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BESOV SPACES ON DOMAINS IN $\mathbb{R}^d$

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ABSTRACT. We study Besov spaces $B_q^\alpha(L_p(\Omega))$, $0 < p, q, \alpha < \infty$, on domains Ω in $\mathbb{R}^d$. We show that there is an extension operator $\mathscr{E}$ which is a bounded mapping from $B_q^\alpha(L_p(\Omega))$ onto $B_q^\alpha(L_p(\mathbb{R}^d))$. This is then used to derive various properties of the Besov spaces such as interpolation theorems for a pair of $B_q^\alpha(L_p(\Omega))$, atomic decompositions for the elements of $B_q^\alpha(L_p(\Omega))$, and a description of the Besov spaces by means of spline approximation.

1. INTRODUCTION

Besov spaces $B_q^\alpha(L_p(\Omega))$ are being applied to a variety of problems in analysis and applied mathematics. Applications frequently require knowledge of the interpolation and approximation properties of these spaces. These properties are well understood when $p \geq 1$ or when the underlying domain Ω is $\mathbb{R}^d$. The purpose of the present paper is to show that these properties can be extended to general nonsmooth domains Ω of $\mathbb{R}^d$ and for all $0 < p \leq \infty$. Besov spaces with $p < 1$ are becoming increasingly more important in the study of nonlinear problems.

To a large extent the present paper is a sequel to [2 and 4] which established various properties of the spaces $B_q^\alpha(L_p(\Omega))$, Ω a cube. Among these are atomic decompositions for the functions in $B_q^\alpha(L_p(\Omega))$, a characterization of $B_q^\alpha(L_p(\Omega))$ through spline approximation, and a description of interpolation spaces for a pair of Besov spaces. We establish similar results for more general domains.

Our approach is to first define an extension operator, $\mathscr{E}$, which extends functions in $B_q^\alpha(L_p(\Omega))$ to all of $\mathbb{R}^d$. Similar extension operators for $p \geq 1$ have been introduced by Calderón and Stein (see [7, Chapter 6]). Our main departure from these earlier approaches is that by necessity our extension operators are nonlinear. Moreover, whereas in the case $p \geq 1$, it is possible to take $\mathscr{E}$ so that $\omega_r(\mathscr{E}f, t)_p \leq C\, \omega_r(f, t)_p$ with ω_r the rth order modulus of smoothness (at least when Ω is minimally smooth [5]), in the case $0 < p < 1$, we only obtain a weak comparison between $\omega_r(\mathscr{E}f, t)_p$ and $\omega_r(f, t)_p$.

We shall establish our results for two important classes of nonsmooth domains: the Lipschitz graph domains, and the (ε, δ) domains introduced by Jones [6]. We begin in §4 with the case of Lipschitz graph domains since the

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geometric arguments in this case are the most obvious. We later generalize these arguments to (ε, δ) domains in §5. Although the results of §5 contain those of §4, we feel that this two tier presentation makes the essential arguments much clearer.

2. Moduli of smoothness and Besov spaces

Let Ω be an open subset of $\mathbb{R}^d$. We can measure the smoothness of a function $f \in L_p(\Omega)$, $0 < p \leq \infty$, by its modulus of smoothness. For any $h \in \mathbb{R}^d$, let I denote the identity operator, $\tau(h)$ the translation operator $(\tau(h)(f, x) := f(x+h))$ and $\Delta_h^r := (\tau(h) - I)^r$, $r = 1, 2, \ldots$, be the difference operators. We shall also use the notation

$$\Delta_h^r(f, x, \Omega) := \begin{cases} \Delta_h^r(f, x), & x, x+h, \ldots, x+rh \in \Omega, \\ 0, & \text{otherwise}. \end{cases}$$

The modulus of smoothness of order r of a function $f \in L_p(\Omega)$ is then

$$\omega_r(f, t)_p := \omega_r(f, t, \Omega)_p := \sup_{|h| \leq t} \|\Delta_h^r(f, \cdot, \Omega)\|_{L_p(\Omega)}. \tag{2.1}$$

For any $h \in \mathbb{R}^d$, we define

$$\Omega(h) := \{x : [x, x+h] \subset \Omega\}.$$

A Besov space is a collection of functions f with common smoothness. If $0 < \alpha < r$ and $0 < q, p \leq \infty$, the Besov space $B_q^\alpha(L_p(\Omega))$ consists of all functions f such that

$$|f|_{B_q^\alpha(L_p(\Omega))} := \left(\int_0^1 [t^{-\alpha} \omega_r(f, t, \Omega)_p]^q \, dt/t \right)^{1/q} < \infty \tag{2.2}$$

with the usual change to sup when $q = \infty$. It follows that (2.2) is a semi(quasi)-norm for $B_q^\alpha(L_p(\Omega))$. (Frequently, the integral in (2.2) is taken over $(0, \infty)$; while this results in a different seminorm, the norms given below are equivalent.) If we add $\|f\|_{L_p(\Omega)}$ to (2.2), we obtain the (quasi)norm for $B_q^\alpha(L_p(\Omega))$. It is well known in the case $p \geq 1$ that different values of $r > \alpha$ give equivalent norms. This remains true for $p < 1$ as well and can be derived from the 'Marchaud inequalities', which compare moduli of smoothness of different orders. These inequalities have been proved for all $p > 0$ and Ω either a cube or all of $\mathbb{R}^d$ in [8] (see also [2]), and for more general domains Ω and $p \geq 1$ by Johnen and Scherer [5] (among others). We address this topic later in §6 for the remaining case $0 < p \leq 1$ and more general Ω.

There are fundamental connections between smoothness and approximation (see [2] and the references therein, especially [8]). We now describe these without proofs (which can be found in [2] or [8]). If $f \in L_p(Q)$, $0 < p \leq \infty$, Q a cube in $\mathbb{R}^d$, we let

$$E_r(f, Q)_p := \inf_{P \in \mathbb{P}_r} \|f - P\|_p(Q) \tag{2.3}$$

be the error of approximation by the elements from the space $\mathbb{P}_r$ of polynomials of total degree less than r where $\|\cdot\|_p(Q)$ denotes the $L_p(Q)$ (quasi)norm. We then have Whitney's inequality

$$E_r(f, Q)_p \leq C \, \omega_r(f, l(Q))_p \tag{2.4}$$

where $l(Q)$ is the side length of Q and C is a constant which depends only on r and d (also p if p is close to 0).

Sometimes (2.4) is not sufficient because it is not possible to add these estimates for different cubes Q. For this purpose, the following *averaged moduli of smoothness* is more convenient. For any domain Ω and $t > 0$, we define

$$\mathbf{w}_r(f, t, \Omega)_p := \left(t^{-d} \int_{|s| \le t} \int_\Omega |\Delta_s^r(f, x, \Omega)|^p \, dx \, ds \right)^{1/p} \tag{2.5}$$

where $p < \infty$. Then, returning once again to cubes Q, ω_r and $\mathbf{w}_r$ are equivalent:

$$C_1 \omega_r(f, t, Q)_p \le \mathbf{w}_r(f, t, Q)_p \le C_2 \omega_r(f, t)_p \tag{2.6}$$

where C_1 and C_2 depend only on d, r and p if p is small. Therefore, the estimate (2.4) can be improved by replacing ω_r by $\mathbf{w}_r$:

$$E_r(f, Q)_p \le C \, \mathbf{w}_r(f, l(Q), Q)_p . \tag{2.7}$$

We shall use the generic notation $P_Q := P_Q(f)$ to denote a polynomial in $\mathbb{P}_r$ which satisfies

$$\|f - P_Q\|_p(Q) \le \lambda \, E_r(f, Q)_p \tag{2.8}$$

where $\lambda \ge 1$ is a constant which we fix. The polynomial P_Q is then called a *near best* approximation to f with constant λ. When $\lambda = 1$, P_Q is a *best approximation.* It follows from (2.7) and (2.8) that

$$\|f - P_Q\|_p(Q) \le C \, \mathbf{w}_r(f, l(Q), Q)_p . \tag{2.9}$$

We shall use the following observation (see [2, Lemma 3.2]) about near best approximation in the sequel. Let $\gamma > 0$. If $P_Q \in \mathbb{P}_r$ is a near best approximation to f with constant λ on Q in the L_γ norm, then it is also a near best approximation to f for all $p \ge \gamma$:

$$\|f - P_Q\|_p(Q) \le C\lambda \, E_r(f, Q)_p \tag{2.10}$$

where the constant C depends only on γ, r, and d.

The estimate (2.10) leads to a characterization of Besov spaces in terms of spline approximation. For $n \in \mathbb{Z}$, let $\mathbb{D}_n$ be the collection of dyadic cubes Q of side length 2^{-n} and let $\mathbb{D} := \bigcup_{n \in \mathbb{Z}} \mathbb{D}_n$ be the collection of all dyadic cubes. For $n \in \mathbb{Z}$, let $\Pi_n := \Pi_{n,r}$ be the space of piecewise polynomials S on $\mathbb{D}_n$ which have degrees less than r. The error of approximation to a function $f \in L_p(\Omega)$ by elements of Π_n is

$$s_n(f)_p := \inf_{S \in \Pi_n} \|f - S\|_p(\Omega) . \tag{2.11}$$

It follows from [2] that a function $f \in L_p(\Omega)$ is in $B_q^\alpha(L_p(\Omega))$, Ω a cube, if and only if

$$\|f\|_{\mathscr{A}_q^\alpha(L_p)} := \left(\sum_{n \in \mathbb{Z}} (2^{n\alpha} s_n(f)_p)^q \right)^{1/q} < \infty . \tag{2.12}$$

Moreover, (2.12) is an equivalent seminorm for $B_q^\alpha(L_p(\Omega))$. Let us emphasize for later use that this same result holds in the case $\Omega = \mathbb{R}^d$ with the same proof.

3. Polynomials

It will be useful to mention briefly some well-known properties of polynomials which we shall use frequently in what follows. If Q is a cube, we let, for $0 < p \le \infty$,

$$\|f\|_p^*(Q) := |Q|^{-1/p}\|f\|_p(Q) \tag{3.1}$$

be the normalized L_p norms. We also introduce the notation ρQ to denote the cube with the same center as Q and side length $\rho l(Q)$ where $l(Q)$ is the side length of Q.

