PIECEWISE SL₂ Z GEOMETRY

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ABSTRACT. Piecewise SL₂ Z geometry studies properties of the plane invariant under pl-homeomorphisms which, locally, have the form \( x \mapsto Ax + b \), with \( A \in \text{SL}_2 \mathbb{Z}, \ b \in \mathbb{Q}^2 \), and whose singular lines are rational. In this paper, invariants of polygons are obtained, relations with Pick’s theorem are described, and a conjecture is posed.

INTRODUCTION

The classic Pick’s theorem (see [GKW]) asserts that if \( P \) is a polygon whose vertices have integral coordinates (an integral polygon) then the number of points of \( \mathbb{Z}^2 \) in the interior of \( P \) is \( \text{area}(P) - \frac{1}{2} \#(\partial P \cap \mathbb{Z}^2) + 1 \) (here \( \# \) denotes cardinality). Looking behind the proof, we are led to consider a certain graph \( G_1 P \) and associated simplicial complex \( K_1 P \) associated to \( P \). The complex \( K_1 P \) can be thought of as the space of triangulations of \( P \); it turns out (1.13) that if \( \text{area}(P) > 1 \), then \( K_1 P \) is a pl-disk.

One motivation for this study is to understand the geometry of integral polygons and the piecewise SL₂ Z maps between them, that is, piecewise linear maps which, in each “piece”, have the form

\[
(*) \quad f(x, y) = A(x, y) + v, \quad A \in \text{SL}_2 \mathbb{Z}, \ v \in \mathbb{Q}^2.
\]

The classifying space of the pseudogroup \( \Gamma \) of such homeomorphisms is rather simple—roughly [Gr] a CW complex with a finite number of cells in each dimension—and it would be interesting to see this reflected in the geometry. We calculated in [Gr] that, in a homological sense, the only quantities of closed integral polygons invariant under \( \Gamma \) are the area and a sort of “length” (1.2). Here we prove this in a stronger, geometric sense (1.3).

The group \( G \) of germs at \((0, 0)\) of the pseudogroup \( \Gamma \) contains a group \( F' \) which is an “algebraic delooping” of the braid group [GS]. Thinking of \( \Gamma \) as a globalization of \( G \), it makes sense to look for connections with the braid groups. As was noted by Devaney in [D], if we restrict the \( v \) in (*) to lie in \( \mathbb{Z}^2 \), then piecewise SL₂ Z maps permute the points \( \frac{1}{N} \mathbb{Z}^2 \) for each \( N \). Thus, if \( \text{Aut}_1(\mathcal{P}, \partial) \) denotes the group of such automorphisms of \( \mathcal{P} \), fixing the boundary, there are evident homomorphisms from \( \text{Aut}_1(\mathcal{P}, \partial) \) to certain braid groups. (See Figure 1.)

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However, perhaps one should look for deeper structural relations between $\text{Aut}_1(P, \partial)$ and braid or mapping class groups. By taking a limit of complexes $K_1NP$, one arrives at a space $K(P)$ on which $\text{Aut}_1(P, \partial)$ acts (1.16). The space $K(P)$ is analogous to the “complexes of curves” which arise in connection with the mapping class groups.

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1. Definitions and main results

We begin by defining certain pseudogroups of $\text{pl}$-homeomorphisms between open subsets of $\mathbb{R}^2$. We will denote by $\frac{1}{N}\mathbb{Z}$ the subgroup of $\mathbb{Q}$ generated by $\frac{1}{N}$, and by $A_N$ (resp. $A_0$) the affine extension of $\frac{1}{N}\mathbb{Z}^2$ (resp. $\mathbb{Q}^2$) generated by $\text{SL}_2 \mathbb{Z}$. A rational line (resp. integral line) is a line passing through two rational (resp. integral) points in the plane.

1.1. Definition. A $p\mathbb{Z}_N$ homeomorphism is an orientation-preserving homeomorphism $g: U \to V$ between open subsets of the plane, such that there exists a finite set of rational lines $\{l_i\}$ such that $g$ agrees with some element $g_c$ of $A_N$ on any component $C$ of $U - \bigcup l_i$. A $p\mathbb{Z}$ homeomorphism is a $p\mathbb{Z}_1$ homeomorphism, in which we require the lines $l_i$ to be integral.

Before discussing invariants, we establish some notation for polygonal curves.

In this paper, a polygonal curve means a curve made of a finite number of rational line segments between rational points of $\mathbb{R}^2$; the endpoints of the segments of an integral polygonal curve are required to lie in $\mathbb{Z}^2$. If $v_i \in \mathbb{Q}^2$, we denote by $\overline{v_0 \cdots v_n}$ the polygonal curve made of segments $\overline{v_k v_{k+1}}$. An (integral) polygon is a simple closed (integral) polygonal curve. We write $\text{int} P$ for the open set enclosed by a polygon $P$, and $\overline{\text{int}} P$ for the closure of $\text{int} P$. Finally, $\mathcal{C}$ and $P$ denote the sets of polygonal curves and polygons.

If $P$ is a polygon, the area $a(P)$ of $\text{int} P$ is invariant under $p\mathbb{Z}_0$ maps. There is also an invariant “length.”

1.2. Proposition. There is a function $L: \mathcal{C} \to \mathbb{Q}$ which takes positive values, such that

(a) (invariance) If $P \in \mathcal{C}$, $P \subseteq U$, and $g: U \to V$ is a $p\mathbb{Z}_0$-homeomorphism, then $L(P) = L(g(P))$.

(b) (subdivision) $L(\overline{v_0 \cdots v_n}) = L(\overline{v_0 \cdots v_k}) + L(\overline{v_k \cdots v_n})$. 

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(c) (homothety) \( L(NP) = NL(P) \), \( P \in \mathcal{P} \), where \( NP \) is the image of \( P \) under the map \( (x, y) \to (Nx, Ny) \).

(d) (no metric) If \( a, b \in \mathbb{Q}^2 \), then \( \inf(a_1 \cdots a_n b) = 0 \).

