. Since K misses J_i there is a boundary-compressing disk D for K such that $D \cap J_i = \varnothing$. Boundary-compression yields two incompressible annuli which miss J_i and are therefore boundary-parallel in Q_i. It follows that K is also boundary-parallel.

Finally, if K misses B_i, then K can be isotoped so that ∂K lies in $H_{i,0}$. The manifold Q_i' obtained by splitting along B_i is homeomorphic to $H_{i,0} \times [0, 1]$ with $H_{i,0} = H_{i,0} \times \{0\}$. The result now follows from Corollary 3.2 of [Wa].

(6) Isotop M so that ∂M lies in $H_{i,0} \cup H_{i,1} \cup T_i$. If M misses B_i, then it lies in Q_i' and can be isotoped in Q_i so that ∂M lies in $H_{i,0}$. By Corollary 3.2 of [Wa] M is parallel to a surface in $H_{i,0}$, but this is impossible since $H_{i,0}$ is a disk with two holes. Thus $M \cap B_i \neq \varnothing$. Intersection arcs which are boundary-parallel in M can be removed as usual.

Suppose $M \cap T_i \neq \varnothing$. Then $M \cap C_i \neq \varnothing$. If there is a component of $M \cap (B_i \cup C_i)$ which joins two components of $M \cap T_i$, then there is a boundary-compressing disk D in $B_i \cup C_i$ such that the surface M' resulting from the boundary-compression has a boundary component which bounds a disk on T_i; this implies that M' and hence M is compressible in Q_i, a contradiction. Thus there are components γ of $M \cap T_i$ and δ of $M \cap H_{i,0}$ which are joined by an arc β of $M \cap C_i$; γ is joined to a component ε of $M \cap H_{i,1}$ by an arc α. Moreover, for some j, δ is parallel in $H_{i,0}$ to $F_{i,j} \cap H_{i,0}$ and one may assume that the corresponding product I-bundle meets M only in δ. This implies that there are boundary-compressing disks D_α and D_β for M with $D_\alpha \cap M = \alpha$, $D_\beta \cap M = \beta$, $D_\alpha \subseteq B_i$, and $D_\beta \subseteq C_i$; D_α and D_β meet $F_{i,j}$ in the arcs $B_i \cap F_{i,j}$ and $C_i \cap F_{i,j}$. The result of boundary-compressing M along these disks is an annulus A which can be isotoped so that it misses J_i. As before this implies that M is boundary-parallel.

Thus we may assume $M \cap T_i = \varnothing$. Then there is a disk D in B_i which is a boundary-compressing disk for M such that ∂D consists of an arc in M and an arc in $H_{i,1}$. The result M' of the boundary-compression is either an annulus and a disk with two holes or a disk with two holes, depending on whether or not the arc separates M. In either case $\partial M'$ misses J_i and so M is boundary-parallel. □

Now let $Q = Q_0 \cup_R Q_1$ as in Figure 3. It is a cube with five handles. Let $J = (J_0 \cup J_1) - \partial R$ and $H_k = H_{0,k} \cup H_{1,k}$. Let $K_{i,k}$ be the disk with two holes obtained from $R \cup H_{i,k}$ by pushing it slightly into Q so that it is properly embedded. In the same way let M_i be the disk with three holes obtained from $H_{i,0} \cup R \cup H_{i,1}$ and let M_i' be the disk with three holes obtained from $H_{i,0} \cup R \cup H_{m,1}$, $i \neq m$. Let F be the union of all the $F_{i,j}$.

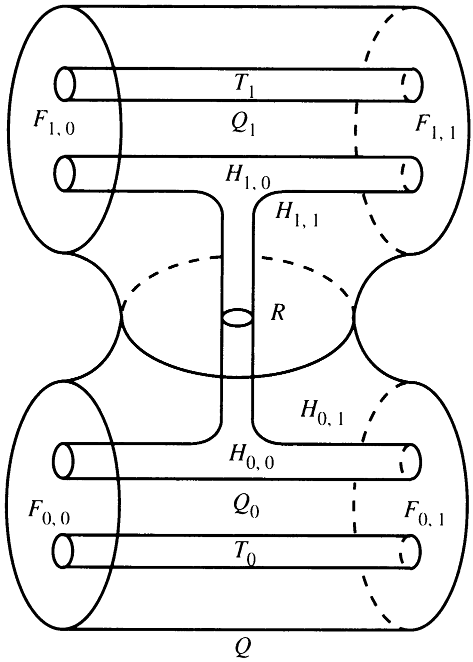

FIGURE 3

Lemma 6.5.

(1) *R and each component of $\partial Q - J$ are incompressible in Q.*
(2) *Every disk D in Q which meets J in at most two points is boundary-parallel.*
(3) *Every disk D in Q such that $D \cap F_{i,j}$ consists of two non-boundary-parallel arcs whose endpoints lie in H_1 and D is otherwise disjoint from J is boundary-parallel.*
(4) *Every incompressible annulus A in Q which misses J is either boundary-parallel or isotopic to R by an isotopy which is fixed on F.*
(5) *Every incompressible disk with two holes K in Q which misses J is either boundary-parallel or isotopic to some $K_{i,k}$.*
(6) *Every incompressible disk with three holes M in Q such that ∂M lies in F is either boundary-parallel or isotopic to some M_i or M_i'.*
(7) *There are no incompressible once-punctured tori L in Q which miss J.*

Proof. (1) This follows from the incompressibility of R, T_i, $F_{i,j}$, and $H_{i,k}$ in Q_i along with the fact that every disk in Q_i which meets J_i in exactly two points is boundary-parallel.

(2) Consider $D \cap R$. An outermost disk D' on D either misses J or meets J in two points. In either case D' is boundary-parallel in some Q_i. Therefore D can be isotoped to reduce the number of intersections with R. Eventually $D \cap R = \varnothing$ and the result follows from the previous lemma.

(3) Isotop D so that each component of $D \cap F_{i,j}$ meets C_i in a single point. Then these points must be joined by an arc α of $D \cap C_i$. Minimality implies

that there are no other intersections. α is parallel in C_i across a disk E to an arc β in $F_{i,j}$. The result of surgery on D along E consists of two disks which can be isotoped so that their boundaries are in H_1. The incompressibility of H_1 implies that they are boundary-parallel in Q and hence that D is boundary-parallel in Q.

(4) Consider $A \cap R$. Any outermost disks on A are boundary-parallel in some Q_i and so the intersection can be reduced by an isotopy.

Suppose $A \cap R$ contains a spanning arc α of A. If α is boundary-parallel on R, then there is a boundary-compressing disk D for A such that $D \cap J = \varnothing$; it follows that A is boundary-parallel in Q. Thus we may assume all the intersection arcs span R. Then there is a component E of $A \cap Q_i$ which is a disk meeting R in two arcs in its boundary. E is parallel in Q_i to a disk E' in ∂Q_i. E' must meet each of R, $H_{i,0}$, and $H_{i,1}$ in a single disk. Therefore the number of intersection curves can be reduced by an isotopy.

Suppose all the componenets of $A \cap R$ are simple closed curves. Let α be an outermost such curve on A. α and a component of ∂A bound a subannulus A' of A which lies in some Q_i. Since A' is boundary-parallel in Q_i, the intersection can be reduced by an isotopy.

Therefore $A \cap R = \varnothing$ and so A lies in some Q_i. Thus A is boundary-parallel in Q_i and hence is either boundary-parallel in Q or isotopic to R in Q.

(5) Consider $K \cap R$.

If $K \cap R = \varnothing$, then K is parallel in some Q_i to a surface K' in ∂Q_i. If R does not lie in K', then K is boundary-parallel in Q. If R does lie in K', then K is parallel to some $K_{i,k}$ in Q.

If $K \cap R$ contains a simple closed curve, then it is boundary-parallel in K. An outermost annulus on K is then boundary-parallel in some Q_i and so the intersection can be reduced by an isotopy. So we may assume every intersection curve is an arc.

Suppose α is an arc of $K \cap R$ which is boundary-parallel in K. Assuming α cuts off an outermost disk D from K, then D lies in some Q_i, α is boundary-parallel in R and D is boundary-parallel in Q_i. Thus α can be removed by an isotopy of K in Q. So we may assume there are no such arcs.

Suppose $K \cap Q_i$ contains a disk D which meets R in two disjoint arcs and is otherwise disjoint from J_i. Then D is parallel in Q_i to a disk D' in ∂Q_i. $D' \cap \partial R$ consists of two arcs which divide D' into two outermost disks D'_0 and D'_1 and another disk D'_2. If D'_2 lies in R, then there is an isotopy of K in Q which takes D to D'_2 and then off R, thus reducing the intersection. So assume D'_2 does not lie in R, while D'_0 and D'_1 do lie in R. Let Z be the 3-cell bounded by $D \cup D'$. Then a disk in Z separating D'_0 from D'_1 is a boundary-compressing disk for K in Q, and Z can be regarded as the regular neighborhood of this disk used in the boundary-compression. Suppose the result of the boundary-compression is an annulus A. If A is boundary-parallel in Q, then it follows that K is boundary-parallel in Q. If A is isotopic to R, then K is isotopic to a surface which boundary-compresses to R via an isotopy fixed on F. The only such surfaces are isotopic to the $K_{i,k}$. Suppose the result of the boundary-compression consists of two annuli A_0 and A_1. If they are both boundary-parallel in Q, then K is boundary-parallel in Q. If, say, A_0 is isotopic to R and A_1 is boundary-parallel in Q, then K is isotopic to a surface which boundary-compresses to R and misses J and so is isotopic to one of the $K_{i,k}$. If A_0 and A_1 are both isotopic to R, then K is isotopic to a surface which misses J and boundary-compresses to two parallel copies of R; it follows that a component of ∂K bounds a disk in ∂Q, contradicting the incompressibility of K in Q.

Suppose no component of $K \cap Q_i$ is a disk which meets R in two arcs. Then $K \cap R$ consists either of a single arc separating K into two annuli A_0 and A_1 or consists of three arcs separating K into two disks D_0 and D_1. In the first case A_0 is parallel in Q_0 to an annulus A_0' in ∂Q_0. $A_0 \cap R$ is an arc which cuts off a disk from R. The remainder of R is contained in A_0'. It follows that K can be isotoped so that K misses R and the result follows. In the second case D_0 is parallel in Q_0 to a disk D_0' in ∂Q_0. If ∂R cuts off a disk E from D_0', then one can isotop K in a regular neighborhood of E to obtain a surface K' which meets R in two arcs which divide K' into an annulus and a disk; the result then follows as above. If ∂R does not cut off a disk from D_0', then ∂R must cut off a disk E from the closure of the complement of D_0' in some $H_{0,k}$. One then performs an isotopy as before to complete the proof.

(6) Consider $M \cap R$.

If $M \cap R = \varnothing$, then M lies in some Q_i and is parallel in Q_i to a surface M' in ∂Q_i. If M' does not contain R, then M is boundary-parallel in Q. If M' contains R, then M' is isotopic to $H_{i,0} \cup R \cup H_{i,1}$, and it follows that M is isotopic to M_i.

Suppose $M \cap R \neq \varnothing$. If A is an annulus component of $M \cap Q_i$, then A must be boundary-parallel in Q_i. Since no component of ∂R is isotopic to a component of $\partial F_{i,j}$, A does not meet ∂M and must be parallel to an annulus in R; it follows that the intersection can be simplified and so we may assume there are no such annuli. It follows that $M \cap Q_i$ is a disk with three holes K_i which is parallel in Q_i to a surface K_i' in ∂Q_i. If each K_i' is isotopic to $H_{i,0}$ or each K_i' is isotopic to $H_{i,1}$, then M is boundary-parallel in Q. If K_i' is isotopic to, say, $H_{i,0}$ and the other K_n' is isotopic to $H_{n,1}$, then M is isotopic to M_i'.

(7) This follows from the facts that the inclusion induced homomorphisms from the first homology groups of the components of $\partial Q - J$ into Q are one-to-one and all these components are planar. □

Let $X = P_0 \cup Q \cup P_1$ as in Figure 4. Let $\partial_+ X = G_0 \cup H_1 \cup G_1$ and $\partial_- X = U_{0,0} \cup U_{1,0} \cup T_0 \cup H_0 \cup T_1 \cup U_{0,1} \cup U_{1,1}$. Let L_i be the once-punctured torus obtained from $R \cup H_{i,0} \cup U_{i,0} \cup U_{i,1}$ by pushing it slightly into X so that it is properly embedded. Let L_j' be the once-punctured torus obtained in a similar fashion from $F_{0,j} \cup U_{0,j}$.

Lemma 6.6.