If r is a nonnegative integer, $\rho > 1$ and P is a polynomial of degree $\le r$, then (see for example [4, §3]) for a constant C depending only on d, r (this constant and other constants in this section also depend on the distance of p to 0), we have for any $q \ge p$:

$$\|P\|_q^*(\rho Q) \le C\|P\|_q^*(Q) \le C\|P\|_p^*(\rho Q). \tag{3.2}$$

We often apply this inequality in the following way. Let Q_1, Q_2 be two cubes with $l(Q_1) \ge l(Q_2)$ and $Q_1 \subset \rho Q_2$ for some $\rho \ge 1$. Then for a constant c depending only on d, ρ, p, r, we have, for all $q \ge p$,

$$\|P\|_q^*(Q_1) \le c\|P\|_p^*(Q_2). \tag{3.3}$$

Indeed, it is enough to compare the left side of (3.3) with $\|P\|_p^*(Q_1)$, compare this with $\|P\|_p^*(\rho Q_2)$, and then finally make a comparison with $\|P\|_p^*(Q_2)$.

4. Extension operators, local approximation, and moduli

We shall define an extension operator $\mathscr{E}$ (similar to that introduced in [4]) which extends each function $f \in L_p(\Omega)$ to all of $\mathbb{R}^d$ and has the property that if $f \in B_q^\alpha(L_p(\Omega))$, then $\mathscr{E}f \in B_q^\alpha(L_p(\mathbb{R}^d))$ (with suitable restrictions on α, p, q, and Ω). We assume at the outset that Ω is a Lipschitz graph domain and treat more general domains in the next section. This means that $\Omega = \{(u, v)\colon u \in \mathbb{R}^{d-1}, v \in \mathbb{R} \text{ and } v > \phi(u)\}$ where ϕ is a fixed Lip 1 function. That is, ϕ satisfies $|\phi(u_1) - \phi(u_2)| \le M|u_1 - u_2|$, for all $u_1, u_2 \in \mathbb{R}^{d-1}$, where M is a fixed constant (which we can assume is greater than one).

We let F denote the Whitney decomposition of Ω into dyadic cubes (see Stein [7, p. 168]). Similarly we denote by F_c the Whitney decomposition of $\Omega^c \backslash \partial\Omega$. Then,

$$\begin{aligned} &\text{(i)} \quad \operatorname{diam}(Q) \le \operatorname{dist}(Q, \partial\Omega) \le 4\operatorname{diam}(Q), \quad Q \in F \cup F_c, \\ &\text{(ii)} \quad \text{if } Q, Q_0 \in F \cup F_c \text{ touch, then } l(Q_0) \le 4\, l(Q), \\ &\text{(iii)} \quad \sup_{(u,v)\in Q} |v - \phi(u)| \le C\, l(Q), \end{aligned} \tag{4.1}$$

where C depends only on the Lipschitz constant M and the dimension d. Here, $\operatorname{diam}(Q) = \sqrt{d}\, l(Q)$ with $l(Q)$ the side length of Q.

For each cube Q in $F \cup F_c$ let $Q^* := \frac{9}{8}Q$. If $Q \in F$, then $Q^* \subset 3Q \subset \Omega$. According to [7, p. 170] there is a partition of unity $\{\phi_Q\}_{Q\in F_c}$ for the open set

$\Omega^c \backslash \partial\Omega$ with the properties:

$$
(4.2)\qquad
\begin{array}{ll}
\text{(i)} & 0 \le \phi_Q \le 1\,, \\
\text{(ii)} & \sum_{Q\in F_c} \phi_Q \equiv 1\,, \text{ on } \Omega\,, \\
\text{(iii)} & \phi_Q \text{ is supported in } \operatorname{int}(Q^*)\,, \\
\text{(iv)} & \|D^\nu \phi_Q\|_\infty \le c[l(Q)]^{-|\nu|}\,,\ |\nu| \le m\,, \\
\text{(v)} & \text{if } Q_1\,,\ Q_2 \in F \cup F_c \text{ with } Q_1^* \cap Q_2^* \ne \varnothing\,, \text{ then } Q_1 \text{ and} \\
 & \quad Q_2 \text{ touch,} \\
\text{(vi)} & \text{at most } N_0 := 12^d \text{ cubes from either } F \text{ or } F_c \text{ may} \\
 & \quad \text{touch a given cube from either family.}
\end{array}
$$

Properties (i)–(iv) and (vi) are proved in [7], while a proof of (v) can be found in [4]. Here m is an arbitrary integer and c depends only on d, Ω, and m. We are using standard multivariate notation for the derivatives $D^\nu := D_{x_1}^{\nu_1} \cdots D_{x_d}^{\nu_d}$.

If $Q \in F_c$ has center (u, v), we let Q^s denote the cube in F which contains the point $(u, 2\phi(u) - v)$. We speak of Q^s as being the cube symmetric to Q across $\partial\Omega$. The symmetric cubes Q^s were introduced in [4, p. 77] and we recall now some of their properties proved in [4]. While Q and Q^s need not have the same size, they are comparable; i.e. there is a constant $C > 0$ for which there holds (for a proof see [4]).

$$
(4.3)\qquad
\begin{array}{ll}
\text{(i)} & C^{-1} l(Q) \le l(Q^s) \le C\, l(Q)\,, \\
\text{(ii)} & \operatorname{dist}(Q, Q^s) \le C\, l(Q)\,, \\
\text{(iii)} & \text{each cube in } F \text{ can be the symmetric cube } Q^s \text{ of at} \\
 & \quad \text{most } C \text{ cubes } Q \in F_c\,.
\end{array}
$$

To define our extension operators $\mathscr{E}$, we fix a value $\gamma > 0$ (which in application is chosen smaller than all p under consideration), and a value r (which in application is larger than all the α under consideration) and we let $\lambda \ge 1$. If $f \in L_\gamma$ (loc) and Q is a cube, we let $P_Q(f)$ be a polynomial which satisfies (2.8). We then define $\mathscr{E}$ by

$$
(4.4)\qquad \mathscr{E} f(x) := \begin{cases} f(x), & x \in \Omega\,, \\ \sum_{Q\in F_c} P_{Q^s} f(x) \phi_Q(x), & x \in \Omega^c \backslash \partial\Omega\,. \end{cases}
$$

Actually, (4.4) defines a family of extension operators, since each choice of near best approximants $P_{Q^s} f$ give an extension $\mathscr{E}$. The results that follow apply to any such extension operator $\mathscr{E}$ with the restriction that the constant $\lambda \ge 1$ of (2.8) is fixed.

We have shown in [4] that $\mathscr{E}$ is a bounded mapping from $L_p(\Omega)$ into $L_p(\mathbb{R}^d)$, $\gamma \le p \le \infty$, and also from $B_q^\alpha(L_p(\Omega))$ into $B_q^\alpha(L_p(\mathbb{R}^d))$ whenever $1 < p \le \infty$. We shall prove now the same result when $0 < p \le 1$. To study the smoothness of $\mathscr{E} f$, we shall need estimates of how well $\mathscr{E} f$ can be approximated by polynomials on cubes R in the L_p norm for $p \ge \gamma$.

We fix $0 < p \le \infty$ and r and use the abbreviated notation $E(Q) := E_r(f, Q)_p$.

Lemma 4.1. *There exists a constant $C > 0$ so that if Q_1, Q_2 belong to F and touch, then*

$$
(4.5)\qquad \|P_{Q_1} - P_{Q_2}\|_\infty(Q_1) \le C|Q_1|^{-1/p}[E(Q_1^*) + E(Q_2^*)]\,.
$$

Proof. By property (4.1)(ii) of the Whitney decomposition, Q_1 and Q_2 have comparable side lengths and so we may select a cube $\widetilde{Q} \subset Q_1^* \cap Q_2^*$ whose side length is comparable to that of either cube:

$$l(\widetilde{Q}) = \tfrac{1}{16} \min\{l(Q_1), l(Q_2)\}.$$

Applying the triangle inequality in $L_\infty(Q_j)$ and using the elementary estimates for polynomials (3.3), we have for $j = 1, 2$

$$\|P_{Q_j} - P_{\widetilde{Q}}\|_\infty(Q_j) \le C[\|P_{Q_j} - P_{Q_j^*}\|_p^*(Q_j) + \|P_{Q_j^*} - P_{\widetilde{Q}}\|_p^*(\widetilde{Q})].$$

Using this inequality and two applications of Lemma 3.3 of [2] (applied once to Q_j and Q_j^* and again to $\widetilde{Q}$ and Q_j^*) gives

$$\|P_{Q_j} - P_{\widetilde{Q}}\|_\infty(Q_j) \le C|Q_j|^{-1/p} E(Q_j^*). \tag{4.6}$$

Again using (3.3), we obtain

$$\|P_{Q_2} - P_{\widetilde{Q}}\|_\infty(Q_1) \le C\|P_{Q_2} - P_{\widetilde{Q}}\|_\infty(Q_2)$$

and so together with (4.6) (applied with $j = 2$) and the modified triangle inequality we obtain the desired result (4.5). □

To estimate the smoothness of $\mathscr{E} f$, we shall approximate $\mathscr{E} f$ on cubes Q from $\mathbb{R}^d$. We consider first the approximation of $\mathscr{E} f$ on cubes close to $\partial\Omega$.

Lemma 4.2. *There exists a constant $c > 0$ so that if $\mathscr{E}$ is any of the extension operators* (4.4) *and R is a cube with* $\operatorname{dist}(R, \partial\Omega) \le \operatorname{diam}(R)$, *then for $f \in L_p(\Omega)$, $\gamma \le p \le 1$, we have*

$$E_r(\mathscr{E} f, R)_p \le C \left(\sum_{\substack{S \in F \\ S \subset cR}} E(S^*)^p \right)^{1/p} \tag{4.7}$$

where c, C depend only on d, r, γ, λ, and Ω.

Proof. For such an R, if (u_0, v_0) denotes its center, then we let R_0 be the member of F containing a point of the form (u_0, v) such that $l(R_0) \ge 16\, l(R)$ and v is smallest. It is clear (see property (4.1)(i)) that R and R_0 have comparable side lengths and so we may choose a constant $c > 0$ so that $cR \supset R_0$. Let $Q \in F$ intersect R. We shall estimate $\|f - P_{R_0}\|_p(Q)$. Since $\operatorname{dist}(Q, \partial\Omega) \le \operatorname{diam}(R) + \operatorname{dist}(R, \partial\Omega) \le 2\operatorname{diam}(R)$, from (4.1)(i) it follows that $l(Q) \le 2l(R)$.

