ON THE BRAUER GROUP OF TORIC VARIETIES

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Abstract. We compute the cohomological Brauer group of a normal toric variety whose singular locus has codimension less than or equal to 2 everywhere.

Associated to each algebraic variety $X$ is the cohomological Brauer group $B'(X) = \text{tors}(H^2(X, \mathbb{G}_m))$ which is the torsion subgroup of the second étale cohomology group of $X$ with coefficients in the sheaf of units. Except in the easiest cases, calculations of this group are scarce. Toric varieties over an algebraically closed field of characteristic 0 provide a nontrivial class of higher dimensional varieties for which calculations of $B'(X)$ can sometimes be made. These calculations are the purpose of this article.

Each toric variety $X$ is determined by a combinatorial object $\Delta$ in real affine space called a fan. Tied into the structure of the fan are arithmetic properties of sublattices of free $\mathbb{Z}$-lattices. Our arguments therefore ultimately reduce questions about $B'(X)$ to calculations with integer matrices.

In §1 we determine the Brauer group $B(X) = B'(X)$ of any nonsingular toric variety $X$ (Theorem 1.1). This group is a direct sum of finitely many copies of finite cyclic groups and copies of $\mathbb{Q}/\mathbb{Z}$. The algebras generating this group are given explicitly as smash products of cyclic Galois extensions of $X$. In §2 we consider toric varieties whose singular locus has codimension at most 2 everywhere in $X$. Let $T_N$ denote the torus identified with an open subset of $X$, $B'(T_N/X)$ the elements in $B'(X)$ split by $T_N$, and $\bar{X}$ a $T_N$-invariant desingularization of $X$. In Theorem 2.2 we construct an exact sequence $0 \to B'(T_N/X) \to B'(X) \to B'(\bar{X}) \to 0$ which reduces the calculation of $B'(X)$ to the calculation of $B'(T_N/X)$. The hypotheses on $X$ in §2 imply we can assume the associated fan $\Delta$ contains cones of dimension at most 2. Corresponding to each cone $\tau_i$ of dimension 2 is an irreducible closed subvariety $V_i = \text{orb} \tau_i$ and an affine neighborhood $U_{\tau_i}$ of $V_i$ which has a finite cyclic divisor class group $\text{Cl}(U_{\tau_i})$. If $\Delta$ has 2-dimensional cones $\tau_1, \ldots, \tau_m$, we construct an exact sequence (Theorem 2.3)

$$0 \to \text{Pic}(X) \to \text{Cl}(X) \to \bigoplus_{i=1}^m \text{Cl}(U_{\tau_i}) \to B'(T_N/X) \to 0.$$
For each prime number \( p \) we find a subset \( \tau_1, \ldots, \tau_s \) (after a suitable relabelling) of \( \{\tau_1, \ldots, \tau_m\} \) such that \( [\bigoplus_{i=1}^s \text{Cl}(U_{\tau_i})]_p \cong [\text{B}'(T_N/X)]_p \). We calculate the Brauer group of any toric surface (Corollary 2.9). In this case \( \text{B}'(T_N/X) \) is nontrivial when there is a cycle of divisors on \( X \) whose pairwise consecutive intersections are singular points on \( X \) whose local rings all have divisor class groups of order divisible by a common prime \( p \). An analogous statement holds for \( X \) of higher dimension. We employ terminology and notation of [12] for toric varieties and [11] for étale cohomology.

Following the notation terminology of [12] let \( r > 0 \) be an integer and \( N = \mathbb{Z}^r \) a free abelian group of rank \( r \). Let \( \Delta \) be a finite fan on \( N_\mathbb{R} \) and \( X = T_N \text{emb}(\Delta) \) the associated toric variety containing the \( r \)-dimensional torus \( T_N \) as an open subset defined over the algebraically closed field \( k \) of characteristic 0. Let \( N' \) be the subgroup of \( N \) generated by \( \bigcup_{\sigma \in \Delta} \sigma \cap N \). The basis theorem for finitely generated abelian groups gives a basis \( n_1, \ldots, n_r \) of \( N \) such that \( N' = \mathbb{Z}a_1n_1 \oplus \mathbb{Z}a_2n_2 \oplus \cdots \oplus \mathbb{Z}a_sn_r \) where the \( a_i \) are nonnegative integers and \( a_1|a_2| \cdots |a_{r-1} \) for \( 1 \leq i \leq r-1 \). Call \( \{a_1, \ldots, a_r\} \) the set of invariant factors of \( \Delta \) (or \( X = T_N \text{emb}(\Delta) \)). Let \( \text{B}(X) \) denote the Brauer group of Azumaya algebras on \( X \) and \( \text{B}'(X) \) the torsion subgroup of \( H^2(X, G_m) \) the cohomological Brauer group of \( X \). Our principal result of §1 is

**Theorem 1.1.** If \( X = T_N \text{emb}(\Delta) \) is nonsingular and \( a_1, \ldots, a_r \) is the set of invariant factors of \( X \), then

\[
\text{B}(X) = \text{B}'(X) \cong \bigoplus_{i=1}^{r-1} \text{Hom}(\mathbb{Z}/a_i, \mathbb{Q}/\mathbb{Z})^{r-i}.
\]

We list two special cases of Theorem 1.1.

**Corollary 1.2.** If \( \Delta \) contains a cone \( \sigma \) such that \( \dim \sigma \geq r-1 \), then \( \text{B}(X) = (0) \).

**Proof.** Since \( X \) is nonsingular, [12, Theorem 1.10] implies there is a basis \( n_1, \ldots, n_r \) of \( N \) such that \( \mathbb{R}_{\geq 0} n_1 + \cdots + \mathbb{R}_{\geq 0} n_{r-1} \subset \sigma \). Since each \( n_i \in \sigma \cap N \), all of the invariant factors \( a_i = 1 \) for \( 1 \leq i \leq r-1 \). So \( \text{B}(X) = (0) \) by Theorem 1.1. \( \Box \)

**Corollary 1.3.** \( \text{B}(X) \) is finite if and only if \( \text{Rank}_{\mathbb{Z}}(N') \geq r - 1 \). In this case

\[
\text{B}(X) \cong \bigoplus_{i=1}^{r-1} (\mathbb{Z}/a_i)^{r-i}.
\]

The rest of this section is devoted to a proof of Theorem 1.1. From now on we assume \( X = T_N \text{emb}(\Delta) \) is nonsingular. Along the way we will obtain explicit information about the Azumaya algebras on \( X \) and show the Brauer group \( \text{B}(X) \) of Azumaya algebras on \( X \) is equal to the cohomological Brauer group \( \text{B}'(X) = H^2(X, G_m) \).

Let \( \Gamma = \{0, \rho_1, \ldots, \rho_n\} \) be the fan on \( N_\mathbb{R} \) consisting of all cones in \( \Delta \) of dimension \( \leq 1 \) and let \( U = T_N \text{emb}(\Gamma) \). The open immersion \( U \hookrightarrow X \) induces the isomorphisms of the next lemma.

**Lemma 1.4.** For each positive integer \( \nu \),

\[
(a) \quad H^1(X, \mathbb{Z}/\nu) \cong H^1(U, \mathbb{Z}/\nu),
\]

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(b) \( H^2(X, \mathbb{Z}/\nu) \cong H^2(U, \mathbb{Z}/\nu) \),
(c) \( B'(X) \cong B'(U) \).

**Proof.** Let \( X, U \) be as above and let \( Z = X - U \). Then part of the long exact sequence of cohomology with supports [11, Proposition 1.25] is

\[
H^2_Z(X, \mathbb{Z}/\nu) \to H^1(X, \mathbb{Z}/\nu) \to H^1(U, \mathbb{Z}/\nu) \to H^2_Z(X, \mathbb{Z}/\nu)
\]

\[
\to H^2(X, \mathbb{Z}/\nu) \to H^2(U, \mathbb{Z}/\nu) \to H^2_Z(X, \mathbb{Z}/\nu)
\]

since the codimension of \( Z \) in \( X \) is \( \geq 2 \), [11, Lemma 9.1, p. 268] implies \( H^2_Z(X, \mathbb{Z}/\nu) = (0) \) for \( s < 4 \), which proves (a) and (b) in our context. There is an exact sequence [3, Theorem 1.c]

\[
0 \to H^2(X, G_m) \to H^2(U, G_m) \to K_Z(X, \mu)
\]

and \( H^2_Z(X, \mu) \cong \lim \longrightarrow H^2_Z(X, \mathbb{Z}/\nu) = (0) \), which proves (c). \( \square \)

Notice that \( N' = \langle \bigcup_{\sigma \in \Delta} \sigma \cap N \rangle = \langle \bigcup_{i=1}^r \rho_i \cap N \rangle \) so as a consequence of Lemma 1.4(c) we can assume that \( \Delta = \{0, \rho_1, \ldots, \rho_n\} \) and \( X = U \). We write \( \rho_k = \mathbb{R}_{\geq 0} \eta_k \) where \( \eta_k \in N \) and \( \eta_k \) is primitive (the GDC of the coordinates of \( \eta_k \) is 1). Let \( n_1, \ldots, n_r \) be a basis for \( N \) with \( N' = \mathbb{Z}a_1n_1 \oplus \cdots \oplus \mathbb{Z}a_nn_r \) and \( a_i|a_{i+1} \) for \( 1 \leq i \leq r \). \( \{a_1, \ldots, a_r\} \) is the set of invariant factors of \( \Delta \).

Let \( m_1, \ldots, m_r \) be a dual basis for \( M = \text{Hom}(N, \mathbb{Z}) \). Then \( T_N = \text{Spec} k[M] \). An element \( \sum a_im_i \) in \( M \) is usually identified with the Laurent monomial \( x_1^{a_1}x_2^{a_2}\cdots x_r^{a_r} \) and \( k[M] \) with \( k[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}] \). Let \( \nu \) be a positive integer and fix a primitive \( \nu \)-th root of unity \( \zeta \). Given units \( \alpha, \beta \) in \( k[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}] \), the symbol algebra \( (\alpha, \beta)_\nu \) is the associative \( k \)-algebra generated by elements \( u, v \) subject to the relations \( u^\nu = \alpha, \nu^\nu = \beta \), and \( uv = \zeta vu \). In what follows, we choose to identify \( (x_i, x_j)_\nu \) as \( (m_i, m_j)_\nu \) and work in \( k[M] \). By [10, Theorem 6], \( \nu B(T_N) \) is a free \( \mathbb{Z}/\nu \)-module with basis given by the set of symbol algebras \( \{(m_i, m_j)_\nu\} \subseteq i < j \leq r \) for each \( \nu \geq 2 \). Since \( T_N \) is an open subset of \( X \) and \( X \) is nonsingular, \( B'(X) \) is a subgroup of \( B(T_N) \) by restriction and our object is to identify this subgroup explicitly.

