A SHORT PROOF OF ZHELUDEV'S THEOREM

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Abstract. We give a short proof of Zheludev's theorem that states the existence of precisely one eigenvalue in sufficiently distant spectral gaps of a Hill operator subject to certain short-range perturbations. As a by-product we simultaneously recover Rofe-Beketov's result about the finiteness of the number of eigenvalues in essential spectral gaps of the perturbed Hill operator. Our methods are operator theoretic in nature and extend to other one-dimensional systems such as perturbed periodic Dirac operators and weakly perturbed second order finite difference operators. We employ the trick of using a selfadjoint Birman-Schwinger operator (even in cases where the perturbation changes sign), a method that has already been successfully applied in different contexts and appears to have further potential in the study of point spectra in essential spectral gaps.

Our main hypothesis reads:
(I) Let \( V \in L^1_{\text{loc}}(\mathbb{R}) \) be real-valued and of period \( a > 0 \), and suppose \( W \in L^1(\mathbb{R}, (1 + |x|)\,dx) \) to be real-valued, \( W \neq 0 \) on a set of positive Lebesgue measure.

Given \( V \), one defines the Hill operator \( H_0 \) in \( L^2(\mathbb{R}) \) as the form sum of the Laplacian in \( L^2(\mathbb{R}) \),

\[
-\frac{d^2}{dx^2} \text{ on } H^2(\mathbb{R}),
\]

and the operator of multiplication by \( V \),

\[
H_0 := -\frac{d^2}{dx^2} + V.
\]

(To be more precise, since \( V \) is not assumed to be continuous, we should define \( H_0 \) as a direct integral over reduced operators on \( L^2([0, a]) \), see [12, §XIII.16].)

Similarly, the perturbed Hill operator \( H_g \) is defined as the form sum in \( L^2(\mathbb{R}) \)

\[
H_g := H_0 + gW, \quad g > 0.
\]

Standard spectral theory [2, 10, 11, 12] then yields that

\[
\sigma(H_0) = \sigma_{\text{ac}}(H_0) = \bigcup_{n \in \mathbb{N}} [E_{2(n-1)}, E_{2n-1}],
\]

\[
-\infty < E_0 < E_1 < E_2 < E_3 \leq E_4 < \cdots
\]
The spectral gaps of $H_0$ (the essential spectral gaps of $H_g$) are denoted by

$$\rho_n := \{E_{2n-1}, E_{2n}\}, \quad E_{2n-1} < E_{2n}, \quad n \in \mathbb{N}. $$

Moreover one has

$$\sigma_p(H_g) \subset \bigcup_{n \in \mathbb{N}_0} \rho_n$$

and all eigenvalues of $H_g$ are simple. (Here $\sigma(\cdot)$, $\sigma_{ac}(\cdot)$, $\sigma_{sc}(\cdot)$, and $\sigma_p(\cdot)$ denote the spectrum, absolutely continuous spectrum, singularly continuous spectrum, and point spectrum (the set of eigenvalues) respectively.) Following the usual terminology we call $\rho_n$ an open spectral gap whenever $\rho_n \neq \emptyset$.

The purpose of this paper is to give a short proof of the following theorem that summarizes results of Firsova, Rofe-Beketov, and Zeludev:

**Theorem 1** [3, 4, 6, 13, 14, 17, 18]. Assume Hypothesis (I). Then

(i) $H_g$ has finitely many eigenvalues in each open gap $\rho_n$, $n \geq 0$.

(ii) $H_g$ has at most two eigenvalues in every open gap $\rho_n$ for $n$ large enough.

(iii) If $\int_{\mathbb{R}} dx W(x) \neq 0$, $H_g$, $g > 0$ has precisely one eigenvalue in every open spectral gap $\rho_n$ for $n$ sufficiently large.

**Remark 2.** Parts (i) and (ii) are due to Rofe-Beketov [13]. Part (iii), under the additional conditions $\text{sgn}(W) = \text{constant}$, $W \in L^1(\mathbb{R}; (1 + x^2) dx)$, $V$ piecewise continuous and $W$ bounded is due to Zeludev [17]. In [18] the condition $\text{sgn}(W) = \text{constant}$ has been replaced by $\int_{\mathbb{R}} dx W(x) \neq 0$ but it has been left open as to whether there are one or two eigenvalues in sufficiently distant spectral gaps $\rho_n$. The present version of (iii) was first proved by Firsova [3, 4] (see also [6]) and Rofe-Beketov [14] on the basis of ODE methods. The case of a perturbed Hill operator on the halfline $(0, \infty)$ has also been studied in [8].

Before we give a short proof of Theorem 1 based on operator theoretic methods we need to prepare various well-known results on Hill operators and establish some further notation.