(1) *X is irreducible and boundary-irreducible; F is incompressible and boundary-incompressible in X.*

(2) *Every incompressible annulus A in X is either boundary-parallel or isotopic to R.*

(3) *There are no incompressible tori in X.*

(4) *Every incompressible, boundary-incompressible disk with two holes K in X such that ∂K has one component on $\partial_+ X$ and two components on $\partial_- X$ is isotopic to a component of F.*

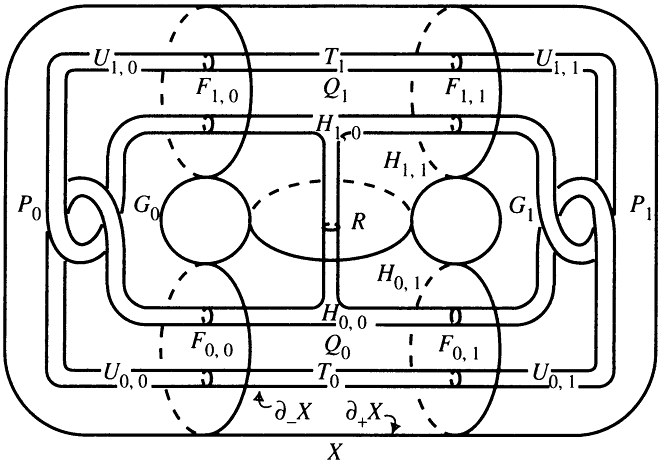

FIGURE 4

(5) *Every incompressible once-punctured torus* L *in* X *such that* ∂L *is contained in* $\partial_+ X$ *is either boundary-parallel or isotopic to an* L_i *or an* L'_j.

(6) *Every closed, orientable, incompressible, genus two surface* S *in* X *is boundary-parallel.*

Proof. (1) This follows from Lemmas 6.2 and 6.5 and Lemma 3.1 of [My1].

(2) If A misses F, then A lies in Q or some P_j. If A lies in Q, then A is isotopic to R or is boundary-parallel in Q and, since no component of F is an annulus, A is boundary-parallel in X. If A lies in P_j, then A is boundary-parallel in P_j and, since no component of F is an annulus, A is boundary-parallel in X.

If $A \cap F$ contains an arc which is boundary-parallel in A, then an outermost disk on A will be boundary-parallel in Q or some P_j, and the intersection can be reduced by an isotopy.

If $A \cap F$ consists of simple closed curves, then there is a curve α which together with a component β of ∂A bounds an outermost annulus A' in A. If A' lies in some P_j, then it is boundary-parallel in P_j and so α can be removed by an isotopy. If A' lies in Q, then it is isotopic to R or is boundary-parallel in Q. The former cannot happen since no component of ∂F is isotopic to a component of ∂R; therefore A' is boundary-parallel in Q and it follows that α can be removed by an isotopy.

If $A \cap F$ consists of spanning arcs of A, then there is a disk component D of $A \cap (P_0 \cup P_1)$ which meets F in exactly two arcs. D is parallel in some P_j to a disk D' in ∂P_j. If $D' \cap F$ consists of two disks, then A is boundary-compressible in X and so is boundary-parallel in X. If $D' \cap F$ consists of a single disk, then there is an isotopy of A which removes $D \cap F$ from $A \cap F$.

(3) $T \cap F \neq \varnothing$ since Q and the P_j are cubes with handles. But every component of $T \cap P_j$ is an annulus A which is boundary-parallel in P_j; the annulus in ∂P_j to which A is parallel must lie in F and so the intersection can be reduced by an isotopy.

(4) Suppose $K \cap F = \varnothing$. If K lies in P_j, then K is parallel in P_j to a surface K' in ∂P_j. K' must contain a component $F_{i,j}$ of F and so K is isotopic to $F_{i,j}$ in X. If K lies in Q, then either K is parallel in Q to a surface K' in ∂Q or K is isotopic to some $K_{i,k}$. In the first case the boundary-incompressibility of K in X implies that K' must contain a component $F_{i,j}$ of F and so K is isotopic to $F_{i,j}$ in X. The second case cannot occur because in order for the components of ∂K to be distributed as required one must have $k = 0$. But then $K_{i,0}$ admits a boundary-compressing disk which misses the interior of F and so K is boundary-compressible in X.

Since F and K are both boundary-incompressible in X we may assume that no arc component of $F \cap K$ is boundary-parallel in F or K.

Suppose $K \cap F$ contains a simple closed curve α. Then α is parallel in K to a component β of ∂K. We may assume that the annulus A in K bounded by $\alpha \cup \beta$ is outermost on K. If A lies in P_j, then A is boundary-parallel in P_j and so α can be removed by an isotopy. If A lies in Q, then either A is boundary-parallel in Q, and so α can be removed by an isotopy, or A is isotopic in Q to R, but this is impossible since no component of ∂F is isotopic to a component of ∂R.

Suppose some component of $K \cap (P_0 \cup P_1)$ is a disk D which meets F in exactly two arcs. Then D is parallel in P_j to a disk D' in ∂P_j. D' cannot meet F in two disks because then K would be boundary-compressible in X. Thus D' meets F in one disk and it follows that $D \cap F$ can be removed from $K \cap F$ by an isotopy of K in X. Therefore we may assume there are no such components.

Suppose some component of $K \cap (P_0 \cup P_1)$ is a disk D which meets F in exactly three arcs. Then either D is boundary-parallel in P_j or is isotopic to some $E_{i,j}$ by an isotopy which preserves F. In the first case suppose D is parallel to D' in ∂P_j. Since no arc of $K \cap F$ is boundary-parallel in F, D' must meet F in a single disk which contains $D \cap F$ in its boundary; it follows that $D \cap F$ can be removed from $K \cap F$ by an isotopy. Thus we may assume that D is isotopic to some $E_{i,j}$. Let $\widetilde{D} = \overline{K - D}$. Suppose $\widetilde{D}$ meets F only in $D \cap \widetilde{D}$. Then since $D \cap T_i = \varnothing$ some component of $\overline{\partial \widetilde{D} - \partial D}$ must be a boundary-parallel arc in T_i. It follows that K can be isotoped to reduce the number of components of $K \cap F$ by one. Thus $\widetilde{D}$ must have other intersections with F and hence with $P_0 \cup P_1$. Since no component of $K \cap (P_0 \cup P_1)$ is a disk meeting F in two arcs, there is exactly one other component; it must be a disk or an annulus. Suppose it is a disk D'. Then D' is isotopic to some $E_{m,n}$. One cannot have $m = i$ and $n = j$ since this would imply the existence of a boundary-parallel arc in T_i as above and so one could reduce the intersection. One cannot have $m \neq i$ and $n = j$ because T_i and $H_{i,1}$ are disjoint. One cannot have $m \neq i$ and $n \neq j$ for the same reason. Therefore $m = i$ and $n \neq j$. However, this forces $K \cap \partial_- X$ to have only one component and so cannot occur. Thus we may assume that the other component of $K \cap (P_0 \cup P_1)$ is an annulus A. It meets F in a single arc. Suppose A is parallel in P_n to an annulus A' in ∂P_n. Then $F \cap A'$ is either a disk or an annulus. Since K is boundary-incompressible it cannot be a disk. Thus it is an annulus and so there is an isotopy which removes $A \cap F$ from $K \cap F$. Therefore A is isotopic to some $A_{m,n}$. Since both components of $\partial A_{m,n}$ meet $\partial_- X$, this implies that

D meets G_j in a single arc, which is impossible since it is isotopic to $E_{i,j}$. Therefore we may assume there are no such components.

Suppose some component of $K \cap (P_0 \cup P_1)$ is a disk D which meets F in exactly four arcs. We may assume that no component of $D \cap (G_j \cup U_{0,j} \cup U_{1,j})$ is boundary-parallel, since otherwise one could reduce the intersection. $\overline{K - D}$ then consists of two disks which do not separate K; these disks must lie in Q. Since $K \cap \partial_- X \neq \varnothing$, D must meet some $U_{i,j}$ and therefore one of the complementary disks must meet T_i in a boundary-parallel arc. But this allows one to reduce the intersection.

Suppose some component of $K \cap P_j$ is an annulus A which meets F in a single arc. If A is boundary-parallel in P_j, then the intersection can be reduced as above. So assume A is isotopic to some $A_{i,j}$. Then $K \cap T_i \neq \varnothing$ and one may assume that it contains a spanning arc of T_i, for otherwise one could use a boundary-parallel arc in T_i to reduce the intersection. This implies that $K \cap P_n \neq \varnothing$ for $n \neq j$. In fact the intersection must be an annulus A' meeting F in a single arc and $\overline{K - (A \cup A')}$ is a disk. If A' is boundary-parallel in P_n, then the intersection can be reduced, as above. So one may assume that A' is isotopic to $A_{i,n}$. But this is impossible since it implies that $K \cap \partial_+ X = \varnothing$.

Suppose $K \cap P_j$ has a component which is an annulus A which meets F in exactly two arcs. Then the arcs lie on the same component of ∂A, and $\overline{K - A}$ is a disk which may be assumed to lie in Q. Since these arcs are not boundary-parallel in F it follows from Lemma 6.3 that one of the complementary arcs in ∂A must be boundary-parallel, which implies that one can reduce the intersection.

This exhausts all the possibilities for $K \cap P_j$.

(5) Suppose $L \cap F = \varnothing$. Then L must lie in some P_j; moreover L is parallel in P_j to a surface L' in ∂P_j. It follows that ∂L lies in G_j and that L' contains $F_{0,j} \cup U_{0,j}$ or $F_{1,j} \cup U_{1,j}$. Thus L is isotopic to L'_j.

Since F is boundary-incompressible we may assume that no component of $L \cap F$ is a boundary-parallel arc in L. We may also assume that no component is a separating simple closed curve α in L. Otherwise α is boundary-parallel in L and we can assume that the corresponding annulus A is outermost on L. If A lies in P_j, then it is boundary-parallel in P_j and so the intersection can be reduced. If A lies in Q then it cannot be isotopic to R for the usual reason and so must be boundary-parallel in Q; thus the intersection can be reduced.

We may assume that L is boundary-incompressible in X. Otherwise L boundary-compresses to an annulus A. A cannot be isotopic to R since the components of ∂R lie on different components of ∂X. Thus A is boundary-parallel and so is L.

Suppose $L \cap F$ contains an arc. Consider the possible components of $L \cap P_j$ containing this arc.

If there is a disk D which meets F in two arcs, then D is parallel to D' in ∂P_j. Since L is boundary-incompressible D' meets F in a single disk and so the intersection can be reduced.

Suppose there is a disk D which meets F in three arcs. If D is parallel to D' in ∂P_j, then again the boundary-incompressibility of L implies that D' meets F in a single disk and so the intersection can be reduced. If D is isotopic to some $E_{i,j}$, then L meets both components of ∂X, an impossibility.

Suppose there is a disk D which meets F in four arcs. Since L is boundary-incompressible we may assume that those arcs lying in $F_{i,j}$ separate the components of $\partial U_{i,j}$. If for some i, $F_{i,j}$ contains at most one of these arcs then some component of $D \cap G_j$ is boundary-parallel in G_j and so one can reduce the intersection. Therefore each $F_{i,j}$ meets D in two arcs. Since we may assume that no component of $D \cap G_j$ is boundary-parallel in G_j, the arcs in $F_{i,j}$ are nonadjacent as one traverses ∂D. Therefore $\overline{L - D}$ consists of two disks D_0 and D_1 such that $D_i \cap D = D \cap F_{i,j}$. It then follows that the D_i are boundary-parallel in Q, and so one can reduce the intersection.

Suppose there is an annulus A such that one component γ of ∂A meets F in a single arc α and all other components of $A \cap F$ lie in the other component of ∂A. Then the complementary arc to α in γ is boundary-parallel in G_j and so the intersection can be reduced.

This exhausts the possibilities if $L \cap F$ contains an arc, so assume $L \cap F$ consists of nonseparating simple closed curves in L. Then some component of $L \cap P_j$ or $L \cap Q$ is a disk with two holes K which contains ∂L, and the remaining components are annuli. For the usual reason none of these annuli are isotopic to R. We may assume that none of them are parallel to annuli in F. It follows that those lying in P_j are parallel to $U_{0,j}$, $U_{1,j}$, or G_j, while those lying in Q are parallel to T_0 or T_1.

Suppose K lies in P_j. Then K is parallel to K' in ∂P_j; K' is isotopic in ∂P_j to some $F_{i,j}$. But this is impossible since it forces $L \cap Q$ to have an annulus component which is not parallel to T_0 or T_1.

Therefore K lies in Q. If K is parallel to K' in ∂Q, then K' must be isotopic in ∂Q to a surface having one boundary component equal to ∂L and the other two equal to components of ∂H_1. It follows that the components of $L \cap (P_0 \cup P_1)$ are parallel to components of $G_0 \cup G_1$. It follows that $L \cap Q = K$ and that $L \cap (P_0 \cup P_1)$ has a single component A. Therefore $L = K \cup A$ is boundary-parallel in X, a contradiction to the boundary-incompressibility of L. So K must be isotopic to some $K_{i,k}$. Since ∂L lies in H_1, $k = 0$. It follows that for each j, $L \cap P_j$ is an annulus parallel to $U_{i,j}$ and $\overline{(L \cap Q) - K}$ is an annulus parallel to T_i. Thus L is isotopic to L_i.

