ON THE THEORY OF FROBENIUS EXTENSIONS
AND ITS APPLICATION TO LIE SUPERALGEBRAS

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ABSTRACT. By using an approach to the theory of Frobenius extensions that emphasizes notions related to associative forms, we obtain results concerning the trace and corestriction mappings and transitivity. These are employed to show that the extension of enveloping algebras determined by a subalgebra of a Lie superalgebra is a Frobenius extension, and to study certain questions in representation theory.

0. INTRODUCTION

The theory of Frobenius extensions, initiated independently by Kasch [10] and Nakayama-Tsuzuku [13], has, aside from its intrinsic value, proven to be a useful tool in the study of groups and modular Lie algebras. The twofold purpose of this note is to extend the abstract theory as well as to point out another application concerning the theory of Lie superalgebras.

In §1 we reformulate some of the basic features of the theory of Frobenius extensions and generalize various results of [2, 15], including results on transitivity of Frobenius extensions. Our approach, which emphasizes associative forms, is particularly useful for the study of the trace and corestriction mappings. The second section is devoted to the investigation of extensions defined by the universal enveloping algebras of a Lie superalgebra L and a subalgebra K containing the space L_0 of even elements. Our main result, namely that their respective enveloping algebras define a Frobenius extension, strikingly parallels recent work on restricted Lie algebras (cf. [3]).

The irreducible modules of the so-called basic classical Lie superalgebras are obtained, according to [8], by inducing from representations of certain subalgebras. By showing that some of the essential features of this theory can be derived from the general principles of the theory of Frobenius extensions, we provide a new conceptual approach to various well-known results.

Frobenius extensions can in particular be used to investigate the cohomology theory of the defining rings. It is, for instance, possible to introduce a complete relative cohomology theory that generalizes well-known principles from the theory of groups (cf. [4, 11]). We shall not dwell on these aspects here, but rather

briefly illustrate in §3 some applications related to more recent developments of the theory.

1. Basic results concerning Frobenius extensions

Throughout this paper all rings are assumed to have an identity that operates via the identity transformation on all modules under consideration. Modules are left modules, unless otherwise stated. Let S be a subring of R, and suppose that α is an automorphism of S. If M is an S-module, we let $_\alpha M$ denote the S-module with a new action defined by $s * m = \alpha(s)m$. Thus, for example, $\mathrm{Hom}_S(R, _\alpha S)$ denotes the set of additive maps $f: R \to S$ such that $f(sr) = \alpha(s)f(r)$ for all $s \in S$, $r \in R$. This is an (R, S)-bimodule via the action $(r \cdot f \cdot s)(x) = f(xr)s$. We say R is an α-*Frobenius extension* of S if

   (i) R is a finitely generated projective S-module, and
   (ii) there exists an isomorphism $\varphi: R \to \mathrm{Hom}_S(R, _\alpha S)$ of (R, S)-bimodules.

Such an extension is referred to as *free* if R is a free S-module. In the literature, Frobenius extensions are frequently defined using right modules. Both notions are equivalent in the sense that every left α-Frobenius extension is a right α^{-1}-Frobenius extension (cf. the remarks after (1.2)).

Given an endomorphism β of S, a β-*associative form from R to S* is a biadditive map $\langle \, , \, \rangle : R \times R \to S$ such that

  (a) $\langle sx, y \rangle = s\langle x, y \rangle$,   (b) $\langle x, ys \rangle = \langle x, y \rangle \beta(s)$,   (c) $\langle xr, y \rangle = \langle x, ry \rangle$

for all $s \in S$, $r, x, y \in R$. We say that $\langle \, , \, \rangle$ is *nondegenerate* if its left and right radicals are trivial.

A β-associative form determines an additive map $\pi: R \to S$; $\pi(r) := \langle r, 1 \rangle = \langle 1, r \rangle$ that satisfies $\pi(sr) = s\pi(r)$ and $\pi(rs) = \pi(r)\beta(s)$ for all $r \in R$ and $s \in S$. Conversely, given any additive map $\pi: R \to S$ with these properties, we can define a β-associative form from R to S by means of $\langle x, y \rangle := \pi(xy)$.

In this section we shall provide conditions pertaining to α^{-1}-associative forms that are tantamount to an extension of rings being an α-Frobenius extension. This approach, which amounts to a refinement of the methods introduced in [15], will be employed later to show that enveloping algebras of Lie superalgebras give rise to Frobenius extensions.

Definition. Let $\langle \, , \, \rangle: R \times R \to S$ be an α^{-1}-associative form. We say that two finite subsets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of R form a *dual projective pair relative to* $\langle \, , \, \rangle$ if

$$r = \sum_{i=1}^{n} y_i \alpha(\langle x_i, r \rangle) = \sum_{i=1}^{n} \langle r, y_i \rangle x_i \quad \text{for all } r \in R.$$

Note that the S-linear mappings $\varphi_i: R \to S$; $\varphi_i(r) := \langle r, y_i \rangle$ together with the set $\{x_1, \ldots, x_n\}$ form a projective basis for R as an S-module.

Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be subsets of R with $R = \sum_{i=1}^{n} Sx_i = \sum_{i=1}^{n} y_i S$. In the ring $M_n(S)$ of $(n \times n)$ matrices with coefficients in S, define $A = [\langle x_i, y_j \rangle]_{1 \leq i, j \leq n}$. If $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ form a dual projective pair relative to the α^{-1}-associative form $\langle \, , \, \rangle$, then the i, j-entry of A^2

is

$$\sum_{k=1}^{n} \langle x_i, y_k \rangle \langle x_k, y_j \rangle = \left\langle \sum_{k=1}^{n} \langle x_i, y_k \rangle x_k, y_j \right\rangle = \langle x_i, y_j \rangle,$$

proving that A is idempotent. In addition, if $r \in R$ and $\langle r, R \rangle = 0$, then $r = \sum_{i=1}^{n} \langle r, y_i \rangle x_i = 0$. Likewise $\langle R, r \rangle = 0$ implies $r = 0$, so $\langle \ , \ \rangle$ is nondegenerate.

Suppose conversely that A is idempotent and $\langle \ , \ \rangle$ is nondegenerate. The above calculation shows $\langle x_i - \sum_{k=1}^{n} \langle x_i, y_k \rangle x_k, y_j \rangle = 0$ for all j. Since $R = \sum_{j=1}^{n} y_j S$, the element $x_i - \sum_{k=1}^{n} \langle x_i, y_k \rangle x_k$ is contained in the left radical of $\langle \ , \ \rangle$. Consequently, $x_i = \sum_{k=1}^{n} \langle x_i, y_k \rangle x_k$ for all i, and since $\sum_{i=1}^{n} S x_i = R$, this implies $r = \sum_{k=1}^{n} \langle r, y_k \rangle x_k$ for any $r \in R$. By the same token, we have $r = \sum_{k=1}^{n} y_k \alpha(\langle x_k, r \rangle)$, so $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ form a dual projective pair.

Theorem 1.1. *Let S be a subring of R and suppose that α is an automorphism of S. Then the following statements are equivalent.*

(a) *R is an α-Frobenius extension of S.*

(b) *There is an α^{-1}-associative form $\langle \ , \ \rangle$ from R to S relative to which a dual projective pair $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ exists.*

(c) *There exist a nondegenerate α^{-1}-associative form $\langle \ , \ \rangle$ from R to S and two subsets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of R with $R = \sum_{i=1}^{n} S x_i = \sum_{i=1}^{n} y_i S$ such that the matrix $A = [\langle x_i, y_j \rangle]_{i,j}$ is equivalent to an idempotent matrix in the sense that there is a left invertible $P \in M_n(S)$ and a right invertible $Q \in M_n(S)$ with PAQ idempotent.*

Proof. (a) $\Rightarrow$ (b) Let $\varphi \colon R \to \mathrm{Hom}_S(R, {}_\alpha S)$ be the defining isomorphism of (R, S)-bimodules. We define a bilinear form $\langle \ , \ \rangle \colon R \times R \to S$ by means of $\langle x, y \rangle := \alpha^{-1}(\varphi(y)(x))$. Using the definition of the bimodule actions, one readily verifies that $\langle \ , \ \rangle$ is an α^{-1}-associative form.

Let $x_1, \ldots, x_n \in R$ and $\varphi_1, \ldots, \varphi_n \in \mathrm{Hom}_S(R, S)$ form a projective basis for the S-module R. Since $\alpha \circ \varphi_i \in \mathrm{Hom}_S(R, {}_\alpha S)$, there exist elements y_i of R for $1 \le i \le n$ such that $\varphi(y_i) = \alpha \circ \varphi_i$. Now

$$r = \sum_{i=1}^{n} \varphi_i(r) x_i = \sum_{i=1}^{n} \alpha^{-1}(\varphi(y_i)(r)) x_i = \sum_{i=1}^{n} \langle r, y_i \rangle x_i$$

for any $r \in R$. Applying $f \in \mathrm{Hom}_S(R, {}_\alpha S)$ to this last equation shows that

$$f = \sum_{i=1}^{n} \varphi(y_i) \cdot f(x_i).$$

If $f = \varphi(r)$, this gives $\varphi(r) = \varphi(\sum_{i=1}^{n} y_i \alpha(\langle x_i, r \rangle))$, whence $r = \sum_{i=1}^{n} y_i \alpha(\langle x_i, r \rangle)$ for any $r \in R$.

