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GEOMETRY OF WEIGHT DIAGRAMS FOR $U(n)$

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ABSTRACT. We study the geometry of the weight diagrams for irreducible representations of $U(n)$. Multiplicity-one weights are shown to have nice geometric characterizations. We then apply our results to study multiplicity-one K-types of principal representations of $U(n, n)$.

1. INTRODUCTION AND NOTATIONS

Let $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{C}$ be the integers, positive integers, reals and complex numbers respectively. Let $GL(n, \mathbb{C})$ be the group of invertible $n \times n$ complex matrices and $U(n)$ be the subgroup of $GL(n, \mathbb{C})$ preserving the positive definite Hermitian form $z_1\overline{z}_1 + z_2\overline{z}_2 + \cdots + z_n\overline{z}_n$ on $\mathbb{C}^n$. The object of this paper is to study the geometry of weight diagrams of representations of $U(n)$. Our techniques are combinatorial and are applicable to representations of $GL(n, \mathbb{C})$ or Lie algebras of type A_n.

Let us first establish some notations. Let $i_1 \geq i_2 \geq \cdots \geq i_k \geq 0$ be a k-tuple of integers such that $\sum_{l=1}^{k} i_l = n$. We say that $U_{i_1} \times U_{i_2} \times \cdots \times U_{i_k}$ is embedded in $U(n)$ is the *standard way* if

$$(A_{i_1}, \ldots, A_{i_k}) \in U_{i_1} \times \cdots \times U_{i_k} \hookrightarrow \text{block diagonal } (A_{i_1}, \ldots, A_{i_k}) \in U(n)$$

where $A_{i_l} \in U_{i_l}$ for $1 \leq l \leq k$. By abuse of notation, we will say that U_k is embedded in the standard way in $U(n)$ if $k < n$ and

$$A \in U_k \hookrightarrow \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \in U(n)$$

where I is the $(n-k) \times (n-k)$ identity matrix. We will denote the tensor product of two representations ρ and σ by $\rho \otimes \sigma$. It could be clear from the context whether the tensor product is an inner or outer tensor. If ρ and σ are representations of a group H, and ρ appears as a constituent of σ, we will write $\rho \hookrightarrow \sigma$. Also we will write $\rho|_{H'}$ to be the restriction of the representation of H to a closed subgroup H'.

Let

$$T = \{\operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \in U(n) | \theta_j \in \mathbb{R}, 1 \leq j \leq n\}$$

be a maximal torus of $U(n)$. We will parametrize characters to T by $(X_1, \ldots, X_n) \in \mathbb{Z}^n$, i.e., $\chi_{(X_1, \ldots, X_n)}$ is the character of T given by

$$\chi_{(X_1, \ldots, X_n)}(\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})) = e^{iX_1\theta_1} e^{iX_2\theta_2} \cdots e^{iX_n\theta_n}. \tag{1.1}$$

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Define a partial ordering on $\mathbb{Z}^n$, by having $(\alpha_1, \dots, \alpha_n) \succ (\beta_1, \dots, \beta_n)$ if and only if $\alpha_i - \beta_i \geq \alpha_{i+1} - \beta_{i+1}$ (for $i = 1, \dots, n-1$), for $(\alpha_1, \dots, \alpha_n)$ and $(\beta_1, \dots, \beta_n)$ in $\mathbb{Z}^n$. In the context of weights we will say $(\alpha_1, \dots, \alpha_n)$ is higher than $(\beta_1, \dots, \beta_n)$ if $(\alpha_1, \dots, \alpha_n) \succ (\beta_1, \dots, \beta_n)$. The Weyl group of $U(n)$ can be identified with S_n, the finite group of permutations on n objects so that S_n acts on T by permuting its diagonal entries. If $(X_1, \dots, X_n) \in \mathbb{Z}^n$ indexes a character of T, then (the induced action) S_n acts on $(X_1, \dots, X_n)$ by permuting its entries.

If we extend the action of S_n on $\mathbb{Z}^n$ to $\mathbb{R}^n$ in the obvious way, this action partitions $\mathbb{R}^n$ into Weyl chambers. Select a fundamental Weyl-chamber $\mathscr{W}$ of $\mathbb{R}^n$ by taking

$$\mathscr{W} = \{x \in \mathbb{R}^n | x \succ (0, \dots, 0)\}.$$

It is well known (see [Zh]) that all irreducible representations of $U(n)$ are finite dimensional and uniquely determined (up to equivalence) by its highest weight; that is, an (equivalence class of) irreducible representation of $U(n)$ can be indexed by an n-tuple of integers $(\alpha_1, \dots, \alpha_n)$ where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ or simply $(\alpha_1, \dots, \alpha_n) \in \mathscr{W}$. We shall denote the irreducible representation of $U(n)$ with highest weight $(\alpha_1, \dots, \alpha_n)$ by $\rho(\alpha_1, \dots, \alpha_n)$. Also denote by $\widehat{U}(n)$ the set of (equivalence classes of) irreducible representations of $U(n)$.

The Weyl group S_n acts on the set of weights of an irreducible representation of $U(n)$, and this is the same action if we index weights by elements of $\mathbb{Z}^n$ as in (1.1). Note that T is chosen to be compatible with the action of S_n. Even more is true; the action of S_n on the weights of an irreducible representation preserves the multiplicities of the weights, that is, all Weyl-group translates of a weight have the same multiplicity (see [Zh]). Identify the set of weights of $\rho(\alpha_1, \dots, \alpha_n) \in \widehat{U}(n)$ as a set of points in $\mathbb{Z}^n \subset \mathbb{R}^n$. We can then plot these points, with the respective multiplicity of the weight attached. This gives us a weighted diagram which we will call the *weight-diagram* (of the representation in consideration).

From now on, we shall not hesitate to refer to *lattice points* as elements in $\mathbb{Z}^n$. Given a finite set S of lattice points, we can define a smallest polytype in $\mathbb{R}^n$ (i.e. a region formed by a finite number of hyperplanes in $\mathbb{R}^n$) containing all the lattice points in S. We shall call this polytope the *hull* of the set S. In particular, when we speak of the *hull of weights of* $\rho \in \widehat{U}(n)$, we mean the hull of the set of lattice points indexing the weights of ρ. We may also sometimes refer to the hull of weights as the *weight polyhedron.*

A brief outline of this paper is as follows. Section 2 discusses the geometry of the weight diagram of an irreducible representation of $U(n)$. By using the standard branching rules for $U(n)$ and the Gel′fand-Cetlin basis, we obtain a 'hereditary' property of the weight polyhedron. In the simplest form, any section of the weight polyhedron looks like the weight polyhedron of some irreducible representation of some subgroup of $U(n)$. In particular, the faces will look like the weight polyhedron of an irreducible representation of some $U(j) \times U(n-j)$, and there is a simple correspondence of multiplicities on each of these faces (see Lemma 2.8). We then digress a little to show an analogous phenomenon for subrepresentations of $\rho|_{U(j)\times U(n-j)}$ where ρ is an irreducible representation of $U(n)$. More precisely, given a $\sigma \in \widehat{U}(j)$, we can find a 'largest' representation

$\Pi \in \widehat{U}(n-j)$ such that $\sigma \otimes \Pi$ is of multiplicity one in $\rho|_{U(j)\times U(n-j)}$ and all other $\sigma \otimes \eta \hookrightarrow \rho|_{U(j)\times(n-j)}$ are such that the highest weight of η is a weight of Π (see Proposition 2.14). It also seems plausible that this property should extend to subrepresentations of $\rho|_{U(k_1)\times\cdots\times U(k_l)}$ where $U(K_1)\times\cdots\times U(k_l)$ is embedded block diagonally in $U(n)$.

Section 3 uses results from §2 to obtain an extremal property of multiplicity-one weights. They must live on some faces of the weights polyhedron, with the exception of simplex representation (symmetric powers of the standard representation of $U(n)$ twisted by some power of the determinant and their duals; see Theorem 3.2). The reason for coining the term "simplex representation" is because all its weights are of multiplicity one, and the weight polyhedron is a regular n-simplex. A closer study reveals that locally (on the weight polyhedron), a multiplicity-one weight lives on some larger codimension facet which is a product of simplices, and all weights on such a facet are multiplicity one (see Theorem 3.7). This result extends easily to weights of tensor proudct of two irreducible representations of $U(n)$. We apply the last result to describe the multiplicity-one K-types in principal series of $U(n, n)$.

The results of this paper are part of the author's thesis [T] under the superversion of Roger Howe. The author is indebted to him for his sustaining interest, encouragement and help.

2. Extremal properties of weights and subrepresentations

This section establishes an interesting hereditary property on the geometry of the weight diagram (see Lemmas 2.5 and 2.8). We will also give an extremal property of the representations of $U(k)\times U(n-k)$ appearing in an irreducible representation of $U(n)$ (see Proposition 2.17). Some results (such as Proposition 2.6, Lemma 2.8, and Lemma 2.14) are probably known to experts, but we could only trace one of them (Proposition 2.6) to an excerise in [Zh]. We will include proofs to provide continuity in the discussion.