Our next step is to construct a 'chain' of cubes $\{R_j\}_0^m$ from F which connect R_0 to $Q = R_m$ with each R_j touching R_{j+1}. We accomplish this as follows. Let $x_1 = (u_1, v_1)$ be the center of R_0 and $x_3 := (u_3, v_3)$ be a point from $Q \cap R$. We consider the path consisting of a 'horizontal' followed by a 'vertical' linear segment which connects first x_1 to the point $x_2 = (u_3, v_1)$ and then x_2 to x_3. The point x_2 is in $\frac{9}{8}R_0 = R_0^*$ and is therefore in a cube $\widetilde{R} \in F$ which touches R_0. If $\widetilde{R} \ne R_0$, we define $R_1 := \widetilde{R}$, otherwise R_1 is not yet defined. The remaining cubes R_j are obtained from the vertical segment which connects x_2 to x_3, namely the cubes we encounter (in order) as v changes from v_1 to

v_3. Since all these cubes are in F, they have disjoint interiors. From property (4.1)(iii), we obtain $\sum_{j=0}^{m} l(R_j)$ is comparable to $l(R_0)$; moreover,

$$l(R_k) \le \sum_{j=k}^{m} l(R_j) \le c\, l(R_k), \qquad 0 \le k \le m. \tag{4.8}$$

In particular, we have $Q \subset cR_j$ and $R_j \subset cR$, where c has been increased as necessary but remains independent of f.

Since $Q \subset cR_j$, the inequalities (3.3) for polynomials, give that for any polynomial P, $\|P\|_\infty(Q) \le C\|P\|_p^*(R_j)$, $j = 0, \dots, m$, for a constant C depending only on p, d, Ω and the degree of P but not on j. We now write $P_Q - P_{R_0} = (P_{R_m} - P_{R_{m-1}}) + \cdots + (P_{R_1} - P_{R_0})$ and find from Lemma 4.1 that

$$\begin{aligned} \|P_Q - P_{R_0}\|_\infty(Q) &\le C \sum_{j=0}^{m-1} \|P_{R_{j+1}} - P_{R_j}\|_\infty(R_j) \\ &\le C \sum_{j=0}^{m-1} |R_j|^{-1/p}[E(R_j^*) + E(R_{j+1}^*)] \\ &\le C \sum_{j=0}^{m} |R_j|^{-1/p} E(R_j^*). \end{aligned} \tag{4.9}$$

Hence, $|Q|^{-1/p}\|P_Q - P_{R_0}\|_p(Q)$ also does not exceed the right side of (4.9). If we write $f - P_{R_0} = (f - P_Q) + (P_Q - P_{R_0})$, we obtain

$$\|f - P_{R_0}\|_p(Q) \le C|Q|^{1/p} \sum_{j=0}^{m} |R_j|^{-1/p} E(R_j^*). \tag{4.10}$$

Since an l_1 norm does not exceed an l_p norm for $0 < p \le 1$, we have

$$\|f - P_{R_0}\|_p^p(Q) \le C|Q| \sum_{j=0}^{m} |R_j|^{-1} E(R_j^*)^p. \tag{4.11}$$

We denote the 'chain' from Q to R_0 by $T_Q := (R_j)_{j=0}^m$. Summing (4.11) over all Q belonging to F such that $Q \cap R \ne \varnothing$, we then obtain

$$\sum_{\substack{Q \in F \\ Q \cap R \ne \varnothing}} \|f - P_{R_0}\|_p^p(Q) \le C \sum_{\substack{Q \in F \\ Q \cap R \ne \varnothing}} \sum_{S \in T_Q} |Q|\, |S|^{-1} E(S^*)^p. \tag{4.12}$$

Next we interchange the order of summation in (4.12) and note that while an S that appears in the sum of (4.12) may occur in more than one T_Q, each such Q is contained in cS and therefore $\sum_{\{Q\,:\,S \in T_Q\}} |Q| \le C|S|$. Since $\mathscr{E} f = f$ on such Q, we obtain

$$\sum_{\substack{Q \in F \\ Q \cap R \ne \varnothing}} \|\mathscr{E} f - P_{R_0}\|_p^p(Q) \le C \sum_{\substack{S \in F \\ S \subset cR}} E(S^*)^p. \tag{4.13}$$

We can prove a similar estimate to (4.13) for cubes $\widetilde{Q} \in F_c$ for which $\widetilde{Q} \cap R \ne \varnothing$:

$$\sum_{\substack{\widetilde{Q} \in F_c \\ \widetilde{Q} \cap R \ne \varnothing}} \|\mathscr{E} f - P_{R_0}\|_p^p(\widetilde{Q}) \le C \sum_{\substack{S \in F \\ S \subset cR}} E(S^*)^p. \tag{4.14}$$

Indeed, for a cube $\widetilde{Q}$ which appears in the left sum of (4.14), we have from the definition of $\mathscr{E}$ in (4.4):

$$\begin{aligned} \|\mathscr{E}f - P_{R_0}\|_p^p(\widetilde{Q}) &\le \sum_{\substack{Q^* \cap \widetilde{Q} \neq \varnothing \\ Q \in F_c}} \|P_{Q^s} - P_{R_0}\|_p^p(\widetilde{Q}) \\ &\le \sum_{\substack{Q^* \cap \widetilde{Q} \neq \varnothing \\ Q \in F_c}} \|P_{Q^s} - P_{R_0}\|_p^p(Q^s) \end{aligned} \tag{4.15}$$

where we have used the fact that the ϕ_Q are positive and sum to one and we have used (3.3) (for $q = p$) to replace $\|P_{Q^s} - P_{R_0}\|_p^p(\widetilde{Q})$ by $\|P_{Q^s} - P_{R_0}\|_p^p(Q^s)$ (recall that Q, $\widetilde{Q}$, and Q^s all have comparable size and the distance between any two of these cubes does not exceed $C\operatorname{diam}(Q)$). Now, by (4.2)(v), $Q^* \cap \widetilde{Q} \neq \varnothing$ only if Q and $\widetilde{Q}$ touch. Therefore by (4.2)(iv) there are at most N terms in the sum (4.15) and N depends only on d and Ω. Also a given Q^s appears for at most C cubes $\widetilde{Q}$ (see the remark following (4.3)). Furthermore Q^s is contained in cR and therefore the estimate (4.9) holds (with the Q there replaced by Q^s). Finally, if we use (3.2) to replace the $L_\infty(Q^s)$ norm by an $L_p(Q^s)$ norm on the left side of (4.9) and then use this in the terms of the right sum of (4.15), we arrive at (4.14) in the same way that we have derived (4.13).

To complete the proof, it is enough to add the estimates (4.13) and (4.14). □

We are now in a position to give an estimate for $\omega_r(\mathscr{E}f, t)_p$ for each of the extension operators $\mathscr{E}$.

Theorem 4.3. *If $\gamma \le p \le 1$ and $t > 0$ then*

$$\omega_r(\mathscr{E}f, t)_p^p \le C^p \left[\sum_{2^j \le c_1 t} \mathbf{w}_r(f, 2^j)_p^p + t^{rp} \sum_{2^j \ge t} 2^{-jrp} \mathbf{w}_r(f, 2^j)_p^p \right] \tag{4.16}$$

where $\mathbf{w}_r$ is the averaged modulus of smoothness (2.5) *and the constants c_1 and C depend only on d, r, γ, λ, and Ω.*

Proof. We write $\mathbb{R}^d \backslash \partial\Omega = \Omega_0 \cup \Omega_- \cup \Omega_+$, where $\Omega_0 := \bigcup\{Q \in F \cup F_c : l(Q) \le 16rt\}$, $\Omega_+ := \Omega \backslash (\Omega_0 \cup \partial\Omega)$, $\Omega_- := \Omega^c \backslash (\Omega_0 \cup \partial\Omega)$. It follows that for each $x \in \Omega_0$ and for the appropriate cube $Q \in F \cup F_c$ which contains x, we have

$$\operatorname{dist}(x, \partial\Omega) \le \operatorname{diam}(Q) + \operatorname{dist}(Q, \partial\Omega) \le 5\operatorname{diam}(Q) \le 80\sqrt{d}rt. \tag{4.17}$$

We shall consider three cases. Let $|h| \le t$.

Case 1 ($x \in \Omega_+$). In this instance, there is a cube $Q \in F$ containing x and $l(Q) > 16rt$. Therefore the expanded cube $Q^* := \frac{9}{8}Q \subset \Omega$ contains the line segment $[x, x + rh]$, which shows for $x \in \Omega_+$, that $\Delta_h^r(\mathscr{E}f, x) = \Delta_h^r(f, x)$. Hence, by (2.6),

$$\begin{aligned} \int_Q |\Delta_h^r(\mathscr{E}f, x, \Omega)|^p dx &\le \int_{Q^*} |\Delta_h^r(f, x, Q^*)|^p dx \le \omega_r(f, t, Q^*)_p^p \\ &\le C\, \mathbf{w}_r(f, t, Q^*)_p^p. \end{aligned}$$

We now sum over all Q which intersect Ω_+ and use the fact that a point $x \in \mathbb{R}^d$ can appear in at most N_0 of the cubes Q^* (see (4.2)(vi)) to find

$$\int_{\Omega_+} |\Delta_h^r(\mathscr{E}f, x)|^p dx \le C\, \mathbf{w}_r(f, t)_p^p. \tag{4.18}$$

Case 2 $(x \in \Omega_0)$. In this case we are near the boundary and employ Lemma 4.2. We take a tiling Λ_0 of $\mathbb{R}^d$ into pairwise disjoint cubes R of side length $80rt$. Next we obtain additional staggered tilings by translating Λ_0 in coordinate directions. Namely, if ν is a vector in $\mathbb{R}^d$ with coordinates 0 or 1, then $\Lambda_\nu := \{40rt\nu + R\}_{R\in\Lambda_0}$ is also a tiling. We let Λ denote the collection of those R such that $R \cap \Omega_0 \neq \varnothing$ and $R \in \Lambda_\nu$ for one of these ν. We note that there are 2^d such ν and for each point $x \in \Omega_0$ there is a cube $R \in \Lambda$ such that $[x, x+rh] \subset R$. Hence,

$$\begin{aligned}\int_{\Omega_0} |\Delta_h^r(\mathscr{E}f, x)|^p dx &\le \sum_{R\in\Lambda} \int_{R(rh)} |\Delta_h^r(\mathscr{E}f, x)|^p dx \\ &\le 2^r \sum_{R\in\Lambda} E(\mathscr{E}f, R)_p^p\end{aligned} \tag{4.19}$$

where the last inequality follows since the rth difference annihilates polynomials of degree less than r. The multiple 80 was chosen so that the cubes R in Λ satisfy $\operatorname{dist}(R, \partial\Omega) \le \operatorname{diam}(R)$ as follows from (4.17) because $\Omega_0 \cap R \neq \varnothing$. We may therefore estimate $E(\mathscr{E}f, R)_p$ by Lemma 4.2 to give

$$\int_{\Omega_0} |\Delta_h^r(\mathscr{E}f, x)|^p dx \le C \sum_{R\in\Lambda} \sum_{\substack{S\in F\\ S\subset cR}} E(S^*)^p. \tag{4.20}$$