(6) $S \cap F \neq \varnothing$ since Q and the P_j are cubes with handles. We may assume that no component of $S \cap Q$ or $S \cap P_j$ is an annulus parallel to an annulus in F. Then every annulus component of $S \cap P_j$ is parallel to $U_{0,j}$, $U_{1,j}$, or G_j, and every annulus component of $S \cap Q$ is parallel to T_0 or T_1.

Suppose L is a once-punctured torus component of $S \cap P_j$. Then L is isotopic to L'_j and so ∂L is parallel to $F_{i,j} \cap \partial_+ X$ in some $F_{i,j}$. Therefore the component of $S \cap Q$ meeting L cannot be an annulus. Since Q contains no incompressible once-punctured tori missing J, it must be a disk with two holes K. If K is parallel in Q to K' in ∂Q, then $\partial K'$ must have a component which is not isotopic to a component of ∂F, contradicting the fact that ∂K lies in F. If K is isotopic to some $K_{i,k}$, then the fact that $\partial K_{i,k}$ has a component which is not isotopic to a component of ∂F again gives a contradiction. Therefore $S \cap P_j$ has no punctured torus components.

Suppose K is a disk with two holes component of $S \cap P_j$. K is parallel to K' in ∂P_j. If K' lies in some $F_{i,j}$, then the intersection can be reduced. If K' does not lie in some $F_{i,j}$, then K' contains G_j and $\partial K'$ has two components

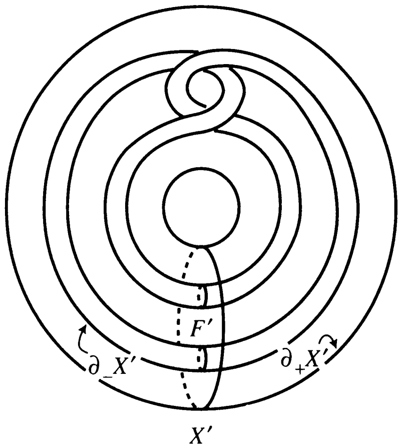

FIGURE 5

on some $F_{i,j}$ parallel to the components of $F_{i,j} \cap U_{i,j}$ and one component on $F_{m,j}$, $m \neq j$, parallel to $F_{i,j} \cap \partial_+ X$. Some component of $\overline{S-K}$ must be an annulus A which either lies in Q or meets Q in a pair of annuli. However, there is only one component of ∂K which can meet an annulus component of $S \cap Q$, so this is impossible.

Thus $S \cap P_j$ consists entirely of annuli. $S \cap Q$ has no disk with two holes component K. This is again because K would be boundary-parallel or isotopic to a $K_{i,k}$, forcing ∂K to have a component not lying in F. $S \cap Q$ also has no punctured torus component and so must have a component M which is a disk with three holes. $\overline{S-M}$ consists of two annuli. These annuli either meet Q in annuli parallel to T_0 or T_1 or do not meet Q at all.

Suppose M is boundary-parallel in Q. Then it is parallel to H_1 or H_0. In the first case S must meet each P_j in a single annulus parallel to G_j and so S is parallel to $\partial_+ X$. In the second case S must meet each P_j in a pair of annuli whose union is parallel to $U_{0,j} \cup U_{1,j}$, which forces the remainder of $S \cap Q$ to consist of a pair of annuli whose union is parallel to $T_0 \cup T_1$, and so S is parallel to $\partial_- X$.

Suppose M is isotopic to some M_i. Then there is an annulus component A of $S \cap P_j$ which is parallel to G_j and meets M in one component of ∂A. However, there is no component of $S \cap Q$ which can meet the other component of ∂A. Thus this situation cannot occur. A similar argument rules out the case of M being isotopic to M_i'. $\square$

Now let X' be the manifold shown in Figure 5. It is obtained from one of the P_j by identifying $F_{0,j}$ and $F_{1,j}$ by the restriction of a reflection in 3-space. Let F' be the image of the $F_{i,j}$ in X'. Let $\partial_+ X'$ be the image of G_j and let $\partial_- X' = \partial X' - \partial_+ X'$.

Lemma 6.7.

(1) *X' is irreducible and boundary-irreducible; F' is incompressible and boundary-incompressible.*
(2) *Every incompressible annulus and torus in X' is boundary-parallel.*

(3) *Every incompressible disk with two holes* K *in* X' *such that* $K \cap \partial_+ X'$ *consists of exactly one component of* ∂K *is isotopic to* F'.

Proof. (1) and (2) follow from Lemma 7.3 of [My1].

(3) K must be boundary-incompressible. Otherwise (2) would imply that K is boundary-parallel, which is impossible since $\partial X'$ consists of tori.

$H_1(X') \cong \mathbf{Z} \oplus \mathbf{Z}$, with the first summand generated by any oriented simple closed curve on $\partial_+ X'$ meeting F' transversely at a single point and the second generated by an oriented component of $F' \cap \partial_- X'$. The images of $H_1(\partial_+ X')$ and $H_1(\partial_- X')$ in $H_1(X')$ intersect in the trivial subgroup. This implies that the component α_+ of ∂K on $\partial_+ X'$ must be isotopic to $F' \cap \partial_+ X'$. One may therefore assume that $\alpha_+ \cap F' = \varnothing$. Let α'_- and α''_- be the other components of ∂K.

If $K \cap F' = \varnothing$, then K may be regarded as lying in P_j. It is parallel in P_j to K' in ∂P_j. K' must be isotopic in ∂P_j to some $F_{i,j}$. It follows that K is isotopic to F' in X'.

Since K and F' are both boundary-incompressible one may assume that no component of $K \cap F$ is a boundary-parallel arc in K or F. Let K' be K split along $K \cap F$. K' may be regarded as lying in P_j.

If $K \cap F$ contains a simple closed curve, then there is an annulus component A of K' which is outermost on K. Since A is boundary-parallel in P_j it follows that the intersection can be reduced.

Suppose α'_- and α''_- are joined by an arc in $K \cap F$. Then some component of K' must be a disk D meeting $F_{0,j} \cup F_{1,j}$ in exactly two arcs, for otherwise at least one of α'_- or α''_- meets F' in exactly one point, which is impossible since it splits to an arc in one of the $U_{i,j}$. D must be parallel in P_j to D' in ∂P_j. $D \cap (\alpha'_- \cup \alpha''_-)$ must lie in one of the $U_{i,j}$ and so D' must lie in $F_{i,j} \cup U_{i,j}$. It follows that one can isotop K in X' to reduce the intersection.

Thus one may assume that there is no such arc. Then there is an annulus component A of K' having one boundary component equal to, say, α'_- and the other equal to the union of an arc β of $K \cap F'$ and an arc γ in α''_-. But then since $\alpha'_- \cap F = \varnothing$ and α'_- and α''_- are parallel in $\partial_- X'$, there is an arc γ' in $F' \cap \partial_- X'$ which is parallel to γ in $\partial_- X'$. One can then isotop K so as to replace β by a simple closed curve, thereby reducing the complexity of the intersection. □

7. Mapping class groups of certain compact 3-manifolds

Let X be the 3-manifold shown in Figures 4 and 6. Let δ be a Dehn twist along the annulus R and φ, θ, and ω rotations of period two about the coordinate axes, as shown in Figure 6.

Proposition 7.1. $\mathscr{H}(X, \partial_+ X) \cong \mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$. $\mathbf{Z}$ *is generated by* $[\delta]$ *and* $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ *by* $[\varphi]$ *and* $[\theta]$. $[\omega] = [\varphi][\theta]$.

Let X' be the 3-manifold shown in Figure 5. It is obtained from one of the P_j shown in Figure 7, by identifying $F_{0,j}$ and $F_{1,j}$ by an involution f which interchanges the two surfaces and is induced by a reflection in 3-space. The period two rotations $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ about the coordinate axes induce involutions $\overline{\alpha}$, $\overline{\beta}$, and $\overline{\gamma}$ of X'.

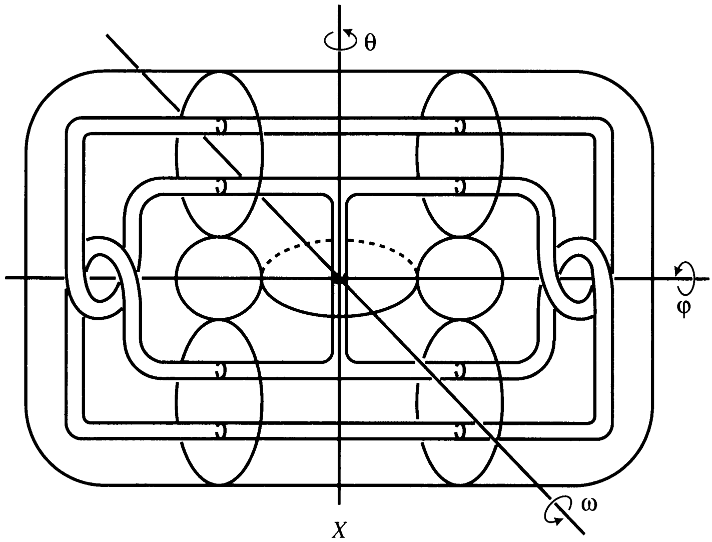

FIGURE 6

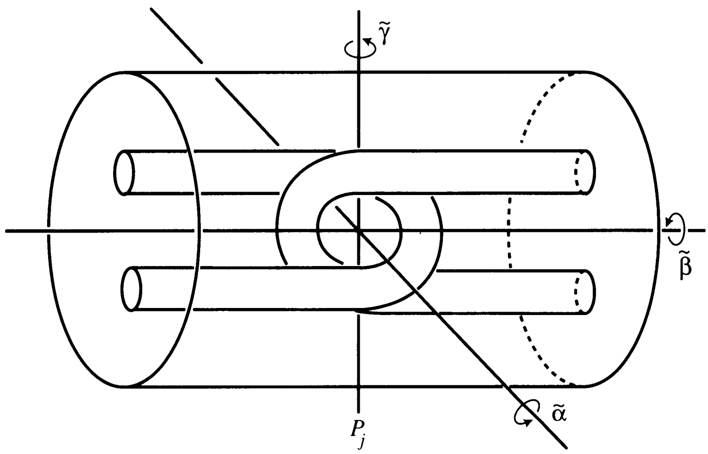

FIGURE 7

Proposition 7.2. *$\mathscr{H}(X', \partial_+X') \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$. It is generated by $[\overline{\alpha}]$ and $[\overline{\beta}]$. $[\overline{\gamma}] = [\overline{\alpha}][\overline{\beta}]$.*

The following two lemmas will be used in the proofs of both propositions.

Lemma 7.3. *Suppose g is an orientation preserving diffeomorphism of P_j which preserves each component of $F_{0,j} \cup F_{1,j}$. Then g is isotopic to the identity via an isotopy which preserves each of these components. If, in addition, g is the identity on $F_{0,j} \cup F_{1,j}$, then the isotopy can be chosen fixed on $F_{0,j} \cup F_{1,j}$.*

Proof. Since $g(F_{i,j}) = F_{i,j}$ one can isotop g so that $g(E_{i,j}) = E_{i,j}$. g is orientation preserving and preserves each component of $\partial F_{i,j}$, so g can be isotoped so that it is the identity on $\partial E_{i,j}$ and so can be further isotoped so that it is the identity on $E_{i,j}$. Since splitting $F_{i,j}$, $U_{i,j}$, and G_j along their intersections with $E_{0,j} \cup E_{1,j}$ yields a set of disks each of which is invariant under g, one can isotop g so that it is the identity on ∂P_j. One checks that if g was originally the identity on $F_{0,j} \cup F_{1,j}$, then each of these isotopies can be chosen fixed on $F_{0,j} \cup F_{1,j}$. Finally, since P_j is a cube with handles one can isotop g rel ∂P_j to the identity. □

Lemma 7.4. *Suppose g is an orientation preserving diffeomorphism of P_j such that $g(F_{0,j} \cup F_{1,j}) = F_{0,j} \cup F_{1,j}$. Then g is isotopic to $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, or the identity. If the restriction of g to $F_{0,j} \cup F_{1,j}$ commutes with f then the isotopy can be chosen so that its restriction also commutes with f.*

Proof. Suppose $g(F_{i,j}) = F_{i,j}$. Then $g(U_{i,j}) = U_{i,j}$ and $g(G_j) = G_j$. Then either g must preserve each boundary component of both the $U_{i,j}$ or must interchange the boundary components of both the $U_{i,j}$. For suppose it preserves each component of, say, $\partial U_{0,j}$ and interchanges the components of $\partial U_{1,j}$. Then there is a basis for the first homology of $F_{0,j} \cup U_{0,j}$ with respect to which the automorphism induced by g has a matrix of the form

$$\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}.$$

For $F_{1,j} \cup U_{1,j}$ the corresponding automorphism has a matrix of the form

$$\begin{bmatrix} -1 & * \\ 0 & -1 \end{bmatrix}.$$

Since g is defined on the product P_j, its restrictions to the ends of the product are homotopic, but this is impossible since the two matrices have different traces.