From [4, Corollary 1.4] there is an exact sequence

\[
0 \to B'(X) \to B(T_N) \to \bigoplus_{i=1}^n H^1(\text{orb } \rho_i, \mathbb{Q}/\mathbb{Z})
\]

where \( \text{orb } \rho_i \) is the \( T_N \)-invariant divisor on \( X = T_N \text{ emb}(\Delta) \) corresponding to the face \( \rho_i \) of \( \Delta \). Given a symbol algebra \( (\alpha, \beta)_\nu \) representing a class in \( B(T_N) \), the ramification map \( a \) agrees with the tame symbol (see the discussion following [4, Remark 1.7] and [14, Theorem 8, p. 155]). This means the \( k \)-th coordinate of \( a((\alpha, \beta)_\nu) \), the ramification of \( (\alpha, \beta)_\nu \) along \( \text{orb } \rho_k \), is identified with a cyclic Galois extension of \( \text{orb } \rho_k \) of degree \( \nu \). Over the function field \( K(\text{orb } \rho_k) \) this extension is given by adjoining the \( \nu \)-th root of \( \alpha^v_k/(\beta)^v_k \) where \( v_k \) is the valuation on \( K(X) \) determined by the prime divisor \( \text{orb } \rho_k \).

From the remarks above, to determine the ramification of an arbitrary algebra \( \Lambda \) representing an element in \( B(T_N) \) along \( \text{orb } \rho_k \) it suffices to determine \( K(\text{orb } \rho_k) \) and \( v_k(m_j) \) for each \( k, j \). The following lemma is well known. We include its short proof for completeness and to fix notation.
Lemma 1.5. Let \( \eta_k \) be the primitive vector in \( N \cap \rho_k \) and \( \langle , \rangle \) the natural inner product from \( M \times N \to \mathbb{Z} \).

(a) \( K(\text{orb } \rho_k) \) is the quotient field of \( k[\eta_k^\perp] \).

(b) \( v_k(m) = \langle m, \eta_k \rangle \).

Proof. Since \( \eta_k \) is primitive there is a primitive \( \mu_k \in M \) with \( \langle \mu_k, \eta_k \rangle = 1 \). Let \( \eta_k^\perp = \{ m \in M | \langle m, \eta_k \rangle = 0 \} \). Then \( M = \eta_k^\perp \oplus \mathbb{Z} \mu_k \) since

\[
0 \to \eta_k^\perp \to M \overset{\varphi}{\to} \mathbb{Z} \to 0
\]
splits, where \( \varphi(m) = \langle m, \eta_k \rangle \). The affine coordinate ring of

\( U_{\rho_k} = T_N \text{emb}\{0, \rho_k\} \)

is \( k[\eta_k^\perp, \mu_k] \). Localizing \( X \) along \( \text{orb } \rho_k \) is equivalent to localizing \( U_{\rho_k} \) along \( \text{orb } \rho_k \). The prime ideal corresponding to \( \text{orb } \rho_k \) is the principal ideal in \( k[\eta_k^\perp, \mu_k] \) generated by \( \mu_k \). Hence \( \mu_k \) is a local parameter along \( \text{orb } \rho_k \). \( K(\text{orb } \rho_k) \) is the quotient field of \( k[\eta_k^\perp, \mu_k]/(\mu_k) \) giving (a). The valuation \( v_k \) of any \( m \in M \) is the \( \mu_k \)-coordinate when \( m \) is written in terms of the decomposition \( M = \eta_k^\perp \oplus \mathbb{Z} \mu_k \). Thus \( v_k(m) = \langle m, \eta_k \rangle \). □

Keeping the notation above, define a homomorphism \( \text{ram}_{\text{orb } \rho_k} : \nu \mathbb{B}(T_N) \to M/\nu M \) by letting \( \text{ram}_{\text{orb } \rho_k}(m_i, m_j)_\nu = \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j + \nu M \) be the assignment on the basis for \( \nu \mathbb{B}(T_N) \), and extending by \( \mathbb{Z}/\nu \)-linearity.

Lemma 1.6. \( (m_i, m_j)_\nu \) is unramified along \( \text{orb } \rho_k \) if and only if

\[
\text{ram}_{\text{orb } \rho_k}(m_i, m_j)_\nu = 0.
\]

Proof. The ramification of \( (m_i, m_j)_\nu \) along \( \text{orb } \rho_k \) corresponds to the cyclic extension of the affine coordinate ring \( k[\eta_k^\perp] \) of \( \text{orb } \rho_k \) obtained by adjoining the \( \nu \)-th root of \( v_k(m_j)m_i - v_k(m_i)m_j = \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j \). (Note \( (\langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j, \eta_k) = 0 \) so \( \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j \in \eta_k^\perp \).) Thus, \( (m_i, m_j)_\nu \) is unramified along \( \text{orb } \rho_k \) if and only if \( \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j \) is a \( \nu \)-th power in \( k[\eta_k^\perp] \) if and only if \( \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j \in \nu M \) if and only if \( \text{ram}_{\text{orb } \rho_k}(m_i, m_j)_\nu = 0 \). □

Let \( \Lambda \) be any Azumaya algebra representing a class in \( \nu \mathbb{B}(T_N) \). We have seen \( \Lambda \) is equivalent to \( \prod_{i \leq j} (m_i, m_j)^{e_{ij}} \) where \( 0 \leq e_{ij} < \nu \). Moreover the class of \( \Lambda \) determines and is determined by the integers \( e_{ij} \). Associate to the class represented by \( \Lambda \) in \( \nu \mathbb{B}(T_N) \) the matrix

\[
M_\Lambda = \\
\begin{bmatrix}
0 & e_{12} & e_{13} & \cdots & e_{1r} \\
-e_{12} & 0 & e_{23} & \cdots & e_{2r} \\
& \ddots & \ddots & \ddots & \vdots \\
& & \ddots & \ddots & \ddots \\
-e_{1r} & -e_{2r} & \cdots & -e_{r-1,r} & 0
\end{bmatrix}
\]

Lemma 1.7. (a) The assignment \( \Lambda \to M_\Lambda \) induces a monomorphism

\[
\phi : \nu \mathbb{B}(T_N) \to \text{Hom}_\mathbb{Z}(N, M/\nu M).
\]

(b) \( \Lambda \) is unramified along \( \text{orb } \rho_k \) if and only if \( M_\Lambda \cdot \eta_k = 0 \).
Proof. (a) The matrix $M_A$ defines the indicated homomorphism $\phi(\Lambda)$ by representing elements in $N$ as column vectors with respect to the basis $n_1, \ldots, n_r$; the elements in $M$ as column vectors with respect to the dual basis $m_1, \ldots, m_r$, and following left multiplication by $M_A$ by reduction modulo $\nu M$. Since multiplication of symbols corresponds to addition of exponents modulo $\nu$, it is clear that $\phi$ is a homomorphism. If $\phi(\Lambda) = M_A = 0$, then each $e_{ij} = 0$, so $\Lambda = 0$ in $B(T_N)$. Thus $\phi$ is a monomorphism.

(b) Write $\eta_k = \sum_{i=1}^r \eta_{ki} n_i$ and let $\Lambda = \prod_{i<j}(m_i, m_j)_{\nu_i}^{e_{ij}}$. Then

$$\text{ram}_{\text{orb} \rho_k}(m_i, m_j)_\nu = \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j = \eta_{kj} m_i - \eta_{ki} m_j.$$  

Hence

$$\text{ram}_{\text{orb} \rho_k}(\Lambda) = \text{ram}_{\text{orb} \rho_k} \left( \prod_{i<j}(m_i, m_j)_{\nu_i}^{e_{ij}} \right) = \sum_{i<j} e_{ij} \eta_{kj} m_i - e_{ij} m_j + \nu M$$

$$= \sum_{i=1}^r \sum_{j=i+1}^r e_{ij} \eta_{kj} m_i - \sum_{i=1}^r \sum_{j=i+1}^r e_{ij} \eta_{ki} m_j + \nu M$$

$$= \sum_{i=1}^r \sum_{j=i+1}^r e_{ij} \eta_{kj} m_i - \sum_{j=1}^r \sum_{i=j+1}^r e_{ji} \eta_{kj} m_i + \nu M$$

$$= M_A \cdot \begin{bmatrix} \eta_{k1} \\ \vdots \\ \eta_{kr} \end{bmatrix}.$$  

As we observed in the proof of Lemma 1.6, $\text{ram}_{\text{orb} \rho_k}(\Lambda) = m + \nu M$ for some $m \in \eta_k^\perp$. The ramification of $\Lambda$ along $\text{orb} \rho_k$ is the cyclic extension of $k[\eta_k^\perp]$ obtained by adjoining the $\nu$th root of $m$ and this extension is split ($\Lambda$ is unramified along $\text{orb} \rho_k$) if and only if $m \in \nu \eta_k^\perp$. Since $\eta_k^\perp$ is a direct summand of $M$, $\Lambda$ is unramified along $\text{orb} \rho_k$ if and only if $m \in \nu M$ if and only if

$$M_A \cdot \begin{bmatrix} \eta_{k1} \\ \vdots \\ \eta_{kr} \end{bmatrix} = 0. \quad \Box$$

Theorem 1.8. Let $X = T_N \text{emb}(\Delta)$ be a nonsingular toric variety and $a_1, \ldots, a_r$ the set of invariant factors of $X$. Then $B'(X)$ is the subgroup of $B(T_N)$ represented by algebra classes $\prod_{i<j}(m_i, m_j)_{\nu_i}^{e_{ij}}$ where $\nu_i|a_i$, $1 \leq i \leq r$.