Proof. If \( a = (p/N, q/N), b = (r/N, s/N), \) and \( p, q, r, s \in \mathbb{Z} \), we define \( L(ab) = \frac{1}{N} \left( \#(\frac{1}{N}\mathbb{Z}^2 \cap \overline{ab}) - 1 \right) \), where \( \#X \) denotes the cardinality of a set \( X \). Observe that \( L(ab) \) is independent of the \( N \) used. Extend \( L \) to all of \( \mathcal{P} \) so as to satisfy (b). Property (a) (invariance) is a consequence of the fact that \( \text{SL}_2 \mathbb{Z} \) preserves the lattices \( \frac{1}{N}\mathbb{Z}^2 \), and property (c) is a quick calculation. To prove (d), by (a) and (c) it suffices to take \( a = (0, 0) \) and \( b = (2, 0) \). Then note that \( L(a(1, 1/k)b) = 2/k \).

The following theorem says that \( a \) and \( L \) are the only invariants of the action of \( \text{pZ}_0 \) homeomorphisms on \( \mathcal{P} \).

1.3. Theorem. Let \( P, Q \in \mathcal{P} \), with \( a(P) = a(Q) \) and \( L(P) = L(Q) \). Let \( p \in P \) and \( q \in Q \) be rational points. Then there exists a \( \text{pZ}_0 \) homeomorphism \( g : \text{int} P \to \text{int} Q \) such that \( g(p) = q \).

If \( P, Q \) are integral polygons, and \( p, q \in \mathbb{Z}^2 \), we may choose \( g \) to be a \( \text{pZ} \) homeomorphism.

The proof of the theorem is somewhat involved, so we postpone it to §2. The main idea, that of a triangulation, will now be applied to reproduce the proof [GKW] of a theorem of Pick.

1.4. Proposition (Pick). Let \( P \) be an integral polygon. The number of points of \( \mathbb{Z}^2 \) in \( \text{int} P \) is

\[
a(P) - \frac{1}{2} \#(P \cap \mathbb{Z}^2) + 1.
\]

(Note that \( \#(P \cap \mathbb{Z}^2) = L(P) \)). The proof requires the following notions.

1.5. Definition. An \( \mathcal{N} \)-segment \( \overline{ab} \) is a segment so that \( \overline{ab} \cap \frac{1}{N}\mathbb{Z}^2 = \{a, b\} \). An \( \mathcal{N} \)-triangle is a triangle \( \overline{abc} \) whose sides are \( \mathcal{N} \)-segments, and whose interior contains no points of \( \frac{1}{N}\mathbb{Z}^2 \). If \( P = \overline{v_0 \cdots v_n} \) is a polygon and \( v_i \in \frac{1}{N}\mathbb{Z}^2 \), then an \( \mathcal{N} \)-triangulation of \( P \) is a triangulation by \( \mathcal{N} \)-triangles.

1.6. Lemma. The length \( L(ab) \) of an \( \mathcal{N} \)-segment is \( \frac{1}{N} \). The area of an \( \mathcal{N} \)-triangle is \( \frac{1}{N^2} \).

Proof. The first statement follows from the definition. For the second, it suffices to take \( N = 1 \). Further, after transformation by an element of \( \mathcal{A}_1 \), we may choose the vertices of the triangle at \( (0, 0), (1, 0), \) and \( (a, b) \), where \( b > 0, a, b \in \mathbb{Z} \).

Consider the parallelogram \( P \) with corners \( (0, 0), (1, 0), (a, b), \) and \( (a - 1, b) \). It suffices to show that \( a(P) = 1 \), or that \( (1, 0), (a - 1, b) \) is a basis for \( \mathbb{Z}^2 \). But since \( \text{int} P \cap \mathbb{Z}^2 \) is empty, \( \text{int} P' \cap \mathbb{Z}^2 \) is also empty for any \( P' \) in the tesselation of \( \mathbb{R}^2 \) by copies of \( P \). Hence, \( (1, 0), (a - 1, b) \) is a basis for \( \mathbb{Z}^2 \).

Proof of 1.4. Let \( P \) be an integral polygon, and let \( V, E, \) and \( T \) be the numbers of vertices, edges, and triangles in a \( \mathcal{N} \)-triangulation. (It will be obvious presently that \( \mathcal{N} \)-triangulations exist.) Then \( T = 2a(P) \) and the number of edges
on $P$ is $L(P) = \#(P \cap \mathbb{Z}^2)$. Each triangle has three edges, and the edges not on the boundary share two triangles, so

$$E = \frac{3T}{2} + \frac{1}{2}L(P) = 3a(P) + \frac{1}{2}L(P).$$

By Euler's formula, $V = 1 + E - T = a(P) + \frac{1}{2}L(P) + 1$. But the number of vertices on $P$ is $L(P)$ so the number of vertices in $\text{int} P$ is $a(P) - \frac{1}{2}\#(P \cap \mathbb{Z}^2) + 1$.

In order to investigate triangulations of polygons, we introduce graphs $G_N P$ associated to integral polygons $P$. The vertices of $G_N P$ are $N$-segments whose interior is contained in the interior of $P$. If two $N$-segments intersect in their interiors, then there is an edge between two vertices and we say the $N$-segments cross. See Figure 2.

Now, if $G$ is a graph, an independent subset of $G$ (see [G]) is a set of vertices $\{v_i\}$ so that there is no edge between any $v_i$ and $v_j$. Let $K(G)$ denote the simplicial complex whose $k$-simplices are independent subsets of $G$ of cardinality $k + 1$; write $K_N P$ for $K(G_N(P))$. We consider $G_N P$ because of the following:

1.8. Remark. A maximal independent set of $G_N P$ is precisely the set of $N$-segments, not in $P$, in an $N$-triangulation of $P$.

1.9. Proposition. Let $P$ be an integral polygon. The maximal independent sets of $G_N P$ have $3N^2a(P) - \frac{N}{2}L(P)$ members.