Next, we observe that F is the disjoint union of the $F_j := F \cap \mathbb{D}_j$ and so (4.20) becomes

$$\int_{\Omega_0} |\Delta_h^r(\mathscr{E}f, x)|^p dx \le C \sum_{j=-\infty}^{\infty} \left(\sum_{R\in\Lambda} \sum_{\substack{S\in F_j\\ S\subset cR}} E(S^*)^p \right) =: C \sum_{j=-\infty}^{\infty} I_j. \tag{4.21}$$

Let $S_j := \bigcup\{S^* : S \in F_j\}$. By properties (4.2)(v) and (vi) of Whitney decompositions, it follows that for each j

$$\sum_{R\in\Lambda} \sum_{\substack{S\subset cR\\ S\in F_j}} \chi_{S^*} \le C N_0\, \chi_{S_j}, \tag{4.22}$$

where N_0 is the constant of (4.2)(vi), and C is a constant which depends only on d and c counting the number of times a cube $S \in F$ can appear in distinct cubes cR, $R \in \Lambda$. Therefore, from (2.7), we obtain for each $j \in \mathbb{Z}$,

$$I_j \le CN\, 2^{jd} \int_{|h|\le \frac{9}{8}2^{-j}} \int_{S_j} |\Delta_h^r(f, x, S_j)|^p dx\, dh \le C\, \mathbf{w}_r(f, 2^{-j+1})_p^p. \tag{4.23}$$

Furthermore, if $S \in F_j$ satisfies $S \subset cR$ for some $R \in \Lambda$, then $l(S) \le cl(R) = 80crt$. Hence, if $c_1 \ge 160cr$ we have from (4.1)(i) that $2^{-j+1} \le c_1 t$. Using this together with inequalities (4.21) and (4.23), we obtain

$$\int_{\Omega_0} |\Delta_h^r(\mathscr{E}f, x)|^p dx \le C \sum_{2^{-j}\le 80crt} I_j \le C \sum_{2^{-j}\le c_1 t} \mathbf{w}_r(f, 2^{-j})_p^p. \tag{4.24}$$

Case 3 $(x \in \Omega_-)$. Let $R \in F_c$ with $R \cap \Omega_- \neq \varnothing$, then $l(R) > 16rt$ and so $[x, x+rh] \subset R^*$ whenever $x \in R$. We consider any other cube $Q \in F_c$

such that Q^* intersects $[x, x+rh]$ for some $x \in R$ and $|h| \le t$. By (4.2)(v), we have that Q and R touch. Next we let $\Lambda_R := \{Q \in F_c : Q \text{ touches } R\}$ denote the collection consisting of R and its neighbors from F_c, then all cubes $Q \in \Lambda_R$ have side length comparable to $l(R)$. The number of cubes in Λ_R does not exceed the constant N_0 of (4.2)(vi). We can use (4.2)(iv) to majorize derivatives of the ϕ_Q. Hence, from the definition of $\mathscr{E}$ and Leibniz' formula, we have for $|\mu| = r$:

$$\begin{aligned} \|D^\mu \mathscr{E} f\|_\infty(R^*) &= \|D^\mu[\mathscr{E} f - P_{R^s}]\|_\infty(R^*) \\ &\le C \max_{0\le k\le r} \sum_{Q\in\Lambda_R} l(R)^{-k} \max_{|\nu|=r-k} \|D^\nu[P_{Q^s} - P_{R^s}]\|_\infty(Q^*) \\ &\le C l(R)^{-r} \sum_{Q\in\Lambda_R} \|P_{Q^s} - P_{R^s}\|_\infty(R^s) \end{aligned} \tag{4.25}$$

where the last inequality uses Markov's inequality and (3.3). We next choose a constant $c > 0$ so large that it exceeds the constant in (4.3) and also cR^s contains each of the cubes Q^s, for $Q \in \Lambda_R$. We shall possibly increase the size of the constant c in the remainder of the proof but it will end up to be a fixed constant depending at most on d, Ω, and previous constants.

For each Q^s, such that $Q \in \Lambda_R$, there is a 'chain' T_Q connecting R^s with Q^s which can be obtained from the proof of Lemma 4.2. Namely, if the constant $C > 0$ is large enough then $\overline{R} := CR$ will contain R^s and all of the Q^s. We choose $R_0 \in F$ as in Lemma 4.2 for the cube $\overline{R}$. The chain T_Q then consists of the cubes in F which connect Q^s to R_0 and then R_0 to R^s. Each cube in the chain T_Q will have side length larger than $c^{-1}l(R)$ where c may have to be increased appropriately. Of course each cube in the chain also has side length $\le Cl(R_0) \le Cl(R)$. Because of the size condition on the cubes in T_Q, the fact that they have disjoint interiors, and $\operatorname{dist}(Q^s, R^s) \le Cl(R^s)$, the number of cubes in T_Q is no larger than a fixed constant depending only on d and Ω. Therefore, we can estimate $P_{Q^s} - P_{R^s}$ as in (4.9) of Lemma 4.2 and obtain

$$\begin{aligned} \|P_{Q^s} - P_{R^s}\|_\infty(R^s) &\le C\|P_{Q^s} - P_{R^s}\|_\infty(Q^s) \\ &\le C|R|^{-1/p} \left(\sum_{S\in T_Q} E(S^*)^p \right)^{1/p}. \end{aligned} \tag{4.26}$$

Now, from (4.25) and (4.26), we obtain for $x \in R$,

$$\begin{aligned} |\Delta_h^r(\mathscr{E} f, x)| &\le \max_{|\mu|=r} \|D^\mu \mathscr{E} f\|_\infty(R^*)\, |h|^r \\ &\le C t^r l(R)^{-r} |R|^{-1/p} \sum_{Q\in\Lambda_R} \left(\sum_{S\in T_Q} E(S^*)^p \right)^{1/p}. \end{aligned} \tag{4.27}$$

Now let $\tilde{\Lambda}_R$ denote the collection of all cubes S from F which are contained in cR^s and have side length $l(S) \ge c^{-1}l(R)$. Then, by again enlarging c if necessary, we can guarantee that any cube S appearing on the right side of (4.27) is contained in $\tilde{\Lambda}_R$. Therefore, if we take pth powers of (4.27) and

integrate over R and then sum over all R, we obtain

$$\int_{\Omega_-} |\Delta_h^r(\mathscr{E}f, x)|^p dx \le Ct^{rp} \sum_{R\cap\Omega_-\neq\varnothing} l(R)^{-rp} \sum_{S\in\tilde{\Lambda}_R} E(S^*)^P \tag{4.28}$$

where we have used the fact that the number of cubes in Λ_R is bounded independent of R.

We now proceed in a similar fashion to the way we derived (4.24). Since (as we have derived earlier) $cl(R) \le l(S) \le Cl(R)$, every cube S appearing in the sum of (4.28) satisfies $ct \le l(S) \le c_1 t$ provided c_1 is sufficiently large. We majorize $E(S^*)$ by (2.5) and (2.7). This gives (compare with the derivation of (4.21) through (4.24)):

$$\begin{aligned} \sum_{R\cap\Omega_-\neq\varnothing} l(R)^{-rp} \sum_{S\in\tilde{\Lambda}_R} E(S^*)^p &= \sum_j \sum_{\substack{R\cap\Omega_-\neq\varnothing \\ R\in\mathbb{D}_j}} 2^{jrp} \sum_{S\in\tilde{\Lambda}_R} E(S^*)^p \\ &\le C \sum_{2^{-j}\ge c_1 t} 2^{jrp} \mathbf{w}_r(f, 2^{-j})_p^p . \end{aligned} \tag{4.29}$$

We use (4.29) in (4.28) to obtain

$$\int_{\Omega_-} |\Delta_h^r(\mathscr{E}f, x)|^p dx \le Ct^{rp} \sum_{2^{-j}\ge c_1 t} \mathbf{w}_r(f, 2^{-j})_p^p . \tag{4.30}$$

The proof of the theorem is completed by adding the estimates (4.18), (4.24), and (4.30) and making the observation that $\mathbf{w}_r(f, s, \Omega) \le a^{d/p}\mathbf{w}_r(f, as, \Omega)_p$ for any $a \ge 1$ to put the resulting sum in the form (4.16). □

5. Extension theorems for (ε, δ) domains

The techniques of §4 also apply to more general domains. We shall indicate in this section the adjustments required in §4 to execute the extension theorem for (ε, δ) domains as introduced by P. Jones [6]. Such domains include as special cases the *minimally smooth* domains in the sense of Stein [S, p. 189]. The latter are equivalent to domains with the uniform cone property [Sh].

We say an open set Ω is called an (ε, δ) *domain* if:

for any $x, y \in \Omega$ satisfying $|x-y| \le \delta$, there exists a rectifiable path Γ, of length $\le C_0|x-y|$, connecting x and y, such that for each $z \in \Gamma$,

$$\operatorname{dist}(z, \partial\Omega) \ge \varepsilon \min(|z-x|, |z-y|). \tag{5.1}$$

We shall also assume that the diameter of Ω is larger than δ which, of course, will be true, if we take δ small enough.

Let F be a Whitney decomposition of Ω and F_c be a Whitney decomposition of $\Omega^c\backslash\partial\Omega$; that is (4.1)(i) and (ii) hold for the cubes $Q \in F \cup F_c$. We shall often make use of the following two properties which hold for a constant C depending only on d:

$$\begin{aligned} &\text{(i)} \quad \text{if } Q, Q' \in F \text{ do not touch, then } C\operatorname{dist}(Q, Q') \ge \operatorname{diam}(Q), \\ &\text{(ii)} \quad \text{if } Q \in F, \text{ then } C\operatorname{dist}(Q, \partial\Omega) \ge \sup_{z\in Q} d(z, \partial\Omega). \end{aligned} \tag{5.2}$$

The first of these properties follow from the fact that the neighbors of Q all have size comparable to that of Q (property (4.1)(ii)), while the second is a consequence of (4.1)(i).

For a cube $Q \in F_c$, we let Q^s be any cube from F of maximal diameter such that $\operatorname{dist}(Q^s, Q) \leq 2\operatorname{dist}(Q, \partial\Omega)$. The cube Q^s will be called the *reflection* of Q and plays the same role as the reflected cubes for the Lipschitz graph domains of §4. We note for further use that from (4.1)(i) and the definition of reflected cubes, it follows that if $Q_1, Q_2 \in F_c$, then

$$\operatorname{dist}(Q_1^s, Q_2^s) \leq C(\operatorname{dist}(Q_1, Q_2) + \max(\operatorname{diam}(Q_1), \operatorname{diam}(Q_2))) \tag{5.3}$$

with C depending only on d.