If g preserves each boundary component of both the $U_{i,j}$ its restriction to each $F_{i,j}$ is isotopic to the identity. If g commutes with f on $F_{0,j} \cup F_{1,j}$, then clearly the isotopy can be chosen so that it also commutes with f. In any case, Lemma 7.3 now allows one to isotop g to the identity as required.

If g interchanges the boundary components of both the $U_{i,j}$, then $\tilde{\beta} \circ g$ preserves each boundary component and it follows that g is isotopic to $\tilde{\beta}$. Since $\tilde{\beta}$ commutes with f, it follows that if g commutes with f then the isotopy can be chosen to commute with f.

Finally, if g interchanges the $F_{i,j}$, then $\tilde{\alpha} \circ g$ preserves each of them and it follows that g is isotopic to $\tilde{\alpha}$ or $\tilde{\gamma}$. As above, if g commutes with f, then so does the isotopy. □

Lemmas 7.5–7.7 will be used only in the proof of Proposition 7.1.

Lemma 7.5. *Let g_i be an orientation preserving diffeomorphism of Q_i such that $g_i(R) = R$ and the restriction of g_i to $F_{i,0} \cup F_{i,1}$ is the identity. Then g_i is isotopic to the identity via an isotopy which preserves R and is fixed on $F_{i,0} \cup F_{i,1}$.*

Proof. For homological reasons $g_i(\partial E_i)$ must be isotopic to ∂E_i; it follows that there is an isotopy satisfying the given conditions such that afterward g_i is the identity on E_i. Split Q_i along E_i to get the product $Q_i' = H_{i,0} \times [0, 1]$ with $H_{i,0} = H_{i,0} \times \{0\}$. Let g_i' be the induced diffeomorphism of Q_i'. $H_{i,1}$ is split to an annulus on one boundary component of which g_i' is the identity. T_i is split to a disk on whose boundary g_i' is the identity. It follows that g_i can be isotoped rel E_i, satisfying the requirements, so that g_i' is the identity on $H_{i,0} \times \{1\}$. By Lemma 8.4 of [Wa] g_i' can be isotoped rel $(H_{i,0} \times \{1\}) \cup$

$((\partial H_{i,0}) \times [0, 1])$ to the identity of Q_i'. Therefore g_i is isotopic to the identity as required. □

Lemma 7.6. *Let g be a diffeomorphism of Q which is the identity on F. Then g is isotopic* rel *F to a product of Dehn twists about R.*

Proof. First isotop g so that $g(R) = R$ and therefore $g(Q_i) = Q_i$. Since the restriction g_i of g to Q_i is the identity on $F_{i,0} \cup F_{i,1}$, the previous lemma implies that g is isotopic rel F to a product of Dehn twists about R. □

Lemma 7.7. *Let g be an orientation preserving diffeomorphism of X which preserves each $F_{i,j}$. Then g is isotopic to a power of δ.*

Proof. $g(P_j) = P_j$ and $g(Q) = Q$. Since $g(T_i) = T_i$ the restriction of g to P_j cannot be isotopic to $\tilde{\beta}_j$. It follows that g can be isotoped so that its restriction to $P_0 \cup P_1$ is the identity. It then follows from the previous lemma that the restriction of g to Q is isotopic rel F to some δ^k and therefore so is g. □

Proof of Proposition 7.1. The diffeomorphisms δ, φ, and θ generate a subgroup of $\mathrm{Diff}(X, \partial_+X)$ isomorphic to $\mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$. For homological reasons this subgroup injects into $\mathscr{H}(X, \partial_+X)$. If g is any element of $\mathrm{Diff}(X, \partial_+X)$, then it can be isotoped so that $g(R) = R$ and $g(F) = F$. g must then either interchange the P_j or leave each P_j invariant. By composing g with θ one may assume that the latter is the case.

By considering $g(T_i)$ it can be seen that g must either interchange $F_{0,j}$ and $F_{1,j}$ for both $j = 0$ and $j = 1$ or leave all the $F_{i,j}$ invariant. By composing g with φ one may assume that the latter is the case. It then follows from the previous lemma that g is isotopic to a power of δ. Thus the original $[g]$ is an element of the $\mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ subgroup. □

Proof of Proposition 7.2. The $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ subgroup of $\mathrm{Diff}(X', \partial_+X')$ generated by $\overline{\alpha}$ and $\overline{\beta}$ must for homological reasons inject into $\mathscr{H}(X', \partial_+X')$. Any element g of $\mathrm{Diff}(X', \partial_+X')$ can be isotoped so that $g(F') = F'$. Then g induces a diffeomorphism g' of P_j which preserves $F_{0,j} \cup F_{1,j}$ and commutes with f. By Lemma 7.4 g' is isotopic to $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, or the identity via an isotopy which commutes with f and so g is isotopic to an element of the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ subgroup of $\mathscr{H}(X', \partial_+X')$. □

8. A genus two example

In this section the mapping class group of a certain genus two Whitehead manifold is computed. The manifold is very similar to an example of McMillan [Mc]. That example is presented as an open subset of $\mathbf{R}^3$. The present example is constructed by gluing together a sequence of compact 3-manifolds along their boundaries. McMillan's example can of course be constructed in this fashion, and in fact the compact 3-manifolds are the same, but the gluing maps are more complicated.

For $n \geq 1$ let X_n be a copy of the manifold X, as shown in Figure 8. Let $h_n: X_n \to X_{n+1}$ be the diffeomorphism which identifies two successive copies of X. Let φ_n, θ_n, and ω_n be the involutions of X_n defined in the previous section. It will be assumed that $h_n \circ g_n \circ h_n^{-1} = g_{n+1}$, where

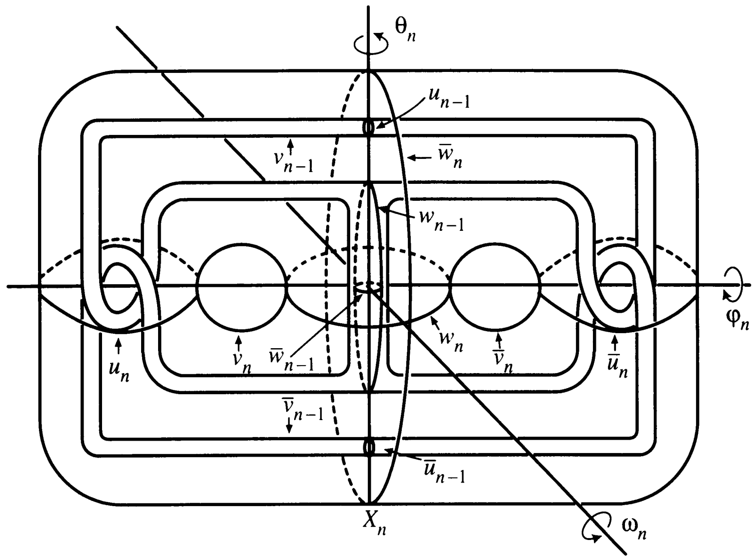

FIGURE 8

$g = \varphi$, θ, or ω, and that h_n carries each of the oriented simple closed curves u_n, v_n, w_n, $\overline{u}_n$, $\overline{v}_n$, $\overline{w}_n$ to u_{n+1}, v_{n+1}, w_{n+1}, $\overline{u}_{n+1}\overline{v}_{n+1}$, $\overline{w}_{n+1}$, respectively. Let $i_n \colon \partial_+ X_n \to \partial_- X_{n+1}$ be a diffeomorphism which carries u_n, etc., in $\partial_+ X_n$ to the curve in $\partial_- X_{n+1}$ having the same label. It will be assumed that $i_{n+1} \circ h_n = h_{n+1} \circ i_n$, $i_n \circ \varphi_n = \theta_{n+1} \circ i_n$, and $i_n \circ \theta_n = \varphi_{n+1} \circ i_n$ on $\partial_+ X_n$.

Let V_0 be a cube with two handles. Let φ_0, θ_0, and ω_0 be the period two rotations about the coordinate axes as indicated in Figure 9. Choose a diffeomorphism $i_0 \colon \partial V_0 \to \partial_- X_1$ which carries the curves u_0, etc., in ∂V_0 to the corresponding curves in $\partial_- X_1$. It will be assumed that $i_0 \circ \varphi_0 = \theta_1 \circ i_0$ and $i_0 \circ \theta_0 = \varphi_1 \circ i_0$ on ∂V_0.

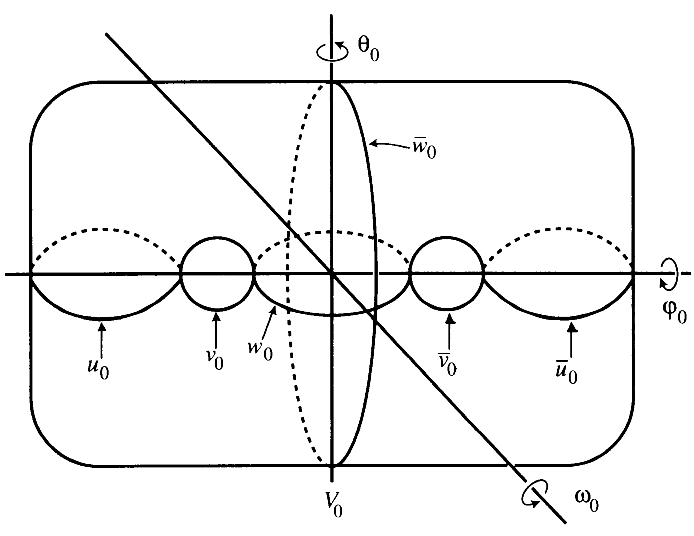

FIGURE 9

Let $V_1 = V_0 \cup X_1$, the identification being via i_0; it is a cube with two handles. There is a diffeomorphism $h_0 \colon V_0 \to V_1$ such that h_0 carries u_0, etc. to u_1, etc., $i_1 \circ h_0 = h_1 \circ i_0$ on ∂V_0, and $h_0 \circ g_0 \circ h_0^{-1} = g_1$ on $\partial_+ X_1$, where $g = \varphi, \theta$, or ω. It is not assumed that this last equation holds on all of X_1.

Let $V_n = V_0 \cup X_1 \cup \cdots \cup X_n$, where boundary components are identified via the maps $i_0, \ldots, i_{n-1}$. Let $W = \bigcup_{n \geq 0} V_n$. Each V_n is a cube with two handles and $V_n \to V_{n+1}$ is null-homotopic. Therefore W is a Whitehead manifold. S_n is incompressible in $\overline{W - V_0}$ and so W is not homeomorphic to $\mathbf{R}^3$. It follows easily from the facts that X contains no incompressible tori and every incompressible annulus in X is either boundary-parallel or isotopic to R that W is not a monotone union of solid tori and so has genus two.

Let s be the involution of W which is equal to φ_0 on V_0, θ_n on X_n for each odd n, and φ_n on X_n for each even n. Let t be the involution of W which is equal to θ_0 on V_0, φ_n on X_n for each odd n, and θ_n on X_n for each even n. s and t generate a $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ subgroup of $\mathrm{Diff}(W)$ which will be denoted G.

Let h be the diffeomorphism of W defined to be h_0 on V_0 and h_n on X_n.

Theorem 8.1. *$\mathscr{H}(W) = \mathscr{H}(W; V) \cong (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \times_\xi \mathbf{Z}$, where $\mathbf{Z}$ is generated by $[h]$, $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $[s]$ and $[t]$, and ξ interchanges the summands of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.*

Proof. First, the structure of $\mathscr{H}(W; V)$ will be established in the next two lemmas.

Lemma 8.2. *$\overline{\mathscr{F}}(W; V) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and is generated by $r(q^{-1}([s]))$ and $r(q^{-1}([t]))$.*

Proof. $\mathscr{H}(X_n, S_n) \cong \mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$, with the summands generated by $[\delta_n]$, $[\varphi_n]$, and $[\theta_n]$. The restriction of δ_n to S_n is a Dehn twist about w_n, while that of δ_{n+1} is a Dehn twist about $\overline{w}_n$. Since these curves are not isotopic in S_n the only elements of $\mathscr{H}(X_n, S_n)$ and $\mathscr{H}(X_{n+1}, S_{n+1})$ having representatives whose restrictions to S_n are isotopic are those lying in the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ summands. For homological reasons the only possible matchings are those of $[id]$ and $[id]$, $[\varphi_n]$, and $[\theta_{n+1}]$, $[\theta_n]$ and $[\varphi_{n+1}]$, and $[\omega_n]$ and $[\omega_{n+1}]$, i.e., those obtained by restricting elements of G.