The Green’s function $G_0(z, x, x')$ (the integral kernel of the resolvent $(H_0 - z)^{-1}$) reads

$$G_0(z, x, x') = W(\psi_+(z, \cdot, x_0), \psi_-(z, \cdot, x_0))^{-1} \times \begin{cases} \psi_-(z, x, x_0)\psi_+(z, x', x_0), & x \leq x', \\ \psi_+(z, x, x_0)\psi_-(z, x', x_0), & x \geq x', \end{cases}$$

$$x_0 \in [0, a], \quad z \in \mathbb{R}. $$

Here $W(f, g)$ denotes the Wronskian of $f$ and $g$,

$$W(f, g)(x) := f(x)g'(x) - f'(x)g(x),$$

and $\psi_\pm$ are the Floquet solutions of $H_0$ defined by

$$\psi_\pm(z, x, x_0) := c(z, x, x_0) + \phi_\pm(z, x_0)s(z, x, x_0), \quad z \in \mathbb{R}, \quad x \in \mathbb{R},$$

$$\psi_\pm(z, x_0, x_0) = 1, \quad z \in \mathbb{R},$$

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\[
\phi_{\pm}(z, x_0) := \{\Delta(z) \pm [\Delta(z)^2 - 1]^{1/2} \}
\]

\[ -c(z, x_0 + a, x_0)s(z, x_0 + a, x_0)^{-1}, \quad z \in \mathcal{R}, \]

where \( \Delta \) denotes the discriminant (Floquet determinant) of \( H_0 \),

\[
\Delta(z) := [c(z, x_0 + a, x_0) + s'(z, x_0 + a, x_0)]/2, \quad z \in \mathbb{C},
\]

and \( s, c \) is a fundamental system of distributional solutions of \( H_0f = zf \), \( z \in \mathbb{C} \), with

\[
s(z, x_0, x_0) = 0, \quad s'(z, x_0, x_0) = 1,
\]

\[
c(z, x_0, x_0) = 1, \quad c'(z, x_0, x_0) = 0, \quad z \in \mathbb{C}.
\]

Moreover, \( \psi_{\pm} \) are meromorphic functions on the two-sheeted Riemann surface \( \mathcal{R} \) of \( [\Delta(z)^2 - 1]^{1/2} \) obtained by joining the upper and lower rims of two copies of the cut plane \( \mathbb{C}\setminus\sigma(H_0) \) (or \( \mathbb{C}\setminus\{\rho(H) \cap \mathbb{R}\} \), \( \rho(\cdot) \) the resolvent set) in the usual (crosswise) way. \( \mathcal{R} \) is assumed to be compactified if only finitely many spectral gaps of \( H_0 \) are open, otherwise \( \mathcal{R} \) is noncompact. Since we do not need this Riemann surface explicitly in the following considerations we assume that a suitable choice of cuts has been made and omit further details.

We note that \( s, c \), and \( \Delta \) are entire with respect to \( z \in \mathbb{C} \), and \( \Delta \) and \( G_0 \) are independent of the chosen reference point \( x_0 \in [0, a] \). Especially, by considering a particular open gap \( \rho_n = (E_{2n-1}, E_{2n}) \), \( n \geq 1 \), one can always choose \( x_0 \) in such a way that the zeros of \( s(z, x_0+a, x_0) \) (there is precisely one simple zero in each \( \rho_n \), \( n \geq 1 \), they constitute the Dirichlet eigenvalues of \( H_0 \) restricted to \( (x_0, x_0+a) \) are not at \( \partial \rho_n = \{E_{2n-1}, E_{2n}\} \). (This fact is relevant in (11) and will be needed later on in (20).) From now on, when considering a particular gap \( \rho_n \), we always assume that \( \rho_n \) is open, i.e., \( \rho_n \neq \emptyset \). For simplicity we shall also assume that \( E_0 \geq 1 \) and for notational convenience we introduce \( E_{-1} = 1 \) (in order not to distinguish \( n = 0 \) and \( n \geq 1 \) in the following).

We also note that

\[
W(\psi_+(z, \cdot, x_0), \psi_-(z, \cdot, x_0)) = -2[\Delta(z)^2 - 1]^{1/2}s(z, x_0 + a, x_0)^{-1}, \quad z \in \mathcal{R},
\]

and

\[
-2[\Delta(z)^2 - 1]^{1/2}G_0(z, x, x) = s(z, x + a, x), \quad z \in \mathbb{C}, \quad x \in \mathbb{R}.
\]

Moreover, restricting \( z \) to the upper sheet \( \mathcal{R}_+ \) of \( \mathcal{R} \) from now on, the Floquet solutions \( \psi_{\pm} \) have the particular structure

\[
\psi_{\pm}(z, x, x_0) = e^{\mp a(z)(x-x_0)p_{\pm}(a(z), x, x_0)},
\]

\[
p_{\pm}(a(z), x + a, x_0) = p_{\pm}(a(z), x, x_0), \quad z \in \mathcal{R}_+, \quad x \in \mathbb{R},
\]

where \( a(z) \) is given by

\[
a(z) := a^{-1}\ln\{\Delta(z) + [\Delta(z)^2 - 1]^{1/2}\}, \quad z \in \mathcal{R}_+,
\]

\[
cosh[a(z)a] = \Delta(z), \quad \sinh[a(z)a] = [\Delta(z)^2 - 1]^{1/2},
\]

and the branch of \( [\Delta(z)^2 - 1]^{1/2} \) on \( \mathcal{R}_+ \) is chosen such that

\[
\psi_{\pm}(z, \cdot, x_0) \in L^2(0, \pm\infty), \quad z \in \mathcal{R}_+ \setminus \sigma(H_0).
\]

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\( \alpha \) (resp. \( \alpha - \pi i \)) is positive on open gaps \( \rho_{2n} \) (resp. \( \rho_{2n+1} \)), \( n \in \mathbb{N}_0 \), and monotonic near \( E_0, E_{4n-1}, E_{4n} \) (resp. \( E_{4n-3}, E_{4n-2} \)), \( n \in \mathbb{N} \).