Proof. For $N = 1$, this is just equation (1.7), with the observation that $P$ contains $L(P)$ edges. For general $N$, apply the homothety $(x, y) \rightarrow (Nx, Ny)$ to change an $N$-triangulation of $P$ to a 1-triangulation of $NP$.

A graph with the property that all of its maximal independent sets have the same cardinality is called well-covered (see [G]). The $G_N(P)$ seem to be new examples of well-covered graphs.

In some sense, the structure of $G_N P$ stabilizes as $N$ gets large.

1.10. Theorem. Let $P$ be an integral polygon. There is a number $N_p$ such that if $N > N_p$, then $G_N P$ is composed of a connected component, together with a set of isolated vertices whose number depends only on $P$.

Indeed, the isolated vertices are associated to the corners of $P$ (as we shall see in §3 in the proof of 1.10).

Not every well-covered graph is $G_1 P$ for some polygon $P$. The following is proved in §4.
1.11. **Theorem.** If $P$ is an integral polygon and $a(P) > 1$, then $K_1 P$ is a $pl$-disk of dimension $3a(P) - \frac{1}{2}L(P) - 1$.

Figure 3 shows that not every well-covered $G$ has $K(G)$ a disk.

Let $P$ be an integral polygon, and let $\text{Aut}_P$ be the group of $p\mathbb{Z}_N$ homeomorphisms of $\text{int} P$. By 1.3, $\text{Aut}_i P$ surjects to $\mathbb{Z}/L(P)$, with kernel $\text{Aut}_1(P, \partial)$ the elements fixing $P$. Now $\text{Aut}_1(P, \partial)$ is clearly related to braid groups: as noted by Delaney in [D], an element of $\text{Aut}_1 P$ permutes the elements of $\text{int} P \cap \frac{1}{N}\mathbb{Z}^2$, so we have homomorphisms from $\text{Aut}_1(P, \partial)$ to the braid group on $\#(\text{int} P \cap \frac{1}{N}\mathbb{Z})$ strings.

We now show that $\text{Aut}_0 P$ acts on $K(P)$ for any integral polygon $P$ and that one can define the limit $K(P) = \lim K_N P$ of the $K_N P$. Thus, $K(P)$ is a sort of "complex of curves" [I] for the group $\text{Aut}_0 P$.

1.12. **Proposition.** Let $P$ be an integral polygon. Then for all $n, N \in \mathbb{Z}$ there is a $pl$-embedding $i: K_N P \to K_{nN} P$. Further, for all $n, m, N \in \mathbb{Z}$ the following diagram commutes:

$$
\begin{array}{ccc}
K_{nN} P & \to & K_{nN} P \\
\downarrow & & \downarrow \\
K_{nN} P & \to & K_{nN} P
\end{array}
$$

**Proof.** We first define $i$ on vertices. If $t = \overline{ab}$ is an $N$-segment, then $S_0(t) = \frac{aa + (b-a)/N}{n}, \ldots, S_{n-1}(t) = \frac{b - (b-a)/N}{n}$ are $Nn$-segments. Define $i(t)$ to be the barycenter of the $n - 1$ simplex $s(t) = (S_0(t), \ldots, S_{n-1}(t))$. Now if $t = (t_0, \ldots, t_k)$ is a $k$-simplex in $K_N P$, then $i(t)$ is defined to be the convex closure of the $i(t_j)$ in the simplex $s(t_0) \cdots s(t_k)$. Naturality follows from the definition.

1.13. **Definition.** $K(P)$ is the direct limit of the $K_N P$, the limit taken over the natural numbers with maps $N \to nN$.

1.14. **Proposition.** The length function extends to a function $L: K(P) \to \mathbb{R}$.

**Proof.** First we define $L$ restricted to $K_N P$. On each vertex $t$ of $K_N P$, we have $L(t) = \frac{1}{N}$. Suppose that $L$ is defined on $(k-1)$-simplices of $K_N P$. If $t = (t_0, \ldots, t_k)$ is a $k$-simplex, then $L$ is defined on $\partial t$. Define $L$ to be $(k+1)/N$ on the barycenter of $t$, and extend to the rest of $t$ by “coning off”.

It is evident that $L$ commutes with the $i_{N, nN}$ and is therefore defined on $K(P)$. 

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1.15. **Proposition.** The group $\text{Aut}_0 P$ of $p\mathbb{Z}_0$ homeomorphisms of $\text{int} \, P$ acts continuously on $K(P)$, and $L$ is invariant under the action.

**Proof.** Let $g \in \text{Aut}_0 P$, and let $s = (s_0, \ldots, s_k)$ be a $k$-simplex in $K_N P$. If $g$ is linear on each $s_i$, define $gs$ to be the simplex $(gs_0, \ldots, gs_k)$. If $g$ is not linear on the $s_i$, there is some subdivision of the $s_i$ on which $g$ is linear, and can thus be defined.

## 2. Triangulations and $p\mathbb{Z}$ homeomorphisms

We begin with a simple observation.

2.1. **Lemma.** Let $T_1$ and $T_2$ be $1$-triangles, with vertices $a_i \in T_i$. There is a unique element $g \in A_1$ such that $gT_1 = T_2$ and $ga_1 = a_2$.

**Proof.** Composing with translations, we can assume that $a_1 = a_2 = (0, 0)$. Recall from the proof of Lemma 1.6 that the remaining sides of each of the $T_i$ form a basis for $\mathbb{Z}^2$. The lemma follows.

Lemma 2.1 gives an interesting way to construct $p\mathbb{Z}$ homeomorphisms. Suppose that $P$ and $Q$ are integral polygons with $1$-triangulations which are combinatorially the same. Then (see Figure 4) applying Lemma 2.1 to each pair of corresponding $1$-triangles constructs a well-defined $p\mathbb{Z}$ homeomorphism from $\text{int} \, P$ to $\text{int} \, Q$ (Figure 4) which we call a *simple* homeomorphism.