The implication (b) $\Rightarrow$ (c) is a direct consequence of the remarks preceding the theorem. To verify (c) $\Rightarrow$ (a) we define $a_1, \ldots, a_n, b_1, \ldots, b_n \in R$ by $(a_1, \ldots, a_n)^t = P(x_1, \ldots, x_n)^t$ and $(b_1, \ldots, b_n) = (y_1, \ldots, y_n)\alpha(Q)$. The left invertibility of P and the right invertibility of $\alpha(Q)$ ensure that $R = \sum_{i=1}^{n} S a_i = \sum_{i=1}^{n} b_i S$. Clearly $[\langle a_i, b_j \rangle]_{1 \le i, j \le n} = PAQ$, and so by the remarks preceding the theorem $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ form a dual projective pair. Define an (R, S)-bimodule map $\varphi \colon R \to \mathrm{Hom}_S(R, {}_\alpha S)$ by

$\varphi(y)(x) := \alpha(\langle x, y \rangle)$. To prove the invertibility of φ we consider the mapping $\psi \colon \mathrm{Hom}_S(R, {}_\alpha S) \to R$; $\psi(f) := \sum_{i=1}^n b_i f(a_i)$. For any $r \in R$, we have

$$\psi(\varphi(r)) = \sum_{i=1}^n b_i \varphi(r)(a_i) = \sum_{i=1}^n b_i \alpha(\langle a_i, r \rangle) = r.$$

Similarly, we obtain for $f \in \mathrm{Hom}_S(R, {}_\alpha S)$:

$$\varphi(\psi(f))(r) = \alpha(\langle r, \psi(f) \rangle) = \sum_{i=1}^n \alpha(\langle r, b_i f(a_i) \rangle)$$

$$= \sum_{i=1}^n \alpha(\langle r, b_i \rangle) f(a_i) = f\left(\sum_{i=1}^n \langle r, b_i \rangle a_i\right) = f(r).$$

Thus φ and ψ are inverses, proving that (a) holds. $\square$

Let $R : S$ be an α-Frobenius extension, let $\langle\ ,\ \rangle$ be the α^{-1}-associative form from R to S in (1.1), and define $\pi \colon R \to S$ by $\pi(r) := \langle r, 1 \rangle = \langle 1, r \rangle$. The maps $\alpha \circ \pi$ and π are customarily referred to as the Frobenius homomorphisms associated to this Frobenius extension. In this paper we shall only refer to π as the Frobenius homomorphism. Our remarks above show the form $\langle\ ,\ \rangle$ is determined by π.

Definition. Let $\langle\ ,\ \rangle \colon R \times R \to S$ be an α^{-1}-associative form. Two subsets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ of R are said to form a *dual free pair relative to* $\langle\ ,\ \rangle$ if

 (i) $R = \sum_{i=1}^n Sx_i = \sum_{i=1}^n y_i S$,
 (ii) $\langle x_i, y_j \rangle = \delta_{ij}$ for $1 \le i, j \le n$.

Note that $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ are bases for R as a left and right S-module, respectively. A direct computation shows that $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ form a dual free pair if and only if they form a dual projective pair and they are bases for R as a left and right S-module, respectively.

The analogue of (1.1) for free Frobenius extensions is the following.

Corollary 1.2. *Let S be a subring of R and let α be an automorphism of S. Then the following statements are equivalent.*

 (a) *R is a free α-Frobenius extension of S.*

 (b) *There is an α^{-1}-associative form $\langle\ ,\ \rangle$ from R to S relative to which a dual free pair $\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\}$ exists.*

 (c) *There exist an α^{-1}-associative form $\langle\ ,\ \rangle$ from R to S and elements $x_1, \dots, x_n, y_1, \dots, y_n \in R$ such that $R = \sum_{i=1}^n Sx_i = \sum_{i=1}^n y_i S$ and the $n \times n$ matrix $[\langle x_i, y_j \rangle]_{i,j}$ has a two-sided inverse in the ring $M_n(S)$.*

Proof. (a) $\Rightarrow$ (b) Define an α^{-1}-associative bilinear form $\langle\ ,\ \rangle \colon R \times R \to S$ as in the proof of (1.1). Let $\{x_1, \dots, x_n\}$ be a basis of the left S-module R. Since φ is surjective there exist $y_1, \dots, y_n$ in R such that $\alpha^{-1} \circ \varphi(y_j)$ is the jth projection from R onto S relative to this basis. Clearly $\langle x_i, y_j \rangle = \delta_{ij}$ for $1 \le i, j \le n$. Since $\{y_1, \dots, y_n\}$ generates the right S-module R as in the proof of (1.1), we obtain (b).

The implication (b) $\Rightarrow$ (c) is trivial. As regards (c) $\Rightarrow$ (a), it follows directly from the invertibility of $[\langle x_i, y_j \rangle]$ that the form $\langle\ ,\ \rangle$ is nondegenerate. An application of (1.1) now yields the desired result. $\square$

Examples. (A) If S is any ring and $R = M_n(S)$, define $\langle x, y \rangle := \mathrm{Tr}(xy)$. If we let e_{ij} denote the matrix with 1 in the i, j position and 0's elsewhere, then $\{e_{ij}\}_{1 \le i, j \le n}$ and $\{e_{ji}\}_{1 \le i, j \le n}$ form a dual free pair. Thus R is a free Frobenius extension of S by (1.2(b)).

(B) It is well known that if G is a group and H is a subgroup of finite index, the group ring $K[G]$ is a free Frobenius extension of $K[H]$. This result can be generalized to the case of any strongly G-graded ring (for example a crossed product) in place of $K[G]$. Recall that a ring R is graded by G if $R = \bigoplus_{g \in G} R_g$, where each R_g is an Abelian subgroup of R and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. We say R is *strongly graded* by G if equality always holds. This condition is easily seen to be equivalent to the existence of $x_{1,g}, \ldots, x_{m(g),g} \in R_g$, $y_{1,g}, \ldots, y_{m(g),g} \in R_{g^{-1}}$ with $\sum_{i=1}^{m(g)} y_{i,g} x_{i,g} = 1$ for each $g \in G$. If R is strongly graded by G and $S = \bigoplus_{g \in H} R_g$, then $R : S$ is an id_S-Frobenius extension.

To see this, we define the Frobenius homomorphism by $\pi(\sum_{g \in G} r_g) := \sum_{g \in H} r_g$. If we let $\{Hg : g \in C\}$ be a complete set of right cosets for H in G, then it is clear that $\{x_{i,g} : g \in C, 1 \le i \le m(g)\}$ and $\{y_{i,g} : g \in C, 1 \le i \le m(g)\}$ form a dual projective pair.

(C) We give an example which employs a method very much like our proof in §2 that enveloping algebras of Lie superalgebras give rise to Frobenius extensions. Let S be a subring of R and suppose that there exist an $x \in R$, a positive integer m, and an automorphism α of S such that

(i) $1, x, \ldots, x^m$ is a basis for R as a left S-module, and
(ii) $sx - x\alpha(s) \in S$ for all $s \in S$.

Then $R : S$ is a free α^m-Frobenius extension.

To see this, first note that we can use (ii) and induction to prove that for any positive integer k,

$$(*) \quad sx^k - x^k \alpha^k(s) \in S + xS + \cdots + x^{k-1}S = S + Sx + \cdots + Sx^{k-1} \quad \text{for all } s \in S.$$

Thus $1, x, \ldots, x^m$ is also a basis for R as a right S-module. Now define $\pi : R \to S$ by $\pi(\sum_{i=0}^m s_i x^i) := s_m$. Clearly π is left S-linear, and $(*)$ implies that for any $s \in S$, there exist $s_0', \ldots, s_{m-1}' \in S$ such that $\sum_{i=0}^m s_i x^i s = s_m \alpha^{-m}(s) x^m + \sum_{i=0}^{m-1} s_i' x^i$. It follows that the form $\langle \, , \, \rangle : R \times R \to S$; $\langle r_1, r_2 \rangle := \pi(r_1 r_2)$, is $(\alpha^m)^{-1}$-associative.

If $0 \le i < j \le m$, then $\langle x^i, x^{m-j} \rangle = 0$, while $\langle x^i, x^{m-i} \rangle = 1$. Consequently the matrix $[\langle x^i, x^{m-j} \rangle]_{0 \le i, j \le m}$ is a lower triangular matrix all of whose diagonal entries are 1. Such a matrix is invertible, and so $R : S$ is a free α^m-Frobenius extension by (1.2).

(D) As a special case of example (C), let K be any ring and consider the extension $R : S$ given by $R = M_2(K)$, and $S = \{[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}] : a, b \in K\}$. This satisfies the hypotheses of (C) if we set $x = [\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}]$ and define $\alpha([\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}]) := [\begin{smallmatrix} a & -b \\ 0 & a \end{smallmatrix}]$. Thus $R : S$ is a free α-Frobenius extension. (The Frobenius homomorphism in this case is defined by $\pi([\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]) = [\begin{smallmatrix} c & a-d \\ 0 & c \end{smallmatrix}]$. If we were to define $\pi' : R \to S$ by $\pi'([\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]) = [\begin{smallmatrix} c & a+d \\ 0 & c \end{smallmatrix}]$, then $R : S$ could be shown to be an id_S-Frobenius extension as well.)

One interesting facet of this example is that $S \cong K[t]/(t^2)$, where t is an

indeterminate, so the global dimension gl.dim S is infinite, while gl.dim $R =$ gl.dim K. Since $R : S$ is a Frobenius extension, the relative global dimension of $R : S$ is 0 or ∞. Consequently, this example shows no simple relation need hold between the actual global dimensions of R and S. In addition, if we take K of finite global dimension, we obtain an example of a finite free ring extension where the extension ring has smaller global dimension than the base ring. This is a simpler example than those usually cited in the literature, as well as being valid in any characteristic (cf. [12, 7.2.7]).