Since V_N is a cube with handles the isotopy class of a diffeomorphism of V_N is determined by its restriction to S_N. Thus the only elements of $\mathscr{H}(V_N)$ which have representatives agreeing with representatives of $\mathscr{H}(X_{N+1}, S_{N+1})$ are the classes obtained by restricting elements of G to V_N.

It follows that $\overline{\mathscr{F}}_N(W; V)$ is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and consists of the sequences

$$([g|_{V_N}], [g|_{X_{N+1}}], \ldots, [g|_{X_n}], \ldots), \qquad g \in G.$$

$\overline{f}_{N,P} \colon \overline{\mathscr{F}}_N(W; V) \to \overline{\mathscr{F}}_P(W; V)$ is clearly onto. It is also one-to-one. For if $g|_{V_P}$ is isotopically trivial, then so is $g|_{S_P}$, and therefore so is $g|_{X_P}$, and hence so is $g|_{S_{P-1}}$. Continuing in this fashion one eventually gets that $g|_{V_N}$ is isotopically trivial. Thus $\overline{\mathscr{F}}(W; V)$ is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and clearly has the generators indicated. $\square$

Lemma 8.3. *$\mathscr{H}(W; V) \cong (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \times_\xi \mathbf{Z}$ as above.*

Proof. Recall that h induces an isomorphism $\overline{\xi}_N \colon \overline{\mathscr{F}}_N(W; V) \to \overline{\mathscr{F}}_{N+1}(W; V)$ given by $\overline{\xi}_N(([g_N], [g_{N+1}], \ldots, [g_n], \ldots)) = ([h \circ g_N \circ h^{-1}], [h \circ g_{N+1} \circ h^{-1}], \ldots,$

$[h \circ g_n \circ h^{-1}], \ldots)$. We may take g_n to be the restriction of an element of G. If $n > N$, then $h \circ g_n \circ h^{-1} = h_n \circ g_n \circ h_n^{-1} = g_{n+1}$, where g is id, φ, θ, or ω. If $n = N$, then the equation may not hold on all of V_{N+1}. But it does hold on S_{N+1}, which is enough to imply that $[h \circ g_N \circ h^{-1}] = [g_{N+1}]$. It is apparent from the definitions of s and t that

$$\overline{\xi}_N(([s|_{V_N}], [s|_{X_{N+1}}], \ldots, [s|_{X_n}], \ldots)) = ([t|_{V_{N+1}}], [t|_{X_{N+2}}], \ldots, [t|_{X_{n+1}}], \ldots),$$

with a similar formula holding with s and t interchanged.

Passing to the limit one has $\overline{\xi}(r(q^{-1}([s]))) = r(q^{-1}([t]))$ and $\overline{\xi}(r(q^{-1}([t]))) = r(q^{-1}([s]))$. It follows that $\xi([s]) = [t]$ and $\xi([t]) = [s]$. □

Now $\mathscr{H}(W) = \mathscr{H}(W; V)$ will be proven by imitating the proof of the "Shift Lemma" of [My2].

Lemma 8.4. *Suppose for some $q > p \geq 0$ that S is a closed, incompressible genus two surface in the interior of $V_q - V_p$. Then S is isotopic to some S_m, $p \leq m \leq q$, via an isotopy with support in the interior of $V_q - V_p$ if $p < m < q$ and support in a regular neighborhood of $\overline{V_q - V_p}$ if $m = p$ or q.*

Proof. Assume S has transverse minimal intersection with $\bigcup_{n=p}^{q} S_n$. If the intersection is empty, then S is boundary-parallel in some X_n and the result follows, so assume the intersection is nonempty.

No component of $S \cap X_n$ is a disk or an annulus with its boundary in one component of ∂X_n, since such a disk or annulus would be boundary-parallel and one could reduce the intersection. This implies that no component of $S \cap X_n$ can be a disk with two or three holes because this would force $S \cap X_k$ to have such an annulus for some k. It thus follows that the only possible components of $S \cap X_n$ are once-punctured tori and annuli isotopic to R_n. For some n there must be a once-punctured torus component L of $S \cap X_n$ such that ∂L lies in S_n, for otherwise there would again be a forbidden annulus in some X_k. Let N be the component of $S \cap X_{n+1}$ meeting L. If N is a once-punctured torus, then ∂L is null-homologous in X_{n+1}. If N is an annulus, then ∂L is isotopic to $R_{n+1} \cap S_n$, i.e., to $\overline{w}_n$. We may assume that L is not boundary-parallel and so is isotopic to one of the L_i or L'_j of Lemma 6.6. But this implies that ∂L is isotopic to w_n, u_n, or $\overline{u}_n$, none of which satisfy the above conditions on ∂L. □

Lemma 8.5. *Let $f: W \to W$ be a diffeomorphism such that $f(V_p) = V_r$ and $f(V_q) = V_s$, where $p < q$ and $r < s$. Then $q - p = s - r$ and f is isotopic to f' such that $f'(S_i) = S_{r-p+i}$ for $p \leq i \leq q$. This isotopy can be chosen constant outside $V_q - V_p$.*

Proof. If $q - p = 1$, then this follows from the facts that every closed, incompressible genus two surface in X_q is boundary-parallel and none of the X_n are product I-bundles. No isotopy is necessary.

If $q - p > 1$, then $f(S_{p+1})$ is isotopic to some S_m, $r \leq m \leq s$. Since S_{p+1} is not parallel to S_p or S_q, one has $r < m < s$, and so the isotopy is constant outside $V_q - V_p$. One then applies induction to complete the proof. □

Lemma 8.6. *Every diffeomorphism f of W is isotopic to a diffeomorphism which is eventually carried by V.*

Proof. First choose $p_0 > 0$ such that $(V_0 \cup f(V_0)) \subseteq V_{p_0}$. Next choose $N_0 > 0$ such that $V_{p_0} \subseteq f(V_{N_0})$. Then choose $q_0 > p_0$ such that $f(V_{N_0}) \subseteq V_{q_0}$. This ensures that $f(S_{N_0})$ is incompressible in $f(\overline{W - V_0})$ and so is incompressible in $\overline{W - V_{p_0}}$ and hence in $V_{q_0} - V_{p_0}$. It can therefore be isotoped to some S_{M_0}, $p_0 \leq M_0 \leq q_0$, via an isotopy constant outside a regular neighborhood of $\overline{V_{q_0} - V_{p_0}}$.

Let $p_1 = q_0 + 1$. Choose $N_1 > N_0$ such that $V_{p_1} \subseteq f(V_{N_1})$ and $q_1 > p_1$ such that $f(V_{N_1}) \subseteq V_{q_1}$. This ensures that $f(S_{N_1})$ is incompressible in $V_{q_1} - V_{p_1}$ and so, as above, is isotopic to some S_{M_1}, $p_1 \leq M_1 \leq q_1$, via an isotopy fixed outside a regular neighborhood of $\overline{V_{q_1} - V_{p_1}}$.

Continuing in this way one obtains a sequence of isotopies with disjoint supports (and thus a single isotopy) whose composition with f gives a diffeomorphism f' such that $f'(S_{N_k}) = S_{M_k}$ for some increasing sequences of integers $\{N_k\}$ and $\{M_k\}$. So $f'(V_{N_k}) = V_{M_k}$. Therefore $M_{k+1} - M_k = N_{k+1} - N_k$ and there is an isotopy fixed outside $f'(V_{N_{k+1}} - V_{N_k})$ taking $f'(S_i)$ to S_j for $N_k < i < N_{k+1}$ and $j = i + M_k - N_k$. The composition of this isotopy with f' gives the desired diffeomorphism f'', with $N = N_0$ and $s = M_0 - N_0$. □

9. A GENUS ONE EXAMPLE

In this section the mapping class group of Whitehead's original example [Wh] of a contractible open 3-manifold which is not homeomorphic to $\mathbf{R}^3$ is computed. The manifold will be constructed by gluing together a sequence of compact manifolds. The result is equivalent to Whitehead's description of it as an open subset of $\mathbf{R}^3$.

Let X_n now be a copy of the manifold X' defined in Figure 10. For $n \geq 1$ let $h_n\colon X_n \to X_{n+1}$ be the diffeomorphism which identifies two successive copies of X'. Let $\overline{\alpha}_n$, $\overline{\beta}_n$, and $\overline{\gamma}_n$ be the involutions of X_n defined in §7. It will be assumed that $h_n \circ \overline{g}_n \circ h_n^{-1} = \overline{g}_{n+1}$ for $n \geq 1$, where $\overline{g} = \overline{\alpha}, \overline{\beta}$, or $\overline{\gamma}$. Each component of ∂X_n is parametrized as $S^1 \times S^1$ in such a way that $m_n = S^1 \times \{1\}$ and $l_n = \{1\} \times S^1$. It will be assumed that the restriction of h_n to ∂X_n is the

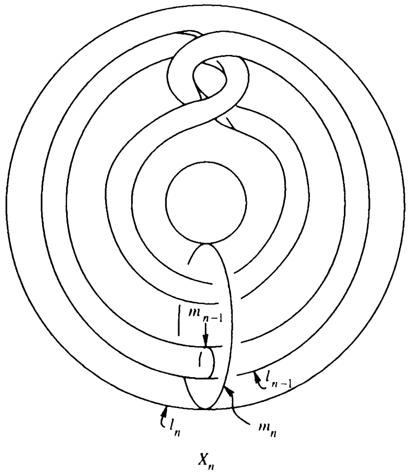

FIGURE 10

identity with respect to these parametrizations. Moreover on $\partial_+ X_n$,

$$\overline{\alpha}_n(z_0, z_1) = (\overline{z}_0, \overline{z}_1), \quad \overline{\beta}_n(z_0, z_1) = (-z_0, z_1), \quad \overline{\gamma}_n(z_0, z_1) = (-\overline{z}_0, \overline{z}_1),$$

while on $\partial_- X_n$,

$$\overline{\alpha}_n(z_0, z_1) = (z_0, -z_1), \quad \overline{\beta}_n(z_0, z_1) = (\overline{z}_0, \overline{z}_1), \quad \overline{\gamma}_n(z_0, z_1) = (\overline{z}_0, -\overline{z}_1).$$

Define $i_n\colon \partial_+ X_n \to \partial_- X_{n+1}$ to be the identity with respect to these parametrizations.

Let V_0 be a solid torus, parametrized as $D^2 \times S^1$. Let

$$\overline{\alpha}_0(rz_0, z_1) = (r\overline{z}_0, \overline{z}_1), \qquad \overline{\beta}_0(rz_0, z_1) = (-rz_0, z_1),$$
$$\overline{\gamma}_0(rz_0, z_1) = (-r\overline{z}_0, \overline{z}_1).$$

Define $i_0\colon \partial V_0 \to \partial_- X_1$ to be the identity with respect to these parametrizations.

Let V_1 be the union of V_0 and X_1 via i_0. It is a solid torus and there is a diffeomorphism $h_0\colon V_0 \to V_1$ which restricts to the identity on ∂V_0 with respect to the given parametrizations. It follows that $i_1 \circ h_0 = h_1 \circ i_0$ on ∂V_0.

Let $V_n = V_0 \cup X_1 \cup \cdots \cup X_n$, where the identifications are carried out via the maps $i_0, \ldots, i_{n-1}$. Let $W = \bigcup_{n \geq 0} V_n$. It is easily checked that W is a genus one Whitehead manifold and V is a very good exhaustion.

Let h be the diffeomorphism of W defined to be h_0 on V_0 and h_n on X_n. Note that unlike the previous example the involutions $\overline{\alpha}_n$, $\overline{\beta}_n$, and $\overline{\gamma}_n$ do *not* piece together to give diffeomorphisms of W. These involutions will be used to describe $\overline{\mathscr{F}}(W; V)$. A set of diffeomorphisms whose isotopy classes generate $\mathscr{H}(W)$ will be described later.

Theorem 9.1. *The exhaustion V of the classical Whitehead manifold W is very good; it is periodic of period $\sigma = 1$ and has minimal shift h.*

(1) $\mathscr{H}(W) = \mathscr{H}(W; V) \cong \mathscr{F}(W; V) \times_\xi \mathbf{Z}$.

(2) *There is an exact sequence*

$$0 \to \mathscr{D}(W; V) \to \mathscr{F}(W; V) \times_\xi \mathbf{Z} \xrightarrow{\hat{r}} \overline{\mathscr{F}}(W; V) \times_{\overline{\xi}} \mathbf{Z} \to 1,$$

where

$$\ker \hat{r} = \mathscr{D}(W; V) \cong \prod_{n=0}^{\infty} \mathbf{Z}^2 / \bigoplus_{n=0}^{\infty} \mathbf{Z}^2,$$

$$\overline{\mathscr{F}}(W; V) \cong \prod_{n=0}^{\infty} \mathbf{Z}_2 / \bigoplus_{n=0}^{\infty} \mathbf{Z}_2,$$

and $\hat{r}$ preserves the semidirect product structure.