Proof. The exact sequence (1) and Lemma 1.7 imply $\nu B'(X)$ consists of those algebra classes $\Lambda$ in $\nu B(T_N)$ such that $M_A \cdot \eta_k = 0$ for the primitive vector $\eta_k$ on each 1-dimensional cone $\rho_k$ in $\Delta$ ($1 \leq k \leq n$). If $N'$ is the subgroup of $N$ generated by $\bigcup_{\sigma \in \Delta} \sigma \cap N$ then $N'$ is generated by $\{\eta_k\}_{k=1}^n$ so $\Lambda$ represents a class in $\nu B'(X)$ if and only if $M_A$ vanishes on $N'$. For each $\nu > 0$ we have
the commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & \longrightarrow & \nu B(T_N) \\
\uparrow & & \uparrow \\
0 & \longrightarrow & \nu B'(X)
\end{array}
\longrightarrow
\begin{array}{ccc}
\Hom_{\mathbb{Z}}(N, M/\nu M) & \longrightarrow & \Hom_{\mathbb{Z}}(N'/N, M/\nu M) \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}
\]

Taking the direct limit over all $\nu \geq 2$ gives a monomorphism $B'(X) \to \Hom_{\mathbb{Z}}(N'/N, M \otimes \mathbb{Q}/\mathbb{Z})$.

Let $n_1, \ldots, n_r$ be a basis for $N$ such that $N' = \mathbb{Z}a_1n_1 \oplus \cdots \oplus \mathbb{Z}a_rn_r$ and $a_i|a_{i+1}$ for $1 \leq i \leq r - 1$. That is, $a_1, \ldots, a_r$ is the set of invariant factors for $X$. Then $\Hom_{\mathbb{Z}}(N'/N, M \otimes \mathbb{Q}/\mathbb{Z}) \cong \Hom_{\mathbb{Z}}(\bigoplus \mathbb{Z}a_1n_1, M \otimes \mathbb{Q}/\mathbb{Z})$. This means $B'(X)$ is contained in the subgroup of $B(T_N)$ of algebra classes $\prod\langle m_i, m_j \rangle_{v_i}$ where $0 < v_i$ and $v_i|a_i$, $1 \leq i \leq r$. Conversely, if $v_i|a_i$ and $v_i \geq 1$ then the matrix $M_\Lambda$ for $(m_i, m_j)_v$ has a $+1$ in the $i$th entry and a $-1$ in the $j$th entry. A typical element in $N'$ is $x = \lambda_1a_1n_1 + \cdots + \lambda_r a_r n_r$ and $M_\Lambda \cdot x = \lambda_j a_j m_i - \lambda_i a_i m_j \in \mathbb{Z}M_\Lambda$. Thus $(m_i, m_j)_v$ represents an element in $B'(X)$. So $B'(X) = \{ \prod\langle m_i, m_j \rangle_{v_i} | 0 < v_i$ and $v_i|a_i \}$. □

Now it follows that $B'(X) \cong \bigoplus_{i=1}^{r-1} \Hom(\mathbb{Z}/a_i, \mathbb{Q}/\mathbb{Z})^{r-i}$. To complete the proof of Theorem 1.1 it suffices to show $B(X) = B'(X)$. It suffices to find an Azumaya algebra $\Lambda$ on $X$ such that $K(X) \otimes \Lambda$ is equivalent to $(m_i, m_j)_v$ for each $v_i|a_i$.

**Lemma 1.9.** Let $X$ be as in Theorem 1.8 and let $N' = \langle \bigcup_{\sigma \in \Delta} \sigma \cap N \rangle$. Let $\nu \geq 2$ and let $M_\nu = \{ m \in M | \langle m, n' \rangle \equiv 0 \pmod{\nu} \text{ for all } n' \in N' \}$. If $(\nu, a_i)$ is the greatest common divisor of $\nu$ and $a_i$, then

\[
H^1(X, \mathbb{Z}/\nu) \cong M_\nu/\nu M \cong \bigoplus_{i=1}^{r} \mathbb{Z}/(\nu, a_i).
\]

**Proof.** Restriction induces an embedding $H^1(X, \mathbb{Z}/\nu) \to H^1(T_N, \mathbb{Z}/\nu)$. The correspondence which assigns to each element $m \in M$ the cyclic extension of $T_N$ obtained by adjoining the $\nu$th root of $m$ induces an isomorphism $M/\nu M \cong H^1(T_N, \mathbb{Z}/\nu)$. An element $m + \nu M$ corresponds to an element of $H^1(X, \mathbb{Z}/\nu)$ if and only if $K(X)(m^{1/\nu})$ is unramified along orb $\rho_k$ for $1 \leq k \leq n$ if and only if the restriction of $m$ to orb $\rho_k$ is a unit in the coordinate ring $k[\eta_k, \mu_k]$ of orb $\rho_k$ if and only if $v_k(m) \equiv 0 \pmod{\nu}$ if and only if $\langle m, \eta_k \rangle \equiv 0 \pmod{\nu}$ where unexplained notation is as in Lemma 1.5. Thus

\[
H^1(X, \mathbb{Z}/\nu) = \{ m \in M | \langle m, \eta_k \rangle \equiv 0 \pmod{\nu} \text{ for all } 1 \leq k \leq n \} + \nu M \\
= \{ m \in M | \langle m, n' \rangle \equiv 0 \pmod{\nu} \text{ for all } n' \in N' \} + \nu M \\
= M_\nu/\nu M.
\]

But $N' = \mathbb{Z}a_1n_1 \oplus \cdots \oplus \mathbb{Z}a_r n_r$ so it is easy to check that

\[
M_\nu/\nu M \cong \bigoplus_{i=1}^{r} \mathbb{Z} \left( \frac{\nu}{(a_i, \nu)} m_i \right) / \mathbb{Z}(\nu m_i) \cong \bigoplus_{i=1}^{r} \mathbb{Z}/(a_i, \nu). \quad \Box
\]
Lemma 1.10. If $X$ is as in Theorem 1.8, then $\mathcal{B}(X) = \mathcal{B}'(X)$.

Proof. It suffices to show each $(m_i, m_j)_{\nu_i}$ is in the image of the cup product map $H^1(X, \mathbb{Z}/\nu_i) \times H^1(X, \mu_{\nu_i}) \rightarrow \mathcal{B}(X)$ when $\nu_i \mid a_i$, since cup products correspond to taking smash products of cyclic Galois extensions and thus are Azumaya algebras (e.g. [6]).

If $a_j = 0$, both $m_i, m_j \in M_{\nu_i}$ since $\nu_i/(a_i, \nu_i) = \nu_i/(0, \nu_i) = 1 = \nu_i/(0, \nu_i)$. If $a_j \neq 0$, then $(m_i, m_j)_{\nu_i} \sim (m_i, m_j)_{a_j/\nu_i} \sim ((a_j/\nu_i)m_i, m_j).$ But $a_j/(a_i, a_j) = a_j/a_i$ which divides $a_j/\nu_i$ since $\nu_i \mid a_i$. Thus $(a_j/\nu_i)m_i$ and $m_j$ are both in $M_{a_j}$ and $(m_i, m_j)_{\nu_i}$ is equivalent to an algebra in the image of the cup product map $H^1(X, \mathbb{Z}/a_j) \times H^1(X, \mu_{a_j}) \rightarrow a_j\mathcal{B}(X)$. So $\mathcal{B}(X) = \mathcal{B}'(X)$. □

As a result of observations made so far, we can show the following proposition.

Proposition 1.11. Let $\prod_{i<j}(m_i, m_j)_{e_{ij}}$ represent a class in $\mathcal{B}(T_N)$ of order $\nu$. Let $M_\Lambda$ be the matrix transformation in $\text{Hom}(N, M/\nu M)$ defined in Lemma 1.7 and let $t$ be the rank of $\text{kernel}(M_\Lambda)$. Then there exists a direct summand $P$ of $M$ with rank($P$) = $r - t$ and an Azumaya algebra $L$ over $k[P]$ with $\Lambda \cong k[M] \otimes_{k[P]} L$. No direct summand of $M$ of smaller rank has this property.

Proof. Find a basis $n_1, \ldots, n_r$ of $N$ such that $\text{ker}(M_\Lambda) = \mathbb{Z}b_1n_1 \oplus \mathbb{Z}b_2n_2 \oplus \cdots \oplus \mathbb{Z}b_rn_r$ and $b_i|b_{i+1}$ for $1 \leq i \leq r - 1$. Since $\text{ker}(M_\Lambda)$ has rank $t$, $b_i \neq 0$ and $b_{t+i} = 0$ for $i \geq 1$.

Let $P$ be a direct summand of $M$ and assume $\Lambda$ is obtained by extending an algebra over $k[P]$. Let $m_1', \ldots, m_r'$ be a basis for $P$ and extend this basis to a basis for $M$. We can assume $\Lambda = \prod_{i<j \leq s}(m_i', m_j')_{e_{ij}}$. If $n_1', \ldots, n_r'$ is the dual basis to $m_1', \ldots, m_r'$, then the matrix of the transformation $M_\Lambda$ with respect to this new basis pair has a kernel which contains a direct summand of $N$ of rank $r - s$. Therefore $t \geq r - s$ so $s \geq r - t$. Now let $m_1, \ldots, m_r$ be a dual basis for $M$ with respect to $n_1, \ldots, n_r$. The matrix $M_\Lambda$ with respect to this new basis is

$$M_\Lambda = \begin{bmatrix}
0 & 0 & e_{t+1,t} & \cdots & e_{t+1,r} \\
0 & -e_{t+1,t} & 0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & -e_{t+1,r} & \cdots & 0 & 0
\end{bmatrix}.$$

So $\Lambda$ is defined on the torus $k[m_{t+1}, \ldots, m_r, -m_{t+1}, \ldots, -m_r]$ and we can take $P = \langle m_{t+1}, \ldots, m_r \rangle$. The rank of $P$ is $r - t$. □

2

In this section we continue to let $\Delta$ be a finite fan on $N_R$ and $X = T_N \text{emb}(\Delta)$ the associated toric variety containing the $r$-dimensional torus $T_N$ as an open subset. Assume $\Delta$ consists of cones of dimension $\leq 2$ and let $\Delta(2) = \{\tau_1, \ldots, \tau_m\}$. Let $U_i = U_{\tau_i}$, $V_i = V(\tau_i) = \text{orb}(\tau_i)$ and let $V = V_1 \cup \cdots \cup V_m$. Then $X - V = T_N \text{emb}(\Delta - \Delta(2))$ is nonsingular. In this situation our first lemma gives information about the étale cohomology groups of the affine open subsets $U_i$ of $X$. 