We also note the asymptotic relations

\[
(19) \quad s(\lambda, x_0 + a, x_0) = \lambda^{-1/2} \sin[\lambda^{1/2}a] + O(\lambda^{-1}),
\]

and [18]

\[
(20) \quad p_{\pm}(\alpha(E_{r(n)}), x, x_0)^2 = -\frac{1}{2} \left[ 1 + \frac{a^2}{4n^2\pi^2} \frac{c'(E_{r(n)}), x_0 + a, x_0}{s(E_{r(n)}), x_0 + a, x_0} \right] \cdot \left\{ 1 - \cos((4\pi n/a)(x - x_0) + 2\delta_r(n)) + O(n^{-1}) \right\},
\]

\[
(21) \quad \delta_r(n) := \arctan \left( \frac{2n\pi/a}{s(E_{r(n)}), x_0 + a, x_0} \right) \left| c'(E_{r(n)}), x_0 + a, x_0 \right|^{1/2},
\]

\[ r(n) = 4n - 1, 4n, \]

and similarly for the odd open gaps \( \rho_{2n+1} \), \( n \in \mathbb{N}_0 \). (In order to avoid that \( s(E_{r(n)}), x_0 + a, x_0) = 0 \) in (20), we tacitly made use of the fact that we may choose \( x_0 = x_0(n) \) appropriately without affect \( \Delta \) and the Green's function \( G_0 \) in (8). Such a choice will always be assumed in the following.)

Given these preliminaries we can split the Green's function \( G_0 \) into two parts as follows. For simplicity we only consider even open gaps \( \rho_{2n} \), \( n \in \mathbb{N}_0 \), in details. The analysis for odd gaps \( \rho_{2n+1} \), \( n \in \mathbb{N}_0 \), is completely analogous.

\[
G_0(\lambda, x, x') = -[s(\lambda, x_0 + a, x_0)/2 \sinh(\alpha(\lambda))]p(\alpha(E_{4n}), x, x_0)
\]

\[
\cdot p(\alpha(E_{4n}), x', x_0) + R_0(\lambda, x, x'),
\]

\[
(22) \quad p(\alpha(E_{4n}), x, x_0) := p_+(E_{4n}), x, x_0)
\]

\[
= p_-(\alpha(E_{4n}), x, x_0), \quad x \in \mathbb{R},
\]

for \( \lambda \in [E_{4n} - \varepsilon_n, E_{4n}] \) (\( \lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \)) with \( \varepsilon_n > 0 \) sufficiently small, \( n \in \mathbb{N}_0 \). One has the bound [13, 17]

\[
|R_0(\lambda, x, x')| \leq C|E_{4n-1}|^{-1/2}(1 + |x| + |x'|),
\]

\[ \lambda \in \rho_{2n}, \quad \alpha(\lambda) \in [0, \varepsilon_n], \quad x, x' \in \mathbb{R}, \]

with \( C \) independent of \( n \in \mathbb{N}_0 \). Since Zheludev [17, 18] relies on the estimate (23), he is forced to assume \( W \in L^1(\mathbb{R}; (1 + x^2)^{-1}d\chi) \) in order to make the integral kernel \( |W(x)|^{1/2}R_0(\lambda, x, x')|W(x')|^{1/2} \) to be the integral kernel of a bounded (in fact Hilbert-Schmidt) operator in \( L^2(\mathbb{R}) \). In order to avoid this limitation we shall employ instead a device from [1] and use a different splitting of \( G_0 \):

\[
G_0(\lambda, z, x, x') = G_0(z, x_0, x_0)^{-1}G_0(z, x, x_0)G_0(z, x_0, x')
\]

\[
= G_0, x_0(z, x, x') + G_0, x_0(z, x, x'),
\]

\[
\gamma(z) := -\{s(z, x_0 + a, x_0)/2 \sinh(\alpha(z)a)\},
\]

\[ r(n) = 4n - 1, 4n, \]
where \( G_{0,x_0}(z, x, x') \) denotes the integral kernel of the resolvent of the Dirichlet operator \( H_{0,x_0}^D \) obtained from \( H_0 \) by imposing an additional Dirichlet boundary condition at \( x_0 \). Explicitly we have

\[
P_{x_0}(\lambda, x, x') = \left\{ \begin{array}{ll}
\psi_- (\lambda, x, x_0), & x \leq x_0 \\
\psi_+ (\lambda, x, x_0), & x \geq x_0 \\
\psi_- (\lambda, x', x_0), & x' \leq x_0 \\
\psi_+ (\lambda, x', x_0), & x' \geq x_0 \\
\end{array} \right., \quad \lambda \in \rho_{2n}, \ n \in \mathbb{N}_0,
\]

and, similar to (3.7) in [1],

\[
|G_{0,x_0}(\lambda, x, x')| \leq C |E_{2n-1}|^{-1/2} |x| \leq C |E_{2n-1}|^{-1/2} |x|^1/2 |x'|^{1/2}, \quad \lambda \in \rho_{2n}, \ n \in \mathbb{N}_0, \ \alpha(\lambda) \geq 0 \text{ small enough,}
\]

where \( C \) is independent of \( n \) and

\[
|x_0| := \begin{cases} 
0, & x \leq x_0 \leq x' \text{ or } x' \leq x_0 \leq x, \\
\min(|x - x_0|, |x' - x_0|), & \text{otherwise.}
\end{cases}
\]