To do this, we will need to work with a set of weight vectors of an irreducible representation of $U(n)$. The Gel′fand-Cetlin basis is one such set (see [Zh]). We index a Gel′fand-Cetlin weight vector, which is also an element of a basis for the representation space of $\rho(\alpha_1, \dots, \alpha_n)$ by an array

$$
\begin{array}{ccccccccccc}
\alpha_1 & & \alpha_2 & & \alpha_3 & & & & \cdots & & \alpha_n \\
& \alpha_1^1 & & \alpha_2^1 & & \alpha_3^1 & & & \cdots & \alpha_n^1 & \\
& & \alpha_1^2 & & \alpha_2^2 & & & & \iddots & & \\
& & & \ddots & & & & & & & \\
& & & & \alpha_1^{n-2} & & \alpha_2^{n-2} & & & & \\
& & & & & \alpha_1^{n-1} & & & & &
\end{array}
\tag{2.1}
$$

where

$$
\alpha_j^{i-1} \geq \alpha_j^i \geq \alpha_{j+1}^{i-1} \tag{2.2}
$$

for $i = 1, \dots, n-1$ and $j = 1, \dots, n-i$ and all entries are integers. The weight of this Gel′fand-Cetlin basis vector (2.1) is given by $(X_1, \dots, X_n)$ where

$X_1 = \alpha_1^{n-1}$ and

$$X_j = \sum_{k=1}^{j} \alpha_k^{n-j} - \sum_{k=1}^{j-1} \alpha_k^{n-j+1} \quad \text{for } j = 2, \dots, n \tag{2.3}$$

where $\alpha_k^0 = \alpha_k$ for $k = 1, \dots, n$. The multiplicity of this weight $(X_1, \dots, X_n)$ will then be the number of Gel'fand-Cetlin basis vectors satisfying the relations (2.2) and (2.3).

Definition 2.4. Let $\rho(\alpha) = \rho(\alpha_1, \dots, \alpha_n) \in \widehat{U}(n)$ and $X \in \mathbb{Z}$ with $\alpha_1 \geq X \geq \alpha_n$. If $\alpha_{i+1} \leq X \leq \alpha_i$, define

$$\rho(\alpha) \setminus X = \rho(\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_i + \alpha_{i+1} - X, \alpha_{i+2}, \alpha_{i+3}, \dots, \alpha_n) \in \widehat{U}(n-1).$$

Take $U(n-1) \times U(1)$ embedded in $U(n)$ in the standard way. The following lemma suggest that we call $\rho(\alpha) \setminus X_n$ the *largest* representation in $\rho(\alpha_1, \dots, \alpha_n)$ corresponding to X_n.

Lemma 2.5. *Let $\rho(\alpha) = \rho(\alpha_1, \dots, \alpha_n) \in \widehat{U}(n)$ and fix $X_n \in \mathbb{Z}$ with $\alpha_1 \geq X_n \geq \alpha_n$. If $\rho(X_1, \dots, X_{n-1}) \otimes \rho(X_n)$ sits in $\rho(\alpha)|_{U(n-1)\times U(1)}$, then $(X_1, \dots, X_{n-1})$ is a weight of $\rho(\alpha) \setminus X_n$.*

Proof. Let X_n be given as above. Suppose $\alpha_i \geq X_n \geq \alpha_{i+1}$. Since $\alpha_i \geq \alpha_i + \alpha_{i+1} - X_n \geq \alpha_{i+1}$, we observe that $(\rho(\alpha) \setminus X_n) \otimes \rho(X_n) \hookrightarrow \rho(\alpha)|_{U(n-1)\times U(1)}$ from standard branching theorems (see [Zh]).

If $\rho(X_1, \dots, X_{n-1}) \otimes \rho(X_n) \hookrightarrow \rho(\alpha)|_{U(n-1)\times U(1)}$, then $(X_1, \dots, X_{n-1})$ satisfies

$$\alpha_j \geq X_j \geq \alpha_{j+1} \quad \text{for } j = 1, \dots, n-1$$

and so

$$X_{i-1} \geq \alpha_i \geq \alpha_i + \alpha_{i+1} - X_n \geq \alpha_{i+1} \geq X_{i+1}$$

i.e. (since $\sum_{j=1}^{n-1} X_j = \sum_{j=1}^{n} \alpha_j - X_n$), we have

$$\rho(X_1, \dots, X_{i-1}, X_{i+1}, X_{i+2}, \dots, X_{n-1}) \hookrightarrow (\rho(\alpha) \setminus X_n)|_{U(n-2)}.$$

This shows that $(X_1, \dots, X_{n-1})$ is a weight of $\rho(\alpha) \setminus X_n$. □

Proposition 2.6. *The weights of $\rho(\alpha_1, \dots, \alpha_n)$ are exactly all the integer-tuples $(X_1, \dots, X_n)$ satisfying the following conditions:*

$$\begin{cases} (1) & \alpha_1 \geq X_i \\ (2) & \alpha_1 + \alpha_2 \geq X_i + X_j \\ & \cdot \\ & \cdot \\ (k) & \sum_{i=1}^{k} \alpha_i \geq \sum_{j=1}^{k} X_{i_j} \\ & \cdot \\ & \cdot \\ (n-1) & \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} \geq \sum_{l=1}^{n} X_l - X_i \\ (n) & X_1 + X_2 + \cdots + X_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n. \end{cases} \tag{2.7}$$

Here the indices i, j, k are distinct and run from 1 *through* n, *and* $\{i_j\}_{j=1}^{k}$ *is a set of* k *distinct numbers between* 1 *and* n.

Remarks. (1) The lemma says that the weight polyhedron of an irreducible representation of $U(n)$ is bounded and convex and *all* lattice points in it correspond to weights of that representation. When one of the inequalities in (2.7) is an equality, we get an equation of a plane which we will call a *supporting hyperplane.* By a *face* of the weight polyhedron, we will mean the intersection of a supporting hyperplance with the weight polyhedron.

(2) Let X_{n-1} and X_n be given so that both of them satisfy

(i) $\alpha_1 \geq X_{n-1} \geq \alpha_n$ and $\alpha_1 \geq X_n \geq \alpha_n$,

(ii) $\alpha_1 + \alpha_2 \geq X_{n-1} + X_n \geq \alpha_{n-1} + \alpha_n$.

This guarantees that X_{n-1} and X_n can be the last two coordinates of some weight of $\rho(\alpha_1, \dots, \alpha_n) \in \widehat{U}(n)$ (in fact any two coordinates of some weight). It is easy to see that

$$(\rho(\alpha) \setminus X_n) \setminus X_{n-1} = (\rho(\alpha) \setminus X_{n-1}) \setminus X_n .$$

Further $(\rho(\alpha) \setminus X_n) \setminus X_{n-1}$ is the 'largest representation' corresponding to X_{n-1} and X_n in the sense that all $\Pi \otimes \rho(X_{n-1}) \otimes \rho(X_n) \hookrightarrow \rho(\alpha)\big|_{U(n-2)\times U(1)\times U(1)}$ must satisfy the highest weight of Π being a weight of $(\rho(\alpha) \setminus X_n) \setminus X_{n-1}$. This is obtained by a simple iteration of Lemma 2.5. In general, given $1 < j \leq n$ we could iterate Lemma 2.5, provided we have $(X_j, \dots, X_n)$ satisfy conditions (2.7), to get a 'largest' representation $(\rho(\alpha) \setminus X_j) \setminus X_{j+1} \setminus \dots \setminus X_n$ and the order of the X's is immaterial.

(3) There is a simple geometric meaning to remark (2). First observe that Lemma 2.5 effectively describes the weights living on a hyperplane parallel to one of the faces of the weight polyhedron. Although we fix X_n in the lemma, there is really not much difference if we were to fix an arbitrary X_i, since we could consider different embeddings of $U(n-1)\times U(1)$ in $U(n)$. The situation is symmetric because of the action of the Weyl group S_n. Lemma 2.5 therefore implies the existence of a representation of $U(n-1)$ which accounts for all the weights of $\rho(\alpha_1, \dots, \alpha_n) \in \widehat{U}(n)$ with an entry fixed. This also means that the hull of the weights living on a hyperplane parallel to one of the faces will look like the weight polyhedron corresponding to the largest representation. Repeated application of this lemma, for $1 < j < n$, simply gives us a description of the weights of $\rho(\alpha_1, \dots, \alpha_n)$ living on a hyperplane of larger codimension (that one) in $\mathbb{R}^n$, and once again we can find a representation of $U(n-j)$ that accounts for all these weights. It is clear that the resulting shape of the weights living in these hyperplanes will look exactly like the weight diagram of some irreducible representation of $U(n-j)$ (if we choose the axes appropriately). This lemma thus brings out a sort of 'hereditary' property of irreducible representations of $U(n)$. Of course, this fact can be appropriately generalised for representations of arbitrary compact Lie groups, but we will not discuss it here. We refer the reader to the figures at the end of the next section for the case $n = 2$, 3, and 4.

Proof of Proposition 2.6. We will prove by induction on n. It is clearly true for $n = 2$. By Lemma 2.5, $(X_1, \dots, X_n)$ is a weight of $\rho(\alpha_1, \dots, \alpha_n)$ if and only if $(X_1, \dots, X_{n-1})$ is a weight of $\rho(\alpha_1, \dots, \alpha_n)\setminus X_n$ and $\alpha_1 \geq X_n \geq \alpha_n$. By the

induction hypothesis, $(X_1, \dots, X_{n-1})$ is a weight of $\rho(\alpha_1, \dots, \alpha_n) \setminus X_n$ if and only if $\alpha_1 \geq X_n \geq \alpha_n$ and conditions (2.7) are satisfied by $(X_1, \dots, X_{n-1})$ relative to $\rho(\alpha_1, \dots, \alpha_n) \setminus X_n$. But a simple rearrangement shows that $(X_1, \dots, X_n)$ satisfy conditions (2.7) relative to $\rho(\alpha_1, \dots, \alpha_n)$ if and only if $(X_1, \dots, X_{n-1})$ satisfy conditions (2.7) relative to $\rho(\alpha_1, \dots, \alpha_n) \setminus X_n$ and $\alpha_1 \geq X_n \geq \alpha_n$. This completes our proof. □

Our next statement describes the weights of an irreducible representation $\rho(\alpha_1, \dots, \alpha_n)$ of $U(n)$, living on one of the faces determined by (2.7). We shall denote the tuple with the i_1th, i_2th, $\dots$, i_jth entries deleted by $(X_1, \dots, \widehat{X}_{i_1}, \dots, \widehat{X}_{i_2}, \dots, \widehat{X}_{i_j}, X_{i_{j+1}}, \dots, X_n)$.