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Lemma 2.1. (a) For each \( i \) and each \( p \geq 0 \), we have a short exact sequence

\[
0 \to H^p(U_i, G_m) \to H^p(U_i - V_i, G_m) \to H^{p+1}_{V_i}(U_i, G_m) \to 0.
\]

(b) \( H^p(U_i, G_m) \cong H^p(T_{r-2}, G_m) \) where \( T_{r-2} \) is a torus of dimension \( r - 2 \).

Proof. First we check that \( H^p(U_i, G_m) \) is torsion for \( p \geq 2 \). For notational simplicity we suppress the subscript \( i \) from \( \tau_i, U_i, \) and \( V_i \). Now \( \tau \) is a two-dimensional cone in \( N_R \). Let \( \bar{\tau} \) be \( \tau \) viewed as a two-dimensional cone in \( \mathbb{R}^r \). Then \( U = U_{\bar{\tau}} \times T_{r-2} \) where \( T_{r-2} \) is an \( (r - 2) \)-dimensional torus. Let \( R \) be the affine coordinate ring of \( U_{\bar{\tau}} \), and \( R^h \) the henselization of \( R \) at the maximal ideal \( m \) corresponding to the closed point \( \text{orb} \bar{\tau} \). Let \( R[X, X^{-1}] \) denote the affine coordinate ring of \( U \) and let \( U^h = \text{Spec} R^h[X, X^{-1}] \). Let \( V^h = V \times U^h \). Then \( V^h \) is the closed set corresponding to \( I = mR^h[X, X^{-1}] \).

The completion of \( R^h[X, X^{-1}] \) in the \( I \)-adic topology is \( \hat{R}[X, X^{-1}] \) where \( \hat{R} \) is the \( m \)-adic completion of \( R \). By [13, p. 127], we see that \( (R^h[X, X^{-1}], I) \) is a Hensel pair. By [5, p. 35] \( \text{Cl}(R^h[X, X^{-1}]) \) embeds into \( \text{Cl}(\hat{R}[X, X^{-1}]) \).

Since the singularity of \( U \) is given by a finite cyclic group action [12, p. 30], it is well known that \( \text{Cl}(\hat{R}[X, X^{-1}]) = \text{Cl}(\hat{R}) \) is also finite cyclic [2, Satz 2.11]. Thus \( \text{Cl}(U^h) \) is finite. The long exact sequences for the pairs \( V \subseteq U \) and \( V^h \subseteq U^h \) give the commutative diagram

\[
\begin{array}{c}
0 \to H^{p-1}(U - V, G_m) \to H^p(U, G_m) \to H^p(U - V, G_m) \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \to H^{p-1}(U^h - V^h, G_m) \to H^p(U^h, G_m) \to H^p(U^h - V^h, G_m)
\end{array}
\]

with exact rows. By excision \( H^p(U, G_m) \cong H^p_{V^h}(U^h, G_m) \) [11, p. 92]. By [15] \( H^p(U^h, G_m) \cong H^p(U^h, G_m) = H^p((\text{orb} \bar{\tau}) \times T_{r-2}, G_m) \) which is torsion for \( p \geq 2 \) since \( T_{r-2} \) is smooth [7, p. 71]. Again by [7, p. 71] \( H^p(U^h - V^h, G_m) \) and \( H^p(U - V, G_m) \) are torsion for \( p \geq 2 \). But \( H^1(U^h - V^h, G_m) = \text{Pic}(U^h - V^h) = \text{Cl}(U^h - V^h) = \text{Cl}(U^h) \) is torsion. It now follows that \( H^p(U, G_m) \) is torsion for \( p \geq 2 \).

The natural map \( U \times A^1 \to U \) and Kummer theory induce the commutative diagram

\[
\begin{array}{cccc}
0 \to H^{p-1}(U \times A^1, G_m) \otimes \mathbb{Z}/n & \to & H^p(U \times A^1, \mu_n) & \to & \gamma
\end{array}
\]

\[
\begin{array}{cc}
\alpha & \beta
\end{array}
\]

\[
\begin{array}{c}
0 \to H^{p-1}(U, G_m) \otimes \mathbb{Z}/n \\
0 \to H^p(U, \mu_n) \\
0 \to H^p(U, G_m)
\end{array}
\]

for all \( p \geq 2 \) and \( n \geq 2 \). By [11, p. 240] \( \beta \) is an isomorphism for \( p \geq 2 \). Since \( \text{Pic} U = 0 = \text{Pic}(U \times A^1) \), \( \alpha \) is an isomorphism for \( p = 2 \). Therefore, \( \gamma \) is an isomorphism for \( p = 2 \) and all \( n \geq 2 \). Taking the inductive limit over all \( n \), we have \( H^2(U \times A^1, G_m) \cong H^2(U, G_m) \). By induction on \( p \) we see that \( H^p(U \times A^1, G_m) \cong H^p(U, G_m) \) for all \( p \geq 2 \).

We can give the coordinate ring \( k[\mathcal{O}_U] \) of \( U \) a grading by the nonnegative integers such that the degree = 0 subring is the coordinate ring of \( T_{r-2} \). Since \( H^p(U \times A^1, G_m) = H^p(U, G_m) \), [8, Theorem 1.1] implies \( H^p(U, G_m) \cong \)}
\[ H^p(T_{r-2}, G_m), \] which proves (b). We have a commutative diagram
\[
\begin{array}{ccc}
H^p(T_{r-2}, G_m) & \xrightarrow{\beta} & H^p(T_r, G_m) \\
\xrightarrow{\gamma} & & \xleftarrow{\delta}
\end{array}
\]
where the maps \( \beta, \gamma, \delta \) are induced from restriction. Since \( \beta \) is injective, \( \alpha \) is injective and Lemma 2.1 now follows. □

**Theorem 2.2.** Let \( \Delta \) be a fan which consists of cones of dimension \( \leq 2 \). Let \( \Delta' \) be a nonsingular fan obtained from \( \Delta \) by subdividing the two-dimensional faces of \( \Delta \) and let \( \tilde{X} = T_N \text{emb}(\Delta') \). Then the sequence \( 0 \to B'(T_N/X) \to B'(X) \to B'((\tilde{X}) \to 0 \) (with natural maps) is exact.

**Proof.** Let \( \pi: \tilde{X} \to X \) be the desingularization resulting from the subdivision \( \Delta' \) of \( \Delta \) [12, Corollary 1.18] and let \( \tilde{U}_i = \pi^{-1}(U_i) \). From the long exact sequence of cohomology with supports, and the observation that \( V \) is a disjoint union of closed sets \( V_i \) (see pp. 92–93 of [11]) we have a commutative diagram with exact rows
\[
\begin{array}{cccc}
0 & \to & B'(T_N/X) & \to & B'(X) & \to & B'(X-V) & \to & H^2(T_N, G_m) \\
& & \downarrow{\alpha} & & \downarrow{=} & & \downarrow{\beta} & & \\
0 & \to & B'(\tilde{X}) & \to & B'((\tilde{X} - \pi^{-1}(V))) & \to & \bigoplus_{i=1}^{m} H^2_{\pi^{-1}(V_i)}(\tilde{U}_i, G_m)
\end{array}
\]
The second row is exact since \( \tilde{X} \) is nonsingular. First check \( \beta \) is injective. For each \( i \), Lemma 2.1 yields the commutative diagram with exact rows
\[
\begin{array}{cccc}
0 & \to & B'(U_i) & \to & B'(U_i-V_i) & \to & H^3_{V_i}(U_i, G_m) & \to & 0 \\
& & \downarrow{\alpha_i} & & \downarrow{=} & & \downarrow{\beta_i} & & \\
0 & \to & B'(\tilde{U}_i) & \to & B'(\tilde{U}_i-\pi^{-1}(V_i)) & \to & H^3_{\pi^{-1}(V_i)}(\tilde{U}_i, G_m)
\end{array}
\]
Here \( B'(U_i) = B'(U_{\tau_i} \times T_{r-2}) = B(T_{r-2}) \) by Lemma 2.1 and \( B'(U_i-V_i) = H^2(U_i-V_i, G_m) \) since \( U_i-V_i \) is nonsingular. If \( \Delta'(\tau_i) \) is the fan whose cones are the cones of \( \Delta' \) contained in \( \tau_i \), then \( \Delta'(\tau_i) \) is a nonsingular fan whose one-dimensional faces lie in a plane. The invariants for \( \tilde{U}_i = T_N \text{emb}(\Delta'(\tau_i)) \) are \( \{1, 1, 0, \ldots, 0\} \) and Theorem 1.1 implies \( B'(\tilde{U}_i) = B'(\tilde{U}_{\tau_i} \times T_{r-2}) = B(T_{r-2}) \) so \( \alpha_i \) is an isomorphism. Since \( \ker \beta_i = \coker \alpha_i \), \( \beta_i \) is injective so \( \beta \) is injective. But \( \ker \beta = \coker \alpha \), so \( \alpha \) is an epimorphism and the theorem follows. □

As a result of Theorem 2.2 and our analysis of the Brauer groups of nonsingular toric varieties in §1, we are left with the study of \( B'(T_N/X) \).