2.2. **Definition.** Let $P$ and $Q$ be integral polygons. Then $f: \text{int} \, P \rightarrow \text{int} \, Q$ is a $1$-triangulated homeomorphism if

(i) $f$ is simple,

(ii) $f$ is a composite of $1$-triangulated homeomorphisms or

(iii) $\text{int} \, P = \text{int} \, P_1 \cup \text{int} \, P_2$, $\text{int} \, Q = \text{int} \, Q_1 \cup \text{int} \, Q_2$, where $P_i$ and $Q_i$ are integral polygons, $\text{int} \, P_i \cap \text{int} \, P_2 = \overline{v_0 \cdots v_n}$, $\text{int} \, Q_1 \cap \text{int} \, Q_2 = \overline{w_0 \cdots w_n}$, with $\overline{v_i v_{i+1}}$ and $\overline{w_i w_{i+1}}$ $1$-segments, and $f_i: \text{int} \, P_i \rightarrow \text{int} \, Q_i$, $i = 1, 2$, are $1$-triangulated homeomorphisms such that $f_i(v_j) = w_j$. Then, defining $f: \text{int} \, P \rightarrow \text{int} \, Q$ by setting $f |_{P_i} \equiv f_i$, $f$ is a $1$-triangulated homeomorphism.

2.3. **Remarks.** (a) Condition (iii) could be replaced by defining "immersed polygons".

(b) $1$-triangulated homeomorphisms are clearly $p\mathbb{Z}_1$ homeomorphisms, but the reverse is not true: let $P$ be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$ (see Figure 5). The $1$-triangulated homeomorphisms form a cyclic group of order 3. However, the homeomorphism pictured in the figure is $p\mathbb{Z}_1$ for all $n$.

The following is evidently stronger than Theorem 1.3.
2.4. **Theorem.** Let $P$ and $Q$ be integral polygons of equal area and length, and let $p \in P \cap \mathbb{Z}^2$ and $q \in Q \cap \mathbb{Z}^2$. Then there exists a 1-triangulated homeomorphism $f: \text{int} P \to \text{int} Q$, with $f(p) = q$.

**Conjecture.** The group of $p\mathbb{Z}$ homeomorphisms of the interior of an integral polygon $P$ is the same as the group of 1-triangulated homeomorphisms. In particular, the group is finitely generated, and the group of $p\mathbb{Z}$ homeomorphisms of a 1-triangle is simply the group $\mathbb{Z}/3$ of rotations.

Note that the group of $p\mathbb{Z}_1$ homeomorphisms of a 1-triangle is not finitely generated (see Figure 5).

Several preliminary notions are necessary for the proof. We shall write the integral points of $P$ and $Q$ in counterclockwise order as $p = p_0, \ldots, p_{L-1}$ and $q = q_0, \ldots, q_{L-1}$, where $L = L(P) = L(Q)$. If $a$, $b$, and $c$ are points on a polygon, then $a < b < c$ means that $c$ follows $b$, which follows $a$, in counterclockwise order. If $0 < i, j < L - 1$, then we take $j - i$ to mean the element of $j - i + L\mathbb{Z}$ between 0 and $L - 1$.

If $S$ is an integral polygon with vertices $S \cap \mathbb{Z}^2 = \{s_0, \ldots, s_n\}$, then a **side triangle** is a 1-triangle of the form $s_i s_{i+1} s_{i+2}$, and an **inner triangle** is a 1-triangle of the form $s_i s_{i+1} v s_i$, where $v \in \text{int} S \cap \mathbb{Z}^2$.

2.5. **Lemma.** In any 1-triangulation of an integral polygon, either a side triangle or an inner triangle must occur.

**Proof.** Let $S$ be an integral polygon with $S \cap \mathbb{Z}^2 = \{s_0, \ldots, s_n\}$. Suppose there is no inner triangle in a given triangulation. Then each $s_i s_{i+1}$ is the edge of a triangle $s_i s_{i+1} s_{f(i)}$ with $s_i \leq s_{i+1} < s_{f(i)}$. Let $j$ be an index minimizing $f(i) - i$. If $f(j) \neq j + 2$, then $s_{j+1} < f(j+1) \leq f(j)$, whence $f(j+1) - (j+1) < f(j) - j$, a contradiction.

2.6. **Corollary.** (a) If $\#(\text{int} S \cap \mathbb{Z}^2) = 0$, then any triangulation contains a side triangle.

(b) If $L(S) = 3$, then any triangulation has an inner triangle.

**Proof of 2.4.** The proof is by induction on $2a(P)$. When $a(P) = \frac{1}{2}$, $P$ and $Q$ are 1-triangles, and we apply Lemma 2.1. In the general case, we will apply Lemma 2.5 to reduce the area of $P$ and $Q$.

Assume first that $\#(\text{int} P \cap \mathbb{Z}^2) = 0$. By Corollary 2.6, $P$ and $Q$ have side triangles $T_P = \overrightarrow{p_i p_{i+1} p_{i+2} p_i}$ and $T_Q = \overrightarrow{q_j q_{j+1} q_{j+2} q_j}$. Let $S$ be an integral polygon with $L(S) = L(P)$, $\#(\text{int} S \cap \mathbb{Z}^2) = 0$, $S \cap \mathbb{Z}^2 = \{s_0, \ldots, s_{L-1}\}$,
Figure 6

Figure 7. (a) Two inner triangles; (b) one inner, one side; (c) two side triangles (begin with the inner triangle, which contains all of \( \text{int} S \cap \mathbb{Z}^2 \), and then add side triangles).

which has side triangles \( s_i s_{i+1} s_{i+2} s_i \) and \( s_j s_{j+1} s_{j+2} s_j \) (Figure 6 indicates the construction of \( S \)).

Now use Definition 2.2(iii) and induction to construct 1-triangulated homeomorphisms \( \text{int} P \to \text{int} S \) and \( \text{int} S \to \text{int} Q \), which take \( p \) to \( S_0 \), and \( S_0 \) to \( q \), and we are done.