Lemma 2.8. *Consider $\rho(\alpha_1, \dots, \alpha_n)$ and let $1 \leq j < n$. Let $S = \{i_1 > \cdots > i_j\}$ be a set of integers from $\{1, \dots, n\}$. Then the weight vectors with weights living on the face*

$$X_{i_1} + X_{i_2} + \cdots + X_{i_j} = \alpha_1 + \cdots + \alpha_j$$

span an irreducible $U(j) \times U(n-j)$ submodule of $\rho(\alpha_1, \dots, \alpha_n)|_{U(j)\times U(n-j)}$ isomorphic to

$$\sigma_S^j \otimes \sigma_S^{n-j} = \rho(\alpha_1, \dots, \alpha_j) \otimes \rho(\alpha_{j+1}, \dots, \alpha_n).$$

(Here it is important to note that $U(j) \times U(n-j)$ need not necessarily be embedded in the standard way in $U(n)$.) Consequently,

(a) *a weight $(X_1, \dots, X_n)$ on such a face corresponds bijectively to the weight $((X_{i_1}, \dots, X_{i_j}), (X_1, \dots, \widehat{X}_{i_1}, \dots, \widehat{X}_{i_j}, \dots, X_n))$ of $\sigma_S^j \otimes \sigma_S^{n-j}$, and*

(b) *the multiplicity of $(X_1, \dots, X_n)$ in $\rho(\alpha_1, \dots, \alpha_n)$ is the product of the multiplicities of $(X_{i_1}, \dots, X_{i_j})$ in σ_S^j and $(X_1, \dots, \widehat{X}_{i_1}, \dots, \widehat{X}_{i_j}, \dots, X_n)$ in σ_S^{n-j}.*

Remarks. (1) Because of Lemma 2.8, we can say (loosely) that the face $X_{i_1} + X_{i_2} + \cdots + X_{i_j} = \alpha_1 + \cdots + \alpha_j$ defines the representation $\sigma_S^j \otimes \sigma_S^{n-j}$ of $U(j) \times U(n-j)$.

(2) We have noted above that the subgroup $U(j) \times U(n-j)$ need not be embedded in the standard way. A little thought will tell us that $U(j)$ needs to be embedded so that the matrix $(x_{k,l}) \in U(j)$ is identified with the matrix $(y_{p,g}) \in U(n)$ where

$$y_{p,q} = \begin{cases} x_{k,l} & \text{if } p = i_k,\ q = i_l, \\ 1 & \text{if } p = q \text{ and } p \neq i_k \text{ for any } k, \\ 0 & \text{otherwise}. \end{cases}$$

(3) We will make a special note here on the case $\{i_1, \dots, i_j\} = \{1, 2, \dots, j\}$, i.e., the face $X_1 + X_2 + \cdots + X_j = \alpha_1 + \alpha_2 + \cdots + \alpha_j$ for some $1 \leq j < n$. In this case, $U(j) \times U(n-j)$ is embedded in the standard way in $U(n)$, and the representation of $U(j) \times U(n-j)$ corresponding to this face is $\rho(\alpha_1, \dots, \alpha_n) \otimes \rho(\alpha_{j+1}, \dots, \alpha_n)$. We will need this in our discussion of multiplicity one weights of $\rho(\alpha_1, \dots, \alpha_n)$.

Proof of Lemma 2.8. Consider the plane

$$X_{i_1} + \cdots + X_{i_j} = \alpha_1 + \cdots + \alpha_j.$$

Appealing to the action of the Weyl group S_n allows us to just consider the plane:

$$X_1 + \cdots + X_j = \alpha_1 + \cdots + \alpha_j \,. \tag{2.9}$$

For a weight $(X_1, \ldots, X_n)$ of $\rho(\alpha_1, \ldots, \alpha_n)$ living on this face, we have from (2.3)

$$\sum_{i=1}^{j} X_i = \sum_{i=1}^{j} \alpha_i^{n-j} \,. \tag{2.10}$$

Also by the way we have set up the Gel′fand-Cetlin basis vector, we have (cf. (2.1))

$$\sum_{i=1}^{j} \alpha_i \geq \sum_{i=1}^{j} \alpha_i^{n-j} \,. \tag{2.11}$$

Thus (2.9), (2.10), and (2.11) thereby forces $\alpha_i^{n-j} = \alpha_i$ for $1 \leq i \leq j$. And so the Gel′fand-Cetlin basis vector looks like

$$\begin{array}{ccccccccccccc}
\alpha_1 & & \alpha_2 & & \cdots & & \alpha_j & & \alpha_{j+1} & & \cdots & & \alpha_n \\
& \ddots & & A & & \iddots & & \alpha_{j+1} & & \cdots & & \alpha_n & \\
& & \ddots & & \iddots & & \iddots & & \cdots & & \iddots & & \\
& & & X_1 & & \alpha_{j+1} & \cdots & \cdots & \cdots & \alpha_n & & & \\
& & & & \alpha_{j+1} & & \cdots & & \alpha_n & & & & \\
& & & & & \ddots & B & \iddots & & & & & \\
& & & & & & X_{j+1} & & & & & &
\end{array}$$

where the arrays A and B would correspond to Gel′fand-Cetlin basis vectors of weights $(X_1, \ldots, X_j)$ and $(X_{j+1}, \ldots, X_n)$ from representation $\rho(\alpha_1, \ldots, \alpha_j)$ and $\rho(\alpha_{j+1}, \ldots, \alpha_n)$ respectively. This concludes the proof for the major part of the lemma. The consequence (a) is obvious and (b) follows from the fact that the multiplicity of a weight is the number of distinct Gel′fand-Cetlin basis vectors with that weight. □

Following remark (1) of Lemma 2.8, we see immediately that we could generalize it to any facet of the weight polyhedron. More precisely, let $S(k_j)$, $1 \leq j \leq l$, be a collection of subsets of $\{1, \ldots, n\}$ where each $S(k_j)$ contains k_j elements such that

$$\{1, \ldots, n\} = \bigcup_{j=1,\ldots,l} S(k_j) \quad \text{(a disjoint union)}$$

where $k_1 + \cdots + k_l = n$.

Definition 2.12. A $(S(k_1), S(k_2), \ldots, S(k_l))$-facet of the weight polyhedron of $\rho(\alpha_1, \ldots, \alpha_n)$ is defined to be the set

$$\Bigg\{(X_1, \dots, X_n) \in \{\text{weights of } \rho(\alpha_1, \dots, \alpha_n)\} \Big| \sum_{i \in S(k_j)} X_i = \alpha_{n_j+1} + \alpha_{n_j+2} + \cdots + \alpha_{n_j+k_j} \text{ for } 1 \le j \le l \text{ and } n_j = \sum_{s=1}^{j-1} k_s \Bigg\}.$$

An easy generalisation of Lemma 2.8 says that every $(S(k_1), \dots, S(k_l))$-facet defines a $U(k_1) \times U(k_2) \times \cdots \times U(k_l)$ irreducible submodule given by

$$\rho(\alpha_1, \alpha_2, \dots, \alpha_{k_1}) \otimes \rho(\alpha_{k_1+1}, \dots, \alpha_{k_1+k_2}) \otimes \cdots \otimes \rho(\alpha_{k_1+\cdots+k_{l-1}+1}, \dots, \alpha_n).$$

Having studied the geometry of the weight polyhedron, we digress a little to consider an extremal property of the representations of subgroups (of the form $U(k_1) \times U(k_2) \times \cdots \times U(k_l)$ embedded in the standard way in $U(n)$) appearing in $\rho(\alpha_1, \dots, \alpha_n) \in \widehat{U}(n)$. (This would not have much to do with §3, but is useful when we are considering K-types of principal series of $U(p, q)$ (see §5).) We can think of Lemma 2.5 as giving an 'extremal' representation of $U(n-1)$ complementary to a representation of $U(1)$.

Definition 2.13. Let $(X_1, \dots, X_n) \in \mathbb{Z}^n$ and $(\alpha_1, \dots, \alpha_p) \in \mathbb{Z}^p$. If $\alpha_1 \ge \alpha_2 \ge \alpha_3 \ge \cdots \ge \alpha_p$ and $X_1 \ge X_2 \ge \cdots \ge X_n$, we write $(\alpha_1, \alpha_2, \dots, \alpha_p) \prec (X_1, X_2, \dots, X_n)$ if $X_i \ge \alpha_i \ge X_{n-p+i}$ for $i = 1, \dots, p$. (We hope that the reader will not confuse the symbol '$\prec$' with the order $\prec$ used in §1.)