**Theorem 2.3.** Let \( \Delta \) be a fan which consists of cones of dimension \( \leq 2 \). Let \( \Delta(2) = \{\tau_1, \ldots, \tau_m\} \). Let \( X = T_N \text{emb}(\Delta) \) and let \( U_i = U_{\tau_i} \) be the open subsets
of $X$ associated to the $\tau_i$. Then there is an exact sequence

$$0 \to \text{Pic}(X) \to \text{Cl}(X) \to \bigoplus_{i=1}^{m} \text{Cl}(U_i) \to \text{B}'(T_N/X) \to 0.$$  

Proof. Let $V_i = V(\tau_i) = \text{orb}(\tau_i)$ and let $V = V_1 \cup \cdots \cup V_m$. From the long exact sequence of cohomology with supports in the closed set $V$ we have (since $V$ is the disjoint union of the closed sets $V_i$)

$$\cdots \to H^1(X, G_m) \to H^1(X-V, G_m) \to \bigoplus_{i=1}^{m} H^2_{V_i}(X, G_m) \to \cdots$$

(1)

$$\to H^2(X, G_m) \to H^2(X-V, G_m) \to \bigoplus_{i=1}^{m} H^3_{V_i}(X, G_m) \to \cdots.$$  

Since $V$ has codimension 2 in $X$, and $X-V$ is nonsingular, $H^1(X-V, G_m) = \text{Pic}(X-V) = \text{Cl}(X-V) = \text{Cl}(X)$. Since $U_i$ is an open neighborhood of $V_i$, $H^p_{V_i}(X, G_m) = H^p_{V_i}(U_i, G_m)$ for all $p \geq 0$ [11, p. 93]. From Lemma 2.1 with $p = 1$ we get the exact sequences

$$0 \to \text{Pic} U_i \to \text{Cl} U_i \to H^2_{V_i}(U_i, G_m) \to 0 \quad (1 \leq i \leq m).$$

Lemma 2.1(b) gives $\text{Pic} U_i = \text{Pic} T_{r-2} = 0$ so $\text{Cl}(U_i) = H^2_{V_i}(U_i, G_m)$. Since $\tau_i$ is simplicial, $\text{Cl}(U_i) = \text{Pic}(U_i - V_i)$ is torsion [12, Proposition 2.1]. Since $X-V$ is nonsingular, $\text{B}'(X-V) = H^2(X-V, G_m)$ and $\text{B}'(X-V) \to \text{B}(T_N)$ is injective. But $\text{Pic} X \to \text{Cl}(X)$ is injective [5]. With these identifications (1) reduces to the sequence of the theorem. □

Corollary 2.4. In the context of Theorem 2.3, if $\text{rank}_Z(N) = r \leq 3$ and $m \geq 1$, then

$$0 \to \text{Pic}(X) \to \text{Cl}(X) \to \bigoplus_{i=1}^{m} \text{Cl}(U_i) \to \text{B}'(X) \to 0$$

is exact.

Proof. We need to check $\text{B}'(X) = \text{B}'(T_N/X)$. $H^2(U_i, G_m) = H^2(T_{r-2}, G_m) = 0$ for $r \leq 3$ (Lemma 2.1 and [10]). From Lemma 2.1(a) we have (since $U_i - V_i$ is nonsingular) $\text{B}'(U_i - V_i) = H^2(U_i - V_i, G_m) = H^3_{V_i}(U_i, G_m) = H^3_{V_i}(X, G_m)$ so (1) becomes

$$0 \to \text{Pic} X \to \text{Cl}(X) \to \bigoplus_{i=1}^{m} \text{Cl}(U_i) \to \text{B}'(X) \to \text{B}'(X-V) \to \bigoplus_{i=1}^{m} \text{B}'(U_i - V_i).$$

Since $X-V$ is nonsingular, restriction induces a monomorphism $\text{B}'(X-V) \to \text{B}'(U_i - V_i)$ for each $i$ and the corollary follows. □

The object of the rest of this section is to give an algorithm for finding for each prime $p$ a subset $\tau_1, \ldots, \tau_p$ of the two-dimensional faces of $\Delta$ such that $[\bigoplus_{i=1}^{p} \text{Cl}(U_i)]_p \cong [\text{B}'(T_N/X)]_p$. (For $G$ a finite abelian group, $G_p$ is the Sylow $p$-subgroup.) In particular, the exact sequence of Theorem 2.3 is split-exact. To the fan $\Delta$ we associated a bipartite graph $\Gamma$. The vertex set of $\Gamma$ is $\Delta(1) \cup \Delta(2) = \{\rho_1, \ldots, \rho_n\} \cup \{\tau_1, \ldots, \tau_m\}$ and there is an edge in $\Gamma$ connecting $\rho_j$ and $\tau_j$ if and only if $\rho_j$ is a face of $\tau_j$. If $Y$ is the $T_N$-invariant divisor $X-T_N$ on $X$, then $\Gamma$ is the graph associated to $Y$ in the sense of [4]. A
cycle $Z$ in $\Gamma$ (i.e., $Z$ is homeomorphic to the unit circle) determines a finite set $\tau_1, \ldots, \tau_t$ of two-dimensional cones and $\rho_1, \ldots, \rho_t$ of one-dimensional faces of $\Delta$ configured as follows:

If $\Delta_Z$ is the subfan of $\Delta$ consisting of the cones $\{0, \rho_1, \ldots, \rho_t, \tau_1, \ldots, \tau_t\}$, we will show that the cohomological Brauer group of $T_N \text{emb}(\Delta_Z)$ is cyclic of order the greatest common divisor of $\{\vert \text{Cl}(U_i) \vert \}_{i=1}^t$. Of course, there may be many such cycles in $\Gamma$ and the last step in the analysis is to choose for each prime $p$ a list of cycles $(Z_i)_{i=1}^t$ and for each $Z_i$ a face $\tau_i$ such that $[\bigoplus_{i=1}^t \text{Cl}(U_i)]_p \cong [B'(T_N/X)]_p$.

We adopt the following notation: for each two-dimensional cone $\tau_i$ in $\Delta$ (1 ≤ $i ≤ m$) let $\rho_{i1}$ and $\rho_{i2}$ be the one-dimensional faces of $\tau_i$ so $\tau_i = \rho_{i1} + \rho_{i2}$. We have observed $\text{Cl}(X) = \text{Cl}(X - V) = \text{Pic}(X - V)$ and $\text{Cl}(U_1) = \text{Cl}(U_1 - V_1) = \text{Pic}(U_1 - V_1)$. Now we want to present $\text{Pic}(X - V)$ and $\text{Pic}(U_1 - V_1)$ in terms of support functions on the fan $\Delta - \{\tau_1, \ldots, \tau_m\}$. If we let $\rho_1, \ldots, \rho_n$ be the one-dimensional cones in $\Delta$, then we can identify the support functions on $\Delta - \{\tau_1, \ldots, \tau_m\}$ with the direct sum of copies of $\mathbb{Z}$ indexed by the $\rho_i$. If $\Delta_i = \{0, \rho_{i1}, \rho_{i2}\}$, then $U_i - V_i = T_N \text{emb}(\Delta_i)$. It follows from [12, Corollary 2.5] that the sequences

\begin{align*}
M &\rightarrow \bigoplus_{i=1}^n \mathbb{Z}\rho_i \rightarrow \text{Cl}(X) \rightarrow 0, \\
M &\xrightarrow{\beta} \bigoplus_{i=1}^n \mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2} \rightarrow \text{Cl}(U_i) \rightarrow 0
\end{align*}
It is routine to check that \( \text{im} \alpha + \text{im} \beta = \ker \delta \gamma \). As a result we have a fundamental exact sequence which we exploit for the remainder of this section: (in this sequence \( \psi = \delta \gamma \)),

\[
(3) \quad \bigoplus_{j=1}^{n} \mathbb{Z} \rho_j \oplus \bigoplus_{i=1}^{m} M \xrightarrow{\alpha + \beta} \bigoplus_{i=1}^{m} (\mathbb{Z} \rho_{i1} \oplus \mathbb{Z} \rho_{i2}) \xrightarrow{\psi} B'(T_N/X) \to 0.
\]

Let \( \Gamma \) be the graph associated to \( \Delta \). Observe that \( \Gamma \) has \( 2m \) edges since each \( \tau_i \) has exactly two one-dimensional faces \( \rho_{i1} \) and \( \rho_{i2} \). The free abelian group \( \bigoplus_{i=1}^{m} (\mathbb{Z} \rho_{i1} \oplus \mathbb{Z} \rho_{i2}) \) is called the edge space of \( \Gamma \). If we write \( \Gamma \) as a union of its connected components \( \Gamma_i \) we get a corresponding decomposition of \( \Delta \) into subfans \( \Delta_i \) with \( \Delta_i \cap \Delta_j = \{0\} \) whenever \( i \neq j \). The decomposition of \( \Delta \) gives an open cover of \( X \) where the elements in the open cover are \( T_N \text{emb}(\Delta_i) = X_i \) and \( X_i \cap X_j = T_N \) whenever \( i \neq j \). With this notation we can prove

**Proposition 2.5.** The natural map \( B'(X) \to \bigoplus_i B'(X_i) \) induces an isomorphism \( B'(T_N/X) \cong \bigoplus_i B'(T_N/X_i) \).

**Proof.** Assume \( \Delta = \Delta_1 \cup \Delta_2 \) where \( \Delta_1 \) and \( \Delta_2 \) are fans with \( \Delta_1 \cap \Delta_2 = \{0\} \). It is sufficient to prove \( B'(T_N/X) \cong B'(T_N/X_1) \oplus B'(T_N/X_2) \) where \( X_i = T_N \text{emb}(\Delta_i) \) (\( i = 1, 2 \)). Let \( \Delta_1(1) = \{\rho_1, \ldots, \rho_{n_1}\} \) and \( \Delta_1(2) = \{\tau_1, \ldots, \tau_{m_1}\} \) and \( \Delta_2(1) = \{r_1, \ldots, r_{n_2}\} \) and \( \Delta_2(2) = \{t_1, \ldots, t_{m_2}\} \). Also let \( \tau_i = \rho_{i1} + \rho_{i2} \) and \( t_j = r_{j1} + r_{j2} \) where \( \rho_{ik} \in \Delta_1(1) \) and \( r_{jl} \in \Delta_2(1) \). With respect to this decomposition the exact sequence (3) decomposes as

\[
\begin{align*}
\bigoplus_{j=1}^{n_1} \mathbb{Z} \rho_j \oplus \bigoplus_{i=1}^{m_1} M & \quad \bigoplus_{j=1}^{n_2} \mathbb{Z} r_j \oplus \bigoplus_{i=1}^{m_2} M \xrightarrow{\alpha_1 + \beta_1} B'(T_N/X_1) \oplus B'(T_N/X_2) \\
\bigoplus_{i=1}^{m_1} (\mathbb{Z} \rho_{i1} \oplus \mathbb{Z} \rho_{i2}) & \quad \bigoplus_{i=1}^{m_2} (\mathbb{Z} r_{i1} \oplus \mathbb{Z} r_{i2}) \xrightarrow{\psi} B'(T_N/X) \to 0.
\end{align*}
\]

But \( \text{coker}(\alpha_1 + \beta_1) \oplus (\alpha_2 + \beta_2) = B'(T_N/X_1) \oplus B'(T_N/X_2) \) by (3) so \( B'(T_N/X) = B'(T_N/X_1) \oplus B'(T_N/X_2) \). \( \square \)

Notice in Proposition 2.5 that if \( X_i \) corresponds to a connected component of \( \Gamma \) containing no two-dimensional faces \( \tau_i \) as vertices, then \( X_i = T_N \text{emb}\{0, \rho\} \) for some one-dimensional cone \( \rho \) in \( \Delta \). In this case \( X_i \) is nonsingular and \( B'(T_N/X_i) = 0 \). Thus, as a result of Proposition 2.5 we can assume \( \Gamma \) is connected and at least one vertex of \( \Gamma \) is a two-dimensional cone in \( \Delta \).