If \( #(\text{int} P \cap \mathbb{Z}^2) > 0 \), we reason as above; the situation is more complicated because \( P \) and \( Q \) have either a side or inner triangle, and we must show that there exist integral polygons \( S \) with \( L(S) = L(P) \), \( #(\text{int} S \cap \mathbb{Z}^2) = #(\text{int} P \cap \mathbb{Z}^2) \), which admit both sorts of triangles in all possible positions (these \( S \) are displayed in Figure 7). Repeating the argument above concludes the proof.

3. Local and Global Structure of \( G(P) \)

Recall the graph \( G_1(P) \) (see §1) whose vertices are 1-segments whose interiors lie in the interior of the integral polygon \( P \), and with an edge between two vertices if the corresponding 1-segments cross. Our goal in this section is to prove Theorem 1.10, which we paraphrase as follows: for each \( P \) there is some \( N_P \) such that, if \( N > N_P \), \( G_1(NP) \) consists of a connected graph with some isolated vertices whose number depends on \( P \). Our approach is inspired by ideas from analysis. As it turns out, the isolated vertices in \( G_1(NP) \), \( N > N_p \), are associated to the corners of \( P \); we make a brief study of the graphs of sectors between two rays. Then, a family of “patches”—integral polygons with connected graphs—is produced. The proof of Theorem 1.10 involves these large and small scales.

We begin with the small-scale picture.

3.1. Definition. A patch is an integral polygon \( P \) so that \( G_1(P) \) is connected, and, if \( \overline{ab} \) is any 1-segment in \( \mathbb{R}^2 \) such that \( \overline{ab} \cap \text{int} P \) is nonempty, then some 1-segment in \( \text{int} P \) crosses \( \overline{ab} \).
Let $R_{n,k}$ be the rectangle with corners $(0,0)$, $(n,0)$, $(n,k)$, $(0,k)$.

3.2. **Proposition.** For any $g \in A_1$, $g R_{n,k}$ is a patch.

*Proof.* Since elements of $A_1$ preserve graphs, it suffices to show that $R_{n,k}$ is a patch. If $\overline{ab}$ is a 1-segment and $\overline{ab} \cap \text{int} \, R_{n,k} \neq \emptyset$, then there is some square $S = (x,y)(x+1,y)(x+1,y+1),(x,y+1)$ whose interior is contained in $\text{int} \, R_{n,k}$, such that $\overline{ab} \cap \text{int} \, S \neq \emptyset$. But then one of the diagonals of $S$ crosses $\overline{ab}$.

3.3. **Corollary** (of the proof). If $P$ is an integral polygon such that $\text{int} \, P$ is the union of interiors of squares, then $P$ is a patch.

Such a $P$ is called a block polygon.

3.4. **Proposition.** The union of two overlapping patches is a patch: Let $P$, $P_1$, and $P_2$ be integral polygons, let $P_1$ and $P_2$ be patches, and let $\text{int} \, P = \text{int} \, P_1 \cup \text{int} \, P_2$. Then $P$ is a patch.

*Proof.* Since $P_1$ and $P_2$ overlap, there is some 1-segment whose interior is contained in $\text{int} \, P_1 \cap \text{int} \, P_2$. Thus $G_1(P)$ is connected. If $\overline{ab}$ is a 1-segment which has nonempty intersection with $\text{int} \, P$, then it has nonempty intersection with $\text{int} \, P_1$ or $\text{int} \, P_2$. Thus $P$ is a patch.

Let us now discuss graphs associated to noncompact regions. If $R$ is the closure of an open region in $\mathbb{R}^2$ whose boundary is the union of 1-segments, then we denote by $G_1(R)$ the graph whose vertices are 1-segments whose interiors are contained in $\text{int} \, R$, with an edge between two vertices if the corresponding 1-segments cross.

Consider first a half-plane, that is, $R = \{(x, y) : ax + by > c, a, b, c \in \mathbb{Q}\}$. Applying an element of $A_1$, we can assume that $R = \{(x, y) : y > 0\}$. Then $R = \bigcup \text{int} \, P_n$, where $P_n$ is the rectangle with corners $(\pm n, 0)$, $(\pm n, n)$. Since each $P_n$ is a patch, $G_1(R)$ is connected, thus:

3.5. **Proposition.** $R$ is a patch.

That is to say, if $\overline{ab}$ is a 1-segment of $\mathbb{R}^2$ whose interior has nonempty intersection with the interior of $R$, then some 1-segment in the interior of $R$ crosses $\overline{ab}$.

If $v = (a, b)$ and $w = (c, d)$, with $a, b,$ and $c, d$ relatively prime, let $r$ and $s$ be the rays from $(0, 0)$ through $v$ and $w$ respectively. Then the angle $A(w, v)$ is the region swept out by a ray sweeping counterclockwise from $s$ to $r$. If $A(w, v)$ (properly) contains a half-plane it is called (strictly) concave, and if not, convex.

Every concave angle $A(w, v)$ is the union of two overlapping half-planes. Applying Propositions 3.4 and 3.5, we find

3.6. **Proposition.** If $A(w, v)$ is concave, it is a patch.

The image of a concave, strictly concave, or convex angle under an element of $A_1$ which takes $(0,0)$ to a point $p$ will also be called a concave, strictly concave, or convex angle at $p$.

3.7. **Definition.** Let $A(w, v)$ be a strictly concave angle, and let $M \in \mathbb{Z}$, $M > 0$. The $M$-cap for $A(w, v)$ is the polygon $P = P_M(w, v)$ so that $\text{int} \, P =$
\[ \text{int} R_1 \cup \text{int} R_2, \text{ where} \]
\[ R_1 = Mv, -Mv, -Mv - Mw, Mv - Mw, Mv \]

and
\[ R_2 = Mw, Mw - Mv, -Mw - Mv - Mw, Mw \]

(see Figure 8). An \( M \)-cap for an angle at \( p \) is the image under some element of \( A_1 \) of an \( M \)-cap at \( (0, 0) \).

Note that since \( v = (a, b) \) and \( w = (c, d) \), with \( a, b \) and \( c, d \) relatively prime, \( (0, 0)v \) and \( (0, 0)w \) are 1-segments. Hence, for example, \( L(P_M(w, v)) = 8M \).