Since there are not necessarily arbitrarily large cubes in Ω, for large cubes $Q \in F_c$, the reflected cube Q^s may have small diameter compared to that of Q. On the other hand, if $\mathscr{F}_c$ denotes the collection of cubes $Q \in F_c$ whose diameters are no larger than δ, then for each Q in $\mathscr{F}_c$ its reflection will satisfy properties (4.3) for a fixed constant C depending only on ε, δ, and d. To see this, we take a point $x_0 \in \partial\Omega$ which is closest to Q from the boundary and let $x \in \Omega$ be a point close to x_0 (to be described in more detail shortly). Since $\operatorname{diam}(\Omega) \geq \delta \geq \operatorname{diam}(Q)$, there is a $y \in \Omega$ such that $\delta \geq |x - y| \geq \delta/2 \geq \operatorname{dist}(Q, \partial\Omega)/8$. Let Γ be a path connecting x to y satisfying the (ε, δ) property. Then, we can find a point $z \in \Gamma$ such that $|x - z| = \operatorname{dist}(Q, \partial\Omega)/16$ and $|y - z| \geq \operatorname{dist}(Q, \partial\Omega)/16$. Therefore, by (5.1), $\operatorname{dist}(z, \partial\Omega) \geq C\operatorname{dist}(Q, \partial\Omega)$. Now let $Q' \in F$ be the cube which contains z. Then by (4.1)(ii) and (5.2)(ii) $\operatorname{diam}(Q') \geq C\operatorname{dist}(Q, \partial\Omega) \geq C\operatorname{diam}(Q)$. If x is close enough to x_0 (e.g., $|x - x_0| < \frac{1}{2}\operatorname{dist}(Q, \partial\Omega)$ will be fine), then $\operatorname{dist}(Q', Q) \leq 2\operatorname{dist}(Q, \partial\Omega)$. Hence Q' is one of the candidates for Q^s which means that $\operatorname{diam}(Q^s) \geq \operatorname{diam}(Q') \geq C\operatorname{diam}(Q)$ from which the properties in (4.3) easily follow.

The key to generalizing the extension theorem from Lipschitz graph domains to (ε, δ) domains is to find chains which connect cubes of F. For this we shall use the following.

Lemma 5.1. *Let R_0 and Q be two cubes from F with* $\operatorname{diam}(Q) \leq \operatorname{diam}(R_0)$ *and* $\operatorname{dist}(Q, R_0) \leq \min(\delta, C_1\operatorname{diam}(R_0))$ *with C_1 a fixed constant. Then, there is a sequence of cubes $Q =: R_m, R_{m-1}, \ldots, R_0$, from F, such that each R_j touches R_{j-1}, $j = 1, \ldots, m$, and for each $j = 1, \ldots, m$, $R_j \subset cR_0$ and for each $j = 0, \ldots, m-1$, $Q \subset cR_j$ with c depending only on C_1 and Ω.*

Proof. Let $z \in Q$ and $z_0 \in R_0$ satisfy $|z - z_0| \leq \delta$ and let $\Gamma(t)$, $0 \leq t \leq 1$, be a path connecting z_0 to z guaranteed by the definition of (ε, δ) domains. We claim that any cube $S \in F$ which intersects Γ has diameter $\geq C\operatorname{diam}(Q)$. Indeed, if S touches Q or R_0, this is clear. If S does not touch Q or R_0 and $w \in \Gamma \cap S$, then, by (4.1)(ii), $|w - z_0| \geq l(R_0)/4$ and $|w - z| \geq l(Q)/4$. Hence, by (5.1), $\operatorname{dist}(w, \partial\Omega) \geq \varepsilon l(Q)/4$ and therefore our claim follows from (5.2)(ii) and (4.1)(i).

We let $S_0, S_1, S_2, \ldots$ be the cubes from F met by the path Γ as t increases; by the above remarks this sequence is finite. If two cubes are identical, $S_i = S_j$, we delete $S_{i+1}, \ldots, S_j$ from this sequence. It is clear that R_j touches R_{j-1} for each $j = 1, 2, \ldots, m$. We take points $z_j \in \Gamma \cap R_j$, $j = 0, \ldots, m$. Since the path Γ has length $\leq C|z_0 - z| \leq C\operatorname{diam}(R_0)$, all points z_j satisfy

$\operatorname{dist}(z_j, \partial\Omega) \le C \operatorname{diam}(R_0)$. Therefore, properties (4.1)(i) and (5.2)(ii) give that $\operatorname{diam}(R_j) \le C \operatorname{diam}(R_0)$. Hence $R_j \subset cR_0$ for some constant depending only on C_1 and Ω. We also claim that $Q \subset cR_j$. This is clear if R_j touches Q or R_0 (see (4.1)(ii)). On the other hand, if R_j does not touch Q or R_0, then by (5.1) and (4.1)(ii), we have

$$\operatorname{dist}(z_j, \partial\Omega) \ge \varepsilon \min(|z - z_j|, |z_j - z_0|) \ge C\, l(Q).$$

Hence, by (5.1)(ii) and (4.1)(i), $\operatorname{diam}(R_j) \ge C \operatorname{diam}(Q)$ and our claim follows in this case as well. □

We shall now define our extension operator for the (ε, δ) domain Ω. Let ϕ_Q, $Q \in F_c$, be a partition of unity for Ω^c which satisfies (4.2). Recall that $\mathscr{F}_c$ is the collection of all cubes $Q \in F_c$ for which $\operatorname{diam}(Q) \le \delta$. If $\gamma > 0$ and r is a positive integer, we define

$$\mathscr{E} f := f\chi_\Omega + \sum_{Q \in \mathscr{F}_c} P_{Q^s} \phi_Q \tag{5.4}$$

where as before P_{Q^s} denotes a near best approximation to f in the metric $L_\gamma(Q^s)$. We let $\Omega_1 := \{x \in \mathbb{R}^d\colon \operatorname{dist}(x, \Omega) \le \delta/4\}$ and $\Omega_2 := \{x \in \mathbb{R}^d\colon \operatorname{dist}(x, \Omega) \le 6\delta\}$. Then, $\mathscr{E} f(x) = 0$, for $x \in \Omega_2^c$, while on Ω_1, we have $\sum_{Q \in \mathscr{F}_c} \phi_Q(x) = 1$. For example, to prove the first of these statements, let $Q \in \mathscr{F}_c$. Then $\operatorname{supp}(\phi_Q) \subset Q^*$. Since any point $x \in Q^*$ satisfies

$$\operatorname{dist}(x, \partial\Omega) \le \tfrac{9}{8} \operatorname{diam}(Q) + \operatorname{dist}(Q, \Omega) \le \tfrac{41}{8} \operatorname{diam}(Q),$$

our claim follows. A similar argument proves the second statement.

The proof of the smoothness preserving property of the extension operator $\mathscr{E}$ is now very similar to the proof in §4. We first consider the analogue of Lemma 4.2.

Lemma 5.2. *Let Ω be an (ε, δ) domain, $\gamma > 0$, r be a positive integer and $\mathscr{E}$ be any extension operator defined by* (5.4). *Let R be a cube with* $\operatorname{dist}(R, \partial\Omega) \le \operatorname{diam}(R) \le a\delta$ *where a is a fixed sufficiently small constant depending only on ε, δ, and d. Then for $f \in L_p(\Omega)$, $\gamma \le p \le 1$, we have*

$$E_r(\mathscr{E} f, R)_p^p \le C \sum_{\substack{S \in F \\ S \subset cR}} E(S^*)^p \tag{5.5}$$

where c, C depend only on d, r, γ, λ, ε, and δ.

Proof. Let

$$\mathscr{Q} := \{Q\colon Q \in F \text{ and } Q \cap R \ne \varnothing\} \cup \{Q^s\colon Q \in \mathscr{F}_c \text{ and } Q \cap R \ne \varnothing\}.$$

If a is small enough then the properties (4.1) and (5.3) give that $\operatorname{dist}(x_0, x_1) \le \sqrt{a}\delta$ for the centers x_0, x_1 of Q_0, Q_1 respectively with these cubes chosen arbitrarily from $\mathscr{Q}$. We want to find a cube R_0 to be used in conjunction with Lemma 5.1. Let Q_0 be the largest cube in $\mathscr{Q}$. If all other cubes in $\mathscr{Q}$ touch Q_0, we can take $R_0 := Q_0$. Otherwise, we pick a cube $Q_1 \in \mathscr{Q}$ such that the centers x_0, x_1 of Q_0, Q_1 respectively have the largest distance, say $|x_0 - x_1| = \eta$. If Γ is a path that connects the centers x_0, x_1 of these two cubes and satisfies the (ε, δ) condition, then there is a point $z \in \Gamma$ such that

$|z - x_i| \geq \eta/2$, $i = 0, 1$. If S is the cube in F which contains z, then we can take R_0 as the largest of the cubes S, Q_0.

We next check that R_0 satisfies the conditions of Lemma 5.1 relative to any $Q \in \mathscr{Q}$. It is clear that $\operatorname{diam}(Q) \leq \operatorname{diam}(Q_0) \leq \operatorname{diam}(R_0)$ for all $Q \in \mathscr{Q}$. Since $\eta := |x_0 - x_1| \leq \sqrt{a}\delta$ and the length of Γ is $\leq C\eta$, we have

$$\operatorname{dist}(Q, R_0) \leq \operatorname{dist}(Q, Q_0) + \operatorname{diam}(Q_0) + \operatorname{dist}(Q_0, R_0) \leq \eta + 2C\eta \leq \delta \tag{5.6}$$

provided a is sufficiently small. Also, by (4.1)(i) and (5.1)

$$\operatorname{diam}(R_0) \geq \operatorname{diam}(R) \geq \operatorname{dist}(R, \partial\Omega)/4 \geq \varepsilon\eta/8.$$

Hence, as in (5.6) $\operatorname{dist}(Q, R_0) \leq (C+1)\eta \leq C_1 \operatorname{diam}(R_0)$ with C_1 a fixed constant.