(E) If we let K be a field and V a vector space of dimension n, then one can easily iterate example (C) and show that the Grassmann algebra $\Lambda(V)$ is a Frobenius algebra over K, with the Frobenius homomorphism $\pi : R \to K$ determined by the projection onto $\Lambda^n(V)$.

Remarks. (1) If $R : S$ is an α-Frobenius extension with form $\langle\ ,\ \rangle$, then one readily sees from (1.1) that the extension defined by their opposite rings is an α^{-1}-Frobenius extension with defining form $(x, y) := \alpha(\langle y, x \rangle)$. This in particular implies the well-known fact that Frobenius extensions can be equally well defined in terms of right modules.

(2) Suppose R_i is an α_i-Frobenius extension of S_i for $1 \le i \le n$, and each S_i and each R_i is an algebra over the commutative ring K, and every α_i is K-linear. Then the tensor product over K of the R_i is an α-Frobenius extension of the tensor product over K of the S_i, where α is the tensor product of the α_i. This is easy to see using (1.1): a dual projective pair for $R : S$ is obtained by tensoring together the elements of the dual projective pairs for the extensions $R_i : S_i$.

It now readily follows that if $R : S$ is an α-Frobenius extension, then $R \otimes_{\mathbb{Z}} R^{\mathrm{op}}$ is an $(\alpha \otimes \alpha^{-1})$-Frobenius extension of $S \otimes_{\mathbb{Z}} S^{\mathrm{op}}$. This enables us to apply module-theoretic results on Frobenius extensions to bimodules.

Now let $R : S$ be an α-Frobenius extension and consider two R-modules M, N. Given a dual projective pair $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ the mapping $\mathrm{Tr}_{[R : S]} : \mathrm{Hom}_S(M, {}_\alpha N) \to \mathrm{Hom}_R(M, N)$, which is defined by

$$\mathrm{Tr}_{[R : S]}(f)(m) = \sum_{i=1}^{n} y_i f(x_i m) \quad \text{for } m \in M,$$

is customarily referred to as the *trace map*.

The ensuing result elaborates on [15, Satz 6] by providing a factorization of the trace map.

Proposition 1.3. *Let $T \subset S \subset R$ be rings such that*

    (a) *$R : S$ is an α-Frobenius extension,*
    (b) *$S : T$ is a γ-Frobenius extension, and*
    (c) *$\alpha(T) = T$.*

Then $R : T$ is an $(\alpha \circ \gamma)$-Frobenius extension and $\mathrm{Tr}_{[R : T]} = \mathrm{Tr}_{[R : S]} \circ \mathrm{Tr}_{[S : T]}$.

Proof. Let $\pi : R \to S$ and $\rho : S \to T$ denote the Frobenius homomorphisms of $R : S$ and $S : T$, respectively. We define an $(\alpha \circ \gamma)^{-1}$-associative form $\langle\ ,\ \rangle : R \times R \to T$ via $\langle a, b \rangle := (\rho \circ \pi)(ab)$ for $a, b \in R$. The extensions $R : S$ and $S : T$ have associated dual projective pairs $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ and $\{u_1, \ldots, u_m\}$, $\{v_1, \ldots, v_m\}$, respectively. Then, for every element r

of R,

$$r = \sum_{i=1}^{n} y_i \alpha(\pi(x_i r)) = \sum_{i=1}^{n} y_i \alpha \left(\sum_{j=1}^{m} v_j \gamma(\rho(u_j \pi(x_i r))) \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} y_i \alpha(v_j)(\alpha \circ \gamma)((\rho \circ \pi)(u_j x_i r))$$

$$= \sum_{i,j} y_i \alpha(v_j)(\alpha \circ \gamma)(\langle u_j x_i , r \rangle).$$

Similarly, $r = \sum_{i,j} \langle r, y_i \alpha(v_j) \rangle u_j x_i$. This shows that $\{u_j x_i : 1 \leq i \leq n, \ 1 \leq j \leq m\}$ and $\{y_i \alpha(v_j): 1 \leq i \leq n, \ 1 \leq j \leq m\}$ form a dual projective pair relative to $\langle \ , \ \rangle$. According to (1.1), $R : T$ is therefore an $(\alpha \circ \gamma)$-Frobenius extension.

Now let f be an element of $\mathrm{Hom}_T(M, {}_{\alpha \circ \gamma} N) = \mathrm{Hom}_T(M, {}_\gamma({}_\alpha N))$. Then $\mathrm{Tr}_{[S \,:\, T]}(f) \in \mathrm{Hom}_S(M, {}_\alpha N)$ and

$$\mathrm{Tr}_{[S \,:\, T]}(f)(m) = \sum_{j=1}^{m} v_j * f(u_j m) = \sum_{j=1}^{m} \alpha(v_j) f(u_j m).$$

Consequently,

$$\mathrm{Tr}_{[R \,:\, S]} \circ \mathrm{Tr}_{[S \,:\, T]}(f)(m) = \sum_{i=1}^{n} y_i \mathrm{Tr}_{[S \,:\, T]}(f)(x_i m)$$

$$= \sum_{i,j} y_i \alpha(v_j) f(u_j x_i m) = \mathrm{Tr}_{[R \,:\, T]}(f)(m). \quad \square$$

The following result provides a partial converse to (1.3).

Theorem 1.4. *Let* $T \subset S \subset R$ *be rings and suppose that*
 (a) $R:T$ *is a* β-*Frobenius extension with Frobenius homomorphism* $\sigma: R \to T$,
 (b) $S:T$ *is a* γ-*Frobenius extension with Frobenius homomorphism* $\rho: S \to T$, *and*
 (c) R *is a projective* S-*module.*
Then the following statements hold:
 (1) *The map* $\pi := \mathrm{Tr}_{[S \,:\, T]}(\gamma \circ \sigma)$ *is the unique element of* $\mathrm{Hom}_S(R, S)$ *such that* $\rho \circ \pi = \sigma$.
 (2) *If there exists an automorphism* α *of* S *with* $\pi(rs) = \pi(r)\alpha^{-1}(s)$ *for all* $r \in R, s \in S$, *then* $R : S$ *is an* α-*Frobenius extension with Frobenius homomorphism* π.

Proof. Let $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ be a dual projective pair for $S : T$, so that $s = \sum_{i=1}^{n} y_i \gamma \circ \rho(x_i s) = \sum_{i=1}^{n} \rho(s y_i) x_i$ for all $s \in S$.
 (1) Suppose that $\pi = \mathrm{Tr}_{[S \,:\, T]}(\gamma \circ \sigma)$. For $r \in R$ we obtain

$$(\rho \circ \pi)(r) = \rho \left(\sum_{i=1}^{n} y_i (\gamma \circ \sigma)(x_i r) \right) = \sum_{i=1}^{n} \rho(y_i) \sigma(x_i r)$$

$$= \sigma \left(\sum_{i=1}^{n} \rho(y_i) x_i r \right) = \sigma(r),$$

so $\rho \circ \pi = \sigma$. Assume conversely that $\rho \circ \pi = \sigma$. As $\mathrm{Tr}_{[S\,:\,T]}(\gamma \circ \rho) = \mathrm{id}_S$, it follows that

$$\mathrm{Tr}_{[S\,:\,T]}(\gamma \circ \sigma) = \mathrm{Tr}_{[S\,:\,T]}(\gamma \circ \rho \circ \pi) = \mathrm{Tr}_{[S\,:\,T]}(\gamma \circ \rho) \circ \pi = \pi.$$

(2) Define $\varphi \colon R \to \mathrm{Hom}_S(R, {}_\alpha S)$ by $\varphi(r) = r \cdot (\alpha \circ \pi)$, i.e., $\varphi(r)(r') = \alpha(\pi(r'r))$. Our assumption concerning π ensures that φ is a well-defined (R, S)-bimodule map.

Suppose $r \in \mathrm{Ker}\,\varphi$. Then for every $r' \in R$, we have $\pi(r'r) = 0$, whence $0 = (\rho \circ \pi)(r'r) = \sigma(r'r)$ for all $r' \in R$. Since $R : T$ is a Frobenius extension, it follows that $r = 0$.

Suppose $f \in \mathrm{Hom}_S(R, {}_\alpha S)$. Then $\beta \circ \rho \circ \alpha^{-1} \circ f \in \mathrm{Hom}_T(R, {}_\beta T)$; since $R : T$ is a β-Frobenius extension, there is an $r \in R$ with $\beta \circ \rho \circ \alpha^{-1} \circ f = r \cdot (\beta \circ \sigma) = \beta \circ \rho \circ (r \cdot \pi)$. We can cancel β and conclude $\rho \circ \alpha^{-1} \circ f = \rho \circ (r \cdot \pi)$. Hence

$$r \cdot \pi = \mathrm{Tr}_{[S\,:\,T]}(\gamma \circ \rho) \circ (r \cdot \pi) = \mathrm{Tr}_{[S\,:\,T]}(\gamma \circ \rho \circ (r \cdot \pi))$$
$$= \mathrm{Tr}_{[S\,:\,T]}(\gamma \circ \rho \circ \alpha^{-1} \circ f) = \alpha^{-1} \circ f,$$

proving that $\varphi(r) = r \cdot (\alpha \circ \pi) = f$. This shows φ is an isomorphism. $\square$

The automorphism α in part (2) of the preceding result is uniquely determined. Moreover, its existence is related to the behavior of the Nakayama automorphism of the extension $R : T$. To see this, we first observe that by definition of π, we have

$$\pi(rt) = \pi(r)\gamma(\beta^{-1}(t))$$

for any $t \in T$, $r \in R$. Consequently, $\alpha(t) = (\beta \circ \gamma^{-1})(t)$. In particular, $\alpha(T) = T$, and hence $\alpha(C_S(T)) = C_S(T)$.