The situation for convex angles is more interesting. We will see that \( G_1(A(w, v)) \) consists of a connected piece and a number of isolated vertices.

If \( A(w, v) \) is the image under \( g \in \text{SL}_2 \mathbb{Z} \) of \( A((1, 0), (0, 1)) \), then we call \( A(w, v) \) a right angle.

3.8. **Lemma.** Right angles are patches.

*Proof.* It suffices to prove \( A((1, 0), (0, 1)) \) is a patch. But \( A((1, 0), (0, 1)) \) is the union of \( \text{int} R_{n, n} \), where \( R_{n, n} \) is the square with corners \( (0, 0), (n, 0), (0, n), \) and \( (n, n) \), and \( R_{n, n} \) is a patch by 3.2.

3.9. **Definition.** Let \( A(w, v) \) be convex. A chain from \( w \) to \( v \) is a sequence \( w = v_0, v_1, \ldots, v_n, v_{n+1} = v \) with \( v_i = (a_i, b_i) \), such that the rays from \( (0, 0) \) to \( v_i \) are in \( A(w, v) \) and occur in counterclockwise order, and such that

\[ \det \begin{pmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{pmatrix} = 1, \quad 0 \leq i \leq n, \]

that is, each \( A(v_i, v_{i+1}) \) is a right angle.

3.10. **Lemma.** If \( A(w, v) \) is convex, then there exists a chain from \( w \) to \( v \).

*Proof.* Applying an element of \( \text{SL}_2 \mathbb{Z} \), we can assume that \( v = (0, 1) \) and \( w = (a, b) \), with \( b/a < 1 \). Considering Farey series \([R]\) we can write

\[ \frac{b}{a} = \frac{p_1 + p_r}{q_1 + q_r}, \]

where \( 0 \leq p_1/q_1 < b/a < p_r/q_r \leq 1 \), and \( bq_r - p_ra = -1 \). Taking \( w = v_0, v_1 = (q_r, p_r) \), and iterating, we will eventually arrive at \( v_n = (1, 1) \). Setting \( v_{n+1} = v = (0, 1) \) we have a chain.

As an example, take \( v = (0, 1) \) and \( w = (5, 3) \). Then \( \frac{3}{5} = \frac{14 + 2}{24 + 3} \) and \( \frac{2}{3} = \frac{1 + 1}{2 + 1} \), so the chain is \( w = (5, 3), v_1 = (3, 2), v_2 = (1, 1), v = (0, 1) \).
3.11. **Proposition.** Let $A(w, v)$ be a convex angle and $w = v_0, \ldots, v_{n+1} = v$ be a chain. There exist $N_i$ such that the isolated vertices of $G_1(A(w, v))$ are $1$-segments $kv_i, (k+1)v_i$, $0 \leq k \leq N_i - 1$. The complement of the collection of these vertices is a connected subgraph of $G_1(A(w, v))$.

**Proof.** By 3.8, each $A(v_i, v_{i+1})$ is a patch. Further, it is clear that for each $i$, there is some $M_i$ so that $mv_i, (m+1)v_i$ is connected to $G_1(A(v_i, v_{i+1}))$ and $G_1(A(v_{i-1}, v_i))$ for $m \geq M_i$. Let $N_i$ be the smallest such $M_i$. Then the subgraph of $G_1(A(w, v))$ whose vertices are all but the $mv_i, (m+1)v_i$, $m < N_i$, is connected. We must show that the vertices $mv_i, (m+1)v_i$, $m \leq N_i - 1$, are indeed isolated. But if $ab$ crosses $mv_i, (m+1)v_i$, then $a + v_i, b + v_i$ crosses $(m+1)v_i, (m+2)v_i$, and so on, contradicting the definition of $M_i$.

The $N_i$ in 3.11 is called the weight of the singular vector $v_i$, if $N_i \leq 1$. The number of isolated vertices in $G_1(A(w, v))$ is $\sum N_i$.

Partially order the set of chains from $w$ to $v$ by inclusion.

3.12. **Theorem.** Given a convex angle $A(w, v)$, there is a minimal chain $w = v_0, \ldots, v_{n+1} = v$. Each $v_i$ has positive weight.

**Proof.** Let $w = v_0, \ldots, v_{n+1} = v$ be a chain. We show that if a $v_j$ has weight $0$, then we can replace the chain with a subchain of cardinality strictly less. If $\overline{Ov}_j$ is not isolated in $G_1(A(w, v))$, then some 1-segment $ab$ crosses $\overline{Ov}_j$. Apply an element of $\text{SL}_2\mathbb{Z}$ so that $v_j = (1, 0)$, $v_{j+1} = (0, 1)$, and $v_{j-1} = (n, -1)$ for some $n \in \mathbb{N}$ (see Figure 9), and take $a$ with $y$ coordinate negative, and $b$ with $y$ coordinate positive. The 1-segment $ab$ crosses some number $m$ of $\overline{Ov}_j$. We prove, by induction on $m$, that the size of the chain can be reduced.

If $m = 1$, then $ab$ crosses only $\overline{Ov}_j$. From Figure 9 one sees that $ab = \overline{v}_{j-1}v_{j+1}$, in which case $v_{j-1} = (1, -1)$, so that $A(v_{j-1}, v_{j+1})$ is a right angle, and $v_j$ can be dropped from the chain.

Assume that $m, n > 1$. Then $ab$ crosses $\overline{Ov}_{j-1}$, and it either crosses $\overline{Ov}_{j+1}$ or not. If $ab$ does not cross $\overline{Ov}_{j+1}$, then $\overline{av}_j$ crosses $\overline{Ov}_{j-1}$; by replacing $ab$ with $\overline{av}_j$ we can reduce $m$ and by induction we can reduce the length of the chain. If $ab$ crosses both $\overline{Ov}_{j-1}$ and $\overline{Ov}_{j+1}$, then either $\overline{av}_j$ crosses $\overline{Ov}_{j-1}$, or $\overline{bv}_j$ crosses $\overline{Ov}_{j+1}$. Either way $m$ is reduced, and by induction the chain is reduced.