(3) ξ *restricts to the automorphism of $\mathscr{D}(W; V)$ given by*

$$\xi(\{(a_0, b_0), (a_1, b_1), \ldots\}) = \{(0, 0), (a_0, b_0), (a_1, b_1), \ldots\}.$$

(4) *For $\overline{c} = \{c_n\} \in \overline{\mathscr{F}}(W; V)$, $\overline{\xi}(\{c_0, c_1, \ldots\}) = \{0, c_0, c_1, \ldots\}$.*

(5) *For every $c \in \mathscr{F}(W; V)$ such that $r(c) = \overline{c}$, and for each $\{(a_n, b_n)\} \in \mathscr{D}(W; V)$,*

$$c\{(a_n, b_n)\}c^{-1} = \{(-1)^{c_n}(a_n, b_n)\}.$$

(6) *For every* $\overline{c} \in \overline{\mathscr{F}}(W\,;\,V)$ *there exists* $c \in \mathscr{F}(W\,;\,V)$ *such that* $r(c) = \overline{c}$ *and*

$$c^2 = \left\{ \frac{1+(-1)^{c_n}}{2}(c_{n-1}\,,\,c_{n+1}) \right\}.$$

The element c' *of* $\mathscr{F}(W\,;\,V)$ *satisfies* $r(c') = r(c)$ *and* $(c')^2 = c^2$ *if and only if* c *and* c' *differ by an element of* $\mathscr{D}(W\,;\,V)$ *of the form*

$$\left\{ \frac{1-(-1)^{c_n}}{2}(a_n\,,\,b_n) \right\}.$$

(7) *There is an involution* γ *of* W *such that* $r([\gamma]) = \{(1\,,\,1\,,\,1\,,\,\dots)\}$. *The finite subgroups of* $\mathscr{H}(W)$ *are precisely the* $\mathbf{Z}_2$ *subgroups generated by elements of the form* $[\gamma]\{(a_n\,,\,b_n)\}$. *Each of these elements is represented by an involution of* W.

Proof. The first statement is a consequence of the previous results and the fact that V is a very good exhaustion. The remaining statements can be obtained from the following lemmas together with previous results.

Lemma 9.2. $\overline{\mathscr{F}}(W\,;\,V) \cong \prod_{n=0}^{\infty} \mathbf{Z}_2 / \bigoplus_{n=0}^{\infty} \mathbf{Z}_2$.

Proof. It will first be established that $\overline{\mathscr{F}_N}(W\,;\,V) \cong \prod_{n=N}^{\infty} \mathbf{Z}_2$.

$\mathscr{H}(X_n\,,\,S_n) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and is generated by $[\overline{\alpha}_n]$ and $[\overline{\beta}_n]$. All the relevant diffeomorphisms of S_n are isotopic either to the identity or to $\rho(z_0\,,\,z_1) = (\overline{z}_0\,,\,\overline{z}_1)$. In particular the restrictions of $\overline{\beta}_n$ and $\overline{\alpha}_{n+1}$ are isotopic to id, while those of $\overline{\alpha}_n$, $\overline{\gamma}_n$, $\overline{\beta}_{n+1}$, and $\overline{\gamma}_{n+1}$ are isotopic to ρ. Let $\overline{\rho}_N$ be an involution of V_N which restricts to ρ on S_N. One need only consider the subgroup $\mathscr{H}'(V_N)$ of $\mathscr{H}(V_N)$ generated by $[\overline{\rho}_N]$.

$\overline{\mathscr{F}_N}(W\,;\,V)$ then consists of those elements $([\overline{g}_N], [\overline{g}_{N+1}], \dots, [\overline{g}_n], \dots)$ of $\mathscr{H}'(V_N) \times \prod_{n=N+1}^{\infty} \mathscr{H}(X_n\,,\,S_n)$ such that:

If $\overline{g}_N = id$, then $\overline{g}_{N+1} = id$ or $\overline{\alpha}_{N+1}$.
If $\overline{g}_N = \overline{\rho}_N$, then $\overline{g}_{N+1} = \overline{\beta}_{N+1}$ or $\overline{\gamma}_{N+1}$.
If $\overline{g}_n = id$ or $\overline{\beta}_n$, then $\overline{g}_{n+1} = id$ or $\overline{\alpha}_{n+1}$.
If $\overline{g}_n = \overline{\alpha}_n$ or $\overline{\gamma}_n$, then $\overline{g}_{n+1} = \overline{\beta}_{n+1}$ or $\overline{\gamma}_{n+1}$.

Let k_n be the restriction of $\overline{g}_n$ to S_n and consider the classes $[k_n] \in \mathscr{H}(S_n)$.

If $\overline{g}_N = id$, then $k_N = id$.
If $\overline{g}_N = \overline{\rho}_N$, then $k_N = \rho$.
If $\overline{g}_n = id$, then $k_{n-1} = id$ and $k_n = id$.
If $\overline{g}_n = \overline{\alpha}_n$, then $k_{n-1} = id$ and $k_n = \rho$.
If $\overline{g}_n = \overline{\beta}_n$, then $k_{n-1} = \rho$ and $k_n = id$.
If $\overline{g}_n = \overline{\gamma}_n$, then $k_{n-1} = \rho$ and $k_n = \rho$.

These observations establish an isomorphism between $\overline{\mathscr{F}_N}(W;\,V)$ and $\prod_{n=N}^{\infty} \mathbf{Z}_2 \subseteq \prod_{n=N}^{\infty} \mathscr{H}(S_n)$.

Now consider the homomorphism $\overline{f}_{N\,,\,P} \colon \overline{\mathscr{F}}_N(W\,;\,V) \to \overline{\mathscr{F}}_P(W\,;\,V)$. It is easily seen to be onto. $([g_N], [g_{N+1}], \dots, [g_n], \dots)$ is in the kernel if and only if $[k_n] = [id]$ for $n \geq P$, and so the kernel is $\bigoplus_{n=N}^{P-1} \mathbf{Z}_2$. Passing to the limit one has $\overline{\mathscr{F}}(W\,;\,V) \cong \prod_{n=0}^{\infty} \mathbf{Z}_2 / \bigoplus_{n=0}^{\infty} \mathbf{Z}_2$. $\square$

Lemma 9.3. *For* $\overline{c} = \{c_n\} \in \overline{\mathscr{F}}(W\,;\,V)$, $\xi(\{c_0\,,\,c_1\,,\,c_2\,,\,\dots\}) = \{0\,,\,c_0\,,\,c_1\,,\,\dots\}$.

Proof. As in the proof of Lemma 9.2, the element $([\overline{g}_N], [\overline{g}_{N+1}], \dots, [\overline{g}_n], \dots)$ and its image $([h \circ g_N \circ h^{-1}], [h \circ g_{N+1} \circ h^{-1}], \dots, [h \circ g_n \circ h^{-1}], \dots)$ under $\overline{\xi}_N$ are determined by the sequences $([k_N], [k_{N+1}], \dots, [k_n], \dots)$ and $([h \circ k_N \circ h^{-1}], [h \circ k_{N+1} \circ h^{-1}], \dots, [h \circ k_n \circ h^{-1}], \dots)$. Since $h \circ \rho \circ h^{-1} = \rho$, one has that $\overline{\xi}_N \colon \prod_{n=N}^{\infty} \mathbf{Z}_2 \to \prod_{n=N+1}^{\infty} \mathbf{Z}_2$ is given by $\overline{\xi}_N(c_N, c_{N+1}, \dots, c_n, \dots) = (c_N, c_{N+1}, \dots, c_n, \dots)$. Passing to the limit, $\overline{\xi}$ is induced by the homomorphism $\tilde{\xi} \colon \prod_{n=0}^{\infty} \mathbf{Z}_2 \to \prod_{n=0}^{\infty} \mathbf{Z}_2$ given by $\tilde{\xi}(c_0, c_1, c_2, \dots) = (0, c_0, c_1, \dots)$. □

To prove the remainder of the theorem we will need to construct explicit diffeomorphisms of W which project to $\overline{\mathscr{F}}(W; V)$. This will be done by defining diffeomorphisms α_n, β_n, and γ_n which are isotopic to $\overline{\alpha}_n$, $\overline{\beta}_n$, and $\overline{\gamma}_n$ and equal id or ρ on the boundary tori, so that they piece together to give diffeomorphisms of W.

Recall that C_n^+ is a collar on S_n in X_{n+1}, parametrized as $S_n \times [0, 1]$ and that C_n^- is a collar on S_n in X_n (in V_0 if $n = 0$), parametrized as $S_n \times [-1, 0]$, with $S_n \times \{0\} = S_n$ in both cases. $C_n = C_n^+ \cup C_n^-$, $X_n^0 = \overline{X_n - (C_{n-1}^+ \cup C_n^-)}$, and $V_0^0 = \overline{V_0 - C_0^-}$. These collars can be chosen so that they are invariant under $\overline{g}_n$, where $\overline{g}_n = \overline{\alpha}_n$, $\overline{\beta}_n$, or $\overline{\gamma}_n$. Moreover, the restrictions of $\overline{g}_n$ to C_{n-1}^+ and C_n^- are given by $\overline{g}_n|_{S_{n-1}} \times id$ and $\overline{g}_n|_{S_n} \times id$, respectively.

Define diffeomorphisms $\varepsilon_n^{\pm}$, $\rho_n^{\pm}$, $\mu_n^{\pm}$, and $\lambda_n^{\pm}$ of $C_n^{\pm}$, as follows:

$$\varepsilon_n^+(z_0, z_1, t) = (z_0, e^{i\pi t} z_1, t), \qquad \varepsilon_n^-(z_0, z_1, t) = (e^{-i\pi t} z_0, z_1, t),$$
$$\rho_n^{\pm}(z_0, z_1, t) = (\overline{z}_0, \overline{z}_1, t),$$
$$\mu_n^{\pm}(z_0, z_1, t) = (e^{\pm 2\pi i t} z_0, z_1, t), \qquad \lambda_n^{\pm}(z_0, z_1, t) = (z_0, e^{\pm 2\pi i t} z_1, t).$$

Note that $\mu_n^{\pm}$ and $\lambda_n^{\pm}$ are Dehn twists in $C_n^{\pm}$ with traces $S^1 \times \{1\}$ and $\{1\} \times S^1$, respectively.

Lemma 9.4.

$$(\varepsilon_n^+)^2 = \lambda_n^+, \quad (\varepsilon_n^-)^2 = \mu_n^-, \quad (\rho_n^{\pm})^2 = id,$$
$$\rho_n^{\pm} \circ \varepsilon_n^{\pm} = (\varepsilon_n^{\pm})^{-1} \circ \rho_n^{\pm}, \quad \mu_n^{\pm} \circ \lambda_n^{\pm} = \lambda_n^{\pm} \circ \mu_n^{\pm},$$
$$\rho_n^{\pm} \circ \mu_n^{\pm} = (\mu_n^{\pm})^{-1} \circ \rho_n^{\pm}, \quad \rho_n^{\pm} \circ \lambda_n^{\pm} = (\lambda_n^{\pm})^{-1} \circ \rho_n^{\pm},$$
$$\varepsilon_n^{\pm} \circ \mu_n^{\pm} = \mu_n^{\pm} \circ \varepsilon_n^{\pm}, \qquad \varepsilon_n^{\pm} \circ \lambda_n^{\pm} = \lambda_n^{\pm} \circ \varepsilon_n^{\pm}.$$

Proof. Compute. □

Now define diffeomorphisms α_n, β_n, and γ_n of X_n, as follows.

$$\alpha_n(x) = \begin{cases} \rho_n^-(x) & \text{if } x \in C_n^-, \\ \overline{\alpha}_n(x) & \text{if } x \in X_n^0, \\ \varepsilon_{n-1}^+(x) & \text{if } x \in C_{n-1}^+. \end{cases}$$

$$\beta_n(x) = \begin{cases} \varepsilon_n^-(x) & \text{if } x \in C_n^-, \\ \overline{\beta}_n(x) & \text{if } x \in X_n^0, \\ \rho_{n-1}^+(x) & \text{if } x \in C_{n-1}^+. \end{cases}$$

$$\gamma_n(x) = \begin{cases} \varepsilon_n^-(\rho_n^-(x)) & \text{if } x \in C_n^-, \\ \overline{\gamma}_n(x) & \text{if } x \in X_n^0, \\ \varepsilon_{n-1}^+(\rho_{n-1}^+(x)) & \text{if } x \in C_{n-1}^+. \end{cases}$$

Define diffeomorphisms α_0, β_0, and γ_0 of V_0, as follows.

$$\alpha_0(x) = \begin{cases} \rho_0^-(x) & \text{if } x \in C_0^-, \\ \overline{\alpha}_0(x) & \text{if } x \in V_0^0. \end{cases}$$

$$\beta_0(x) = \begin{cases} \varepsilon_0^-(x) & \text{if } x \in C_0^-, \\ \overline{\beta}_0(x) & \text{if } x \in V_0^0. \end{cases}$$

$$\gamma_0(x) = \begin{cases} \varepsilon_0^-(\rho_0^-(x)) & \text{if } x \in C_0^-, \\ \overline{\gamma}_0(x) & \text{if } x \in V_0^0. \end{cases}$$

Lemma 9.5. *Let* $n \geq 1$.