In order to derive (26) one separately considers the four regions \( x \leq x' \leq x_0 \), \( x' \leq x \leq x_0 \), \( x_0 \leq x' \leq x \), \( x_0 \leq x \leq x' \) (the cases \( x \leq x_0 \leq x' \), \( x' \leq x_0 \leq x \) being trivial) and uses the mean value theorem to bound

\[
|p_+ (\alpha(\lambda), y, x_0) - p_- (\alpha(\lambda), y, x_0)| \leq D \alpha(\lambda) |y - x_0|, \quad \lambda \in \rho_{2n}, \ \alpha(\lambda) \geq 0 \text{ small enough,}
\]

with \( D \) independent of \( n \in \mathbb{N}_0 \).

Finally, we introduce Birman-Schwinger type operators and related quantities. We distinguish three cases and again study even (open) gaps \( \rho_{2n}, \ n \in \mathbb{N}_0 \) for simplicity.

(a) \( W \leq 0 \). We factorize

\[
w := |W|^{1/2}, \quad W = -w^2,
\]

and define the Birman-Schwinger kernel by

\[
k(\lambda) := -g w (H_0 - \lambda)^{-1} w, \quad \lambda \in \rho_{2n}, \ n \in \mathbb{N}_0, \ g > 0.
\]

Then the selfadjoint Birman-Schwinger kernel satisfies \( k(\lambda) \in \mathcal{B}_2 (L^2(\mathbb{R})) \) (\( \mathcal{B}_2 (\cdot) \) the set of Hilbert-Schmidt operators) and due to (24)-(26)

\[
k(\lambda) = -\gamma(\lambda) g P(\lambda) - g M(\lambda), \quad \lambda \in \rho_{2n},
\]

\[
\gamma(\lambda) = \frac{C_{4n}}{(4n-1)} |\alpha(\lambda)|^{-1}, \quad C_{4n-1} < C_{4n}, \ n \in \mathbb{N}_0,
\]

where \( P(\lambda), \lambda \in \rho_{2n}, \) is a positive rank one projection, \( M(\lambda) \in \mathcal{B}_2 (L^2(\mathbb{R})) \), \( \lambda \in \rho_{2n}, \) is selfadjoint, and

\[
\|M(\lambda)\| \leq C E_{4n-1}^{-1/2}, \quad \lambda \in \rho_{2n}, \ n \in \mathbb{N}_0,
\]

with \( C \) independent of \( n \). (One can show that \( \alpha(\lambda) = d_{4n} |\lambda - E_{4n}|^{1/2} \)

for some constants \( d_{4n} > 0 \).)

(b) \( W \geq 0 \). Introducing the factorization

\[
w := |W|^{1/2}, \quad W = w^2,
\]
one defines

\[ (34) \quad \hat{k}(\lambda) := gw(H_0 - \lambda)^{-1}w, \quad \lambda \in \rho_{2n}, \ n \in \mathbb{N}_0, \ g > 0. \]

Then \( \hat{k}(\lambda) \in \mathcal{B}_2(L^2(\mathbb{R})) \) and (31) and (32) (with \( \gamma \to -\gamma \)) hold again.

(c) \( W = W_+ - W_- \), \( W_+ > 0 \) on sets of positive Lebesgue measure. If necessary, we modify \( W_\pm \) such that

\[ W = W_+ - W_- = \tilde{W}_+ - \tilde{W}_-, \]

\[ \tilde{W}_{\pm} \geq (1 + x^2)^{-1-\varepsilon}, \quad \varepsilon > 0, \quad \tilde{W}_{\pm} \in L^1(\mathbb{R}, (1 + |x|)dx), \]

\[ \tilde{w}_\pm := \tilde{W}_\pm^{1/2}. \]

Following a device of Simon [16] we define the selfadjoint Birman-Schwinger kernel by

\[ (35) \quad K(X) := gw^+(H_0 - gW_+ - Xy \rho_+ e^2), \quad X \in \rho_{2n} \setminus \sigma_p(H_0 - g\tilde{W}_-), \ n \in \mathbb{N}_0, \ g > 0. \]

The fact that \( \tilde{K}(\lambda) \) is selfadjoint (as opposed to the usual choice

\[ |W|^{1/2} \text{sgn}(W')(H_0 - \lambda)^{-1}|W'|^{1/2}, \]

even though \( W \) changes sign, will be of crucial importance below. (This trick has also been employed successfully in [7].)

Given all these preliminaries we now turn to the Proof of Theorem 1. It suffices to treat the even open gaps \( \rho_{2n}, \ n \in \mathbb{N}_0 \).