We have verified the hypothesis of Lemma 5.1. Therefore, there is a chain of cubes R_j, $j = 0, \ldots, m$, connecting R_0 to Q. By our assumptions, $Q \subset \Omega_1$ whenever $Q \in \mathscr{F}_c$ and $Q \cap R \neq \varnothing$ (provided a is sufficiently small). Hence $\sum_{Q \in \mathscr{F}_c} \phi_Q \equiv 1$ on R. We can therefore apply exactly the same proof as for Lemma 4.2 (namely from (4.9) on) to derive (5.5). □

Theorem 5.3. *Let Ω be an (ε, δ) domain and let $\gamma > 0$ and r be a positive integer. If $\mathscr{E}$ is any extension operator defined by* (5.4), *then for each $1 \geq p \geq \gamma$ and $f \in L_p(\Omega)$, we have for $0 < t \leq 1$,*

$$\omega_r(\mathscr{E} f, t)_p^p \leq C^p \left[\sum_{2^j \leq c_1 t} \mathbf{w}_r(f, 2^j)_p^p + t^{rp} \left(\|f\|_p^p(\Omega) + \sum_{1 \geq 2^j \geq t} 2^{-jrp} \mathbf{w}_r(f, 2^j)_p^p \right) \right] \tag{5.7}$$

with the constants C and c_1 depending only on d, r, λ, γ, ε, and δ.

Proof. The proof of (5.7) is very similar to that of (4.16) and we shall only highlight the differences. We first observe that (5.7) automatically holds if $t \geq a\delta$ and a is a fixed constant because $\|\mathscr{E} f\|_p \leq C\|f\|_p$. Therefore, we need only consider $t \leq a\delta$ with a a sufficiently small but fixed constant to be prescribed in more detail as we proceed. As in the proof of Theorem 4.3, we write $\mathbb{R}^d \backslash \partial\Omega = \Omega_0 \cup \Omega_- \cup \Omega_+$, where $\Omega_0 := \bigcup\{Q \in F \cup F_c : l(Q) \leq 16rt\}$, $\Omega_+ := \Omega \backslash (\Omega_0 \cup \partial\Omega)$, $\Omega_- := \Omega^c \backslash (\Omega_0 \cup \partial\Omega)$. We estimate $\int_S |\Delta_h^r(\mathscr{E} f)|^p dx$ for the three sets $S = \Omega_\pm$, Ω_0 and for $|h| \leq t$.

We proceed as in the proof of Theorem 4.3 and consider three cases. Case 1 which estimates the integral over Ω_+ is identical to the proof in Theorem 4.3 and yields the estimate (4.18). Case 2 is also the same since if a is small enough the cube R which contains $[x, x + rh]$ will be one of the cubes to which we can apply Lemma 5.2. We obtain in this way the estimate (4.24) for the integral over Ω_0.

In Case 3, that is $x \in \Omega_-$, we let $R \in F_c$ have nontrivial intersection with Ω_-. If $x \in R$, then $[x, x + rh] \subset R^*$. We have two possibilities for R. If $\operatorname{dist}(R, \partial\Omega) \leq a\delta$ and a is small enough, then $\sum_{Q \in \mathscr{F}} \phi_Q \equiv 1$ on R^*. We consider $\mathscr{Q} := \{Q^s : Q \in F_c, Q \text{ touches } R\}$. We can take R_0 as the largest cube in $\mathscr{Q}$. Then R_0 and any other cube Q^s in $\mathscr{Q}$ will satisfy the hypothesis of Lemma 5.1. We take a chain (R_j) connecting Q^s and R_0 and proceed as in Theorem 4.3 to obtain

$$\sum_R \|\Delta_h^r(\mathscr{E} f)\|_p^p(R) \leq C t^{rp} \sum_{1 \geq 2^{-j} \geq t} 2^{jrp} \mathbf{w}_r(f, 2^{-j})_p^p, \tag{5.8}$$

where the sum is taken over all cubes R of this type.

The second possibility is that $\operatorname{dist}(R, \partial\Omega) \geq a\delta$. Whenever $Q \in \mathscr{F}_c$ is such that ϕ_Q does not identically vanish on R, then $6\delta \geq l(Q)$, $Cl(Q) \geq \delta$ and therefore from (4.2), $\|D^\nu \phi_Q\|_\infty \leq C$, $|\nu| \leq r$, with C a constant depending only on δ and r. Also $\|P_{Q^s}\|_p(Q^s) \leq C\|f\|_p(Q^s)$ by the definition of P_{Q^s} as a near best approximation. From this and by Markov's inequality for polynomials, we obtain $\|D^\nu(P_{Q^s})\|_\infty(Q^s) \leq C\|f\|_p(Q^s)$, $|\nu| \leq r$. Therefore, Leibniz' rule for differentiation gives that

$$\|D^\nu(\mathscr{E}f)\|_\infty(R) \leq C\|f\|_p(R')$$

where R' is the union of all the cubes Q^s such that ϕ_Q does not vanish on R. Here we are using the fact that the number of cubes which appear nontrivially in $\mathscr{E}f(x)$ does not exceed a constant which depends only on d. This gives

$$\|\Delta_h^r(\mathscr{E}f)\|_p(R) \leq |h|^r \max_{|\nu|=r} \|D^\nu(\mathscr{E}f)\|_\infty(R) \leq C|h|^r\|f\|_p(R'). \tag{5.9}$$

Since a point $x \in \Omega$ can appear in at most C of the sets R' with C depending only on d, we can raise the inequality (5.9) to the power p and sum over all R of this type and obtain

$$\sum_R \|\Delta_h^r(\mathscr{E}f)\|_p^p(R) \leq C|h|^{rp}\|f\|_p^p(\Omega) \leq Ct^{rp}\|f\|_p^p(\Omega). \tag{5.10}$$

We add (5.8) and (5.10) to obtain that $\int_{\Omega-} |\Delta_h^r(\mathscr{E}f)|^p dx$ does not exceed the sum of the right sides of (5.8) and (5.10). The proof is then completed by adding the estimates in the three cases. □

6. Applications of the extension theorem

In this section, we establish the boundedness of the extension operator $\mathscr{E}$ on Besov spaces and apply this to obtain other characterizations of these spaces. Given $0 < \alpha < \infty$ and $0 < q \leq \infty$ and a sequence $\{a_k\}_{k\in\mathbb{N}}$ of real numbers, we define

$$\|(a_k)\|_{l_q^\alpha} := \left(\sum_{k\in\mathbb{N}} [2^{k\alpha}|a_k|]^q\right)^{1/q}, \tag{6.1}$$

with the usual adjustment when $q = \infty$. We shall need the following discrete Hardy inequalities (for a proof see [2]). If for sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ of real numbers, we have either

$$\begin{aligned} &\text{(i) } |b_k| \leq c2^{-kr}\left(\sum_{j=0}^{k} [2^{jr}|a_j|]^\mu\right)^{1/\mu} \quad \text{or} \\ &\text{(ii) } |b_k| \leq \left(\sum_{j=k}^{\infty} |a_j|^\mu\right)^{1/\mu}, \end{aligned} \tag{6.2}$$

then for all $q \geq \mu$ and $0 < \alpha < r$, in case (i), and all $q \geq \mu$ and $0 < \alpha \leq \infty$, in case (ii), we have

$$\|(b_k)\|_{l_q^\alpha} \leq C\|(a_k)\|_{l_q^\alpha}. \tag{6.3}$$

Therefore, (6.3) holds for $q \geq \mu$ and $0 < \alpha < r$, if $|b_k|$ does not exceed the sum of the right sides of (6.2).

Theorem 6.1. *If Ω is an (ε, δ) domain, $\gamma > 0$, and r is positive integer, then the extension operator $\mathscr{E}$ of (5.4) is a bounded mapping from $B_q^\alpha(L_p(\Omega))$ into $B_q^\alpha(L_p(\mathbb{R}^d))$ for all $\gamma \le p \le 1$, $0 < q \le \infty$, and $\alpha < r$:*

$$\|\mathscr{E} f\|_{B_q^\alpha(L_p(\mathbb{R}^d))} \le C\|f\|_{B_q^\alpha(L_p(\Omega))} \tag{6.4}$$

with the constant C depending only on $d, r, \lambda, \gamma, \varepsilon$, and δ.

Proof. Let $\mu < \min(q, p)$. Since an l_p norm is less than an l_μ norm and since $\mathbf{w}_r \le \omega_r$, from (5.7) for $t = 2^{-k}$, we have

$$\begin{aligned} \omega_r(\mathscr{E} f, 2^{-k}, \mathbb{R}^d)_p &\le C\left[\sum_{j=ck}^{\infty} \omega_r(f, 2^{-j}, \Omega)_p^\mu\right]^{1/\mu} \\ &\quad + C2^{-kr}\left[\|f\|_p^\mu(\Omega) + \sum_{j=0}^{k}[2^{jr}\omega_r(f, 2^{-j}, \Omega)_p]^\mu\right]^{1/\mu}. \end{aligned} \tag{6.5}$$

We can therefore apply (6.3) and obtain

$$\|(\omega_r(\mathscr{E} f, 2^{-k}, \mathbb{R}^d)_p)\|_{l_q^\alpha} \le C[\|f\|_p(\Omega) + \|(\omega_r(f, 2^{-k}, \Omega)_p)\|_{l_q^\alpha}]. \tag{6.6}$$

The monotonicity of ω_r shows that the left side of (6.6) is equivalent to $|f|_{B_q^\alpha(L_p(\mathbb{R}^d))}$ while the right side is equivalent $\|f\|_{B_q^\alpha}(L_p(\Omega))$. Since $\mathscr{E}$ is a bounded map from $L_p(\Omega)$ into $L_p(\mathbb{R}^d)$, (6.6) establishes the theorem. □

If follows from Theorem 6.1 that for each $0 < p \le 1$, $0 < q \le \infty$, $\alpha > 0$ and any (ε, δ) domain Ω, we have

$$\|f\|_{B_q^\alpha(L_p(\Omega))} \le \|\mathscr{E} f\|_{B_q^\alpha(L_p(\mathbb{R}^d))} \le C\|f\|_{B_q^\alpha(L_p(\Omega))} \tag{6.7}$$

with constant C depending only on d, r, γ, λ, and Ω.

We next show that functions in $B_q^\alpha(L_p(\Omega))$ have atomic or wavelet decompositions. Let $N = N_r$ be the tensor product B spline in $\mathbb{R}^d$ obtained from the univariate B spline of degree $r-1$ which has knots at $0, 1, \dots, r$. Let $\mathbb{D}_k$ denote the collection of all dyadic cubes for $\mathbb{R}^d$ which have side length 2^{-k} and $\mathbb{D}_+ := \bigcup_{k\ge 0}\mathbb{D}_k$. With N, we can associate to any dyadic cube $I := [j2^{-k}, (j+1)2^{-k}] \in \mathbb{D}_k$, $j \in \mathbb{Z}^d$, $k \in \mathbb{N}$, the dilated functions $N_I(x) := N(2^k x - j)$. This function has support on an expansion of the cube I.