Associated to any Frobenius extension is the Nakayama automorphism, an automorphism of the centralizer of the subring in the extension ring (cf. [11]). In the situation of (1.4), there are two Nakayama automorphisms, $\mu \colon C_R(T) \to C_R(T)$ and $\nu \colon C_S(T) \to C_S(T)$, whose defining relations are

$$\sigma(cr) = \sigma(r\mu(c)) \quad \text{and} \quad \rho(cs) = \rho(s\nu(c)).$$

Suppose that (a) through (c) of (1.4) hold and put $\pi := \mathrm{Tr}_{[S\,:\,T]}(\gamma \circ \sigma)$. Then the following statements are equivalent:

(i) $\mu(C_S(T)) = C_S(T)$.
(ii) There exists an automorphism α of $C_S(T)$ such that $\pi(rs) = \pi(r)\alpha^{-1}(s)$ for all $r \in R$, $s \in C_S(T)$.

To prove (i) implies (ii), suppose that $\mu(C_S(T)) = C_S(T)$. Then for any $r \in R$, $c \in C_S(T)$, we have

$$\rho \circ \pi(rc) = \sigma(rc) = \sigma(\mu^{-1}(c)r) = \rho(\mu^{-1}(c)\pi(r)) = \rho(\pi(r)(\nu \circ \mu^{-1})(c)).$$

Upon composing both sides with γ and applying the operator $\mathrm{Tr}_{[S\,:\,T]}$, we obtain $\pi(rc) = \pi(r)(\nu \circ \mu^{-1})(c)$. Consequently, $\alpha := \mu \circ \nu^{-1}$ has the requisite property.

Conversely, assume (ii) holds. For $c \in C_S(T)$ and $r \in R$ we obtain

$$\sigma(\mu^{-1}(c)r) = \rho \circ \pi(rc) = \rho(\pi(r)\alpha^{-1}(c)) = \rho((\nu^{-1} \circ \alpha^{-1})(c)\pi(r))$$
$$= \rho \circ \pi((\nu^{-1} \circ \alpha^{-1})(c)r) = \sigma((\nu^{-1} \circ \alpha^{-1})(c)r).$$

Consequently, $\mu^{-1}(c) = (\nu^{-1} \circ \alpha^{-1})(c)$. As a result we have $\mu(C_S(T)) = C_S(T)$.

It follows from our above deliberations that condition (i) is necessary for the existence of α in (1.4)(2).

If T is contained in the center of S, the above calculations show that $\pi(rs) = \pi(r)(\nu \circ \mu^{-1}(s))$ for all $r \in R$, $s \in S$. Thus, in this case (1.4) can be restated as follows.

Corollary 1.5 [15, Satz 7]. *Let $T \subset S \subset R$ be rings such that $T \subset Z(S)$ and suppose that the following conditions hold.*

(a) *$R : T$ is a β-Frobenius extension with Nakayama automorphism $\mu :$ $C_R(T) \to C_R(T)$ such that $\mu(S) = S$,*

(b) *$S : T$ is a Frobenius extension with Nakayama automorphism $\nu : S \to S$, and*

(c) *R is a projective S-module.*

Then $R : S$ is a $(\mu \circ \nu^{-1})$-Frobenius extension. □

Examples. We give an example showing that the technical assumption (c) is essential to the validity of (1.4) and (1.5).

(F) Let K be a field and t an indeterminate, and set $R := K[t]/(t^3) = K[\bar{t}]$. Let S be the subalgebra $K[\bar{t}^2]$ of R, and set $T := K$. Then by example (C), R and S are Frobenius algebras with $\sigma : R \to K$ given by $\sigma(a + b\bar{t} + c\bar{t}^2) = c$ and $\rho : S \to K$ obtained by restricting σ to S. All the hypotheses of (1.4) and (1.5) are satisfied in this setting except projectivity of R as an S-module. Since S is a local ring, if R were projective, it would be free. This is impossible, however, since R is 3-dimensional over K and S is 2-dimensional. (In fact, one can show in this example that $\mathrm{Hom}_S(R, S) \cong R$, so $R : S$ satisfies all the conditions for a Frobenius extension except projectivity.)

The following example illustrates that Frobenius extensions also arise if the technical assumptions of (1.4) and (1.5) do not hold.

(G) Let K be a field whose characteristic is not 2 and let R be the Grassmann algebra over K in the indeterminates x, y. By example (E), R is a Frobenius algebra over K with Frobenius homomorphism $\sigma : R \to K$ defined by $\sigma(a + bx + cy + dxy) = d$ for $a, b, c, d \in K$. The Nakayama automorphism μ of R is given by

$$\mu(a + bx + cy + dxy) = a - bx - cy + dxy.$$

Set $t := x + xy \in R$; then $t^2 = 0$, whence $S := K + Kt$ is a subalgebra of R. By example (C), S is also a Frobenius algebra over K with Frobenius homomorphism $\rho : S \to K$ defined by $\rho(a + bt) = b$. It is easy to see that $\{1, y\}$ is a basis for R as a right or left S-module. If we define an automorphism α of S by $\alpha(t) = -t$, then $sy = y\alpha(s)$ for $s \in S$. It thus follows from example (C) that $R : S$ is an α-Frobenius extension with Frobenius homomorphism $\pi(a + bx + cy + dxy) = c + (d - b)t$.

Since S is not invariant under the Nakayama automorphism μ of R, the hypothesis $\mu(S) = S$ in (1.5) is independent of the other hypotheses of (1.5), and so by the equivalence of (i) and (ii) in the remarks after (1.4), the hypothesis (2) of (1.4) is independent of the other hypotheses in (1.4). Moreover, this example shows the hypotheses of (1.4) and (1.5) are not necessary for $R : S$ to be a Frobenius extension.

2. Frobenius extensions of Lie superalgebras

Let F be a field whose characteristic is not 2 and suppose that L is a Lie superalgebra over F with even part L_0 and odd part L_1. (See [8] for definitions and terminology.) We shall be concerned with subalgebras $K = K_0 \oplus K_1$ of L for which K_0 coincides with L_0. Note that these subalgebras correspond to the L_0-submodules of L_1. Suppose that K is such a subalgebra with $\dim_F L/K = n$. Let $\mathscr{I} = \{0, 1\}^n$ be the set of all multi-indices of length n and put $\tau = (1, 1, \ldots, 1)$. We say that two such multi-indices $I, J \in \mathscr{I}$ are *disjoint* if there is no component in which both are nonzero. The *weight of* I, which is defined to be the sum of the components, will be denoted by $|I|$. Let $\mathscr{U}(L)$ and $\mathscr{U}(K)$ be the universal enveloping algebras of L and K, respectively, and choose $x_1, \ldots, x_n$ to be elements of L_1 whose cosets form a basis for L/K. For $I = (i(1), \ldots, i(n)) \in \mathscr{I}$, we define $\mathbf{x}^I = x_1^{i(1)} \cdots x_n^{i(n)}$. The Poincaré-Birkhoff-Witt Theorem ensures that every element of $\mathscr{U}(L)$ can be written uniquely in the form $u = \sum_{I \in \mathscr{I}} u_I \mathbf{x}^I$, with each u_I lying in $\mathscr{U}(K)$.

We filter the $\mathscr{U}(K)$-module $\mathscr{U}(L)$ by setting $V(m) := \sum_{i=0}^m \mathscr{U}(K) \cdot (L_1^i)$ (so $V(0) = \mathscr{U}(K)$ and $V(n) = \mathscr{U}(L)$). A standard straightening argument shows that $V(m) \cdot \mathscr{U}(K) \subset V(m)$, qualifying $V(m)$ as a $\mathscr{U}(K)$-subbimodule of $\mathscr{U}(L)$. By applying the Poincaré-Birkhoff-Witt Theorem we thus obtain $V(m) = \bigoplus_{|I| \leq m} \mathscr{U}(K)\mathbf{x}^I = \bigoplus_{|I| \leq m} \mathbf{x}^I \mathscr{U}(K)$. Clearly, $V(k)V(l) \subseteq V(k+l)$ for all $k, l \geq 0$.

Lemma 2.1. (1) *For* $m \geq 2$, *the product of any* m *linearly dependent elements of* L_1 *lies in* $V(m-2)$.

(2) *Let* $y_1, \ldots, y_m \in L_1$ *and let* σ *be a permutation in* S_m. *Then*

$$y_{\sigma(1)} \cdots y_{\sigma(m)} - (\mathrm{sgn}\,\sigma)y_1 \cdots y_m \in V(m-2).$$

(3) *If* $I, J \in \mathscr{I}$, *then one of* $\mathbf{x}^I \mathbf{x}^J + \mathbf{x}^{I+J}$ *or* $\mathbf{x}^I \mathbf{x}^J - \mathbf{x}^{I+J}$ *lies in* $V(|I|+|J|-2)$.