(A) \( W \leq 0 \). Since

\[ (37) \quad \frac{d}{d\lambda} k(\lambda) = -gw(H_0 - \lambda)^{-2}w \leq 0, \quad \lambda \in \rho_{2n}, \ n \in \mathbb{N}_0, \]

all eigenvalues of \( k(\lambda) \) are monotonically decreasing with respect to \( \lambda \in \rho_{2n} \). Moreover, by the Birman-Schwinger principle [12], \( H_g = H_0 - g|W| \) has an eigenvalue \( E^* \in \rho_n \) iff \( k(E^*) \) has an eigenvalue \(-1\) of the same multiplicity. Since \( E^* \) is necessarily simple, no eigenvalues of \( k(\lambda) \) can cross in \( \rho_n \). Because of (31), \( k(\lambda) \) has precisely one eigenvalue decreasing from \( +\infty \) at \( E_{4n-1} \) to \( O(E_{4n-1}^{1/2}) \) near \( E_{4n} \) and one eigenvalue branch decreasing from \( O(E_{4n-1}^{1/2}) \) near \( E_{4n-1} \) to \( -\infty \) at \( E_{4n} \) (assuming \( n \) large enough such that \( E_{4n-1} >> 1 \)).

The remaining eigenvalues of \( k(\lambda) \) in \( \rho_{2n} \) are of order \( O(E_{4n-1}^{-1/2}) \) for \( n \) large enough. Thus choosing \( n \) sufficiently large, precisely one eigenvalue of \( K(\lambda) \) (the one diverging to \( -\infty \)) will cross \(-1\). Since \( k(\lambda) \) is compact, only finitely many eigenvalues of \( k(\lambda) \) cross \(-1 \) in each gap \( \rho_n \). This proves (i) and (iii) for \( W \leq 0 \).

Since \( W \geq 0 \) can be dealt with analogously, the only difference being that now \( \frac{d}{d\lambda} \hat{k}(\lambda) \geq 0 \) on \( \rho_n \) and hence the eigenvalues of \( \hat{k}(\lambda) \) are monotonically increasing (accounting for no eigenvalue crossing of the line \(-1\) on \( \rho_0 \) since \( \hat{k}(\lambda) \geq 0 \) on \( \rho_0 \)), we immediately turn to the general case.

(B) \( \text{sgn}(W) \neq \text{constant} \).
Throughout the rest of the proof we assume that $\lambda \in \rho_{2n}$ with $n$ large enough unless otherwise stated. We start with the elementary identity

$$K_-(\lambda) := g w_-(H_0 - \tilde{W}_- - \lambda)^{-1} w_-
(38) = -1 + [1 - g w_-(H_0 - \lambda)^{-1} w_-]^{-1}
$$

$$= -1 + [1 + \tilde{k}_-(\lambda)]^{-1}, \quad \lambda \in \rho_{2n} \setminus \{E^*_2\},$$

where $E^*_2$ denotes the unique eigenvalue of $H_0 - g \tilde{W}_-$ in $\rho_{2n}$ determined in Part A. We note that

$$\tilde{k}_-(\lambda) = -\hat{\gamma}(\lambda) g \tilde{P}_-(\lambda) - g \tilde{M}_-(\lambda), \quad \lambda \in \rho_{2n},
(39)$$

where the selfadjoint rank-one operator $\tilde{P}_-(\lambda), \lambda \in \rho_{2n}$, has the integral kernel

$$\hat{\gamma}(\lambda) := \gamma(\lambda) \int_R dy \tilde{W}_-(y) P_{x_0}(\lambda, y, y)\]

$$w_-(x) P_{x_0}(\lambda, x, x') w_-(x'), \quad \lambda \in \rho_{2n},
(40)$$

$$and \tilde{M}_-(\lambda) \in \mathcal{B}_2(L^2(\mathbb{R})), \lambda \in \rho_{2n}, is selfadjoint with integral kernel

$$\tilde{w}_-(x) G_{0,x_0}(\lambda, x, x') \tilde{w}_-(x'), \quad \lambda \in \rho_{2n}.
(42)$$

Next we introduce the orthogonal projection

$$\tilde{Q}_-(\lambda) := 1 - \tilde{P}_-(\lambda), \quad \lambda \in \rho_{2n},
(43)$$

and insert (39) into (38). Assuming $\varepsilon_n > 0$ sufficiently small, a straightforward computation (inverting 1+rank one + perturbation) then yields for the behavior of $\tilde{K}_-(\lambda)$ near the band edges $E_{4n-1}, E_{4n}$,

$$\tilde{K}_-(\lambda) = -1 - \tilde{P}_-(\lambda) + [1 - g \tilde{Q}_-(\lambda) \tilde{M}_-(\lambda) \tilde{Q}_-(\lambda)]^{-1} + O(\gamma(\lambda)^{-1})
(44)$$

$$= -\tilde{P}_-(\lambda) + \tilde{Q}_-(\lambda) [1 - g \tilde{Q}_-(\lambda) \tilde{M}_-(\lambda) \tilde{Q}_-(\lambda)]^{-1} - 1] \tilde{Q}_-(\lambda) + O(\gamma(\lambda)^{-1})
= \left(-1 \begin{array}{c} O \\[1 - g \tilde{Q}_-(\lambda) \tilde{M}_-(\lambda) \tilde{Q}_-(\lambda)]^{-1} - 1 \end{array} \right) + O(\gamma(\lambda)^{-1}),$$

$$\lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}],$$

with respect to the decomposition $L^2(\mathbb{R}) = \tilde{P}_-(\lambda) L^2(\mathbb{R}) \oplus \tilde{Q}_-(\lambda) L^2(\mathbb{R})$. (Here the symbol $O(\gamma(\lambda)^{-1})$ denotes a compact operator with norm bounded by $C|\gamma(\lambda)|^{-1}$.) In particular,