Theorem 6.2. *Let Ω be an (ε, δ) domain and $0 < p \le 1$, $0 < q \le \infty$, $\alpha > 0$. Then each function $f \in B_q^\alpha(L_p(\Omega))$ has a decomposition*

$$f(x) = \sum_{I \in \mathbb{D}_+} a_I(f) N_I(x), \qquad x \in \Omega, \tag{6.8}$$

where the coefficients $a_I(f)$ satisfy

$$\|f\|_{B_q^\alpha(L_p(\Omega))} \approx \left(\sum_{k=0}^{\infty} 2^{k\alpha q}\left(\sum_{I\in\mathbb{D}_k} |a_I(f)|^p |I|\right)^{q/p}\right)^{1/q} \tag{6.9}$$

with constants of equivalency independent of f and the usual change on the right side of (6.9) *when* $q = \infty$.

Proof. By (6.7), $f \in B_q^\alpha(\Omega)$ if and only if $\mathscr{E} f \in B_q^\alpha(\mathbb{R}^d)$ with equivalent norms. It was shown in [D-P] that $\mathscr{E} f$ has a decomposition (6.8) on $\mathbb{R}^d$ with coefficients $a_I(\mathscr{E} f)$ satisfying (6.9). Since $\mathscr{E} f = f$ on Ω, the theorem follows. □

We next discuss the interpolation of Besov spaces using the real method of Peetre. If X_0 and X_1 are a pair of quasi-normed spaces which are continuously embedded in a linear Hausdorff space $\mathscr{X}$, their K-functional is defined for any $f \in X_0 + X_1$ by

$$K(f, t) := K(f, t; X_0, X_1) := \inf_{f=f_0+f_1} \|f_0\|_{X_0} + t\|f_1\|_{X_1}. \tag{6.10}$$

For each $0 < \theta < 1$, $0 < q \leq \infty$, the space $X_{\theta,q} := (X_0, X_1)_{\theta,q}$ is the collection of all functions $f \in X_0 + X_1$ for which

$$\|f\|_{X_{\theta,q}} := \left(\int_0^\infty (t^{-\theta} K(f, t))^q \frac{dt}{t}\right)^{1/q} \tag{6.11}$$

is finite (with again the usual adjustment on the right side of (6.11) when $q = \infty$). This is an interpolation space since it follows easily from the definition of the K-functional that each linear operator which is bounded on X_0 and X_1 is also bounded on $X_{\theta,q}$.

We are interested in interpolation for a pair of Besov spaces. Suppose that $0 < p_0, p_1 \leq 1$, and $0 < q_0, q_1 \leq \infty$ and $\alpha_0, \alpha_1 \geq 0$. We let $X_i(\Omega) := B_{q_i}^{\alpha_i}(L_{p_i}(\Omega))$, $i = 0, 1$, with the understanding that this space is $L_{p_i}(\Omega)$ when $\alpha_i = 0$. If we choose $r > \max(\alpha_0, \alpha_1)$ and $\gamma \leq \min(p_0, p_1)$ then the extension operators $\mathscr{E}$ of (5.4) are defined and (6.7) holds for each of these extensions. In fact, we observe that

$$\begin{aligned} K(f, t; X_0(\Omega), X_1(\Omega)) &\leq K(\mathscr{E} f, t; X_0(\mathbb{R}^d), X_1(\mathbb{R}^d)) \\ &\leq C K(f, t; X_0(\Omega), X_1(\Omega)). \end{aligned} \tag{6.12}$$

The left inequality in (6.12) is clear. The usual proof of the right inequality relies on the linearity of the operator, which as we have previously mentioned may fail for $\mathscr{E}$ since near best approximations $P_Q(f)$ are used in its definition (5.4). However, given any decomposition $f = f_0 + f_1$, we may decompose $\mathscr{E} f$ as $F_0 + F_1$ where F_i is a norm bounded extension (in X_i) of f_i, $(i = 0, 1)$. To see this, we recall Lemma 6.2 of [2] which established that if $f = f_0 + f_1$ and $P_Q(f)$ is any near best approximation to f, then there exist near best approximations R_Q^i to f_i $(i = 0, 1)$ so that $P_Q(f) = R_Q^0 + R_Q^1$. We then use $R_{Q^s}^i$ in place of P_{Q^s} in (5.4) to define F_i from which we may conclude that (6.12) holds. From (6.12) it follows, therefore, that the interpolation spaces $(X_0(\Omega), X_1(\Omega))_{\theta,q}$ and $(X_0(\mathbb{R}^d), X_1(\mathbb{R}^d))_{\theta,q}$ are identical with equivalent norms. From known results for the latter spaces (see [D-P]) we obtain the following.

Theorem 6.3. *Let Ω be an (ε, δ) domain. If $0 < p \leq 1$ and $\alpha, q_0 > 0$, then for any $0 < \theta < 1$, $0 < q \leq \infty$, we have*

$$(L_p(\Omega), B_{q_0}^\alpha(L_p))_{\theta,q} = B_q^{\theta\alpha}(L_p) \tag{6.13}$$

with equivalent norms. If $0 < p \leq 1$, *we let* $\tau(\beta) := (\beta/d + 1/p)^{-1}$, $\beta > 0$, *then for any* $\alpha > 0$ *and* $0 < \theta < 1$, $0 < q \leq \infty$, *we have*

$$(L_p(\Omega), B^{\alpha}_{\tau(\alpha)}(L_{\tau(\alpha)}(\Omega)))_{\theta, \tau(\theta\alpha)} = B^{\theta\alpha}_{\tau(\theta\alpha)}(L_{\tau(\theta\alpha)}(\Omega)) \tag{6.14}$$

with equivalent norms.

Remark 6.4. The proof in [2] of interpolation of Besov spaces relies on establishing the equivalence of the K-functional of f with that of its retract. We take this opportunity to correct the proof of the lower inequality of that equivalence. The sentences in lines 3 through 7 on page 411 of [2] should be replaced by:

"We may estimate each term of the last sum as

$$\|t_j - g_j\|_{p_0} \leq c(\|t_j - a_j\|_{p_0} + \|a_j - T_j(a_j)\|_{p_0}),$$

and apply Corollary 4.7 to obtain

$$\|a_j - T_j(a_j)\|_{p_0} \leq c s_j(a_j)_{p_0} \leq c\|t_j - a_j\|_{p_0}.$$

Hence,

$$\|t_j - g_j\|_{p_o} \leq c\|t_j - a_j\|_{p_0}."$$

While preparing the present paper, Ridgway Scott posed to us a question concerning interpolation of Besov spaces for $1 \leq p < \infty$. It is rather easy to settle this question given the machinery developed in §4 of the present paper. We shall from here on assume that Ω is a *minimally smooth* domain in the sense of Stein (it may be that Theorem 6.6 that follows also holds for (ε, δ) domains, however our proof does not seem to apply in this generality). A minimally smooth domain in $\mathbb{R}^d$ is an open set for which there is a number $\eta > 0$ and open sets U_i, $i = 1, 2, \ldots$, such that: (i) for each $x \in \partial\Omega$, the ball $B(x, \eta)$ is contained in one of the U_i; (ii) a point $x \in \mathbb{R}^d$ is in at most N of the sets U_i where N is an absolute constant; and (iii) for each i, $U_i \cap \Omega = U_i \cap \Omega_i$ for some domain Ω_i which is the rotation of a Lipschitz graph domain with Lipschitz constant M independent of i (see §4).

We recall the *fractional order Sobolev spaces.* Let $1 \leq p < \infty$ and $\alpha > 0$. If α is not an integer, we write $\alpha = \beta + r$ where $0 < \beta < 1$ and r is a nonnegative integer. Let W^{α}_p be the collection of all functions f in the Sobolev space $W^r_p(\Omega)$, for which

$$|f|^p_{W^{\alpha}_p(\Omega)} := \sum_{|\nu|=r} \int_{\Omega\times\Omega} \frac{|D^\nu f(x) - D^\nu f(y)|^P}{|x-y|^{\beta p + d}} dx dy \tag{6.15}$$

is finite.

If $\Omega = \mathbb{R}^d$ and α is not an integer, then it is well known that (6.15) is equivalent to $|f|^p_{B^{\alpha}_p(L_p)}$. We want to show this remains true for minimally smooth domains Ω. For this purpose, we define for $f \in W^r_p(\Omega)$,

$$\overline{w}_{r+1}(f, t)^p := t^{rp} \sum_{|\nu|=r} \mathbf{w}_1(D^\nu f, t)^p_p \tag{6.16}$$

with $\mathbf{w}_1$, as before, the averaged modulus of smoothness (2.5).

Lemma 6.5. *Let Ω be any open set. For $1 \le p < \infty$ and $\alpha > 0$ not an integer, we have*

$$|f|^p_{W^\alpha_p(\Omega)} = (\beta p + d)^{-1} \int_0^\infty [t^{-\alpha} \overline{w}_{r+1}(f, t)_p]^p \frac{dt}{t} \tag{6.17}$$

where $\alpha = \beta + r$ as above.

Proof. For any $g \in L_p(\Omega)$, we have for $0 < \beta < 1$, by a change of variables and Fubini's theorem,

$$\begin{aligned} &\int_0^\infty [t^{-\beta p} \mathbf{w}_1(g, t)_p]^p \frac{dt}{t} \\ &= \int_0^\infty \int_{|s| \le t} \int_\Omega |\Delta_s(g, x, \Omega)|^p t^{-\beta p - d - 1} dx\, ds\, dt \\ &= \int_\Omega \int_\Omega \int_{|x-y|}^\infty t^{-\beta p - d - 1} dt\, |g(x) - g(y)|^p dx\, dy \\ &= (\beta d + p)^{-1} \int_\Omega \int_\Omega |x - y|^{-\beta p - d} |g(x) - g(y)|^p dx\, dy. \end{aligned} \tag{6.18}$$

We take $g = D^\nu f$, $|\nu| = r$, and add the identities (6.18) to obtain (6.17). □

We shall next show that an analogue of inequality (5.7) holds for $p \ge 1$. It is well known that if $f \in W^{r-1}_p$ then for the error $E(S)_p$ for approximating f in the norm $L_p(S)$ on a cube S by polynomials of degree $< r$, we have

$$\begin{aligned} E(S)^p &\le C\, l(S)^{p(r-1)} \sum_{|\nu| = r-1} \omega_1(D^\nu f, l(S), S)^p_p \\ &\le C\, l(S)^{p(r-1)} \sum_{|\nu| = r-1} \mathbf{w}_1(D^\nu f, l(S), S)^p_p = C\, \overline{w}_r(f, l(S), S)^p_p \end{aligned} \tag{6.19}$$

where as before $\mathbf{w}$ is the averaged modulus of smoothness given by (2.5) and $\overline{w}_r$ is defined by (6.16).