Proof. (1) We may write the factors in the product as a linear combination of some subset of them, and so for the first claim it is enough to prove $y_1 \cdots y_{m-1}y_i \in V(m-2)$ whenever $y_1, \ldots, y_{m-1} \in L_1$, $1 \leq i \leq m-1$. If $i = m-1$, then, observing $[y_{m-1}, y_{m-1}] \in L_0$, we obtain

$$y_1 \cdots y_{m-2}y_{m-1}y_{m-1} = \tfrac{1}{2}y_1 \cdots y_{m-2}[y_{m-1}, y_{m-1}] \in V(m-2).$$

If $i < m-1$, then

$$y_1 \cdots y_{m-2}y_{m-1}y_i = -y_1 \cdots y_{m-2}y_i y_{m-1} + y_1 \cdots y_{m-2}[y_{m-1}, y_i],$$

with the second summand obviously being contained in $V(m-2)$, since $[y_{m-1}, y_i] \in L_0$. By induction on m we may assume $y_1 \cdots y_{m-2}y_i \in V(m-3)$, whence the element $y_1 \cdots y_{m-2}y_i y_{m-1}$ belongs to $V(m-2)$.

(2), (3) To prove (2), we may assume without loss of generality that σ is a transposition of the form $(i\ i+1)$. In this case

$$y_1 \cdots y_{i+1}y_i \cdots y_m + y_1 \cdots y_i y_{i+1} \cdots y_m$$
$$= y_1 \cdots y_{i-1}[y_i, y_{i+1}]y_{i+1} \cdots y_m \in V(m-2). \qquad \square$$

The definition of a Lie superalgebra implies that the map $f: K \to \mathrm{gl}(L/K)$ defined by $f(a)(y+K) := [a, y]+K$ for $a \in K$, $y \in L$ is a homomorphism of Lie superalgebras, and so $\lambda: K \to F$ defined by $\lambda(a) := \mathrm{tr}(f(a)) = -\mathrm{str}(f(a))$

(where str stands for the supertrace) is a linear functional on K which vanishes on $[K, K] + K_1$. Hence there is a unique automorphism α of $\mathscr{U}(K)$ such that

$$(*) \qquad \alpha(a) = \begin{cases} a + \lambda(a)1 & \text{for } a \in K_0, \\ (-1)^n a & \text{for } a \in K_1. \end{cases}$$

Theorem 2.2. *Let L be a Lie superalgebra, and $K \subset L$ a subalgebra of L of codimension n that contains L_0. Then the extension $\mathscr{U}(L) : \mathscr{U}(K)$ is a free α-Frobenius extension, where α is defined by $(*)$.*

Proof. We will use the notation in the discussion preceding the theorem. We first consider the case $K = L_0$. Accordingly, let $x_1, \ldots, x_n$ be a basis for L_1. We define $\sigma: \mathscr{U}(L) \to \mathscr{U}(L_0)$ by $\sigma(\sum_{I \in \mathscr{I}} u_I \mathbf{x}^I) = u_\tau$ and consider the associated biadditive map $\langle \ , \ \rangle: \mathscr{U}(L) \times \mathscr{U}(L) \to \mathscr{U}(L_0)$; $\langle u, v \rangle := \sigma(uv)$. The Poincaré-Birkhoff-Witt Theorem shows that $\mathscr{U}(L)$ is a finitely generated free right and left $\mathscr{U}(L_0)$-module, each on the basis $\{\mathbf{x}^I : I \in \mathscr{I}\}$. The form $\langle \ , \ \rangle$ is evidently associative, and it is clear that if a is an element of $\mathscr{U}(L_0)$, then $\sigma(au) = a\sigma(u)$, so $\langle au, v \rangle = a\langle u, v \rangle$. Defining $\beta: \mathscr{U}(L_0) \to \mathscr{U}(L_0)$ via $\beta(x) = x + \text{tr}(\text{ad } x|_{L_1})$ for $x \in L_0$, we claim $\sigma(ua) = \sigma(u)\beta^{-1}(a)$ for all $u \in \mathscr{U}(L)$, $a \in \mathscr{U}(L_0)$.

The Poincaré-Birkhoff-Witt Theorem shows that the multiplication map of $\mathscr{U}(L)$ induces an isomorphism

$$\mathscr{U}(L_0) \otimes_F \Lambda(L_1) \cong \mathscr{U}(L)$$

of L_0-modules, with L_0 acting on $\mathscr{U}(L_0)$ and the exterior algebra $\Lambda(L_1)$ via the adjoint representation. Recall that $\Lambda(L_1)$ is a graded L_0-module on whose nth component L_0 operates via the map $x \mapsto \text{tr}(\text{ad } x|_{L_1})$. Now let $a \in L_0$ and $u \in \mathscr{U}(L_0)$. Then we have

$$\sigma([a, u\mathbf{x}^I]) = \sigma([a, u]\mathbf{x}^I) + \sigma(u[a, \mathbf{x}^I]) = \delta_{I,\tau}[a, u] + \delta_{I,\tau}\lambda(a)u$$
$$= [a, \sigma(u\mathbf{x}^I)] + \lambda(a)\sigma(u\mathbf{x}^I).$$

Consequently,

$$\sigma([a, r]) = [a, \sigma(r)] + \lambda(a)\sigma(r) \quad \text{for all } a \in L_0, \ r \in \mathscr{U}(L).$$

This implies

$$\sigma(ra) = \sigma(ar - [a, r]) = a\sigma(r) - [a, \sigma(r)] - \lambda(a)\sigma(r) = \sigma(r)\beta^{-1}(a)$$

for every $a \in L_0$ and $r \in \mathscr{U}(L)$. Thus $\sigma(rs) = \sigma(r)\beta^{-1}(s)$ for every $s \in \mathscr{U}(L_0)$ and $r \in \mathscr{U}(L)$.

Next define any total order $\ll$ on $\mathscr{I}$ which has the property that $I \ll J$ implies $|I| \leq |J|$. Let I' denote the complement $\tau - I$ of I, so $|I'| = n - |I|$. If $|I| < |J|$, then $|I| + |J'| < n$, so $\mathbf{x}^I \mathbf{x}^{J'} \in V(n-1)$. If $|I| = |J|$ and $I \neq J$, then $|I| + |J'| = n$, but I and J' are not disjoint (i.e., $\mathbf{x}^I$ and $\mathbf{x}^{J'}$ have some variables in common), so $(2.1)(1)$ yields $\mathbf{x}^I \mathbf{x}^{J'} \in V(n-2)$. This shows that if $I \ll J$ and $I \neq J$, then $\langle \mathbf{x}^I, \mathbf{x}^{J'} \rangle = 0$. Applying (2.1) again we see that $\langle \mathbf{x}^I, \mathbf{x}^{I'} \rangle = \pm 1$.

For $I \in \mathscr{I}$ put $x_I := \mathbf{x}^I$ and $y_I := \mathbf{x}^{I'}$. Then $\{x_I : I \in \mathscr{I}\}$ and $\{y_I : I \in \mathscr{I}\}$ are ordered bases of the left and right $\mathscr{U}(L_0)$-module $\mathscr{U}(L)$, respectively. Consider the matrix $[\langle x_I, y_J \rangle]_{I, J \in \mathscr{I}}$. According to the last paragraph this is

a lower triangular matrix whose diagonal entries are all ± 1. Thus by (1.2)(c), $\mathcal{U}(L)$ is a free β-Frobenius extension of $\mathcal{U}(L_0)$.

Now suppose that K is as given in the theorem. Let $x_1, \ldots, x_n$ be elements of L_1 whose cosets form a basis of L/K and let $w_1, \ldots, w_m$ be a basis for K_1. Adopting the notation introduced above, we define $\pi: \mathcal{U}(L) \to \mathcal{U}(K)$ by $\pi(\sum_{I \in \mathcal{I}} u_I \mathbf{x}^I) := u_\tau$, where the coefficients u_I are in $\mathcal{U}(K)$. By the first part of the proof, $\mathcal{U}(L): \mathcal{U}(L_0)$ is a β-Frobenius extension, with the Frobenius homomorphism σ being the projection of an element of $\mathcal{U}(L)$ onto the coefficient (from $\mathcal{U}(L_0)$) of the monomial $w_1 \cdots w_m x_1 \cdots x_n$. Likewise, $\mathcal{U}(K): \mathcal{U}(L_0)$ is a γ-Frobenius extension, where $\gamma(x) = x + \mathrm{tr}(\mathrm{ad}\, x|_{K_1})$ for $x \in L_0$, with the Frobenius homomorphism ρ being the projection of an element of $\mathcal{U}(K)$ onto the coefficient (from $\mathcal{U}(L_0)$) of the monomial $w_1 \cdots w_m$. Obviously $\rho \circ \pi = \sigma$, and we have noted before that $\mathcal{U}(L)$ is free over $\mathcal{U}(K)$. By (1.4), to show $\mathcal{U}(L): \mathcal{U}(K)$ is an α-Frobenius extension with Frobenius homomorphism π, it suffices to verify

$$\pi(ua) = \pi(u)\alpha^{-1}(a) \quad \text{for all } u \in \mathcal{U}(L), \ a \in K.$$

By definition we have $\alpha(a) = \beta \circ \gamma^{-1}(a)$ for $a \in \mathcal{U}(L_0)$, so it is enough to show this equality for $a \in K_1$.