$$\|\tilde{K}_-(\lambda)\| = O(1), \quad \lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}]
(45)$$

for $\varepsilon_n > 0$ sufficiently small. Noticing that

$$\tilde{K}(\lambda) = (\tilde{w}_+/\tilde{w}_-)^{-1} \tilde{K}_-(\lambda) (\tilde{w}_+/\tilde{w}_-), \quad \lambda \in \rho_{2n} \setminus \{E^*\},
(46)$$

we infer for the behavior of $\tilde{K}(\lambda)$ near the band edges $E_{4n-1}, E_{4n}$ that

$$\tilde{K}(\lambda) = -\tilde{P}(\lambda) + (\tilde{w}_+/\tilde{w}_-) \tilde{Q}_-(\lambda) [1 - g \tilde{Q}_-(\lambda) \tilde{M}_-(\lambda) \tilde{Q}_-(\lambda)]^{-1} - 1]
\tilde{Q}_-(\lambda) (\tilde{w}_+/\tilde{w}_-) + O(\gamma(\lambda)^{-1})
(47)$$

$$\lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}].$$
Here $\tilde{P}(\lambda)$ has the integral kernel

$$\left[ \int \mathbb{R} dy \tilde{W}(y) P_{x_0}(\lambda, y, y) \right]^{-1} \tilde{w}_+(x) P_{x_0}(\lambda, x, x') \tilde{w}_+(x'), \quad \lambda \in \rho_{2n},$$

and by using a geometric series expansion one checks that $\tilde{L}(\lambda)$ indeed extends to a $B_2(\mathbb{L}^2(\mathbb{R}))$-operator for $\lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}]$ with $\varepsilon_n > 0$ sufficiently small. Moreover,

$$\|\tilde{L}(\lambda)\| \sim O(E^{-1/2}_{n}), \quad \lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}].$$

It remains to study $\tilde{K}(\lambda)$ near $E_{2n}$. By (38) we have

$$\tilde{K}_-(\lambda) = -\tilde{k}_-(\lambda)[1 + \tilde{k}_-(\lambda)]^{-1}$$

where we used the spectral representation for $\tilde{k}_-(\lambda)$,

$$\tilde{k}_-(\lambda) = \mu_1(\lambda) g P_1(\lambda) + g R_1(\lambda),$$

with $\mu_1(\lambda) g$ the unique eigenvalue branch of $\tilde{k}_-(\lambda)$ diverging to $-\infty$ as $\lambda \uparrow E_{4n}, P_1(\lambda)$ the associated rank one projection onto the corresponding eigenspace, and

$$\|R_1(\lambda)\| \leq C|E_{4n-1}|^{-1/2}, \quad \lambda \in \rho_{2n},$$

by (32). By (46), (50) yields an analogous formula for $\tilde{K}(\lambda), \lambda \in \rho_{2n}\{E_{2n}^*\}$.

Given these results one can now finish the proof (similar to Part A). Since

$$\frac{d}{d\lambda} \tilde{K}(\lambda) = g \tilde{w}_+(H_0 - g \tilde{W}_- - \lambda)^{-2} \tilde{w}_+ \geq 0, \quad \lambda \in \rho_n,$$

all eigenvalues of $\tilde{K}(\lambda)$ are monotonically increasing with respect to $\lambda \in \rho_n$. By the Birman-Schwinger principle, $H_g = H_0 + g W$ has an eigenvalue $E^* \in \rho_n$ iff $\tilde{K}(E^*)$ has an eigenvalue $-1$ with multiplicities preserved. Since $H_g$ has only simple eigenvalues, again no eigenvalue crossing of $\tilde{K}(\lambda)$ occurs in $\rho_n$.

Due to (47), (49), (50), and its analog for $\tilde{K}(\lambda), \tilde{K}(\lambda)$ has precisely one eigenvalue branch $\nu_1(\lambda)$ in $(E_{2n}^*, E_{4n})$ that is monotonically increasing from $-\infty$ at $E_{2n}^*$ to $O(1)$ near $E_{4n}$, all other eigenvalues of $\tilde{K}(\lambda)$ in $(E_{2n}^*, E_{4n})$ being $O(E_{4n-1}^{-1/2})$. Similarly, there is precisely one monotonically increasing eigenvalue branch $\nu_2(\lambda)$ of $\tilde{K}(\lambda)$ in $(E_{4n-1}, E_{2n}^*)$ that is $O(E_{4n-1}^{-1/2})$ near $E_{4n-1}$ and $+\infty$ at $E_{2n}^*$, and precisely one eigenvalue branch $\nu_3(\lambda)$ that is $O(1)$ near $E_{4n-1}$ and $O(E_{4n-1}^{-1/2})$ near $E_{2n}^*$, all other eigenvalues of $\tilde{K}(\lambda)$ being $O(E_{4n-1}^{-1/2})$ throughout $(E_{4n-1}, E_{2n}^*)$. The $O(1)$ branches near $E_{4n}$ are of course due to $\tilde{P}(\lambda)$ in (47) (see also (48)). Given $n$ sufficiently large we thus have the following distinctions:

(a) If $\int \mathbb{R} dx W(x) > O$, then (20), (25), and (48) imply that only $\nu_3(\lambda)$ crosses $-1$. 