Lemma 2.14. *Let $p < n$ be positive integers. Let $(\alpha_1, \dots, \alpha_p)$ be a p-tuple of integers and let $\rho(X_1, \dots, X_n) \in \widehat{U}(n)$. There exists a $\Pi \in \widehat{U}(n-p)$ such that $\rho(\alpha_1, \dots, \alpha_p) \otimes \Pi \hookrightarrow \rho(X_1, \dots, X_n)|_{U(p)\times U(n-p)}$ if and only if $(\alpha_1, \dots, \alpha_p) \prec (X_1, \dots, X_n)$*

Proof. Suppose $\rho(\alpha_1, \dots, \alpha_p) \otimes \Pi \hookrightarrow (X_1, \dots, X_n)|_{U(p)\times U(n-p)}$ for some integer p with $1 \le p < n$ and where $U(p) \times U(n-p)$ is embedded in the standard way in $U(n)$. A Gel'fand-Cetlin basis vector (or array) coming from such a $U(p) \times U(n-p)$ summand must have its $(p+1)$th row given by "$\alpha_1\alpha_2 \cdots \alpha_p$" and so $(\alpha_1, \dots, \alpha_p) \prec (X_1, \dots, X_n)$. Conversely if $(\alpha_1, \dots, \alpha_p) \prec (X_1, \dots, X_n)$, then we know from the branching theorem that there exist $Y_{p+1}, \dots, Y_n$ such that

$$\rho(\alpha_1, \dots, \alpha_p) \otimes \rho(Y_{p+1}) \otimes \cdots \otimes \rho(Y_n) \hookrightarrow \rho(X_1, \dots, X_n)|_{U(p)\times U(1)^{n-p}}.$$

Now appeal to the fact that $\rho(X_1, \dots, X_n)$ is a finite dimensional representation of $U(n)$ and must decompose into a finite direct sum of $U(p)\times U(n-p)$ representations and so $\rho(\alpha_1, \dots, \alpha_p) \otimes \rho(Y_{p+1}) \otimes \cdots \otimes \rho(Y_n)$ must sit inside some $\rho(\alpha_1, \dots, \alpha_p) \otimes \rho(\beta_1, \dots, \beta_{n-p})$ of $\widehat{U}(p) \times \widehat{U}(n-p)$. This completes the proof. □

With $(\alpha_1, \dots, \alpha_p) \prec (X_1, \dots, X_n)$, look at a Gel'fand-Cetlin vector from a component $\rho(\alpha_1, \dots, \alpha_p) \otimes \Pi$ of $\rho(X_1, \dots, X_n)|_{U(p)\times U(n-p)}$ (following the comments in the proof of Lemma 2.14 we could disregard the rows after "$\alpha_1\alpha_2$

$\cdots \alpha_p$" as they correspond to the Gel'fand-Cetlin vectors of $\rho(\alpha_1, \dots, \alpha_p) \in \widehat{U}(p))$:

$$\begin{array}{ccccccccccc} X_1 & & X_2 & & X_3 & & \cdots & & \cdots & & X_n \\ & \ddots & & & & & & & & \iddots & \\ & & \ddots & & & & X_j^i & & \iddots & & \\ & & & \ddots & & & & \iddots & & & \\ & & & & \alpha_1 & \cdots & \alpha_p & & & & \end{array}$$

For each entry X_j^i we have

$$\max(X_{l_1}, \alpha_j) \le X_j^i \le \min(X_j, \alpha_{l_2}) \tag{2.15}$$

where $l_1 = i + j$ and $l_2 = i + j - (n - p)$ and it is understood (for notational convience) that

$$\begin{aligned} \alpha_i &= +\infty && \text{if } i < l, \\ \alpha_i &= -\infty && \text{if } i > p, \\ X_j^i &= X_j && \text{if } i = 0 \text{ and } j = 1, \dots, n, \\ X_j^i &= \alpha_j && \text{if } i = n - p \text{ and } j = 1, \dots, p, \end{aligned}$$

and

$$\min(\alpha_i, X_j) = 0 \quad \text{if } i > p.$$

Relation (2.15) provides a motivation to give the following definition.

Definition 2.16. If $\alpha = (\alpha_1, \dots, \alpha_p)$ is a p-tuple of integers such that $(\alpha_1, \dots, \alpha_p) \prec (X_1, \dots, X_n)$, define $\Pi^\alpha = (\Pi_1^\alpha, \dots, \Pi_{n-p}^\alpha)$ as follows:

$$\Pi_i^\alpha = \sum_{j=1}^{p+i} \min(\alpha_{j-i}, X_j) = \sum_{j=1}^{p+i-1} \min(\alpha_{j-i+1}, X_j), \qquad 1 \le i \le n - p.$$

Proposition 2.17. *Suppose $\rho(X_1, \dots, X_n) \in \widehat{U}(n)$ and $(\alpha_1, \dots, \alpha_p) \prec (X_1, \dots, X_n)$, we have $\Pi_1^\alpha \ge \Pi_2^\alpha \ge \Pi_3^\alpha \ge \cdots \ge \Pi_{n-p}^\alpha$, and hence $\rho(\Pi^\alpha) \in \widehat{U}(n-p)$. In particular,*

$$\rho(\alpha_1, \dots, \alpha_p) \otimes \rho(\Pi^\alpha) \hookrightarrow \rho(X_1, \dots, X_n)|_{U(p) \times U(n-p)}$$

with multipicity one and if

$$\rho(\alpha_1, \dots, \alpha_p) \otimes \rho(b_1, \dots, b_{n-p}) \hookrightarrow \rho(X_1, \dots, X_n)\big|_{U(p) \times U(n-p)},$$

then $(b_1, \dots, b_{n-p})$ is a weight of $\rho(\Pi^\alpha)$.

Remarks. This implies Lemmas 2.5 and 2.8 immediately, but we refrained from presenting it first because of the complexity of Definition 2.16.

Proof. Checking that $\rho(\Pi^\alpha) \in \widehat{U}(n-p)$ is straightforward but somewhat tedious. We omit the computations. It follows from (2.16) that

$$\sum_{i=1}^{k} \Pi_i^\alpha = \sum_{j=1}^{p+k} \min(X_j, \alpha_{j-k}) - (\alpha_1 + \cdots + \alpha_p). \tag{2.18}$$

If

$$\rho(\alpha_1, \dots, \alpha_p) \otimes \rho(b_1, \dots, b_{n-p}) \hookrightarrow \rho(X_1, \dots, X_n)\big|_{U(p)\times U(n-p)},$$

then by (2.18),

$$\sum_{i=1}^{k} b_i \le \sum_{i=1}^{k} \Pi_i^\alpha \quad \text{and} \quad \sum_{i=1}^{n-p} b_i = \sum_{i=1}^{n-p} \Pi_i^\alpha \tag{2.19}$$

since $(b_1, \dots, b_{n-p})$ must also sit in a Gel'fand-Cetlin basis of similar form (as in the case of $\rho(\alpha_1, \dots, \alpha_p) \otimes \Pi^\alpha$) and Π_i^α is chosen to maximize the entries. Now because $(b_1, \dots, b_p)$ is dominant, (2.7) reduces to (2.19) and so $(b_1, \dots, b_p)$ is a weight of $\rho(\Pi^\alpha)$. Hence all other

$$\rho(\alpha_1, \dots, \alpha_p) \otimes \rho(b_1, \dots, b_{n-p})$$

appearing are such that $(b_1, \dots, b_{n-p})$ is a weight of $\rho(\Pi^\alpha)$.

With $(\alpha_1, \dots, \alpha_p)$ fixed, there is only one Gel'fand-Cetlin basis vector with the $(n-p+1)$th row given by "$\alpha_1 \cdots \alpha_p$" and with weight $(\alpha_1, \dots, \alpha_p, \Pi_1^\alpha, \dots, \Pi_{n-p}^\alpha)$ because of our choice of Π_i^α's. To see this, observe that the $(n-p)$th row can only be filled in one and only one way to get Π_1^α (recall $\Pi_1^\alpha =$ sum of $(n-p)$th row $-\sum_{i=1}^{p} \alpha_i$ from (2.3)) which is by maximizing all entries. The same argument applies inductively to the other Π_i^α's. Hence $\rho(\alpha_1, \dots, \alpha_p) \otimes \rho(\Pi_1^\alpha, \dots, \Pi_{n-p}^\alpha)$ appears with multiplicity one in $\rho(X_1, \dots, X_n)\big|_{U(p)\times U(n-p)}$. □

We have mentioned that $\Pi^{\alpha_1} = \rho(X) \setminus \alpha_1$ is the 'largest representation' of $U(n-1)$ corresponding to α_1. This is an easy computation. However, $\Pi^{(\alpha_1, \alpha_2)}$ is not necessarily $(\rho(X) \setminus \alpha_1) \setminus \alpha_2$. We are concerned about $U(2) \times U(n-2)$ components in the case of $\Pi^{(\alpha_1, \alpha_2)}$, while we are interested in $U(1) \times U(1) \times U(n-2)$ components in the case of $(\rho(X) \setminus \alpha_1) \setminus \alpha_2$. Let $U = U(k_1) \times U(k_2) \times \cdots \times U(k_l)$ be a subgroup of $U(n)$ embedded in the standard way and where $k_1 + k_2 + \cdots + k_l = n$. Let $\rho \in \widehat{U}(n)$, then the restriction of ρ to this subgroup can be written as

$$\rho\big|_U = \sum_D m(D) \sigma_1^D \otimes \sigma_2^D \otimes \cdots \otimes \sigma_l^D$$

where $\sigma_i^D \in \widehat{U}(k_i)$ for $1 \le i \le l$ and D is some parametrizing set, and $m(D)$ is the multiplicity of $\sigma_1^D \otimes \sigma_2^D \otimes \cdots \otimes \sigma_l^D$ in $\rho\big|_U$. Look at the components of $\rho\big|_U$ of the form $\rho_1 \otimes \cdots \otimes \rho_{l-1} \otimes \tilde{\sigma}$ where $\rho_1 \otimes \cdots \otimes \rho_{l-1} \in \widehat{U}(k_1) \times \cdots \times \widehat{U}(k_{l-1})$ is fixed. We will say $\tilde{\sigma}$ is *complementary* to $\rho_1 \otimes \cdots \otimes \rho_{l-1}$ in ρ.