Let $a \in K_1$ and let $u \in \mathcal{U}(K)$. For $I \in \mathcal{I}$, we obtain as a consequence of (2.1) that $u\mathbf{x}^I a \equiv (-1)^n u a \mathbf{x}^I \pmod{V(n-1)}$, so that

$$\pi(u\mathbf{x}^I a) = (-1)^n u a \delta_{I,\tau} = \pi(u\mathbf{x}^I)\alpha^{-1}(a).$$

It now follows that $\pi(rb) = \pi(r)\alpha^{-1}(b)$ for all $r \in \mathcal{U}(L)$ and all $b \in \mathcal{U}(K)$. $\square$

Throughout the remainder of this section $L = L_0 \oplus L_1$ is assumed to be a finite-dimensional, $\mathbb{Z}$-graded Lie superalgebra, i.e., $L = \bigoplus_{i \in \mathbb{Z}} L_{(i)}$ with each $L_{(i)}$ a homogeneous subspace and with $[L_{(i)}, L_{(j)}] \subset L_{(i+j)}$. We consider $L^- := \bigoplus_{i<0} L_{(i)}$ and $L^+ := \bigoplus_{i>0} L_{(i)}$ and define $\mathcal{P} := L_{(0)} \oplus L^+$. Using the Poincaré-Birkhoff-Witt Theorem one readily verifies the following facts for the $\mathbb{Z}$-grading inherited by $\mathcal{U}(L)$:

(i) $\mathcal{U}(\mathcal{P}) = \bigoplus_{i \geq 0} \mathcal{U}(\mathcal{P})_{(i)}$; $\mathcal{U}(\mathcal{P})_{(i)} \subset \mathcal{U}(L_{(0)})\mathcal{U}(L^+)^+$ for all $i > 0$, where $\mathcal{U}(L^+)^+$ is the augmentation ideal of $\mathcal{U}(L^+)$ (that is, the kernel of the canonical supplementation on $\mathcal{U}(L^+)$).

(ii) $\mathcal{U}(L^-) = \bigoplus_{i \leq 0} \mathcal{U}(L^-)_{(i)}$; $\mathcal{U}(L^-)_{(0)} = F \cdot 1$.

Following Shen [16], we refer to a $\mathbb{Z}$-graded L-module $\mathcal{V}$ as *positively graded* if $\mathcal{V} = \bigoplus_{i \geq 0} \mathcal{V}_{(i)}$. A positively graded module $\mathcal{V}$ is said to be *transitively graded* if $\mathcal{V}_{(0)} = \{v \in \mathcal{V}: x \cdot v = 0 \text{ for all } x \in L^-\}$.

Let V be an $L_{(0)}$-module. We extend the operation on V to $\mathcal{P}$ by letting L^+ act trivially. Note that the augmentation ideal $\mathcal{U}(L^+)^+$ of $\mathcal{U}(L^+)$ then annihilates V. We shall be studying the coinduced module $\mathrm{Coind}_{\mathcal{P}}(V) := \mathrm{Hom}_{\mathcal{U}(\mathcal{P})}(\mathcal{U}(L), V)$.

Theorem 2.3. *Let V be an $L_{(0)}$-module and suppose $L_0 \subset \mathcal{P}$.*

(1) *The L-module $\mathcal{V} := \mathrm{Coind}_{\mathcal{P}}(V)$ obtains the structure of a transitively graded L-module by setting $\mathcal{V}_{(i)} := \{f \in \mathcal{V}: f(\mathcal{U}(L)_{(j)}) = (0) \text{ for all } j \neq -i\}$.*

(2) *$\mathcal{V}_{(0)}$ is isomorphic to V as an $L_{(0)}$-module.*
Suppose, in addition, that V is finite dimensional.

(3) *If V is indecomposable, then $\mathcal{V}$ is indecomposable.*

(4) *If V is irreducible, $V = \mathscr{U}(L_{(0)})v_0$, then $\mathscr{V}$ contains a unique irreducible submodule, which is generated by the image of v_0 under the isomorphism in (2).*

Proof. (1) Let f be an element of $\mathscr{V}_{(i)}$, $u \in \mathscr{U}(L)_{(k)}$, and suppose that $j \neq -i - k$. If $x \in \mathscr{U}(L)_{(j)}$, then $xu \in \mathscr{U}(L)_{(j+k)}$ and as $j + k \neq -i$ we obtain $(u \cdot f)(x) = f(xu) = 0$. Consequently, $(u \cdot f)$ belongs to $\mathscr{V}_{(i+k)}$.

Now let $\{u_1, \ldots, u_t\}$ be the basis of $\mathscr{U}(L)$ over $\mathscr{U}(\mathscr{P})$ induced by a basis $\{e_1, \ldots, e_k\}$ of L^-. (Such a basis is finite since $L_0 \subset \mathscr{P}$.) Owing to (ii) we have $u_l \in \mathscr{U}(L)_{(i(l))}$, where $i(l) \leq 0$ for $1 \leq l \leq t$. For an arbitrary element $v \in V$ and $l \in \{1, \ldots, t\}$, let $X_v^{(l)} : \mathscr{U}(L) \to V$ be the $\mathscr{U}(\mathscr{P})$-module homomorphism given by $X_v^{(l)}(u_i) := \delta_{li}v$ for $1 \leq i \leq t$. Any element $u \in \mathscr{U}(L)_{(k)}$ has a representation $u = \sum_{l=1}^t r_l u_l$, where $r_l \in \mathscr{U}(\mathscr{P})_{(k-i(l))}$ for $1 \leq l \leq t$. Given $l \in \{1, \ldots, t\}$, we have $X_v^{(l)}(u) = r_l v$. If $k \neq i(l)$, then (i) in conjunction with the definition of the $\mathscr{U}(\mathscr{P})$-module structure of V forces the vanishing of $X_v^{(l)}(u)$. This qualifies $X_v^{(l)}$ as an element of $\mathscr{V}_{(-i(l))}$. For every element $f \in \mathscr{V}$ we have $f = \sum_{l=1}^t X_{f(u_l)}^{(l)}$. Consequently, $\mathscr{V} = \bigoplus_{l=1}^t \mathscr{V}_{(-i(l))}$, proving that $\mathscr{V}$ is a positively graded module.

To verify the transitivity of $\mathscr{V}$, let f be an element of $\mathscr{V}$ such that $x \cdot f = 0$ for every $x \in L^-$. Since each homogeneous constituent of f enjoys the same property, we may assume f to be homogeneous of degree l. Suppose that $l > 0$: we may write any element of $\mathscr{U}(L)_{(-l)}$ as a sum of elements of the form $u = \sum_{i+k=-l} a_i b_k$, where $a_i \in \mathscr{U}(\mathscr{P})_{(i)}$ and $b_k \in \mathscr{U}(L^-)_{(k)}$. Owing to (i) we have $a_i \cdot V = (0)$ for $i > 0$, and so

$$f(u) = \sum_{i+k=-l} a_i f(b_k) = a_0 f(b_{-l}) = a_0 (b_{-l} \cdot f)(1).$$

By virtue of (ii), b_{-l} is contained in $\mathscr{U}(L^-)^+$ and therefore annihilates f. Thus $f(u) = 0$ and it follows that f is the zero map. This shows $\mathscr{V}$ is transitively graded.

(2) We proceed by showing that $\lambda : \mathscr{V}_{(0)} \to V$; $\lambda(f) := f(1)$ is an isomorphism of $L_{(0)}$-modules. If $x \in L_{(0)}$, then

$$\lambda(x \cdot f) = (x \cdot f)(1) = f(x) = xf(1) = x\lambda(f),$$

and so λ is a homomorphism of $L_{(0)}$-modules. Since $1 = u_{l_0} \in \mathscr{U}(L)_{(0)}$ is, according to definition, contained in $\{u_1, \ldots, u_t\}$, the corresponding map $X_v^{(l_0)}$ is a preimage of $v \in V$ under λ. Suppose that $f \in \ker \lambda$. Owing to the Poincaré-Birkhoff-Witt Theorem, every element of $\mathscr{U}(L)_{(0)}$ is a sum of elements of the form $u = \sum_{i+k=0} a_i b_k$, where $a_i \in \mathscr{U}(\mathscr{P})_{(i)}$ and $b_k \in \mathscr{U}(L^-)_{(k)}$. Since, according to (i), $a_i = 0$ for $i < 0$ and $a_i \in \mathscr{U}(L_{(0)})\mathscr{U}(L^+)^+$ for $i > 0$, we obtain, observing (ii),

$$f(u) = \sum_{i+k=0} a_i f(b_k) = a_0 f(b_0) = a_0 b_0 f(1) = 0.$$

As a result, f vanishes on $\mathscr{U}(L)_{(0)}$ and thereby on all of $\mathscr{U}(L)$.

(3), (4) Assume that V is finite dimensional. As $\mathscr{U}(L)$ is a free module of finite rank over $\mathscr{U}(\mathscr{P})$, $\mathscr{V}$ is finite dimensional. Let $U \subset \mathscr{V}$ be any nonzero submodule. The associative F-algebra $\mathscr{U}(L^-)^+$ obviously acts on $\mathscr{V}$

by nilpotent transformations. The Engel-Jacobson Theorem (cf. [7, p. 33]) thus implies that $U_0(L^-) := \{v \in U : x \cdot v = 0 \text{ for all } x \in L^-\}$ is not trivial.

If $\mathscr{V} = U \oplus W$ is a decomposition of $\mathscr{V}$, (1) ensures that $\mathscr{V}_{(0)} = U_0(L^-) \oplus W_0(L^-)$ is a decomposition of $\mathscr{V}_{(0)}$ into $L_{(0)}$-modules. It now follows from (2) and our present assumption that $U_0(L^-) = (0)$ or $W_0(L^-) = (0)$. Hence the initially given decomposition of $\mathscr{V}$ is trivial, by the observation in the last paragraph.