(b) If \( \int_{\mathbb{R}} dx W(x) < 0 \), then (20), (25), and (48) imply that only \( \nu_1(\lambda) \) crosses \(-1\).

(c) If \( \int_{\mathbb{R}} dx W(x) = 0 \), then \( \nu_1(\lambda), \nu_3(\lambda) \) may or may not cross \(-1\) and we have either 0, 1, or 2 eigenvalues in \( \rho_{2n} \).

Since \( \tilde{K}(\lambda) \) is compact, only finitely many eigenvalues can cross \(-1\) in each gap \( \rho_n \). This completes the proof of Theorem 1. \( \Box \)

Since one can replace the phrase "for \( n \) large enough" by "\( g > 0 \) sufficiently small" in every step of the above proof, Theorem 1 can also be viewed as a "weak-coupling" result in the following sense:

**Theorem 3.** Assume Hypothesis (I). Then

(i) \( H_g \) has at most two eigenvalues in every open gap \( \rho_n, n \in \mathbb{N}_0 \) for \( g > 0 \) sufficiently small.

(ii) Abbreviate

\[
I(E_{2n}) := \int_{(2n-1)}^{(2n)} dx W(x) p(\alpha(E_{2n}), x, x_0)^2, \quad n \in \mathbb{N}_0,
\]

and assume that \( g > 0 \) is small enough. Then \( H_g \) has no eigenvalues in \( \rho_n = (E_{2n-1}, E_{2n}) \), \( n \in \mathbb{N} \) if \( I(E_{2n-1}) < 0 \) and \( I(E_{2n}) > 0 \), \( H_g \) has precisely one eigenvalue in \( \rho_n \) if \( I(E_{2n-1}) < 0 \) and \( I(E_{2n}) < 0 \) or \( I(E_{2n-1}) > 0 \) and \( I(E_{2n}) > 0 \), and \( H_g \) has two eigenvalues in \( \rho_n \) if \( I(E_{2n-1}) > 0 \) and \( I(E_{2n}) < 0 \). Moreover, \( H_g \) has no eigenvalues in \( \rho_0 = (-\infty, E_0) \) if \( I(E_0) > 0 \) and precisely one eigenvalue in \( \rho_0 \) if \( I(E_0) \leq 0 \).

**Proof.** By the paragraph preceding Theorem 3 we only need to demonstrate the last assertion in the case \( I(E_0) = 0 \). For that purpose we first prove that \( R_0(E_0, x, x') \) (see (22) and (24)) is conditionally positive definite, i.e.,

\[
\int_{\mathbb{R}_2} dx dx' W(x) p(\alpha(E_0), x, x_0) R_0(E_0, x, x') W(x') p(\alpha(E_0), x', x_0) > 0
\]

if \( I(E_0) = \int_{\mathbb{R}} dx W(x) p(\alpha(E_0), x, x_0)^2 = 0 \).

(We also note that \( R_0(E_0, x, x') = G^D_0, x_0 (E_0, x, x') \). In order to prove (55) we invoke the eigenfunction expansion associated with \( H_0 \). Let

\[
f(\cdot) = s - \lim_{R \to \infty} (2\pi)^{-1/2} \int_{|\beta| \leq R} d\beta \hat{f}_\pm(\beta) \Psi_\pm(\beta, \cdot),
\]

\[
\hat{f}_\pm(\cdot) = s - \lim_{R \to \infty} (2\pi)^{-1/2} \int_{|y| \leq R} dy f(y) \Psi_\pm(\cdot, y), \quad f \in L^2(\mathbb{R}),
\]

where

\[
\Psi_\pm(\beta, x) := a^{1/2} \left[ \int_{x_0}^{x_0+a} dy \psi_-(z(\beta), y, x_0) \psi_+(z(\beta), y, x_0) \right]^{-1/2}
\]

\[
\cdot \psi_\pm(z(\beta) x, x_0),
\]

(58) \( \Psi_\pm(\beta, x) = \Psi_+(\beta, x) = \overline{\Psi_\pm(\beta, x)} \), \( \beta \in \mathbb{R} \),

and

(59) \( \cosh[\beta(z)a] = \Delta(z), \quad \sinh[\beta(z)a] = [\Delta(z)^2 - 1]^{1/2} \)
with $\beta(z)$ an appropriate analytic continuation of $\text{arc sinh}\{[\Delta(z)^2 - 1]^{1/2}\}$ to the Riemann surface $\mathcal{R}$ (see, e.g., [5] for more details). If $f \in L^1(\mathbb{R})$ then the integral for $\hat{f}_\pm$ in (56) becomes an ordinary Lebesgue integral over $\mathbb{R}$ since $\Psi_\pm(\beta, x)$ is uniformly bounded in $x \in \mathbb{R}$. (If $V = 0$ then $\Psi_\pm(\beta, x) = e^{\pm i \beta x}$.)