Unfortunately, in general, we cannot simply reiterate Proposition 2.17 to try to find a largest representation complementary to $\rho_1 \otimes \cdots \otimes \rho_{l-1}$. For instance, take $l = 3$ and let

$$\rho_1 \otimes \Pi^{\rho_1} \oplus \rho_1 \otimes \Pi_1 \oplus \rho_1 \otimes \Pi_2 \oplus \cdots \oplus \rho_1 \otimes \Pi_k \hookrightarrow \rho\big|_{U(k_1)\times U(n-k_1)}.$$

We have seen that each Π_j is a dominant weight of Π^{ρ_1}. But it is not true that every $\rho_2 \hookrightarrow \Pi_1\big|_{U(k_2)\times U(n-k_1-k_2)}$ satisfies $\rho_2 \prec \Pi^{\rho_1}$ For instance $(6, 6) \prec (10, 6, 5, 2)$ and $(10, 6, 5, 2)$ is a dominant weight of $\rho(11, 5, 5, 2)$, but we do not have $(6, 6) \prec (11, 5, 5, 2)$. This means that we may not be able to find complements of $\rho_1 \otimes \rho_2$ in $\rho_1 \otimes \Pi^{\rho_1}$ by hoping to iterate Proposition 2.17. The only situation where iterations is straightforward is when $k_2 = k_3 = \cdots =$

$k_{l-1} = 1$, in which case we do have a largest representation complementary to a fixed $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_{l-1}$ in ρ. In the general case, we need a better understanding of the set of σ $(\in \widehat{U}(n-j))$ which complements ρ_1 $(\in \widehat{U}(j))$ in $\rho \in \widehat{U}(n)$.

3. Multiplicity-one weights

In this section, multiplicity-one weights are given a description in terms of the geometry of the weight diagram.

Definition 3.1. A point $(X_1, \dots, X_n)$ is an interior point of the set described by (2.7) if *strict* inequality occurs to all the conditions $(1), (2), \dots, (n-1)$ in (2.7).

The following theorem says that multiplicity-one weights must lie on the faces of the weight polyhedron, with the exception of some very special cases.

Theorem 3.2. *A multiplicity-one weight $(X_1, \dots, X_n)$ of $\rho(\alpha_1, \dots, \alpha_n)$ can be an interior point only when the representation $\rho(\alpha_1, \dots, \alpha_n)$ satisfies*

$$\begin{cases} \text{(i)} & \alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} \text{ or} \\ \text{(ii)} & \alpha_2 = \alpha_2 = \cdots = \alpha_n . \end{cases} \tag{3.3}$$

If $\rho(\alpha_1, \dots, \alpha_n)$ satisfies (3.3), *then all its weights are of multiplicity one. Otherwise the multiplicity-one weights must lie on some face of the weight polyhedron described by* (2.7), *that is, there is some k, $1 \le k < n$, and $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, such that*

$$X_{i_1} + \cdots + X_{i_k} = \alpha_1 + \cdots + \alpha_k . \tag{3.4}$$

Remark. We will call the representations referred to in this first statement of Theorem 3.2 *simplex representation*, because the weight polyhedron is actually a regular n-simplex. They are actually symmetric powers of the standard representation of $U(n)$ twisted by some power of the determinant representation and the duals of these. Note that determinant representations, i.e., $\alpha_1 = \cdots = \alpha_n$ above, are also simplex representations; their weight diagrams have only one point.

Proof. For $n = 2$, all irreducible representations have a multiplicity-free weight decomposition and conditions (3.3) are automatically true. So the theorem is true for $n = 2$. Suppose $(X_1, \dots, X_n)$ is a multiplicity-one weight of $\rho(\alpha) = \rho(\alpha_1, \dots, \alpha_n)$ which is an interior point. Observe first that $X_n \neq \alpha_1$ and $X_n \neq \alpha_n$ by definition of interior point.

We consider 3 cases:

Case I: $\alpha_2 \le X_n < \alpha_1$.

Here $\rho(\alpha) \setminus X_n = \rho(\alpha_1 + \alpha_2 - X_n, \alpha_3, \alpha_4, \dots, \alpha_n)$ and $\alpha_1 \ge \alpha_1 + \alpha_2 - X_n > \alpha_2$. Observe that $(X_1, \dots, X_{n-1})$ is also an interior point and weight of $\rho(\alpha) \setminus X_n$ with multiplicity one. By induction, either $\alpha_3 = \alpha_4 = \cdots = \alpha_n$ or $\alpha_1 + \alpha_2 - X_n = \alpha_3 = \alpha_4 \cdots = \alpha_{n-1}$. The latter condition is impossible as $\alpha_1 + \alpha_2 - X_n > \alpha_2 \ge \alpha_3$ by assumption. If $\alpha_2 \neq \alpha_3$, we see from branching theorem that $\rho(\alpha_1 + \alpha_2 - X_n - 1, \alpha_3 + 1, \alpha_4, \dots, \alpha_n) \otimes \rho(X_n) \hookrightarrow \rho(\alpha_1, \dots, \alpha_n)|_{U(n-1) \times U(1)}$. Since $(X_1, \dots, X_{n-1})$ is interior, it will also be a weight of $\rho(\alpha_1 + \alpha_2 - X_n - 1, \alpha_3 + 1, \alpha_4, \dots, \alpha_n)$ by Proposition 2.6. This contradicts the assumption

that $(X_1, \dots, X_n)$ is of multiplicity one. Hence $\alpha_2 = \alpha_3$ and so $\alpha_2 = \alpha_3 = \alpha_4 = \cdots = \alpha_n$.

Case II: $\alpha_{i+1} \le X_n \le \alpha_i$, $i = 2, \dots, n-2$.

Hence $\rho(\alpha) \setminus X_n = \rho(\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_i + \alpha_{i+1} - X_n, \alpha_{i+2}, \dots, \alpha_n)$ and it is easy to see that $(X_1, \dots, X_{n-1})$ is an interior point of $\rho(\alpha) \setminus X_n$. By induction on n, we see that either

(i) $\alpha_1 = \alpha_2 = \cdots = \alpha_{i-1} = \alpha_i + \alpha_{i+1} - X_n = \alpha_{i+2} = \cdots = \alpha_{n-1}$ or

(ii) $\alpha_2 = \alpha_3 = \cdots = \alpha_{i-1} = \alpha_i + \alpha_{i+1} - X_n = \alpha_{i+2} = \cdots = \alpha_n$.

Now $\alpha_{i-1} \ge \alpha_i \ge \alpha_i + \alpha_{i+1} - X_n \ge \alpha_{i+1} \ge \alpha_{i+2}$ which forces $\alpha_{i-1} = \alpha_i = \alpha_{i+1} = \alpha_{i+2}$ and so we are done for this case.

Case III. $\alpha_n < X_n < \alpha_{n-1}$.

This case is similar to Case I. This concludes our proof for the first statement. It is well known that highest weight representations satisfying conditions (3.3) give a multiplicity-free weight decomposition (see [Zh]). We know their standard realisation (see remarks above); it is then easy to see that the weights are all of multiplicity one. This takes care of the second statement. The last statement is a rephrasing of the first statement. □

Definition 3.5. Let $\mathscr{M}(\rho(\alpha_1, \dots, \alpha_k))$ be the set of multiplicity-one weights of $\rho(\alpha_1, \dots, \alpha_k)$. This definition may be extended to arbitrary representations (not necessarily irreducible) of $U(k)$, that is, $\mathscr{M}(\sigma)$ is the set of multiplicity-one weights of a finite dimensional representation σ of $U(k)$.

Theorem 3.6. *If $\rho(\alpha_1, \dots, \alpha_n)$ is not simplex, then we have*

$$\mathscr{M}(\rho(\alpha_1, \dots, \alpha_n)) = S_n \cdot \left\{ \bigcup_{j=1}^{n-1} \{\mathscr{M}(\rho(\alpha_1, \dots, \alpha_j))\mathscr{M}(\rho(\alpha_{j+1}, \dots, \alpha_n))\} \right\}$$

where S_n is understood to act on the set of its right, by permuting its entries.

Proof. If $(X_1, \dots, X_n) \in \mathscr{M}(\rho(\alpha_1, \dots, \alpha_n))$, then all S_n translates of $(X_1, \dots, X_n)$ are in $\mathscr{M}(\rho(\alpha_1, \dots, \alpha_n))$ (see §1). Because of Theorem 3.2, $(X_1, \dots, X_n)$ must lie on a face of the weight polyhedron described by (2.7), and Lemma 2.8 describes the set of such weights to be those arising from some irreducible representation of a certain subgroup $U(j) \times U(n-j)$ of $U(n)$. Now because of the action of S_n, we might as well assume that there is a $1 \le j < n$ such that $X_1 + \cdots + X_j = \alpha_1 + \cdots + \alpha_j$, and Lemma 2.8 immediately implies that the weights on this face comes from the representation $\rho(\alpha_1, \dots, \alpha_j) \otimes \rho(\alpha_{j+1}, \dots, \alpha_n)$ of $U(j) \times U(n-j)$ (with $U(j) \times U(n-j)$ embedded in the standard way) and the multiplicity ones are $\mathscr{M}(\rho(\alpha_1, \dots, \alpha_j)) \times \mathscr{M}(\rho(\alpha_{j+1}, \dots, \alpha_n))$. This completes our proof. □

We have seen (cf. Lemma 2.8 and comments after it) that each facet of the weight polyhedron of $\rho(\alpha_1, \dots, \alpha_n)$ defines an irreducible representation of some subgroup of $U(n)$. In the following corollary, we will see that a multiplicity-one weight of an irreducible representation of $U(n)$ lives in a facet which looks like a product of simplices (and so the representation that this facet defines is a tensor product of simplex representations); and every weight on this facet is of multiplicity one!