We now determine a matrix representation for the map \( \alpha + \beta \) in (3). Let \( \tau = \rho_1 + \rho_2 \in \Delta(2) \) and consider the map

\[
M \xrightarrow{\beta} \mathbb{Z} \rho_1 \oplus \mathbb{Z} \rho_2 \to \text{Cl}(U_\tau) \to 0
\]

as in (2). Pick a basis \( n_1, \ldots, n_r \) for \( N \) and a dual basis \( m_1, \ldots, m_r \) for \( M \). Let \( \eta_i \) be a primitive element in \( N \) with \( \rho_i = R_{\geq 0} \eta_i \). The matrix of \( \beta \) with respect to the basis pair \( \{m_1, \ldots, m_r\}, \{\rho_1, \rho_2\} \) is the \( 2 \times r \) matrix whose \( i, j \)th entry is \( \langle m_j, \eta_i \rangle \). But \( \langle m_j, \eta_i \rangle \) is the \( j \)th coordinate of \( \eta_i \) so we can write this matrix as \( \begin{pmatrix} h_{i1} \\ \vdots \\ h_{ir} \end{pmatrix} \) where we think of \( \eta_i \) as a row vector. Therefore the map \( \beta \) in (3)

\[
\bigoplus_{i=1}^{m} M \xrightarrow{\beta} \bigoplus_{i=1}^{m} (\mathbb{Z} \rho_{i1} \oplus \mathbb{Z} \rho_{i2})
\]
has a matrix representation which is a direct sum of $2 \times r$ matrices $(\eta_{i1}, \eta_{i2})$ where $\eta_{i1}$ and $\eta_{i2}$ are the primitive generators of $\rho_{i1}$ and $\rho_{i2}$ expressed with respect to the basis $\{n_1, \ldots, n_r\}$. To determine the matrix for the homomorphism

$$
\bigoplus_{j=1}^{n} \mathbb{Z}\rho_j \xrightarrow{\alpha} \bigoplus_{i=1}^{m} (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})
$$

given in (3) we observe the $j$th column of this matrix is $\alpha(\rho_j)$. Thus the $j$th column has a 1 in the row determined by $\rho_{ik}$ if $\rho_j = \rho_{ik}$. Otherwise this entry is 0. The matrix of the homomorphism $\alpha + \beta$ of (3) is then

$$
Q = \begin{bmatrix}
\alpha(\rho_1) & \cdots & \alpha(\rho_n)
\end{bmatrix}
\begin{pmatrix}
\eta_{i1} \\
\eta_{i2}
\end{pmatrix}.
$$

Note $Q$ is an integral matrix with $2m$ rows and $n + rm$ columns, and we can identify $\text{im}(\alpha + \beta)$ with the column space of $Q$. Since $B'(T_{N/X}) = \text{coker}(\alpha + \beta)$ from (3), calculating $B'(T_{N/X})$ is reduced to determining the column space of $Q$. Our first observation is a straightforward calculation:

$$
Q \cdot \begin{bmatrix}
\eta_1 \\
\vdots \\
\eta_n \\
-I_r
\end{bmatrix} = (0)
$$

where $I_r$ is the $r \times r$ identity matrix. Thus the last $r$ columns of $Q$ containing $(\eta_{n1}, \eta_{n2})$ are linear combinations of the preceding columns. We now assume $\Gamma$ is connected, and let $T$ be a spanning tree for $\Gamma$. We observe that in $\Gamma$ each vertex $\tau_i$ is joined by edges $\tau_i - \rho_{i1}$, $\tau_i - \rho_{i2}$ to vertices $\rho_{i1}$, $\rho_{i2}$ so there are $2m$ edges in $\Gamma$. Since $\Gamma$ is connected, there are $n + m - 1$ edges in $T$ [1].

Thus, if $c_1, \ldots, c_e$ denote the edges of $\Gamma$ that are not in $T$, then $e = m - n + 1$ and for each $i$ at least one of $\tau_i - \rho_{i1}$, $\tau_i - \rho_{i2}$ is in $T$. By reindexing we can assume $c_1 = \tau_1 - \rho_{i1}$, $\ldots$, $c_e = \tau_e - \rho_{i2}$. We identify $c_j$ with the basis vector $\rho_{j2}$ in the edge space $\bigoplus_{i=1}^{m} (Z\rho_{i1} \oplus Z\rho_{i2})$. For $1 \leq i \leq m$ let $n_{i1}$ be a primitive vector in $N$ with $\mathbb{R}_{\geq 0} n_{i1} = \rho_{i1}$ and choose $n_{i2}$ in $N$ with $\{n_{i1}, n_{i2}\}$ a basis for $\mathbb{R}_{\geq 0} n_{i1} \cap N$. We can extend $\{n_{i1}, n_{i2}\}$ to a basis $\{n_{i1}, n_{i2}, \ldots, n_{ir}\}$ for $N$. With respect to this basis we can write $\eta_{i1} = a_i n_{i1} + b_i n_{i2}$ where the $\eta_{ij}$ are as in $Q$. With respect to these basis choices for $N$ and corresponding dual basis choices for $M$, and after deleting columns consisting of zeros, the matrix $Q$ for $\alpha + \beta$ becomes
$Q = \begin{pmatrix}
\tau_1 - \rho_{11} & \tau_1 - \rho_{12} & \ldots & \rho_n \\
\tau_i - \rho_{11} & \tau_i - \rho_{12} & \ldots & \rho_n \\
\tau_m - \rho_{m1} & \tau_m - \rho_{m2} & \ldots & \rho_n
\end{pmatrix}
\begin{pmatrix}
\rho_1 & \rho_2 & \cdots & \rho_n \\
\alpha(\rho_1) & \alpha(\rho_2) & \cdots & \alpha(\rho_n)
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \beta_1 & \alpha_2 \beta_2 & \cdots & \alpha_m \beta_m
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
[\alpha_i, \beta_i] \\
[\alpha_m, \beta_m]
\end{pmatrix}$

where the first $n$ columns span $\text{im}\alpha$ and the last $2m$ columns span $\text{im}\beta$.

We checked in (4) above that the last 2 columns labeled $\alpha_m$ and $\beta_m$ are linear combinations of the preceding $n + 2(m - 1)$ columns. It follows from (2) that $|b_i| = |\text{Cl}(U_{r_i})| = |\text{Cl} U_i|$. The columns $\beta_1, \ldots, \beta_e$ are $b_1c_1, \ldots, b_ec_e$.

**Theorem 2.6.** Let $\Delta$ be a fan on $N_\mathbb{R}$ and let $X = T_N \text{emb}(\Delta)$. Assume all the cones in $\Delta$ have dimension $\leq 2$. Assume the two-dimensional faces $\tau_1, \ldots, \tau_m$ and one-dimensional faces $\rho_1, \ldots, \rho_n$ of $\Delta$ can be ordered so that $\tau_i \cap \tau_{i+1} = \rho_i$ ($1 \leq i \leq m - 1$) and $\tau_m \cap \tau_1 = \rho_m$. Let $b_i$ be the order of $\text{Cl}(U_{r_i})$. Then $B'(T_N/X)$ is cyclic of order $\gcd\{b_1, \ldots, b_m\}$.

**Proof.** Using Proposition 2.5 and the hypotheses, we can assume that the graph $\Gamma$ is connected and consists of one cycle as shown:

We take the spanning tree $T$ for $\Gamma$ to be the graph obtained from $\Gamma$ by deleting the edge $c_1 = \tau_1 - \rho_m$. Let $C$ be the matrix whose only column is $c_1$ and form the augmented matrix $[Q|C]$:

$$
\begin{pmatrix}
\tau_1 - \rho_{11} & \tau_1 - \rho_{12} & \ldots & \rho_n & \alpha_1 \beta_1 & \alpha_2 \beta_2 & \cdots & \alpha_m \beta_m & c_1 \\
\tau_1 - \rho_{m1} & \tau_1 - \rho_{m2} & \ldots & \rho_n & \alpha_1 \beta_1 & \alpha_2 \beta_2 & \cdots & \alpha_m \beta_m & c_1
\end{pmatrix}
\begin{pmatrix}
\rho_1 & \rho_2 & \rho_3 & \cdots & \rho_{m-2} & \rho_{m-1} & \rho_m & \alpha_1 & \alpha_2 & \cdots & \alpha_{m-1} & \beta_{m-1} & \alpha_m & \beta_m & c_1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & a_1 & b_1 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & a_2 & b_2 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & \cdots & 1 & 0 & 0 & a_{m-1} & b_{m-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & a_m & b_m & 0
\end{pmatrix}$$
Here $c_1$ corresponds to the edge $\tau_{1-p_m}$ and $\bigoplus_{i=1}^{m}(\mathbb{Z}\rho_i \oplus \mathbb{Z}\rho_{i-1})$ is the edge space of $\Gamma$ where $\rho_0 = \rho_m$. We observed that $B'(T_N/X)$ is the quotient of the edge space by the column space of $Q$. Let $[B|C]$ be the matrix whose columns are the columns labeled $\rho_1, \ldots, \rho_m, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}, c_1$. We check that the columns of $[B|C]$ form a basis for the edge space by using column operations to reduce to a permutation matrix. Use column $c_1$ to eliminate $\alpha_1$ from column $\alpha_1$ and the 1 in the entry with row index $\tau_{1-p_m}$ and column index $\rho_m$. Then use the 1 in the new column $\alpha_1$ to eliminate the 1 in the entry with row index $\tau_{1-\rho_1}$ and column index $\rho_1$. Use the new column $\rho_1$ to eliminate the $a_2$ in column $\alpha_2$. Continue inductively, eliminating $a_3, \ldots, a_{m-2}$ from columns indexed $\alpha_3, \ldots, \alpha_{m-2}$ and the ones in the entries with row index $\tau_{i-\rho_i}$ and column index $\rho_i$, for $2 \leq i \leq m-2$. At the last step use the remaining 1 in column $\rho_{m-2}$ to eliminate $a_{m-1}$ in column $\alpha_{m-1}$. Use the new column $\alpha_{m-1}$ to eliminate the 1 in the entry in row $\tau_{m-1-\rho_{m-1}}$ column $\rho_{m-1}$. The result is a matrix whose $\mathbb{Z}$-rank is $2m$, which shows that $C\{c_1\}$ generates the quotient of the edge space $\bigoplus_{i=1}^{m}(\mathbb{Z}\rho_i \oplus \mathbb{Z}\rho_{i-1}) (\rho_0 = \rho_m)$ by the column space of $Q$. Recall that the last two columns of $Q$ are a linear combination of the preceding ones. Thus to calculate this quotient we simply project each of the columns $\beta_1, \ldots, \beta_m$ on $\mathbb{Z}c_1$. These projections follow the recursive pattern:

<table>
<thead>
<tr>
<th>Column vector</th>
<th>Projection on $\mathbb{Z}c_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1 = b_1c_1$</td>
<td>$b_1c_1$</td>
</tr>
<tr>
<td>$\beta_2 = b_2(\rho_1 - (\alpha_1 - a_1c_1))$</td>
<td>$b_2a_1c_1$</td>
</tr>
<tr>
<td>$\beta_3 = b_3(\rho_2 - (\alpha_2 - a_2(\rho_1 - (\alpha_1 - a_1c_1))))$</td>
<td>$b_3a_2a_1c_1$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\beta_m = b_m a_{m-1} \cdots a_2a_1c_1$</td>
<td></td>
</tr>
</tbody>
</table>

The subgroup generated by the projections of the columns $\beta_i$ on $\mathbb{Z}c_1$ is the subgroup generated by $dc_1$ where $d = \gcd\{b_1, b_2a_1, \ldots, b_m a_{m-1} \cdots a_2a_1\}$. Since $\gcd(a_i, b_i) = 1$ for $1 \leq i \leq m$, we see $d = \gcd\{b_1, \ldots, b_m\}$. But $|b_i|$ is the order of $\text{Cl}(U_{t_i})$, so the theorem follows. \(\square\)

To extend Theorem 2.6 it is necessary to introduce some additional notation. Suppose the graph $\Gamma$ we have associated to the fan $\Delta$ is connected and let $T$ be a spanning tree for $\Gamma$. Since each vertex labeled by a two-dimensional face $\tau_i$ is connected by exactly two edges to vertices $\rho_{i1}$ and $\rho_{i2}$ corresponding to the one-dimensional faces of $\tau_i$ in $\Delta$, each $\tau_i$ is a vertex in $T$. If $\Delta(2) = \{\tau_1, \ldots, \tau_m\}$, designate $\tau_m$ as the root node for $T$. Let $C$ be the matrix whose columns are $c_1, \ldots, c_e$ and let $[Q|C]$ be the augmented matrix similar to that used in the proof of Theorem 2.6. Then $\tau_1, \ldots, \tau_e$ are leaf nodes of $T$ and $c_i = \tau_{i-\rho_{i2}}$ for $1 \leq i \leq e$. For $e < i < m$ relabel $\rho_{i1}$ and $\rho_{i2}$ if necessary so the edge $\tau_{i-\rho_{i1}}$ is closer to the root node $\tau_m$ than the edge $\tau_{i-\rho_{i2}}$. In our previous analysis this amounts to permuting the basis of the edge space $\bigoplus_{i=1}^{m}(\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$. This does not affect the columns labeled $\alpha_1, \beta_1, \ldots, \alpha_e, \beta_e$ in $Q$. Let $[B|C]$ be the matrix obtained from $[Q|C]$ by deleting from $Q$ the columns labeled $\beta_1, \ldots, \beta_{m-1}, \beta_m, \alpha_m$. We note that the column space of $B$ depends on the choice of root node $\tau_m$.

**Lemma 2.7.** The columns of $[B|C]$ form a basis for $\bigoplus_{i=1}^{m}(\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$, the edge space, for any choice of root node $\tau_m$. 

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Proof. If \( \Gamma \) is a tree, then \( T = \Gamma \) and \( e = 0 \). In this case we need to show that the columns of \( B \) span the edge space. Each leaf node of \( \Gamma \) must be \( \rho_{i2} \) for some \( i \) since each \( \tau_i \) is incident to two edges in \( \Gamma = T \) and \( \rho_{i1} \) is closer to the root node \( \tau_m \) than \( \rho_{i2} \). We call the pair \((\tau_i, \rho_{i2})\) a leaf node pair. If \( i = m \), then it is possible for \( \rho_{i1} \) to be a leaf node. This is the only exception and will be treated in the basis step for our induction below. Assume \((\tau_i, \rho_{i2})\) is a leaf node pair and \( i \neq m \). In \( \Delta, \rho_{i2} \) is a face of exactly one two-dimensional cone \( \tau_i \). Thus the column indexed by \( \rho_{i2} \) in \( B \) has exactly one nonzero entry which is a 1 in the row indexed \( \tau_i-\rho_{i1} \) as indicated below.

\[
\begin{pmatrix}
\rho_{i1} & \rho_{i2} & 0 \\
0 & 1 & 1 \\
1 & 0 & a_i
\end{pmatrix}
\]

Use the column indexed \( \rho_{i2} \) to eliminate \( a_i \) in the column \( \alpha_i \) by an elementary column operation, then use the new column \( \alpha_i \) to eliminate the 1 in the \( \tau_i-\rho_{i1} \) entry of column \( \rho_{i1} \). After these two steps are performed, we say we have pruned the leaf node pair from the tree \( \Gamma = T \). The two columns indexed \( \rho_{i2} \) and \( \alpha_i \) are now elementary basis vectors in our basis for the edge space and appear in no further column operations. After the columns indexed \( \rho_{i2} \) and \( \alpha_i \) are deleted, the remaining matrix is the matrix we would associate to the fan \( \Delta' \) obtained from \( \Delta \) by deleting the cones \( \rho_{i2} \) and \( \tau_i \). Apply this leaf pruning algorithm iteratively to reduce to the case where \( \Gamma \) is the tree \( \rho_{m1}-\tau_m-\rho_{m2} \). The matrix \( B \) for this tree is \( \begin{bmatrix} 1 & 0 \end{bmatrix} \). Thus our algorithm reduces the original matrix \( B \), using elementary column operations, to a permutation matrix.

If \( \Gamma \) is not a tree, let \( \tau_{1-\rho_{i1}}, \ldots, \tau_{e-\rho_{e2}} \) be the edges of \( \Gamma \) which are not in \( T \). Fix \( i, \ 1 \leq i \leq e \). Since \( \tau_i-\rho_{i1} \) is in \( T \), it follows that \( \tau_i \) is a leaf node of \( T \). We can use the column indexed \( c_i \) and elementary column operations to eliminate the entry \( a_i \) from the column indexed \( \alpha_i \) and the entry 1 in the \( \tau_i-\rho_{i1} \) entry of column \( \rho_{i1} \). After these two steps are performed, we say we have pruned the leaf node pair \((\tau_i, \rho_{i2})\) from the tree \( \Gamma = T \). The remaining matrix is the matrix we would associate to the fan \( \Delta' \) obtained from \( \Delta \) by deleting the cones \( \rho_{i2} \) and \( \tau_i \). Apply this leaf pruning algorithm iteratively to reduce to the case where \( \Gamma \) is the tree \( \rho_{m1}-\tau_m-\rho_{m2} \). The matrix \( B \) for this tree is \( \begin{bmatrix} 1 & 0 \end{bmatrix} \). Thus our algorithm reduces the original matrix \( B \), using elementary column operations, to a permutation matrix.

Corollary 2.8. Let \( \Delta \) be a fan on \( \mathbb{R}^r \) whose cones all have dimension \( \leq 2 \). If the graph \( \Gamma \) associated to \( \Delta \) is a disjoint union of trees and \( X = T_N \text{emb}(\Delta) \), then \( B'(T_N/X) = 0 \).

Proof. By Proposition 2.5 we can assume \( \Gamma \) is connected, so the hypotheses imply \( \Gamma \) is a tree. By Lemma 2.7 the columns of \( B \) span the edge space of \( \Gamma \). But the column space of \( B \) is contained in \( \text{im}(\alpha+\beta) \) in (3) \( B'(T_N/X) = 0 \).

Corollary 2.9. Let \( \Delta \) be a fan on \( \mathbb{R}^2 \) and \( X = T_N \text{emb}(\Delta) \) the associated toric surface.

(a) If \( \Delta = \{0\} \), then \( B(X) \cong \mathbb{Q}/\mathbb{Z} \).

(b) If \( \Delta \neq \{0\} \) and \(|\Delta| \neq \mathbb{R}^2\) (i.e., \( X \) is not complete), then \( B(X) = 0 \).

(c) If \(|\Delta| = \mathbb{R}^2\) (i.e., \( X \) is complete), \( \Delta(1) = \{\rho_1, \ldots, \rho_n\} \) and \( N' = \langle \rho_1 \cap N, \ldots, \rho_n \cap N \rangle \), then \( B(X) \cong N/N' \).
ON THE BRAUER GROUP OF TORIC VARIETIES

Proof. Every toric surface is a normal projective surface [12] so by [9, Corollary 9] $B(X) = B'(X)$. If $\Delta = \{0\}$, then $X = T_N$ is nonsingular and since $r = 2$, $B(X) = \mathbb{Q}/\mathbb{Z}$ [10]. If $\Delta \neq \{0\}$ and $|\Delta| \neq \mathbb{R}^2$, then the graph $\Gamma$ associated to $\Delta$ is a disjoint union of trees. By Proposition 2.5 we assume $\Gamma$ is a tree and contains at least one two-dimensional cone $\tau$. By Corollary 2.8 and Corollary 2.4, $B(X) = 0$. If $|\Delta| = \mathbb{R}^2$, then $\Gamma$ is a cycle. Corollary 2.4 implies $B(X) = B(T_N/X)$. If $\Delta(2) = \{\tau_1, \ldots, \tau_m\}$, then Theorem 2.6 implies $B(X)$ is cyclic of order $\gcd(|\text{Cl}(U_i)|)$ for $1 \leq i \leq m$. Let $\rho_1, \ldots, \rho_m$ be the one-dimensional cones in $\Delta$ with $\tau_i = \rho_i + \rho_{i+1}$ (1 $\leq i \leq m$) where $\rho_{m+1} = \rho_1$ and let $\rho_i = \mathbb{R}_{>0} \eta_i$ for primitive vectors $\eta_1, \ldots, \eta_m$ in $N$. Choose a basis $n_1, n_2$ for $N$ with $n_1 = \eta_1$ and write $\eta_1 = a_1 n_1 + b_1 n_2$. Then $N/N' = N/\langle \eta_1, \ldots, \eta_m \rangle$ is cyclic of order $\gcd(b_1, \ldots, b_m)$. On the other hand for $(1 \leq i \leq m-1)$, $\text{Cl}(U_{i+1}) = \text{Cl}(U_i)$ is cyclic of order $|\det(\begin{smallmatrix} a_i \\ b_i \end{smallmatrix})|$ and $|\text{Cl}(U_m)| = |\det(\begin{smallmatrix} a_m \\ b_m \end{smallmatrix})|$ by (2). Since $\gcd(a_i, b_i) = 1$ for each $i$, an easy calculation shows $\gcd(|\text{Cl}(U_i)|) = \gcd(b_1, \ldots, b_m)$. □