μ_n^-, λ_n^-, μ_{n-1}^+, *and* λ_{n-1}^+ *all commute.*

$\alpha_n^2 = \lambda_{n-1}^+$, $\beta_n^2 = \mu_n^-$, $\gamma_n^2 = id$

$\alpha_n \circ \beta_n = (\mu_n^-)^{-1} \circ \gamma_n$, $\beta_n \circ \alpha_n = (\lambda_{n-1}^+)^{-1} \circ \gamma_n$

$\alpha_n \circ \gamma_n = (\mu_n^-)^{-1} \circ \lambda_{n-1}^+ \circ \beta_n$, $\gamma_n \circ \alpha_n = \beta_n$

$\beta_n \circ \gamma_n = \mu_n^- \circ (\lambda_{n-1}^+)^{-1} \circ \alpha_n$, $\gamma_n \circ \beta_n = \alpha_n$

$\alpha_n \circ \mu_n^- = (\mu_n^-)^{-1} \circ \alpha_n$, $\beta_n \circ \mu_n^- = \mu_n^- \circ \beta_n$, $\gamma_n \circ \mu_n^- = (\mu_n^-)^{-1} \circ \gamma_n$

$\alpha_n \circ \lambda_n^- = (\lambda_n^-)^{-1} \circ \alpha_n$, $\beta_n \circ \lambda_n^- = \lambda_n^- \circ \beta_n$, $\gamma_n \circ \lambda_n^- = (\lambda_n^-)^{-1} \circ \gamma_n$

$\alpha_n \circ \mu_{n-1}^+ = \mu_{n-1}^+ \circ \alpha_n$, $\beta_n \circ \mu_{n-1}^+ = (\mu_{n-1}^+)^{-1} \circ \beta_n$, $\gamma_n \circ \mu_{n-1}^+ = (\mu_{n-1}^+)^{-1} \circ \gamma_n$

$\alpha_n \circ \lambda_{n-1}^+ = \lambda_{n-1}^+ \circ \alpha_n$, $\beta_n \circ \lambda_{n-1}^+ = (\lambda_{n-1}^+)^{-1} \circ \beta_n$, $\gamma_n \circ \lambda_{n-1}^+ = (\lambda_{n-1}^+)^{-1} \circ \gamma_n$

Proof. Compute, using the previous lemma. □

Lemma 9.6. μ_0^- *and* λ_0^- *commute.*

$\alpha_0^2 = id$, $\beta_0^2 = \mu_0^-$, $\gamma_0^2 = id$

$\alpha_0 \circ \beta_0 = (\mu_0^-)^{-1} \circ \gamma_0$, $\beta_0 \circ \alpha_0 = \gamma_0$

$\alpha_0 \circ \gamma_0 = (\mu_0^-)^{-1} \circ \beta_0$, $\gamma_0 \circ \alpha_0 = \beta_0$

$\beta_0 \circ \gamma_0 = \mu_0^- \circ \alpha_0$, $\gamma_0 \circ \beta_0 = \alpha_0$

$\alpha_0 \circ \mu_0^- = (\mu_0^-)^{-1} \circ \alpha_0$, $\beta_0 \circ \mu_0^- = \mu_0^- \circ \beta_0$, $\gamma_0 \circ \mu_0^- = (\mu_0^-)^{-1} \circ \gamma_0$

$\alpha_0 \circ \lambda_0^- = (\lambda_0^-)^{-1} \circ \alpha_0$, $\beta_0 \circ \lambda_0^- = \lambda_0^- \circ \beta_0$, $\gamma_0 \circ \lambda_0^- = (\lambda_0^-)^{-1} \circ \gamma_0$

Proof. Compute a little more. □

Let $\widehat{\mathscr{H}}(X_n, S_n)$ be the group of diffeomorphisms of (X_n, S_n) whose restrictions to each component of ∂X_n lie in the group generated by ρ, modulo isotopies which are constant on ∂X_n. Let $\widehat{\mathscr{H}}(V_N)$ be the group of diffeomorphisms of V_N whose restrictions to S_N lie in the group generated by ρ, modulo isotopies which are constant on S_N. Note that $[\alpha_0] = [\gamma_0] = [\overline{\rho_0}]$ and $[\beta_0] = [id]$ in $\widehat{\mathscr{H}}(V_0)$.

Lemma 9.7.

(1) *There is an exact sequence* $0 \to \mathbf{Z}^4 \to \widehat{\mathscr{H}}(X_n, S_n) \to \mathscr{H}(X_n, S_n) \to 0$.

(2) $\mathbf{Z}^4$ *has basis* $\{[\mu_n^-], [\lambda_n^-], [\mu_{n-1}^+], [\lambda_{n-1}^+]\}$.

(3) $[\alpha_n]$ *and* $[\beta_n]$ *project to* $[\overline{\alpha}_n]$ *and* $[\overline{\beta}_n]$; *they, together with the elements in* (2), *generate* $\widehat{\mathscr{H}}(X_n, S_n)$.

(4) $\widehat{\mathscr{H}}(X_n, S_n)$ *has a presentation with these generators and the following relations.*

$[\mu_n^-]$, $[\lambda_n^-]$, $[\mu_{n-1}^+]$, *and* $[\lambda_{n-1}^+]$ *all commute.*

$[\alpha_n]^2 = [\lambda_{n-1}^+]$, $[\beta_n]^2 = [\mu_n^-]$

$[\alpha_n][\beta_n] = [\mu_n^-]^{-1}[\lambda_{n-1}^+][\beta_n][\alpha_n]$

$$[\alpha_n][\mu_n^-][\alpha_n]^{-1} = [\mu_n^-]^{-1}, \quad [\beta_n][\mu_n^-][\beta_n]^{-1} = [\mu_n^-]$$
$$[\alpha_n][\lambda_n^-][\alpha_n]^{-1} = [\lambda_n^-]^{-1}, \quad [\beta_n][\lambda_n^-][\beta_n]^{-1} = [\lambda_n^-]$$
$$[\alpha_n][\mu_{n-1}^+][\alpha_n]^{-1} = [\mu_{n-1}^+], \quad [\beta_n][\mu_{n-1}^+][\beta_n]^{-1} = [\mu_{n-1}^+]^{-1}$$
$$[\alpha_n][\lambda_{n-1}^+][\alpha_n]^{-1} = [\lambda_{n-1}^+], \quad [\beta_n][\lambda_{n-1}^+][\beta_n]^{-1} = [\lambda_{n-1}^+]^{-1}$$

(5) *Every element of* $\widehat{\mathscr{H}}(X_n, S_n)$ *has unique normal form*

$$[\mu_n^-]^{a_n^-}[\lambda_n^-]^{b_n^-}[\mu_{n-1}^+]^{a_{n-1}^+}[\lambda_{n-1}^+]^{b_{n-1}^+}[\alpha_n]^{p_n}[\beta_n]^{q_n},$$

where $p_n, q_n \in \{0, 1\}$.

Proof. The fact that the Dehn twists about ∂X_n contribute a $\mathbf{Z}^4$ subgroup follows from Lemma 4.7. The remainder of (1), (2), and (3) is clear. The relations in (4) follow from Lemma 9.5, so there is a homomorphism from the abstract group H they present to $\widehat{\mathscr{H}}(X_n, S_n)$. The fact that it is an isomorphism then follows from the following diagram.

$$\begin{array}{ccccccc} 0 & \to & \mathbf{Z}^4 & \to & H & \to & \mathbf{Z}_2 \oplus \mathbf{Z}_2 = 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathbf{Z}^4 & \to & \widehat{\mathscr{H}}(X_n, S_n) & \to & \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to 0. \end{array}$$

Clearly every element of $\widehat{\mathscr{H}}(X_n, S_n)$ can be written in the form given in (5). The exponents p_n and q_n are unique for homological reasons and therefore the remaining exponents are also unique. □

Lemma 9.8. $\widehat{\mathscr{H}}(V_N) \cong \mathbf{Z}_2$, *generated by* $[\overline{\rho}_N]$.

Proof. Omitted. □

Lemma 9.9. *If* $r(c) = \overline{c} = \{c_n\}$, *then* $c\{(a_n, b_n)\}c^{-1} = \{(-1)^{c_n}(a_n, b_n)\}$.

Proof. c is represented by some diffeomorphism g of W such that $g(V_n) = V_n$ for all $n \geq N$. Isotop g so that its restriction to each S_n is in the group generated by ρ. This gives rise to an element $([g_N], [g_{N+1}], \cdots, [g_n], \cdots)$ of $\widehat{\mathscr{H}}(V_N) \times \prod_{n=N+1}^{\infty} \widehat{\mathscr{H}}(X_n, S_n)$. There are infinitely many such elements, but they all differ by Dehn twists about the S_n, and so they all induce the same automorphism of $\mathscr{D}(W; V)$. Let $\{(a_n, b_n)\} \in \mathscr{D}(W; V)$. Consider $n > N$.

If $c_n = 1$, then $g_n = \alpha_n$ or γ_n and $g_{n+1} = \beta_{n+1}$ or $\gamma_{n+1} = \mu_{n+1}^- \circ \alpha_{n+1} \circ \beta_{n+1}$. By Lemma 9.5, $g_n \circ \mu_n^- \circ g_n^{-1} = (\mu_n^-)^{-1}$ and $g_n \circ \lambda_n^- \circ g_n^{-1} = (\lambda_n^-)^{-1}$, while $g_{n+1} \circ \mu_{n+1}^+ \circ g_{n+1}^{-1} = (\mu_{n+1}^+)^{-1}$ and $g_{n+1} \circ \lambda_{n+1}^+ \circ g_{n+1}^{-1} = (\lambda_{n+1}^+)^{-1}$. Thus conjugation by $[g]$ sends Dehn twists about S_n to their inverses.

If $c_n = 0$, then $g_n = id$ or β_n and $g_{n+1} = id$ or α_{n+1}. By Lemma 9.5, $g_n \circ \mu_n^- \circ g_n^{-1} = \mu_n^-$ and $g_n \circ \lambda_n^- \circ g_n^{-1} = \lambda_n^-$, while $g_{n+1} \circ \mu_{n+1}^+ \circ g_{n+1}^{-1} = \mu_{n+1}^+$ and $g_{n+1} \circ \lambda_{n+1}^+ \circ g_{n+1}^{-1} = \lambda_{n+1}^+$. Thus conjugation by $[g]$ sends Dehn twists about S_n to themselves. □

Lemma 9.10. *For every* $\overline{c} \in \overline{\mathscr{F}}(W; V)$ *there exists* $c \in \overline{\mathscr{F}}(W; V)$ *such that* $r(c) = \overline{c}$ *and*

$$c^2 = \left\{ \frac{1 + (-1)^{c_n}}{2} (c_{n-1}, c_{n+1}) \right\}.$$

Proof. Represent $\overline{c}$ by $(c_0, c_1, c_2, \dots) \in \prod_{n=0}^{\infty} \mathbf{Z}_2$. This corresponds to an element $([\overline{g}_0], [\overline{g}_1], [\overline{g}_2], \dots)$ of $\overline{\mathscr{F}}_0(W; V)$, where each $\overline{g}_n$ is one of id, $\overline{\alpha}_n$, $\overline{\beta}_n$, or $\overline{\gamma}_n = \overline{\alpha}_n \circ \overline{\beta}_n$. Let g_n be the corresponding diffeomorphism id, α_n,

β_n, or $\gamma_n = \mu_n^- \circ \alpha_n \circ \beta_n$. One thus obtains an element $([g_0], [g_1], [g_2], \dots)$ of $\widehat{\mathscr{H}}(V_0) \times \prod_{n=1}^{\infty} \widehat{\mathscr{H}}(X_n, S_n)$ which gives rise to a diffeomorphism g of W which is defined up to isotopies constant on the S_n. By allowing isotopies which preserve each V_n but need not be constant on the S_n one obtains an element of $\mathscr{F}_0(W; V)$ and therefrom an element c of $\mathscr{F}(W; V)$.

Since we are working modulo direct sums, it is sufficient to consider $n > 1$.