Theorem 3.7. *If a weight $(X_1, \dots, X_n)$ of $\rho(\alpha_1, \dots, \alpha_n)$ has multiplicity one, then there exists a $(S(k_1), \dots, S(k_l))$-facet of the weight polyhedron containing*

$(X_1, \dots, X_n)$ *with the following properties*:

(1) $k_1 + \cdots + k_l = n$ *and* $\bigcup_{j=1,\dots,l} S(k_j) = \{1, \dots, n\}$ (*disjoint union*),

(2) *it is a product of simplices* (*by an appropriate rearrangement of the coordinates*),

(3) *every weight living on this facet is of multiplicity one.*

In particular, the irreducible representation of $U(k_1) \times U(k_2) \times \cdots \times U(k_l)$ *defined by this facet is a tensor product of simplex representations.*

Proof. If $\rho(\alpha_1, \dots, \alpha_n)$ is simplex, the statement is clearly true by Theorem 3.2. Let $\rho(\alpha_1, \dots, \alpha_n)$ be nonsimplex, and suppose $(X_1, \dots, X_n)$ is a weight of multiplicity one. By Theorem 3.2, after adjusting $(X_1, \dots, X_n)$ by some action of S_n, we have the existence of an integer j $(1 \le j \le n-1)$ such that

$$(X_1, \dots, X_n) \in \mathscr{M}(\rho(\alpha_1, \dots, \alpha_j)) \times \mathscr{M}(\rho(\alpha_{j+1}, \dots, \alpha_n)).$$

If either $\rho(\alpha_1, \dots, \alpha_j)$ or $\rho(\alpha_{j+1}, \dots, \alpha_n)$ is nonsimplex, one could repeat the argument and find the existence of integers $1 \le k_j \le n-1$ for $j = 1, \dots, l$ and for some l such that (again adjusting $(X_1, \dots, X_n)$ by the action of S_n).

(i) $(X_1, \dots, X_n) \in \mathscr{M}(\rho(\alpha_1, \dots, \alpha_{k_1})) \times \mathscr{M}(\rho(\alpha_{k_1+1}, \dots, \alpha_{k_1+k_2})) \times \cdots \times \mathscr{M}(\rho(\alpha_{k_1+\cdots+k_{l-1}+1}, \dots, \alpha_n))$. (The sum of k_j must necessarily be n.)

(ii) $\rho(\alpha_{k_1+\cdots+k_j+1}, \dots, \alpha_{k_1+\cdots+k_{j+1}})$ are simplex representations of $U(k_{j+1})$ for $j = 1, \dots, l-1$.

Now we could replace $(X_1, \dots, X_n)$ by any $S_{k_1} \times S_{k_2} \times \cdots \times S_{k_l}$ translates of $(X_1, \dots, X_n)$ and such weights are necessarily of multiplicity one. This set of weights lies precisely on a $(S(k_1), \dots, S(k_l))$ facet of the weight polyhedron, containing $(X_1, \dots, X_n)$, and the corresponding weight vectors span an irreducible representation of $U(k_1) \times U(k_2) \times \cdots \times U(k_l)$ given by

$$\rho(\alpha_1, \dots, \alpha_{k_1}) \otimes \rho(\alpha_{k_1+1}, \dots, \alpha_{k_1+k_2}) \otimes \cdots \otimes \rho(\alpha_{k_1+\cdots+k_{l-1}+1}, \dots, \alpha_n).$$

This completes the proof. □

Corollary 3.8. *If* $(X_1, \dots, X_n)$ *is a dominant multiplicity-one weight of* $\rho(\alpha_1, \dots, \alpha_n)$, *then there exists a partition of* n, $(k_1, \dots, k_l)$, *such that*

$$\sum_{i=1}^{k_j} X_{n_j+i} = \sum_{i=1}^{k_j} \alpha_{n_j+i} \quad \textit{for } 1 \le j \le l-1$$

where $n_1 = 0$, $n_j = \sum_{s=1}^{j-1} k_s$ *for* $2 \le j \le l$, *and* $\rho(\alpha_{n_j+1}, \alpha_{n_j+2}, \dots, \alpha_{n_j+k_j})$ *is a simplex representation.*

Proof. If $(X_1, \dots, X_p)$ is a dominant weight of a representation $\rho(\alpha_1, \dots, \alpha_n)$ then we may write (2.7) as

$$\sum_{j=1}^{k} X_j \le \sum_{j=1}^{k} \alpha_j, \qquad k = 1, \dots, n-1,$$

and

$$\sum_{j=1}^{n} X_j = \sum_{j=1}^{n} \alpha_j.$$

Along with Theorem 3.7, this gives our result. □

FIGURE 1

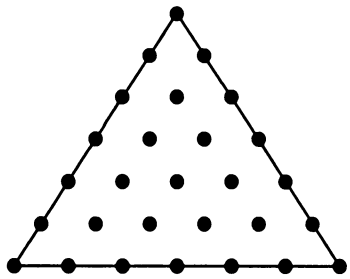

FIGURE 2a

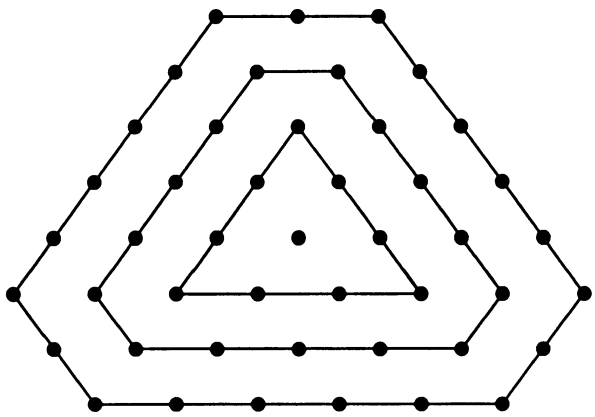

FIGURE 2b

We conclude this section with some simple weight diagrams, illustrating the theorems just proven above. It is well known (see [H]) that the weight diagrams for irreducible representations of $U(2)$ and $U(3)$ are given in Figures 1, 2a, and 2b.

In the case of $U(3)$, we get a "nonregular hexagon", whose opposite sides are parallel though not necessarily equal, with weights given by the set of lattice points on it. This is the general case, and the multiplicity-one weights fall on the boundary. The simplex representations correspond to triangles in which all weights are of multiplicity one. Less known are the weight diagrams for $U(4)$.

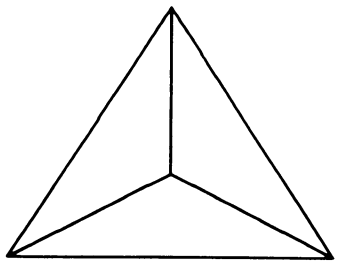

FIGURE 3a

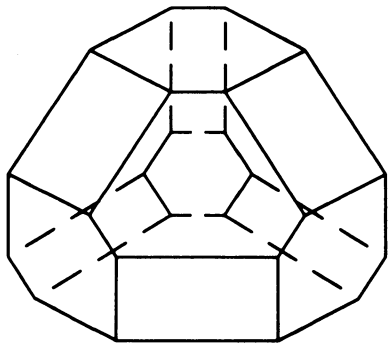

FIGURE 3b

Figure 3a corresponds to the weight polyhedrons of simplex representations of $U(4)$. The determinant representation, of course, has only a point for its weight diagram. Figure 3b is the weight polyhedron of $\rho(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$. This looks like a 'truncated' tetrahedron. The hexagonal faces would correspond to representations $\rho(\alpha_1, \alpha_2, \alpha_3) \otimes \rho(\alpha_4)$ and $\rho(\alpha_1) \otimes \rho(\alpha_2, \alpha_3, \alpha_4)$, while the rectangular faces would correspond to representations of $\rho(\alpha_1, \alpha_2) \otimes \rho(\alpha_3, \alpha_4)$ (cf. Lemma 2.8). From Theorem 3.7, the multiplicity-one weights live on the sides of the hexagonal faces as well as the rectangular faces.

The remaining three figures are slight degenerations of Figure 3b. Figure 3c corresponds to $\rho(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where $\alpha_1 = \alpha_2 > \alpha_3 > \alpha_4$ or $\alpha_1 > \alpha_2 > \alpha_3 = \alpha_4$. The rectangular faces $\rho(\alpha_1, \alpha_2) \otimes \rho(\alpha_3, \alpha_4)$ collapses into lines and four of the faces are triangular (for instance, if $\alpha_1 = \alpha_2 > \alpha_3 > \alpha_4$, then the faces defined by representations $\rho(\alpha_1, \alpha_2, \alpha_3) \otimes \rho(\alpha_4)$ are triangular, and those defined by $\rho(\alpha_1) \otimes \rho(\alpha_2, \alpha_3, \alpha_4)$ remains hexagonal). From Theorem 3.7, the multiplicity-one weights live on the triangular faces and the sides of the hexagonal faces.