To prove 1.10, we need a finite version of 3.12. If $A(w, v)$ is a right angle, the $M$-square $S_M(w, v)$ at $A(w, v)$ is the parallelogram with corners $(0, 0)$, $Mw$, $Mv$, $M(w + v)$. If $A(w, v)$ is convex, let $w = v_0, \ldots, v_{n+1} = v$ be
the minimal chain. The polygon $P_M(w, v)$ so that $\text{int} P_M = \bigcup \text{int} S_M(v_i, v_{i+1})$ is called the $M$-pencil point at $A(w, v)$ (see Figure 10).

3.13. Lemma. Let $A(w, v)$ be a convex angle, and let $w = v_0, \ldots, v_{n+1} = v$ be a minimal chain with weights $N_i$. If $M > \max N_i$, then $G_1(P_M(w, v))$ consists of a connected component with $\sum N_i$ isolated vertices $kv_i, (k+1)v_i$, $0 \leq k < N_i$, $1 \leq i \leq n$.

Proof. The $M$-squares $S_M(v_i, v_{i+1})$ are patches, so it suffices to show that the vertices $kv_i, (k+1)v_i$, $k \geq N_i$, are connected to the $G_1(S_M(v_i, v_{i+1}))$ and $G_1(S_M(v_{i-1}, v_i))$. With an element of $\text{SL}_2 \mathbb{Z}$, we can take $v_i = (1, 0)$, $v_{i+1} = (0, 1)$, and $v_{i-1} = (2m + \epsilon, -1)$ with $\epsilon = 0$ or 1. It is not hard to check that $v_i$ has weight $N_i = m - 1$, and that $(n; 0)(m + 1, 0)$ is crossed by $(2n + \epsilon, -1)(1 - \epsilon, 1)$.

Proof of Theorem 1.10. We prove that if $P$ is an integral polygon, then there is some $N_p$ such that, if $N > N_p$, $G_1(NP)$ consists of a connected component and $m_p$ isolated vertices. Here $m_p = \sum_i \sum_j N_i$, the sum over the weights of the singular vectors associated to minimal chains of each convex angle in $P$.

Begin by taking $N$ large enough so that $M$-caps or $M$-pencil points can be placed at each angle in $P$, where $M$ is larger than $\max M_i$ (Figure 11(a)).
Now (Figure 11(b)) enlarging $N$ if necessary, translate the outer $M$-squares or rectangles of the pencil points and caps along their respective sides, so that each point in $P \cap \mathbb{Z}^2$ is contained in one of the translated squares or rectangles. Finally, enlarge $N$ to an $N_p$ so that (possibly increasing $M$) there is a block polygon (recall 3.3) which overlaps the union of the squares and rectangles (Figure 11(c)). Applying 3.4, we are done.

4. THE COMPLEX $K_1 P$

The object of this section is to prove that if $P$ is an integral polygon whose area is at least $\frac{3}{2}$, then $K_1 P$ is a combinatorial disk. We know from 1.9 that $K_1 P$ is a pure simplicial complex (that is, all maximal simplices have the same dimension) of dimension

$$\dim K_1 P = 2a(P) + N(P) - 1,$$

where $N(P) = \#(\text{int } P \cap \mathbb{Z}^2)$. Also, from §3, $G_1 P$ often has isolated vertices, so that $K_1 P$ is a cone. With some simple examples, these remarks lead to the suspicion that $K_1 P$ is a (piecewise-linear) disk.

The proof that $K_1 P$ is a disk is by induction and requires a generalization of the idea of polygon, which we approach as follows. If $K_1 P$ is a disk, it is first of all a manifold, so that the link $Lk(s)$ of each simplex $s$ should be a disk or a sphere. These $Lk(s)$ can be seen as $K_1 P_s$, where $P_s$ is a generalized polygon, called a "slit polygon."

Suppose $s = (s_0, \ldots, s_k)$, where the $s_i$ are 1-segments in $\text{int } P$. Then an $m$-simplex $t = (t_0, \ldots, t_m)$ is in $Lk(s)$ if and only if $s * t = (s_0, \ldots, s_k, t_0, \ldots, t_m)$ is a simplex in $K_1 P$; in other words, none of the $t_i$ cross any $s_j$. We think then of $t$ as a simplex in $K_1 P_s$, where $P_s$ is the polygon $P$, slit at each $s_i$.

By $a(P_s)$ we mean $a(P)$; $N(P_s)$ is $N(P)$ less the number of points in $\text{int } P \cap \mathbb{Z}^2$ which are endpoints of some $s_i$. Then equation (4.1) holds for slit polygons. By $\text{int } P_s$ we mean $\text{int } P - \bigcup S_i$, and $h(P_s) = \text{rank } H_1(\text{int } P_s)$. If $t$ is a simplex in $K_1(P_s) = Lk(s)$, we define $(P_s)_t = P_{s+t}$, so we can "slit" slit polygons. The number of components of $P_s$ means the number of components of $\text{int } P_s$; each component of $\text{int } P_s$ is $\text{int } Q$ for some slit polygon $Q$, and we speak of the components $Q_1, \ldots, Q_n$ of $P_s$. The following remark is important for the sequel.

4.2. Lemma. If the components of $P_s$ are $Q_1, \ldots, Q_n$, then $K_1(P_s) = K_1(Q_1) \ast \cdots \ast K_1(Q_n)$.

Let $P$ be an integral polygon, $s$ a simplex in $K_1 P$, and consider the slit polygon $P_s$. Then $\text{int } P_s$ is the interior of a pl-manifold with boundary which submerges onto $\text{int } P$ (see Figure 12). We will call this closed manifold $\text{int } P_s$. By $\partial P_s$ is meant the component of the boundary of $\text{int } P_s$ whose image in $\text{int } P$ contains $P = \partial \text{ int } P$. Note that $\partial P_s$ can be described as a series $P_1 P_2 \cdots P_n P_1$ of points in $\mathbb{Z}^2$ such that each $p_i p_{i+1}$ is a 1-segment in counterclockwise order (Figure 12). By an angle of $P_s$ is meant a 3-point fragment $p_i p_{i+1} p_{i+2}$ of $\partial P_s$.