To determine the $p$-subgroups of $B'(T_N/X)$ for each prime number $p$, we introduce some additional notation and terminology. Let $\Gamma$ be a finite edge-weighted graph such that to each edge $E$ is associated the positive integer $\text{weight}(E)$. Let $v_p$ be the $p$-adic valuation on $\mathbb{Z}$ and set the $p$-weight of $E = \text{weight}_p(E) = v_p(\text{weight}(E))$. If $\Gamma_1$ is a subgraph of $\Gamma$, let $\text{weight}_p(\Gamma_1) = \sum \text{weight}_p(E)$ where the summation is over all edges $E$ in $\Gamma_1$. A $p$-maximal spanning tree for $\Gamma$ is a spanning tree $T$ for $\Gamma$ such that $\text{weight}_p(T)$ is maximal among the $p$-weights of all spanning trees. It is clear that every connected graph has a $p$-maximal spanning tree. Let $T$ be a $p$-maximal spanning tree for $\Gamma$ and let $c$ denote an edge of $\Gamma$ not present in $T$. Since $T$ is a spanning tree, the subgraph $\Gamma_1$ of $\Gamma$ obtained by adding the edge $c$ to $T$ contains a cycle $Z$ which is unique since there is a unique path between any two vertices of the tree $T$. Suppose there is some edge $E$ in $Z$ with $\text{weight}_p(E) < \text{weight}_p(c)$. Then we could obtain a spanning tree of larger $p$-weight by deleting the edge $E$ from $\Gamma_1$. This means that if $T$ is a $p$-maximal spanning tree for $\Gamma$, $c$ is an edge of $\Gamma$ not in $T$ and $Z$ is the unique cycle in the graph $T \cup \{c\}$, then $c$ is an edge of minimal $p$-weight in $Z$.

Let $\Gamma$ be the (connected) graph we have associated to the fan $\Delta$ whose cones all have dimension $\leq 2$. Assign weights to the edges $\tau_{i-\rho_{ij}}$ of $\Gamma$ by setting $\text{weight}(\tau_{i-\rho_{ij}}) = b_i = |\text{Cl}(U_{i})|$ (recall $\text{Cl}(U_{i}) = \mathbb{Z}/b_i$ is cyclic from (2)). Let $T$ be a $p$-maximal spanning tree for $\Gamma$. We have labeled the edges of $\Gamma$ not in $T$ as $\tau_1 - \rho_{12}, \tau_2 - \rho_{22}, \ldots, \tau_e - \rho_{ec}$. We call the set of 2-dimensional cones $\{\tau_1, \ldots, \tau_e\}$ in $\Delta$ a $p$-minimal set of cones in $\Delta$. If $\Gamma$ is not connected, then we can decompose $\Delta$ as a union of fans $\Delta_i$ with $\Delta_i \cap \Delta_j = \{0\}$ when $i \neq j$ and the graphs $\Gamma_i$ associated to $\Delta_i$ are connected. We define a $p$-minimal set of cones in $\Delta$ to be the union of $p$-minimal sets of cones in each $\Delta_i$.

Theorem 2.10. Let $\Delta$ be a fan on $N_R$ and assume every cone in $\Delta$ has dimension $\leq 2$. Let $\{\tau_1, \ldots, \tau_e\}$ be a $p$-minimal set of cones in $\Delta$ and let $|\text{Cl}(U_{i})| = b_i$. If $X = T_N \text{emb}(\Delta)$, then $B'(T_N/X)_p \cong \bigoplus_{i=1}^e \mathbb{Z}/b_i$. This isomorphism is induced by the epimorphism $\psi$ of (3).

Proof. By Proposition 2.5 and the discussion preceding the theorem, we can assume the graph $\Gamma$ associated to the fan $\Delta$ is connected. Let $T$ be a $p$-maximal spanning tree for $\Gamma$. Continuing the analysis that was begun in
the proof of Theorem 2.6 we consider the matrix \([Q|C]\) defined there. If \(\psi: \bigoplus_{i=1}^{m} (\mathbb{Z}_p \rho_1 \oplus \mathbb{Z}_p \rho_2) \to B'(T_N/X)\) is the epimorphism given in (3) and \(\mathbb{Z}_p\) is the \(p\)-adic integers, we have an epimorphism

\[
\psi_p: \bigoplus_{i=1}^{m} (\mathbb{Z}_p \rho_1 \oplus \mathbb{Z}_p \rho_2) \to B'(T_N/X)_p.
\]

It follows from Lemma 2.7 that \(\{c_1, \ldots, c_e\}\) generates \(\text{coker}(\alpha+\beta)\) so \(\{\psi_p(c_1), \ldots, \psi_p(c_e)\}\) generates \(B'(T_N/X)_p\). We check

\[
0 = \langle \psi_p(c_j) \rangle \cap \langle \psi_p(c_1), \ldots, \psi_p(c_{j-1}), \psi_p(c_{j+1}), \ldots, \psi_p(c_e) \rangle
\]

and \(\psi_p(c_j)\) has order \(p^{v_p(b_j)}\) for \(1 \leq j \leq e\) by identifying these elements in \(B'(T_N/X)\) with their corresponding preimages \(c_j + \text{image}(\alpha+\beta) \in \text{coker}(\alpha+\beta)\) in (3) and then checking the corresponding statements in \(\text{coker}(\alpha+\beta)\).

Fix \(j\) and let \(\pi\) be a permutation of \(\{1, \ldots, m\}\) with \(\pi\) chosen so \(\pi(1) = j\) where the edge \(c_j = \tau_j - \rho_s\) and the cycle in \(T \cup \{c_j\}\) is

\[
\tau_j = \tau_{\pi(1)} - \rho_1 - \tau_{\pi(2)} - \rho_2 - \cdots - \tau_{\pi(s)} - \rho_s - \tau_{\pi(1)}.
\]

Choose the vertex \(\tau_{\pi(s)}\) as the root node for \(T\). By Lemma 2.7 we know the columns of \([B|C]\) form a basis for \(\bigoplus_{i=1}^{m} (\mathbb{Z}_p \rho_1 \oplus \mathbb{Z}_p \rho_2)\). The column space of \(B\) is a submodule of \(\text{image}(\alpha+\beta)\). Project the submodule \(\mathbb{Z}_p \beta_1 + \cdots + \mathbb{Z}_p \beta_m\) of \(\text{image}(\alpha+\beta)\) onto a \(\mathbb{Z}_p\)-submodule of the column space of \(C\) over \(\mathbb{Z}_p\). Then \(\text{coker}(\alpha+\beta)_p\) is the quotient module.

If \(s+1 \leq i \leq m\) we check the projection of \(\beta_{\pi(i)}\) on \(\mathbb{Z}_p c_j\) is 0. The selection of \(\tau_{\pi(s)}\) as the root node for \(T\) gives a partial order on the vertices of \(T\). Let \(T_i\) be the subtree of \(T\) with root node \(\rho_{\pi(i)}\). This means the vertices \(v\) in \(T_i\) are those for which the unique path from \(v\) to \(\tau_{\pi(s)}\) contains \(\rho_{\pi(i)}\). In the expression for \(\beta_{\pi(i)}\) as a linear combination of the columns of \([B|C]\), the columns of \(C\) that appear are those \(c_k\) which when considered as edges of \(\Gamma\) are incident to some vertex in \(T_i\) (see the proof of Theorem 2.6). But neither \(\tau_j\) nor \(\rho_s\) are in \(T_i\) since \(i \geq s+1\) so the projection of \(\beta_{\pi(i)}\) on \(c_j\) has coefficient = 0.

If \(1 \leq i \leq s\) let the projection of \(\beta_{\pi(i)}\) on \(\bigoplus_{k=1}^{e} \mathbb{Z}_p c_k\) be \(\sum_{k=1}^{e} d_{ki} c_k\). We say in the proof of Theorem 2.6 that \(b_{\pi(i)} = \langle Cl(U_{\pi(i)})\rangle\). The projections of \(\beta_{\pi(1)}, \ldots, \beta_{\pi(m)}\) on \(\bigoplus_{k=1}^{e} \mathbb{Z}_p c_k\) are the columns of the \(e \times m\) matrix \((d_{ki})\). We have observed above that \(d_{j(s+1)} = \cdots = d_{jm} = 0\). The definitions of \(\beta_j\) and \(c_j\) imply \(\beta_j = b_j c_j\) so \(d_{k1} = 0\) if \(k \neq j\). Also \(b_{\pi(i)} | d_{ji}\) for \(2 \leq i \leq s\) and we chose \(j\) with \(v_p(b_j) = \min \{v_p(b_{\pi(1)}), \ldots, v_p(b_{\pi(s)})\}\). Thus after elementary column operations over \(\mathbb{Z}_p\) the column space of \((d_{ki})\) is equal to the column space of

\[
\begin{bmatrix}
0 & d_{12} & \cdots & d_{1m} \\
\vdots & \vdots & & \vdots \\
0 & d_{(j-1)2} & \cdots & 0 \\
b_j & 0 & \cdots & 0 \\
0 & d_{(j+1)2} & & \\
\vdots & & \ddots & \vdots \\
0 & d_{e2} & \cdots & d_{em}
\end{bmatrix}
\]
Therefore \( c_1, \ldots, c_e \) represents a basis for \( \ker(\alpha + \beta)_p \), \( B'(T_N/X)_p = \langle \psi_p(c_1) \rangle \oplus \cdots \oplus \langle \psi_p(c_e) \rangle \), and \( \langle \psi_p(c_j) \rangle \) is cyclic of order \( p^{\nu_p(b_j)} \). \( \square \)

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**References**


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