We also note that

$$z(\beta) = E_0 + (2\mathbb{R}_0)^{-1} \beta^2 + O(\beta^4)$$

for some $\mathbb{R}_0 > 0$. Next we define

$$\omega(\cdot) := W(\cdot) p(\alpha(E_0), \cdot, x_0)$$

and compute for $\lambda < E_0$,

$$\int_{\mathbb{R}^2} dx \, dx' \omega(x) R_0(\lambda, x, x') \omega(x') = \int_{\mathbb{R}^2} dx \, dx' \omega(x) G_0(\lambda, x, x') \omega(x')$$

$$= \int_{\mathbb{R}} d\beta |\omega_+(\beta)|^2 |z(\beta) - \lambda|^{-1},$$

where we used (22) together with $I(E_0) = 0$ in the first equality and

$$((H_0 - \lambda)^{-1} \Psi_\pm(\beta(z), x) = [z(\beta) - \lambda]^{-1} \Psi_\pm(\beta(z), x),$$

$$z(\beta) \geq E_0, \beta \in \mathbb{R},$$

$$\omega(\cdot) := W(\cdot) p(\alpha(E_0), \cdot, x_0)$$

and hence

$$\int_{\mathbb{R}^2} dx \, dx' \omega(x) R_0(E_0, x, x') \omega(x')$$

$$= \int_{\mathbb{R}} d\beta |\omega_+(\beta)|^2 |z(\beta) - E_0| > 0$$

by (23) and the monotone convergence theorem. This proves (55). It remains to go through the proof of Theorem 1 step-by-step. In fact, let $E^*_0$ be the unique eigenvalue of $H_0 - g \overline{W}_-$ in $\rho_0 = (-\infty, E^*_0)$ determined by Part A of the proof of Theorem 1. Since (53) remains valid for $n = 0$, and

$$(H_0 - g \overline{W}_- - \lambda)^{-1} \geq 0$$

for $\lambda \in (-\infty, E^*_0)$, we have

$$\tilde{K}(\lambda) \geq 0$$

for $\lambda \in (-\infty, E^*_0)$. Thus no eigenvalue branch of $\tilde{K}(\lambda)$ can cross $-1$ for $\lambda < E^*_0$. In the interval $(E^*_0, E_0)$ there is precisely one eigenvalue branch $\nu_1(\lambda)$ that is monotonically increasing from $-\infty$ at $E^*_0$ to $O(1)$ near $E_0$, all other eigenvalues of $\tilde{K}(\lambda)$ being $O(g)$ throughout $[E^*_0, E_0]$. In order to prove that $\nu_1(\lambda)$ actually crosses
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-1 for \( g > 0 \) small enough we next consider \( \tilde{K}(E_0) = n - \lim_{\lambda \uparrow E_0} \tilde{K}(\lambda) \). In analogy to (44) one proves

\[
\tilde{K}_-(E_0) = -\tilde{P}_-(E_0) + g\tilde{Q}_-(E_0)\tilde{M}_-(E_0)\tilde{Q}_-(E_0) + O(g^2),
\]

where \( O(g^2) \) denotes a compact operator with norm bounded by \( Cg^2 \). This yields

\[
\tilde{K}(E_0) = -\tilde{P}(E_0) + g(\tilde{w}_+/\tilde{w}_-)\tilde{Q}_-(E_0)\tilde{M}_-(E_0)\tilde{Q}_-(E_0)(\tilde{w}_+ / \tilde{w}_-) + O(g^2),
\]

where \( \tilde{P}(E_0) \) is an orthogonal projection with integral kernel (see (22), (25) and (48))

\[
\left[ \int dy \tilde{W}_+(y) p(\alpha(E_0), y, x_0)^2 \right]^{-1} \tilde{w}_+(x)p(\alpha(E_0), x, x_0)p(\alpha(E_0), x', x_0)\tilde{w}_+(x')
\]

since \( I(E_0) = 0 \), and \( \tilde{M}_-, \tilde{Q}_- \) have been introduced in (42), (43). A simple computation then yields

\[
(\tilde{w}_+ p(\alpha(E_0), \cdot, x_0), \tilde{K}(E_0)\tilde{w}_+ p(\alpha(E_0), \cdot, x_0))/||\tilde{w}_+ p(\alpha(E_0), \cdot x_0)||^2
\]

\[
= -1 + g \int \int_{\mathbb{R}^2} dx dx' \omega(x) R_0(E_0, x, x') \omega(x') + O(g^2).
\]

By (55) this indeed proves that \( \nu_1(\lambda) \) crosses \(-1\) for \( g > 0 \) sufficiently small. \( \square \)

Remark 4. To the best of our knowledge the fact that \( R_\infty(E_0, x, x') \) is conditionally positive definite (in the sense of (55)) and that for \( g > 0 \) small enough \( H_g \) has precisely one eigenvalue in \( \rho_0 = (-\infty, E_0) \) if \( I(E_0) = 0 \) appears to be new. It generalizes a corresponding result of [15] (extended in [9]) in the special case where \( V \equiv 0 \).

Evidently, our strategy of using a selfadjoint Birman-Schwinger kernel, even if \( \text{sgn}(W) \neq \text{constant} \), extends to perturbed one-dimensional periodic Dirac operators and weakly perturbed second-order finite difference operators.

Finally, we remark that Theorem 1, in particular, implies that \( N \)-soliton solutions of the Korteweg-de Vries equation relative to a periodic background solution (i.e., relative reflectionless solutions) will in general not decay as \( x \to +\infty \) and \( x \to -\infty \) since by definition they are associated with the insertion of \( N \) eigenvalues in the spectral gaps of the periodic background Hamiltonian.

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REFERENCES