In case V is irreducible and $U \neq (0)$ is an L-submodule of $\mathscr{V}$, we obtain from (2) that $U_0(L^-)$ coincides with $\mathscr{V}_{(0)}$. Thus the submodule generated by $\lambda^{-1}(v_0)$ is contained in U and hence is the unique irreducible submodule of $\mathscr{V}$. $\square$

We now specialize our considerations to basic classical Lie superalgebras over an algebraically closed field of characteristic 0. According to [9] such a Lie superalgebra is simple with a reductive Lie algebra as subalgebra of even elements. Let H be a Cartan subalgebra of L_0. Then there exists a triangular decomposition $L = N^- \oplus H \oplus N^+$ of subalgebras of L. An L-module V is said to be a *highest weight module of L with highest weight* $\lambda \in H^*$, if there is $v_\lambda \in V$ such that

(a) $V = \mathscr{U}(L)v_\lambda$,
(b) $h \cdot v_\lambda = \lambda(h)v_\lambda$ for every $h \in H$, and
(c) $N_0^+ \cdot v_\lambda = (0)$.

If L is of type I, it admits a so-called distinguished $\mathbb{Z}$-gradation of the form $L = L_{(-1)} \oplus L_{(0)} \oplus L_{(1)}$, for which $\mathscr{P} = L_0 + N^+ = L_{(0)} \oplus L_{(1)}$ and $N_1^- = L_{(-1)}$. If ρ_1 denotes the half sum of all odd positive roots, then $\mathrm{tr}(\mathrm{ad}\, h|_{N_1^-}) = -2\rho_1(h)$ for all $h \in H$. Given an irreducible highest weight module $V(\lambda)$ for L_0, Kac [9] studied the induced modules $\mathrm{Ind}_{\mathscr{P}}(V(\lambda)) := \mathscr{U}(L) \otimes_{\mathscr{U}(\mathscr{P})} V(\lambda)$. Note that $\mathrm{Ind}_{\mathscr{P}}(V(\lambda))$ is a highest weight module of L with highest weight λ and highest weight vector $1 \otimes v_\lambda$.

Corollary 2.4. *Let $L = L_0 \oplus L_1$ be a basic classical Lie superalgebra of type I and $V(\lambda)$ a finite-dimensional irreducible highest weight module for L_0. Then $\mathrm{Ind}_{\mathscr{P}}(V(\lambda))$ contains a unique irreducible submodule, which is of highest weight $\lambda - 2\rho_1$.*

Proof. Since L_0 is contained in $\mathscr{P}$, $\mathscr{U}(L) : \mathscr{U}(\mathscr{P})$ is, in accordance with (2.2), an α-Frobenius extension. In view of the isomorphism

$$\mathrm{Ind}_{\mathscr{P}}(V(\lambda)) \cong \mathrm{Coind}_{\mathscr{P}}(_\alpha V(\lambda))$$

(cf. §3) it suffices to establish the assertion for the coinduced module. Since N_0^+ operates nilpotently on L, we have $\alpha(x) = x$ for every $x \in N_0^+$. Hence the highest weight vector v_λ of the L_0-module $V(\lambda)$ is a highest weight vector of highest weight $\lambda - 2\rho_1$ for the irreducible L_0-module $_\alpha V(\lambda)$. Since L_0 is contained in $\mathscr{P}$ and $V(\lambda)$ is finite dimensional, (2.3)(4) applies and the coinduced module $\mathrm{Coind}_{\mathscr{P}}(_\alpha V(\lambda))$ contains a unique irreducible submodule, which is generated by a highest weight vector of highest weight $\lambda - 2\rho_1$. $\square$

3. COHOMOLOGICAL OBSERVATIONS

Suppose $R : S$ is an α-Frobenius extension, and let V be an R-module. The theory of Frobenius extensions[13, p. 96f] provides a natural equivalence

$$R \otimes_S V \cong \mathrm{Hom}_S(R, {}_\alpha V).$$

Consequently, there exists a projection $\mathrm{Hom}_S(R, {}_\alpha N) \to N$ for every R-module N. By Frobenius reciprocity there result corestriction maps

$$\mathrm{Cor}^n_{[R:S]}: \mathrm{Ext}^n_S(M, {}_\alpha N) \to \mathrm{Ext}^n_R(M, N)$$

for any two R-modules M and N (cf. [2, §3]). Note that by [2, 3.1(1)], $\mathrm{Cor}^0_{[R:S]} = \mathrm{Tr}_{[R:S]}$, so the following result generalizes part of (1.3).

Theorem 3.1. *Let* $T \subset S \subset R$ *be rings such that*

    (a) $R:S$ *is an* α-*Frobenius extension*,
    (b) $S:T$ *is a* γ-*Frobenius extension*, *and*
    (c) $\alpha(T) = T$.

If M *and* N *are* R-*modules, then* $\mathrm{Cor}^n_{[R:T]} = \mathrm{Cor}^n_{[R:S]} \circ \mathrm{Cor}^n_{[S:T]}$ *for all* $n \geq 0$.

Proof. We shall proceed by induction on n, the case $n = 0$ following from (1.3). Now suppose that $n \geq 1$. Let $(0) \to X \to P \to M \to (0)$ be a projective presentation of the R-module M. Since the rings involved are projective over the relevant subrings, the above sequence is also a projective presentation of M viewed as an S- and T-module, respectively. The corestriction mappings are natural transformations that commute with connecting homomorphisms. The long exact cohomology sequence thus yields the following commutative diagram:

$$
\begin{array}{ccccc}
\mathrm{Ext}^{n-1}_T(X, {}_{\alpha \circ \gamma} N) & \longrightarrow & \mathrm{Ext}^n_T(M, {}_{\alpha \circ \gamma} N) & \longrightarrow & (0) \\
\downarrow & & \downarrow & & \\
\mathrm{Ext}^{n-1}_S(X, {}_\alpha N) & \longrightarrow & \mathrm{Ext}^n_S(M, {}_\alpha N) & \longrightarrow & (0) \\
\downarrow & & \downarrow & & \\
\mathrm{Ext}^{n-1}_R(X, N) & \longrightarrow & \mathrm{Ext}^n_R(M, N) & \longrightarrow & (0),
\end{array}
$$

where the vertical arrows represent the relevant corestriction mappings. By the inductive hypothesis our claim holds for the corestriction maps of level $n - 1$. The assertion now follows from a standard argument. $\square$

We let $\mathrm{Res}^n_{[R:S]}: \mathrm{Ext}^n_R(M, N) \to \mathrm{Ext}^n_S(M, N)$ denote the natural change of rings map. The following result provides a generalization of [2, (3.1)].

Theorem 3.2. *Let* $R:S$ *be an* α-*Frobenius extension and* X, M, N *be* R-*modules. Let* f *be an element of* $\mathrm{Hom}_S(M, {}_\alpha N)$. *Then the following statements hold*:

    (1) $\mathrm{Ext}^n_R(\mathrm{id}_X, \mathrm{Tr}_{[R:S]}(f)) = \mathrm{Cor}^n_{[R:S]} \circ \mathrm{Ext}^n_S(\mathrm{id}_X, f) \circ \mathrm{Res}^n_{[R:S]}$ *for all* $n \geq 0$. $\mathrm{Ext}^n_R(\mathrm{Tr}_{[R:S]}(f), \mathrm{id}_X) = \mathrm{Cor}^n_{[R:S]} \circ \mathrm{Ext}^n_S(f, \mathrm{id}_X) \circ \mathrm{Res}^n_{[R:S]}$ *for all* $n \geq 0$.

    (2) *Suppose that* $\mathrm{Ext}^{n+1}_R(X, \ker \mathrm{Tr}_{[R:S]}(f)) = (0)$ *for some* $n \geq 0$ *and* $\mathrm{Tr}_{[R:S]}(f)$ *is surjective.*

        (a) *If* $\mathrm{Ext}^n_S(X, M) = (0)$ *or* $\mathrm{Ext}^n_S(X, {}_\alpha N) = (0)$, *then* $\mathrm{Ext}^n_R(X, N) = (0)$.
        (b) *If* $\mathrm{Ext}^{n+1}_S(X, M) = (0)$ *or* $\mathrm{Ext}^{n+1}_S(X, {}_\alpha N) = (0)$, *then* $\mathrm{Ext}^{n+1}_R(X, M) = (0)$.

Proof. (1) We proceed by induction on n. For $n = 0$, both identities follow via direct computation using $\mathrm{Cor}^0_{[R\,:\,S]} = \mathrm{Tr}_{[R\,:\,S]}$. Since all maps involved are natural in X, the argument of the proof of (3.1) may now be adopted mutatis mutandis to yield our assertions.

(2)(a) By considering the long exact cohomology sequence induced by $(0) \to \ker \mathrm{Tr}_{[R\,:\,S]}(f) \to M \to N \to (0)$ in conjunction with the hypothesis

$$\mathrm{Ext}^{n+1}_R(X, \ker \mathrm{Tr}_{[R\,:\,S]}(f)) = (0),$$

we obtain the surjectivity of the map $\mathrm{Ext}^n_R(\mathrm{id}_X, \mathrm{Tr}_{[R\,:\,S]}(f)): \mathrm{Ext}^n_R(X, M) \to \mathrm{Ext}^n_R(X, N)$. If $\mathrm{Ext}^n_S(X, M)$ or $\mathrm{Ext}^n_S(X, {}_\alpha N)$ is trivial, then $\mathrm{Ext}^n_S(\mathrm{id}_X, f)$ vanishes and the above factorization identifies $\mathrm{Ext}^n_R(\mathrm{id}_X, \mathrm{Tr}_{[R\,:\,S]}(f))$ as the zero map. Consequently, $\mathrm{Ext}^n_R(X, N) = (0)$.