Suppose $c_n = 0$. This implies that the restrictions of $\overline{g}_n$ and $\overline{g}_{n+1}$ to S_n are isotopic to the identity and thus that $g_n = id$ or β_n ($c_{n-1} = 0$ or 1, respectively) and $g_{n+1} = id$ or α_{n+1} ($c_{n+1} = 0$ or 1). Therefore $[g_n]^2 = [id]$ or $[\mu_n^-]$, contributing $(0, 0)$ or $(1, 0)$, i.e., $(c_{n-1}, 0)$, to the group of Dehn twists about S_n, and $[g_{n+1}]^2 = [id]$ or $[\lambda_n^+]$, contributing $(0, 0)$ or $(0, 1)$, i.e., $(0, c_{n+1})$. Thus the total contribution is (c_{n-1}, c_{n+1}).

Suppose $c_n = 1$. Then the restrictions of $\overline{g}_n$ and $\overline{g}_{n+1}$ to S_n are isotopic to ρ and thus $g_n = \alpha_n$ or γ_n and $g_{n+1} = \beta_{n+1}$ or γ_{n+1}. Therefore $[g_n]^2 = [\lambda_{n-1}^+]$ or $[id]$ and $[g_{n+1}]^2 = [\mu_{n+1}^-]$ or $[id]$, in all cases contributing $(0, 0)$. $\square$

Lemma 9.11. *Let c be as in the previous lemma. Then $c' \in \mathscr{F}(W; V)$ satisfies $r(c') = r(c)$ and $(c')^2 = c^2$ if and only if c and c' differ by an element of $\mathscr{D}(W; V)$ of the form*

$$\left\{\frac{1 - (-1)^{c_n}}{2}(a_n, b_n)\right\}.$$

Proof. c' is represented by a diffeomorphism g' of W such that $g'(V_n) = V_n$ for $n \geq N$. We may assume that the restriction of g' to S_n is id or ρ. This gives an element $([g'_N], [g'_{N+1}], \dots, [g'_n], \dots)$ of $\widehat{\mathscr{H}}(V_N) \times \prod_{n=N+1}^{\infty} \widehat{\mathscr{H}}(X_n, S_n)$. (There are of course infinitely many such elements.)

Suppose $c' = c\{(1 - (-1)^{c_n})(a_n, b_n)/2\}$. Consider $n > N$.

If $c_n = 0$, then in $X_n^0 \cup C_n \cup X_{n+1}^0 [g'_n] = [g_n] = [id]$ or $[\beta_n]$ and $[g'_{n+1}] = [g_{n+1}] = [\alpha_{n+1}]$ or $[\gamma_{n+1}]$. It then follows as in the proof of Lemma 9.10 that $[c']^2$ has the required nth coordinate.

If $c_n = 1$, then in $X_n^0 \cup C_n \cup X_{n+1}^0$ one can take $[g'_n] = [g_n][\mu_n^-]^{a_n}[\lambda_n^-]^{b_n}$ and $[g'_{n+1}] = [g_{n+1}]$, where $[g_n] = [\alpha_n]$ or $[\gamma_n]$ and $[g_{n+1}] = [\beta_{n+1}]$ or $[\gamma_{n+1}]$. Since conjugation of $[\mu_n^-]$ and $[\lambda_n^-]$ by $[\alpha_n]$ or $[\gamma_n]$ sends these elements to their inverses, one computes that $[g'_n]^2 = [g_n]^2$ and so the result follows from Lemma 9.10. Note that one could alternatively take $[g'_n] = [g_n]$ and $[g'_{n+1}] = [g_{n+1}][\mu_n^+]^{a_n}[\lambda_n^+]^{b_n}$. The results are the same.

Now assume that c' satisfies $r(c') = r(c)$ and $(c')^2 = c^2$. Then for $n > N$, $[g'_n]$ has normal form

$$[\mu_n^-]^{a_n^-}[\lambda_n^-]^{b_n^-}[\mu_{n-1}^+]^{a_{n-1}^+}[\lambda_{n-1}^+]^{b_{n-1}^+}[\alpha_n]^{p_n}[\beta_n]^{q_n}.$$

One computes that $[g'_n]^2$ has normal form

$$\begin{array}{ll}
[\mu_n^-]^{2a_n^-}[\lambda_n^-]^{2b_n^-}[\mu_{n-1}^+]^{2a_{n-1}^+}[\lambda_{n-1}^+]^{2b_{n-1}^+} & \text{if } p_n = 0 \text{ and } q_n = 0, \\
{[\mu_{n-1}^+]}^{2a_{n-1}^+}[\lambda_{n-1}^+]^{2b_{n-1}+1} & \text{if } p_n = 1 \text{ and } q_n = 0, \\
{[\mu_n^-]}^{2a_n^-+1}[\lambda_n^-]^{2b_n^-} & \text{if } p_n = 0 \text{ and } q_n = 1, \\
{[id]} & \text{if } p_n = 1 \text{ and } q_n = 1.
\end{array}$$

It follows that $[g'_n]^2$ contributes the following to the Dehn twists about S_n.

$$\begin{array}{ll} (2a_n^-, 2b_n^-) & \text{if } [\overline{g}_n] = [id], \\ (0, 0) & \text{if } [\overline{g}_n] = [\overline{\alpha}_n]. \end{array}$$

$$\begin{array}{ll} (2a_n^- + 1, 2b_n^-) & \text{if } [\overline{g}_n] = [\overline{\beta}_n], \\ (0, 0) & \text{if } [\overline{g}_n] = [\overline{\gamma}_n]. \end{array}$$

The contributions to the Dehn twists about S_{n-1} are as follows.

$$\begin{array}{ll} (2a_{n-1}^+, 2b_{n-1}^+) & \text{if } [\overline{g}_n] = [id], \\ (2a_{n-1}^+, 2b_{n-1}^+ + 1) & \text{if } [\overline{g}_n] = [\overline{\alpha}_n], \\ (0, 0) & \text{if } [\overline{g}_n] = [\overline{\beta}_n], \\ (0, 0) & \text{if } [\overline{g}_n] = [\overline{\gamma}_n]. \end{array}$$

Suppose $c_n = 0$. Then the total contributions to the Dehn twists about S_n are as follows.

$$\begin{array}{ll} (2a_n^-, 2b_n^-) + (2a_n^+, 2b_n^+) & \text{if } [\overline{g}_n] = [id] \text{ and } [\overline{g}_{n+1}] = [id], \\ & ((c_{n-1}, c_{n+1}) = (0, 0)); \\ (2a_n^- + 1, 2b_n^-) + (2a_n^+, 2b_n^+) & \text{if } [\overline{g}_n] = [\overline{\beta}_n] \text{ and } [\overline{g}_{n+1}] = [id], \\ & ((c_{n-1}, c_{n+1}) = (1, 0)); \\ (2a_n^-, 2b_n^-) + (2a_n^+, 2b_n^+ + 1) & \text{if } [\overline{g}_n] = [id] \text{ and } [\overline{g}_{n+1}] = [\overline{\alpha}_{n+1}], \\ & ((c_{n-1}, c_{n+1}) = (0, 1)); \\ (2a_n^- + 1, 2b_n^-) + (2a_n^+, 2b_n^+ + 1) & \text{if } [\overline{g}_n] = [\overline{\beta}_n] \text{ and } [\overline{g}_{n+1}] = [\overline{\alpha}_{n+1}], \\ & ((c_{n-1}, c_{n+1}) = (1, 1)). \end{array}$$

In all cases for the total contribution to equal (c_{n-1}, c_{n+1}) one must have $(a_n^-, b_n^-) = -(a_n^+, b_n^+)$. Thus the contributions of g'_n and g'_{n+1} to the Dehn twists about S_n cancel, and so one may take $[g'] = [g]$. Therefore c' and c can only differ for $c_n = 1$, as required. □

Lemma 9.12. *There is an involution γ of W such that $r([\gamma]) = \{(1, 1, 1, \dots)\} \in \overline{\mathscr{F}}(W; V)$. The finite subgroups of $\mathscr{H}(W)$ are precisely the $\mathbf{Z}_2$ subgroups generated by elements of the form $[\gamma]\{(a_n, b_n)\}$. Each of these elements is represented by an involution of W.*

Proof. Let $\gamma = \gamma_n$ for all $n \geq 0$. It is easily checked that γ is an involution and $r([\gamma]) = \{(1, 1, 1, \dots)\}$.

Suppose c' is an element of $\mathscr{H}(W)$ of finite order. Then $r(c') = \overline{c} = \{c_n\} \neq 0$. Let c be as in Lemma 9.10. Then $c' = cd$ for some $d \in \mathscr{D}(W; V)$. Suppose $d = \{(a_n, b_n)\}$. Then by Lemma 9.9 one computes that $(c')^2 = \{(1 + (-1)^{c_n})(a_n, b_n)\}$. Thus for large n $(a_n, b_n) = (0, 0)$ if $c_n = 0$, and so d is of the form given in Lemma 9.11. Therefore

$$(c')^2 = c^2 = \left\{\frac{1 + (-1)^{c_n}}{2}(c_{n-1}, c_{n+1})\right\}.$$

If $c_n = 0$ for infinitely many n, then since $\overline{c} \neq 0$ one must have infinitely many n such that $c_n = 0$ and $c_{n+1} = 1$. But then the nth coordinate of c^2 must be $(c_{n-1}, 1)$, which is impossible. Hence $\bar{c} = \{(1, 1, 1, \ldots)\}$, and it follows that c' has the form $[\gamma]\{(a_n, b_n)\}$. It is represented by the diffeomorphism γ' whose restriction to X_n is $\gamma_n \circ (\mu_n^-)^{a_n} \circ (\lambda_n^-)^{b_n}$ and whose restriction to V_0 is $\gamma_0 \circ (\mu_0^-)^{a_0} \circ (\lambda_0^-)^{b_0}$. Using Lemmas 9.5 and 9.6 one computes that $(\gamma')^2 = id$.

Now suppose $\tilde{c}$ is another element of finite order in $\mathscr{H}(W)$. Then $\tilde{c} = [\gamma]\{(\tilde{a}_n, \tilde{b}_n)\}$. One computes that $c'\tilde{c} = \{(\tilde{a}_n - a_n, \tilde{b}_n - b_n)\}$. For this to have finite order, one must have $(\tilde{a}_n, \tilde{b}_n) = (a_n, b_n)$ for large n, and so $\tilde{c} = c'$. Thus the only finite subgroups are the $\mathbf{Z}_2$ subgroups described above. □

10. Torus bundle groups

Theorem 10.1. *Let W be a periodic genus one Whitehead manifold. Then for every torus bundle M over the circle there is a subgroup of $\mathscr{H}(W)$ which is isomorphic to $\pi_1(M)$.*

Proof. For notational convenience we shall assume that V has period one.

$\pi_1(M) \cong (\mathbf{Z} \oplus \mathbf{Z}) \times_\tau \mathbf{Z}$, that is, it is the semidirect product of $\mathbf{Z} \oplus \mathbf{Z}$ and $\mathbf{Z}$ with respect to the automorphism τ of $\pi_1(S^1 \times S^1)$ induced by the monodromy. For a generator t of $\mathbf{Z}$, $t(a, b)t^{-1} = \tau(a, b)$.

This group will be embedded in the subgroup $\mathscr{D}(W; V) \times_\xi \mathbf{Z}$ of $\mathscr{H}(W)$. The $\mathbf{Z}$ subgroup is generated by the class $[h]$ of a minimal shift.

$$\begin{aligned}
&[h]\{(a_0, b_0), (a_1, b_1), (a_2, b_2), \ldots\}[h]^{-1} \\
&\quad = \xi(\{(a_0, b_0), (a_1, b_1), (a_2, b_2), \ldots\}) \\
&\quad = \{(0, 0), (a_0, b_0), (a_1, b_1), \ldots\}.
\end{aligned}$$

Define $\alpha\colon (\mathbf{Z} \oplus \mathbf{Z}) \times_\tau \mathbf{Z} \to \mathscr{D}(W; V) \times_\xi \mathbf{Z}$ by

$$\alpha(a, b) = \{(a, b), \tau^{-1}(a, b), \tau^{-2}(a, b), \ldots\} \in \mathscr{D}(W; V),$$

for $(a, b) \in \mathbf{Z} \oplus \mathbf{Z}$ and $\alpha(t) = [h]$. The restriction of α to each factor is clearly one-to-one.

$$\begin{aligned}
\alpha(\tau(a, b)) &= \{\tau(a, b), (a, b), \tau^{-1}(a, b), \ldots\} \\
&= \{(0, 0), (a, b), \tau^{-1}(a, b), \ldots\} \\
&= \xi(\{(a, b), \tau^{-1}(a, b), \tau^{-2}(a, b), \ldots\}) \\
&= \xi(\alpha(a, b)).
\end{aligned}$$

Thus α is a well-defined monomorphism. □

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