Theorem 6.6. *Let Ω be a minimally smooth domain, let r be a positive integer and let $1 \le p < \infty$. Then for any $f \in W^{r-1}_p(\Omega)$ and $0 < t < 1$, we have*

$$\begin{aligned} \omega_r(\mathscr{E} f, t)^p_p \le C^p \Bigg[&\sum_{2^j \le t} \overline{w}_r(f, 2^j)^p_p \\ &+ t^{rp} \left(\|f\|^p_p(\Omega) + \sum_{2^j \ge t} 2^{-jrp} \overline{w}_r(f, 2^j)^p_p \right) \Bigg], \end{aligned} \tag{6.20}$$

with C a constant depending only on d, r, λ and Ω.

Proof. We first recall that a minimally smooth domain is an (ε, δ) domain. Since Ω will be an (ε, δ) domain for any ε and δ sufficiently small, we can assume that η in the definition of minimally smooth domains is $\ge C_0 \delta$ with C_0 arbitrary but fixed. We shall prescribe η in more detail as we continue through the proof.

We proceed as in Theorems 4.3 and 5.3. The first case, namely the estimate of $\int_{\Omega_+} |\Delta^r_h(\mathscr{E} f, x)|^p dx$ is as before, but we use standard estimates of rth differences in terms of a first order difference of $(r-1)$th derivatives. This gives that the integral does not exceed $\overline{w}_r(f, t, \Omega)^p_p$.

For the estimate in the second case, that is over Ω_0, we need first to derive an analogue of Lemma 5.2 for $\overline{w}_r$. With the same constructions and notation as in Lemma 5.2 and the same argument, we arrive at the estimate (4.10), where now $1 \le p < \infty$. We need to observe that for each k, at most C of the cubes R_j appearing in (4.10) belong to $\mathbb{D}_k$. To see this, we recall that these cubes meet the path Γ which connects a point $z \in Q$ to a point $z_0 \in R_0$. From (4.1)(i), letting S be such an R_j, any point $w \in S \cap \Gamma$ satisfies

$$\operatorname{dist}(w, \partial\Omega) \le \operatorname{diam}(S) + \operatorname{dist}(S, \partial\Omega)) \le 5 \operatorname{diam}(S) = 5\sqrt{d}2^{-k}.$$

Therefore, by the definition of (ε, δ) domain (property (5.1)), we have

$$\min(|w - z|, |w - z_0|) \le \varepsilon^{-1} \operatorname{dist}(w, \partial\Omega) \le 5\sqrt{d}\varepsilon^{-1}2^{-k}.$$

That is, each of these cubes S meets one of the balls of radius $5\sqrt{d}\varepsilon^{-1}2^{-k}$ about z and z_0. Since the cubes S are disjoint there are at most C of them with C depending only on ε and d.

We now write $|R_j|^{-1/p} = |R_j|^{-a/p}|R_j|^{-b/p}$ where $a + b = 1$ and $ad > d - 1$. We then apply Hölder's inequality to (4.10) and use the observation above for $l(R_j) = 2^k l(Q)$ to conclude that

$$\begin{aligned} \|f - P_{R_0}\|_p^p(Q) &\le |Q| \left(\sum_{j=0}^m |R_j|^{-bp'/p}\right)^{p/p'} \left(\sum_{j=0}^m |R_j|^{-a} E(R_j^*)^p\right) \\ &\le C|Q|^{1-b} \left(\sum_{j=0}^m |R_j|^{-a} E(R_j^*)^p\right) = C|Q|^a \left(\sum_{j=0}^m |R_j|^{-a} E(R_j^*)^p\right). \end{aligned} \tag{6.21}$$

We now sum over all $Q \in F$ such that $Q \cap R \ne \varnothing$ in (5.12), reverse the order of summation to obtain that (5.5) is valid for this range of p provided that we can show that for fixed $S = R_j$, we have

$$\sum_{\substack{Q \in F \\ Q \subset cS}} |Q|^a \le C|S|^a \tag{6.22}$$

with $c \ge 1$ a fixed constant and C depending only on d, ε, δ and η.

We postpone for a moment the proof of (6.22) and conclude the proof of the theorem. Now that we have established (5.5) of Lemma 5.2 for $1 \le p < \infty$, the estimate of $\int_{\Omega_0} |\Delta_h^r(f, x)|^p dx$ can be made exactly as in the proof of Theorem 4.3 with (6.19) used in place of (2.7) and $\overline{\omega}_r$ used in place of $\mathbf{w}_r$. Finally, the proof in Case 3, that is the estimate of $\int_{\Omega_-} |\Delta_h^r(f, x)|^p dx$, can be made exactly as in the proof of Theorem 4.3 because the number of cubes in the sums appearing in (4.26), (4.27), and (4.28) is bounded by a constant C depending only on d, ε, and δ. This then completes the proof of the theorem subject to the verification of (6.22).

To prove (6.22), we count the number N_k of cubes $Q \in F$ with $Q \subset cS$ and $l(Q) = 2^{-k}l(S)$. There are only a finite number of values of $k \le 0$ and for each of these $N_k \le C$ with C depending only on d (because the cubes Q are pairwise disjoint). Therefore, this portion of the sum appearing in (6.22) does not exceed the right side of (6.22).

To estimate N_k for $k \geq 1$, we recall that the cubes S have side length $\leq l(R_0) \leq Cl(R) \leq C\delta$. Therefore, by choosing δ sufficiently small, we can assume that $2c\,\mathrm{diam}(S) \leq \eta$ with c the constant in the summation index of (6.22) and η of course the constant in the definition of minimally smooth domains. Therefore, by property (ii) of minimally smooth domains, we may assume that $(4cdS) \cap \Omega = (4cdS) \cap \Omega_j$ for one of the domains Ω_j. Since $c \geq 1$ and $\mathrm{dist}(Q, \partial\Omega) \leq 4\,\mathrm{diam}(Q) \leq 2\,\mathrm{diam}(S)$, we have $\mathrm{dist}(Q, \partial\Omega) = \mathrm{dist}(Q, \partial\Omega_j)$. From property (4.1)(i) of Whiney cubes, we have $Q \subset A_k := \{x\colon \mathrm{dist}(x, \partial\Omega_j) \leq 5\ 2^{-k}\,\mathrm{diam}(S)\} \cap cS$. Now from the fact that Ω_j is a Lipschitz graph domain, we have that $|A_k| \leq C2^{-k}|S|$ with the constant C depending only on d and the Lipschitz constant M. Hence A_k can contain at most $C2^{k(d-1)}$ cubes Q of side length $2^{-k}l(S)$. This shows that $N_k \leq C2^{k(d-1)}$. Using this estimate for N_k, we find that the portion of the sum on the left side of (6.22) that remains to be estimated does not exceed

$$\sum_{k=1}^{\infty} N_k(2^{-k}l(S))^{da} \leq C\sum_{k=1}^{\infty} 2^{k(d-1)}2^{-kda}|S|^a \leq C|S|^a$$

because $ad > d-1$. □

Using Theorem 6.6 we are able to easily establish the equivalent of the fractional Sobolev spaces $W_p^\alpha(\Omega)$with the special family of Besov spaces $B_p^\alpha(L_p(\Omega))$.

Theorem 6.7. *Let Ω be a minimally smooth domain in $\mathbb{R}^d$, and $1 \leq p < \infty$, $0 < \alpha$, then $W_p^\alpha(\Omega) = B_p^\alpha(L_p(\Omega))$ and there exist positive constants c_1, c_2 independent of f so that*

$$c_1\|f\|_{W_p^\alpha(\Omega)} \leq \|f\|_{B_p^\alpha(L_p(\Omega))} \leq c_2\|f\|_{W_p^\alpha(\Omega)}. \tag{6.23}$$

Proof. The upper inequality in (6.23) is obtained by applying the l_p^α norm to both sides of inequality (6.20) and using Hardy's inequality (6.3) together with Lemma 6.5. The lower inequality is confirmed by recalling that the corresponding result holds on $\mathbb{R}^d$, and then following with an application of Theorem 6.1:

$$\|f\|_{W_p^\alpha(\Omega)} \leq \|\mathscr{E}f\|_{W_p^\alpha(\mathbb{R}^d)} \leq c\|\mathscr{E}f\|_{B_p^\alpha(L_p(\mathbb{R}^d))} \leq c\|f\|_{B_p^\alpha(L_p(\Omega))}. \quad \square$$

As we previously mentioned, when $1 \leq p$ the extension operators may be taken to be linear. It then follows that $\|\mathscr{E}f\|_{B_p^\alpha(L_p(\mathbb{R}^d))}$ is equivalent (within constants independent of f) to $\|f\|_{B_q^\alpha(L_p(\Omega))}$. Applying the interpolation theorem Corollary 6.3 of [2] to $B_q^\alpha(L_p(\mathbb{R}^d))$, we obtain the following interpolation result for the fractional order Sobolev spaces $W_p^\alpha(\Omega)$:

Corollary 6.8. *Let Ω be a minimally smooth domain in $\mathbb{R}^d$, and $1 \leq p_0, p_1 < \infty$, $0 < \alpha_0, \alpha_1$, then for p satisfying $1/p = (1-\theta)/p_0 + \theta/p_1$ and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, we have*

$$(W_{p_0}^{\alpha_0}(\Omega), W_{p_1}^{\alpha_1}(\Omega))_{\theta,p} = W_p^\alpha(\Omega) \tag{6.24}$$

with equivalent norms.

Remark 6.9. While preparing this paper, we were informed by O. V. Besov that Ju. A. Brudnyi and P. A. Shvartzman have also considered extension theorems for Besov spaces on domains (including the case $0 < p < 1$). We have not been able yet to obtain a publication of those results to compare to ours.

REFERENCES

1. M. Christ, *The extension problem for certain function spaces involving fractional orders of differentiability*, Ark. Mat. **22** (1984), 63–81.
2. R. DeVore and V. Popov, *Interpolation of Besov spaces*, Trans. Amer. Math. Soc. **305** (1988), 397–414.
3. ——, *Free multivariate splines*, Constr. Approx. **3** (1987), 239–248.
4. R. DeVore and R. Sharpley, *Maximal functions measuring smoothness*, Mem. Amer. Math. Soc., No. 293, 1983.
5. H. Johnen and K. Scherer, *On the equivalence of the K-functional and moduli of continuity and some applications*, Lecture Notes in Math., vol. 571, Springer-Verlag, Berlin, 1976, 119–140.
6. P. W. Jones, *Quasiconformal mappings and extendability of functions in Sobolev spaces*, Acta Math. **147** (1981), 71–88.
7. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970.
8. E. A. Storozhenko and P. Oswald, *Jackson's theorem in the spaces $L_p(\mathbb{R}^k)$, $0 < p < 1$*, Siberian Math. J. **19** (1978), 630–639.

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