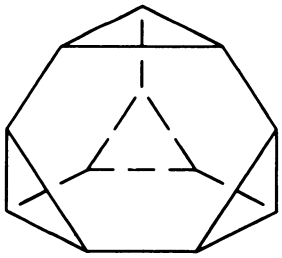

FIGURE 3c

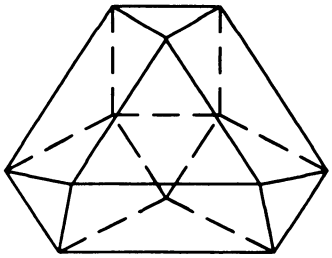

FIGURE 3d

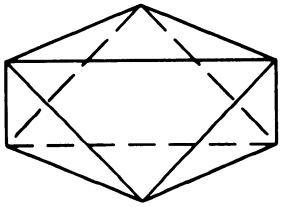

FIGURE 3e

In a similar way, it is easy to see that Figures 3d and 3e are weight polyhedrons for $\rho(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where $\alpha_1 > \alpha_2 = \alpha_3 > \alpha_4$ and $\alpha_1 = \alpha_2 > \alpha_3 = \alpha_4$ respectively. In both cases, weights living on any faces are necessarily multiplicity one.

4. Multiplicity-one weights of tensor products

Let us extend the analysis in §3 to weights of tensor products of two representations of $U(n)$. First observe that the weights of the inner tensor product of two irreducible representations of $U(n)$ given by $\rho(\alpha_1, \dots, \alpha_n) \otimes \rho(\beta_1, \dots, \beta_n)$ are of the form $(X_1 + Y_1, X_2 + Y_2, \dots, X_n + Y_n)$ where $(X_1, \dots, X_n)$ is a weight of $\rho(\alpha_1, \dots, \alpha_n)$ and $(Y_1, \dots, Y_n)$ is a weight of $\rho(\beta_1, \dots, \beta_n)$.

The representation $\rho(\alpha_1, \dots, \alpha_n) \otimes \rho(\beta_1, \dots, \beta_n)$ decompose into a direct sum of representations $\rho(\gamma_1, \dots, \gamma_n)$ where

$$(\gamma_1, \dots, \gamma_n) = (\alpha_1, \dots, \alpha_n) + (X_1, \dots, X_n)$$

and $(X_1, \dots, X_n)$ is a weight of $\rho(\beta_1, \dots, \beta_n)$ (cf. [H] or [Zh]). Amongst these, the representation $\rho(\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$ appears with multiplicity one. Therefore the set of weights of $\rho(\alpha_1, \dots, \alpha_n) \otimes \rho(\beta_1, \dots, \beta_n)$ is also the set of weights of $\rho(\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$. In particular, multiplicity-one weights of $\rho(\alpha_1, \dots, \alpha_n) \otimes \rho(\beta_1, \dots, \beta_n)$ must necessarily be of multiplicity one in $\rho(\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$, but not conversely.

Next we give a simple decomposition of multiplicity-one weights of tensor products of two irreducible representations of $U(n)$.

Proposition 4.1. *Consider the inner tensor product $\rho(\alpha_1, \dots, \alpha_n) \otimes \rho(\beta_1, \dots, \beta_n)$ of $U(n)$.*

(a) *If neither $\rho(\alpha_1, \dots, \alpha_n)$ nor $\rho(\beta_1, \dots, \beta_n)$ is a determinant representation and $\rho(\alpha_1 + \beta_1, \dots, \alpha_n + \beta)$ is not a simplex representation, we have*

$$\begin{aligned}\mathscr{M}&(\rho(\alpha_1, \dots, \alpha_n) \otimes \rho(\beta_1, \dots, \beta_n)) \\ &= S_n \cdot \bigcup_{j=1}^{n-1} \{\mathscr{M}(\rho(\alpha_1, \dots, \alpha_j) \otimes \rho(\beta_1, \dots, \beta_j)) \\ &\qquad \times \mathscr{M}(\rho(\alpha_{j+1}, \dots, \alpha_n) \otimes \rho(\beta_{j+1}, \dots, \beta_n))\}.\end{aligned}$$

(b) *If $\rho(\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$ is a simplex representation, then*

$$\mathscr{M}(\rho(\alpha_1, \dots, \alpha_n) \otimes \rho(\beta_1, \dots, \beta_n)) = S_n \cdot \{(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n)\}.$$

(c) $\mathscr{M}(\rho(\alpha_1, \dots, \alpha_n) \otimes \det^\beta) = \mathscr{M}(\rho(\alpha_1 + \beta, \alpha_2 + \beta, \dots, \alpha_n + \beta))$ *for $\beta \in \mathbb{Z}$, and where $\det^\beta = \rho(\beta, \dots, \beta)$ is a determinant representation.*

Proof. We have noted that a multiplicity-one weight of

$$\rho(\alpha_1, \dots, \alpha_n) \otimes \rho(\beta_1, \dots, \beta_n)$$

is necessarily a multiplicity-one weight of $\rho(\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$. If $\rho(\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$ is not a simplex representation, then Theorem 3.2 says that if $(X_1, \dots, X_n)$ is a multiplicity-one weight of $\rho(\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$, then (after adjusting $(X_1, \dots, X_n)$ by some action of S_n) we have, for some $1 \le j < n$

$$X_1 + \cdots + X_j = \alpha_1 + \cdots + \alpha_j + \beta_1 + \cdots + \beta_j.$$

From Lemma 2.8 the weights on this face looks like the weights of

$$\rho(\alpha_1+\beta_1,\dots,\alpha_j+\beta_j)\otimes\rho(\alpha_{j+1}+\beta_{j+1},\dots,\alpha_n+\beta_n)$$

(of $\rho(\alpha_1+\beta_1,\dots,\alpha_n+\beta_n)|_{U(j)\times U(n-j)}$). To index the Gel'fand-Cetlin basis of a tensor product, we have to have a pair of Gel'fand-Cetlin vectors (u_α, u_β) where u_α corresponds to $\rho(\alpha_1,\dots,\alpha_n)$ and u_β corresponds to $\rho(\beta_1,\dots,\beta_n)$. Write as in (2.1) and (2.2),

$$u_\alpha=(\alpha_i^j)_{\substack{j=0,1,\dots,n-1\\ i=1,\dots,n-j}}\quad\text{and}\quad u_\beta=(\beta_i^j)_{\substack{j=0,1,\dots,n-1\\ i=1,\dots,n-j}}. \tag{4.2}$$

Let the weight of u_α be $(X_1^\alpha,\dots,X_n^\alpha)$ and the weight of u_β be $(X_1^\beta,\dots,X_n^\beta)$. Then the weight of $u_\alpha\otimes u_\beta$ is $(X_1^\alpha+X_1^\beta,\dots,X_n^\alpha+X_n^\beta)$. If such a weight were to satisfy

$$(X_1^\alpha+X_1^\beta)+\cdots+(X_j^\alpha+X_j^\beta)=\alpha_1+\cdots+\alpha_j+\beta_1+\cdots+\beta_j,$$

then by the relations (2.7), we must have

$$X_1^\alpha+\cdots+X_j^\alpha=\alpha_1+\cdots+\alpha_j\quad\text{and}\quad X_1^\beta+\cdots+X_j^\beta=\beta_1+\cdots+\beta_j.$$

This means that $(X_1^\alpha,\dots,X_j^\alpha)$ and $(X_1^\beta,\dots,X_j^\beta)$ are both multiplicity-one weights of $\rho(\alpha_1,\dots,\alpha_j)$ and $\rho(\beta_1,\dots,\beta_j)$ respectively, and in particular, $(X_1^\alpha+X_1^\beta,\dots,X_j^\alpha+X_j^\beta)$ is a multiplicity-one weight of $\rho(\alpha_1,\dots,\alpha_j)\otimes\rho(\beta_1,\dots,\beta_j)$. This completes the proof for (a). Statement (c) is trivial.

It is easy to see that given $\alpha_1\geq\cdots\geq\alpha_n$ and $\beta_1\geq\cdots\geq\beta_n$, $\rho(\alpha_1+\beta_1,\dots,\alpha_n+\beta_n)$ is simplex if and only if one of the following is satisfied:

(i) $\alpha_1=\cdots=\alpha_{n-1}$ and $\beta_1=\cdots=\beta_{n-1}$,

(ii) $\alpha_2=\cdots=\alpha_n$ and $\beta_2=\cdots=\beta_n$.

Look at a multiplicity-one weight $(X_1,\dots,X_n)$ of a simplex representation $\rho(\alpha_1+\beta_1,\dots,\alpha_n+\beta_n)$ satisfying (i) above (the other case is similar). Assume that neither $\rho(\alpha_1,\dots,\alpha_n)$ nor $\rho(\beta_1,\dots,\beta_n)$ is the determinant representation (these cases are covered by statement (c)). Write the corresponding Gel'fand-Cetlin basis vector as in (4.2). The weight $(X_1^\alpha,\dots,X_n^\alpha)$ for u_α may be written as (just look at a typical Gel'fand-Cetlin vector)

$$(\alpha_1-(\gamma_1+\cdots+\gamma_{n-1}),\alpha_2+\gamma_1,\alpha_2+\gamma_2,\dots,\alpha_2+\gamma_{n-1})$$

and likewise the weight $(X_1^\beta,\dots,X_n^\beta)$ for u_β as

$$(\beta_1-(\delta_1+\cdots+\delta_{n-1}),\beta_2+\delta_1,\beta_2+\delta_2,\dots,\beta_2+\delta_{n-1})$$

where all γ_j, δ_j are nonnegative integers satisfying

$$\gamma_1+\cdots+\gamma_{n-1}\leq\alpha_1-\alpha_2\quad\text{and} \tag{4.3}$$

$$\delta_1+\cdots+\delta_{n-1}\leq\beta_1-\beta_2. \tag{4.4}$$

Observe that if some γ_j, say γ_1, is nonzero, then

$$(\alpha_1+(\gamma_1+\cdots+\gamma_{n-1}-1),\alpha_2+\gamma_1-1,\alpha_2+\gamma_2,\dots,\alpha_2+\gamma_{n-1})$$

is still a weight of $\rho(\alpha_1,\alpha_2,\dots,\alpha_2)$ and if equality holds in (4.4), then

$$(\beta_1-(\delta_1+\cdots+\delta_{n-1}+1),\beta_2+\delta_1+1,\beta_2+\delta_2,\dots,\beta_2+\delta_{n-1})$$

is also a weight of $\rho(\beta_1, \beta_2, \dots, \beta_2)$. In particular, the tensor product of the corresponding weight vectors has the same weight as $u_\alpha \otimes u_\beta$. Likewise if equality holds in (4.4), some δ_j must be nonzero and we can still produce another vector in the tensor product with the same weight as $u_\alpha \otimes u_\beta$, unless equality holds in (4.3).