We will prove the following version of 1.11.

4.3. Theorem. If $P$ is a connected slit polygon and $a(P) > 1$ or $N(P) \leq 1$, then $K_1(P)$ is a pl-disk.
To begin, consider the connected slit polygons with area $\frac{3}{2}$ or less. If $a(P) = \frac{1}{2}$, then $P$ is a 1-triangle and $K_1(P)$ is empty. If $a(P) = 1$, then $P$ is two 1-triangles joined at a face, so $K_1(P)$ is a 0-sphere $S^0$ or a point, that is, a 0-disk.

4.4. **Lemma.** If $P$ is a connected slit polygon and $a(P) = \frac{3}{2}$, then $K_1(P)$ is a 1-disk or a 2-disk.

**Proof.** Suppose first that $P$ is not slit. Then, by Pick's theorem, either $N(P) = 1$, $L(P) = 3$ or $N(P) = 0$, $L(P) = 5$. In the former case, there is a 1-segment from the interior vertex to each of the three vertices of $P$, so $K_1(P)$ is a 2-simplex.

Suppose that $N(P) = 0$ and $L(P) = 5$. Label the points of $P \cap \mathbb{Z}^2$ in counterclockwise order $a, b, c, d, e$. By 2.6, $P$ has a side triangle; without loss of generality we can assume that $ac$ is a 1-segment in $\text{int} P$ and that $abca$ is a 1-triangle. Composing with an element of $A_1$ we can assume that $a = (0, 1)$, $b = (0, 0)$ and $c = (1, 0)$.

If neither $bd$ nor $be$ are 1-segments in $\text{int} P$, then $K_1(P)$ is the cone at the vertex $ac$ of $K_1(acde)$, which is an $S^0$ or a $D^0$, and thus $K_1(P)$ is a 1-disk.

If at least one of $bd$ or $be$ is a 1-segment in $\text{int} P$, then one of $d$ or $e$ must be $(1, 1)$ (Figure 13); without loss of generality, put $e = (1, 1)$, whence $d = (2, k)$ for some $k \in \mathbb{Z}$ (Figure 13(a)). Then Figure 13(b), (c), (d) show that $K_1(P)$ is a 1-disk.

Now, if $P$ is a connected slit polygon with area $\frac{3}{2}$, it must be $Q$, where $L(Q) = 3$ and $s$ is a 1-segment from a point of $Q \cap \mathbb{Z}^2$ to the interior vertex. Thus $K_1(P) = K_1(Q)$ is a 1-simplex.

**Proof of 4.3.** Put the triple $(a(P), N(P), h(P))$ in lexicographic order (e.g., $(2, 1, 1) > (1, 4, 6) > (1, 3, 8)$); the proof is by induction on the triples. Lemma 4.4 deals with the initial case $(\frac{3}{2}, 0, 0)$. Let $\overline{abc}$ be a convex angle in $\partial P$ (e.g., in Figure 12, $gfe$). Applying an element of $A_1$, we can put $b$ at $(0, 0)$, $a$ at $(0, 1)$, and $c$ at $(n, -m)$, $n, m \geq 0, m < n$. 

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Suppose that $m > 0$. Then (as in 3.12) the edge $s = (0,0)(1,0)$ is an isolated vertex in $G_1P$, and so $K_1P$ is the cone at $s$ on $K_1P_s$. Now $a(P_s) = a(P)$, but either $h(P_s) < h(P)$ or $N(P_s) < N(P)$, so, by induction, $K_1P_s$ is a disk and hence $K_1P$ is a disk.

Now suppose that $m = 0$, so that $c$ is at $(1,0)$. Let $s = (0,1)(1,0)$. If no 1-segments of $\text{int}P$ have an endpoint at $(0,0)$, then $s$ is an isolated vertex in $G_1P$, and so $K_1P$ is the cone at $s$ on $K_1Q$, where $Q$ is $P$ with $abc$ replaced by $\overline{abc}$, that is, with the triangle $\overline{abc}$ excised. (Since $\overline{bc} \subset P$, $Q$ is still a slit polygon.) Since $a(Q) < a(P)$, by induction $K_1P$ is a disk.

If, on the other hand, some edge crosses $s$, then $(1,1) \in \text{int}P$ and $t = (0,0)(1,1)$ must be a 1-segment in $P$. I claim any triangulation of $P$ must contain either $s$ or $t$ as an edge. For suppose some triangulation does not include $t$. Then some 1-segment $x$ of the triangulation crosses $s$. But any edge crossing $s$ would also cross $x$, and consequently $s$ is an edge in the triangulation.

Since either $s$ or $t$ is in any triangulation of $P$, it follows that $K_1P$ is the union of two cones: the cone at $s$ of $K_1P_s$ and the cone at $t$ of $K_1P_t$. The two cones intersect in $X = K_1P_s \cap K_1P_t$. If we can show that $X$ is a disk we are done (by, for example, Corollary II.16 of [GL]).

$X$ is the subcomplex of $K_1P$ whose simplices are partial triangulations of $P$ which contain no edges crossing $s$ or $t$. Let $s_1 = (0,1)(1,1)$ and $s_2 = (1,0)(1,1)$. Either $s_1$ or $s_2$ or both are vertices in $K_1P$. If only one is a vertex in $K_1P$, say $s_1$, then $X$ is the cone at $s_1$ on $K_1Q$, where $Q$ is the slit polygon obtained by replacing $\overline{abc}$ with $\overline{a(1,1)c}$ in $\partial P$. Since $a(Q) < a(P)$, by induction $K_1P$ is disk.

If both $s_1$ and $s_2$ are in $K_1P$, then $K_1P$ is the join of $K_1Q$ with the 1-simplex $(s_1, s_2)$, and again is a disk.

**Bibliography**


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