(b) In this situation $\mathrm{Ext}^{n+1}_S(\mathrm{id}_X, f)$ is trivial and $\mathrm{Ext}^{n+1}_R(\mathrm{id}_X, \mathrm{Tr}_{[R\,:\,S]}(f))$ is injective. Consequently, $\mathrm{Ext}^{n+1}_R(X, M) = (0)$. $\square$

Remark. If α can be extended to an endomorphism of R, then the first part of the preceding result, applied to $\mathrm{id}_N \in \mathrm{Hom}_S({}_\alpha N, {}_\alpha N)$, readily yields [2, (3.1)(2)].

Given an R-module M, we let $\mathrm{inj.dim}_R M$ denote its injective dimension.

Corollary 3.3. *Let $R : S$ be an α-Frobenius extension, X and M be R-modules, and suppose that M is (R, S)-injective. Then the following statements hold:*

(1) The restriction map $\mathrm{Res}^n_{[R\,:\,S]}: \mathrm{Ext}^n_R(X, M) \to \mathrm{Ext}^n_S(X, M)$ is injective.

(2) $\mathrm{inj.dim}_R M = \mathrm{inj.dim}_S M$. In particular, M is R-injective if and only if it is S-injective.

Proof. (1) Owing to [15, Satz 11] there exists $f \in \mathrm{Hom}_S(M, {}_\alpha M)$ such that $\mathrm{Tr}_{[R\,:\,S]}(f) = \mathrm{id}_M$. It thus follows from (3.2)(1) that $\mathrm{Cor}^n_{[R\,:\,S]} \circ \mathrm{Ext}^n_S(\mathrm{id}_X, f) \circ \mathrm{Res}^n_{[R\,:\,S]}$ is the identity operator on $\mathrm{Ext}^n_R(X, M)$, thereby proving the injectivity of $\mathrm{Res}^n_{[R\,:\,S]}$.

(2) It suffices to prove that $\mathrm{inj.dim}_R M \le \mathrm{inj.dim}_S M$. This is a direct consequence of (1). $\square$

An analogous result holds for projective dimensions.

We shall next study the relation between the cohomology theories of L and L_0 under the hypothesis that L is finite dimensional. Our attention will be focused on the case where the Lie algebra L_0 is *semireductive* in the sense that it is the direct sum of a nilpotent and a semisimple ideal. Assuming the characteristic of the underlying base field to be zero, we then have $\mathrm{tr}(\mathrm{ad}_{L_0}(x)) = 0$ for every $x \in L_0$. Consequently, the automorphism α of the Frobenius extension $\mathscr{U}(L) : \mathscr{U}(L_0)$ is the restriction of the automorphism of $\mathscr{U}(L)$ that maps an element x of L onto $x - \mathrm{str}(\mathrm{ad}_L(x))1$.

In general, consider a subalgebra K of L that contains L_0. If the automorphism α of $\mathscr{U}(K)$ can be extended to an endomorphism α of $\mathscr{U}(L)$ (this is not always the case) and $\{x_1, \ldots, x_m\}, \{y_1, \ldots, y_m\}$ is a dual free pair for the Frobenius extension $\mathscr{U}(L) : \mathscr{U}(K)$, we let $c_0 := \sum_{i=1}^m y_i \alpha(x_i)$ denote the Casimir element of the extension (cf. [2]). This element has the property that $r c_0 = c_0 \alpha(r)$ for all $r \in \mathscr{U}(L)$.

Given an L-module M, we shall consider $H^n(L, M) := \mathrm{Ext}^n_{\mathscr{U}(L)}(F, M)$,

the cohomology groups of L with coefficients in M. Let $\varepsilon \colon \mathcal{U}(L) \to F$ be the canonical supplementation.

Proposition 3.4. *Suppose that* $\operatorname{char}(F) = 0$ *and let* L_0 *be semireductive. Given a finite-dimensional L-module M, define $M^{L_0} := \{m \in M \mid x \cdot m = 0$ for all $x \in L_0\}$. Then the following statements hold*:

(1) *Suppose that* $M^{L_0} = \{0\}$. *If there exists* $f \in \operatorname{Hom}_{\mathcal{U}(L_0)}(M, {}_\alpha M)$ *such that* $\operatorname{Tr}(f)$ *is a bijection, then* $H^n(L, M) = (0)$ *for all* $n \geq 0$.

(2) *Suppose that* $M^{L_0} = \{0\}$. *If* $\varepsilon(c_0) \neq 0$, *then* $H^n(L, M) = (0)$ *for all* $n \geq 0$.

(3) *If L_0 is semisimple and c_0 operates invertibly on M, then M is completely reducible.*

(4) *If L_0 is semisimple and if either c_0 operates invertibly on M or $\varepsilon(c_0) \neq 0$, then $H^n(L, M) = (0)$ for $1 \leq n \leq 2$.*

Proof. (1) We apply (3.2) to the case $X := F$ and $N := M$. Since $\ker \operatorname{Tr}(f) = (0)$, we have that $\operatorname{Ext}^n_{\mathcal{U}(L)}(F, \ker \operatorname{Tr}(f))$ vanishes for all n. On the other hand, [1, (3.1)] shows that $H^n(L_0, M) = (0)$ for $n \geq 0$. Consequently, (3.2) gives the desired result.

(2) According to [2, (3.1), (3.2); 1, (3.1)], the map $\operatorname{Cor}_n \circ \operatorname{Res}_n$ is invertible while Res_n is the zero map. Consequently, $H^n(L, M)$ is trivial for all n.

(3) Let $V \subset M$ be a submodule. By the first Whitehead Lemma, we have $\operatorname{Ext}^1_{\mathcal{U}(L_0)}(M/V, V) \cong H^1(L_0, \operatorname{Hom}_F(M/V, V)) = (0)$. Since c_0 operates invertibly on V, the map $\operatorname{Cor}_n \circ \operatorname{Res}_n$ is invertible, and we obtain the vanishing of $\operatorname{Ext}^1_{\mathcal{U}(L)}(M/V, V)$. Hence V is a direct summand of M.

(4) The proof is analogous to (2) with the exception that the Whitehead Lemmas (cf. [7, pp. 77, 89]) are used in lieu of [1, (3.1)]. $\quad\square$

Remarks. (1) If $\alpha = \operatorname{id}_{\mathcal{U}(L_0)}$, then c_0 lies in the center of $\mathcal{U}(L)$, and any finite-dimensional L-module M admits a decomposition $M = M_0 \oplus M_1$ into L-stable Fitting components relative to c_0. By the preceding result, M_1 is a completely reducible L-module.

(2) Suppose that L is a basic classical Lie superalgebra of type I. Let $\operatorname{Ind}_{\mathcal{P}}(V(\lambda))$ be the highest weight module defined by an irreducible L_0-module $V(\lambda)$. Then $\operatorname{Ind}_{\mathcal{P}}(V(\lambda))$ is indecomposable and the Casimir element operates on $\operatorname{Ind}_{\mathcal{P}}(V(\lambda))$ by a scalar λ_0. If λ_0 is not zero, then the above observations ensure the irreducibility of $\operatorname{Ind}_{\mathcal{P}}(V(\lambda))$.

Example. The Lie superalgebra $\operatorname{osp}(1, 2)$ is the set of matrices of the form

$$\begin{bmatrix} 0 & d & -c \\ \hline c & r & s \\ d & t & -r \end{bmatrix}$$

with the usual supercommutator; note that $L_0 \cong \operatorname{sl}(2)$. A basis for L_1 is given by $x = e_{21} - e_{13}$, $y = e_{31} + e_{12}$. Thus the left $\mathcal{U}(L_0)$-module $\mathcal{U}(L)$ has a basis consisting of $x_1 = 1$, $x_2 = x$, $x_3 = y$, $x_4 = xy + 1 - [x, y]$. A dual basis with respect to $\langle \, , \, \rangle$ for $\mathcal{U}(L)$ as a right $\mathcal{U}(L_0)$-module is given by $y_1 = xy$, $y_2 = y$, $y_3 = -x$, $y_4 = 1$. Now

$$c_0 = \sum_{i=1}^{4} y_i x_i = xy + yx - xy + xy + 1 - [x, y] = 1.$$

Thus (3.4)(3) applies and every finite-dimensional osp(1, 2)-module is completely reducible. This fact was first observed by Hochschild in [6].

Corollary 3.5. *Suppose that* $L = L_0 \oplus L_1$ *is a Lie superalgebra with semireductive even part over a field* F *of characteristic* 0. *Let* K *be a subalgebra of* L *containing* L_0 *and* V *be a finite-dimensional* K-*module. Suppose that*

(a) $\{v \in V : x \cdot v = -\mathrm{tr}(\mathrm{ad}_{L/K}(x))v \ \text{for all} \ x \in L_0\} = (0)$, *and*

(b) *the Casimir element* c_0 *of the extension* $\mathscr{U}(K) : \mathscr{U}(L_0)$ *operates nontrivially on* F.

Then $H^n(L, \mathrm{Ind}_K(V)) = (0)$ *for all* $n \geq 0$.

Proof. Let α be the automorphism of the Frobenius extension $\mathscr{U}(L) : \mathscr{U}(K)$. By Frobenius reciprocity in conjunction with the natural equivalence

$$\mathscr{U}(L) \otimes_{\mathscr{U}(K)} V \cong \mathrm{Hom}_{\mathscr{U}(K)}(\mathscr{U}(L), {}_\alpha V),$$

we have $H^n(L, \mathrm{Ind}_K(V)) = H^n(K, {}_\alpha V)$ for every $n \geq 0$. Now condition (a) implies $({}_\alpha V)^{L_0} = (0)$. Consequently, (3.4)(2) along with (b) forces the vanishing of $H^n(K, {}_\alpha V)$ for every $n \geq 0$. $\square$

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