Suppose equality holds for both (4.3) and (4.4), then some δ_j and δ_k is nonzero. Suppose $j \neq k$, say $\gamma_1 \neq 0$ and $\delta_2 \neq 0$. Then

$$(\alpha_1 - (\gamma_1 + \cdots + \gamma_{n-1}), \alpha_2 + \gamma_1 - 1, \alpha_2 + \gamma_2 + 1, \alpha_2 + \gamma_3, \dots, \alpha_2 + \gamma_{n-1})$$

and

$$(\beta_1 - (\delta_1 + \cdots + \delta_{n-1}), \beta_2 + \delta_1 + 1, \beta_2 + \delta_2 - 1, \beta_2 + \delta_3, \dots, \beta_2 + \delta_{n-1})$$

are both weights of $\rho(\alpha_1, \alpha_2, \dots, \alpha_2)$ and $\rho(\beta_1, \beta_2, \dots, \beta_2)$ respectively. Further, the tensor product of the corresponding weight vectors will have the same weight as $u_\alpha \otimes u_\beta$. Thus, our comments above tell us that if the weight of $u_\alpha \otimes u_\beta$ is of multiplicity one, then either $\gamma_j = \delta_j = 0$ for all $j = 1, \dots, n-1$ or $\gamma_j = \alpha_1 - \alpha_2$ and $\beta_j = \beta_1 - \beta_2$ for some $j = 1, \dots, n-1$. Hence the weight of $u_\alpha \otimes u_\beta$ must be a permutation of $(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_2 + \beta_2)$. Such weights are necessarily of multiplicity one in the product, and so we have (b). □

5. On K-types of principal series representations of $U(n, n)$

Let $U(n, n)$ be the subgroup of $GL(2n, \mathbb{C})$ stabilizing the indefinite Hermitian form $z_1\overline{z}_1 + \cdots + z_n\overline{z}_n - (z_{n+1}\overline{z}_{n+1} + \cdots + z_{2n}\overline{z}_{2n})$ on $\mathbb{C}^{2n}$. A maximal compact subgroup K can be chosen as $U(n) \times U(n)$ which is embedded as diagonal blocks in $U(n, n)$. Consider a minimal parabolic subgroup $P_{\min}$ of $U(n, n)$. The compact part of the Levi-factor of the minimal parabolic $P_{\min}$ can then be identified as the subgroup $M = U(1)^n$ embedded as follows:

$$(u_1, \dots, u_n) \in \underbrace{U(1) \times \cdots \times U(1)}_{n \text{ copies}} \hookrightarrow \operatorname{diag}(u_1, \dots, u_n, u_1, \dots, u_n) \in U(2n).$$

We have the Langlands decomposition $P_{\min} = MAN$. Let σ be an irreducible representation of M, and $\nu \in \mathfrak{a}^*$ where $\mathfrak{a}$ is the complexified Lie algebra of A. Consider the principal series representation $(\Pi_{\sigma,\nu}, \operatorname{Ind}_{P_{\min}}^{U(n,n)} \sigma \otimes e^\nu \otimes 1)$ of $U(n, n)$. We want to know the K-types that appear in $\Pi_{\sigma,\nu}|_K$. By Mackey's Subgroup Theorem (see [M]), we have

$$\Pi_{\sigma,\nu}|_K \cong \operatorname{Ind}_M^K \sigma. \tag{5.1}$$

Therefore it suffices to try to understand $\operatorname{Ind}_M^K \sigma$. The decomposition above implies that each K-type can only appear finitely many times in $\Pi_{\sigma,\nu}|_K$. The multiplicity ones are extremely interesting and has been the subject of much research.

An irreducible representation σ of M can be written as

$$\sigma = \rho(\mu_1) \otimes \rho(\mu_2) \otimes \cdots \otimes \rho(\mu_n) \tag{5.2}$$

where $\rho(\mu_i) \in \widehat{U}(1)$ for $i = 1, \dots, n$. Since $K \cong U(n) \times U(n)$ we may parametrize irreducible representations of K (via their highest weights) by

$(X_1, \dots, X_n, Y_1, \dots, Y_n)$ where $\rho(X_1, \dots, X_n)$ and $\rho(Y_1, \dots, Y_n)$ are in $\widehat{U}(n)$. By Frobenius Reciprocity, we reduced our study of (5.1) to the study of weights in the tensor product of two $U(n)$ modules.

By parametrizing K-types in this way, we can construct a K-type diagram in $\mathbb{R}^{2n}$, in an analogous fashion as the weight diagrams. If we take σ as in (5.2), then the set of K-types of 5.1 will be lattice points $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ in $\mathbb{R}^{2n}$ satisfying

(a) $X_1 \geq X_2 \geq \cdots \geq X_n$;
(b) $Y_1 \geq Y_2 \geq \cdots \geq Y_n$;
(c) $\mu_{i_1} + \cdots + \mu_{i_j} \leq X_1 + \cdots + X_j + Y_1 + \cdots + Y_j$, $j = 1, 2, \dots, n-1$;
(d) $\sum_{s=1}^n \mu_s = \sum_{s=1}^n X_s + \sum_{s=1}^n Y_s$.

Here $\{i_j\}$ are distinct elements in $\{1, 2, \dots, n\}$. Conditions (c) and (d) merely asserts that $(\mu_1, \dots, \mu_n)$ is a weight of $\rho(X_1 + Y_1, \dots, X_n + Y_n)$ (cf. Proposition 2.6).

Theorem 5.3. *The set of multiplicity one $(U(n) \times U(n))$-types of*

$$\mathrm{Ind}_{U(1)^n}^{U(n)\times U(n)} \rho(\mu_1) \otimes \cdots \otimes \rho(\mu_n),$$

where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, is given by the set of integer tuples $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ satisfying the following conditions:

There exists a set $\{0 = i_1, i_2, \dots, i_{l-1}, i_l = n\} \subset \{1, \dots, n\}$ such that one of the following holds for any $1 \leq j \leq l-1$:

(a)
$$\mu_{i_j+1} + \cdots + \mu_{i_{j+1}} = (X_{i_j+1} + Y_{i_j+1}) + \cdots + (X_{i_{j+1}} + Y_{i_{j+1}})$$
and $\rho(X_{i_j+1} + Y_{i_j+1}, \dots, X_{i_{j+1}} + Y_{i_{j+1}})$ is a simplex representation and either $\rho(X_{i_j+1}, \dots, X_{i_{j+1}})$ or $\rho(Y_{i_j+1}, \dots, Y_{i_{j+1}})$ is a determinant representation.

(b) $\mu_k = X_k + Y_k$ *for* $i_j + 1 \leq k \leq i_{j+1}$ *and either*

$$\begin{cases} X_{i_j+1} \geq X_{i_j+2} = \cdots = X_{i_{j+1}-1} = X_{i_{j+1}} \text{ and} \\ Y_{i_j+1} \geq Y_{i_j+2} = \cdots = Y_{i_{j+1}-1} = Y_{i_{j+1}} \end{cases}$$

or

$$\begin{cases} X_{i_j+1} = X_{i_j+2} = \cdots = X_{i_{j+1}-1} \geq X_{i_{j+1}} \text{ and} \\ Y_{i_j+1} = Y_{i_j+2} = \cdots = Y_{i_{j+1}-1} \geq Y_{i_{j+1}}. \end{cases}$$

Remarks. Geometrically, we observe that the multiplicity-one K-types in this case would live on facets of the hull of K-types which correspond to tensor products of simplex representations. More accurately, in both cases (a) and (b), we note that they correspond to simplex representations such that each $\rho(X_{i_j+1} + Y_{i_j+1}, \dots, X_{i_{j+1}} + Y_{i_{j+1}})$ is simplex.

Proof. This is an easy consequence of Theorem 3.7, Proposition 4.1, and Frobenius Reciprocity. □

It is possible to apply the above techniques to study the K-types of principal series representations of $U(p, q)$ where $p > q > 0$. However, it becomes very combinatorial and messy. We have worked out certain cases, for instance, when $q = 1, 2$ and the spherical principal series. In the latter case, the multiplicity-one K-types are very 'degenerate' in the sense that they sit on codimension greater than one surfaces on the boundaries of the K-type diagram. We omit the details but hope to publish our results when they are more complete. Also it

is possible to obtain an analogue of the hereditary property of weight diagrams for $U(n)$ for K-type diagrams of principal series representations of $U(p, q)$; although the statement is not very enlightening geometrically. In theory, one could therefore use this reduction process to obtain all the multiplicity-one K-types.

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