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WEAK SOLUTIONS OF THE POROUS MEDIUM EQUATION IN A CYLINDER

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ABSTRACT. We show that if $D \subset \mathbf{R}^n$ is a bounded domain with smooth boundary, and $u \in L^m(D \times (\varepsilon, T))$, $u \geq 0$, solves $\frac{\partial u}{\partial t} = \Delta u^m$, $m > 1$, in the sense of distributions on $D \times (0, T)$, and vanishes on $\partial D \times (0, T)$ in a suitable weak sense, then u is Hölder continuous in $\overline{D} \times (0, T)$.

1. INTRODUCTION

The initial value problem for the porous medium

$$\frac{\partial u}{\partial t} = \Delta u^m, \qquad m > 1, \ u \geq 0, \tag{1}$$

has recently received considerable attention. An important direction was the study of the solvability properties of the initial value problem in $\mathbf{R}^n \times (0, \infty)$. We want to mention here the results of Aronson and Caffarelli [1], Benilan, Crandall and Pierre [2] Dahlberg and Kenig [3, 4].

Combining the results of the above papers we have the following picture. Let $\mu \geq 0$ be a measure on $\mathbf{R}^n$. Then equation (1) has a solution in $\mathbf{R}^n \times (0, \infty)$ with initial data μ if and only if

$$\mu(|x| < R) = o(R^n + 2/(m-1))$$

as $R \to \infty$. Furthermore, the solution is unique in the class of nonnegative continuous weak solutions.

For the heat equation $\partial u/\partial t = \Delta u$ we remark that the corresponding growth condition is that

$$\int_{\mathbf{R}^n} e^{-\sigma|x|^2} d\mu < \infty$$

for all $\sigma > 0$.

The theory of the initial value problem for a bounded domain has been carried out in Dahlberg and Kenig [5]. Denote by $\mathscr{P}$ the class of all $u \in C(\overline{D} \times (0, \infty))$, $u \geq 0$, such that $u = 0$ on $\partial D \times (0, \infty)$ and u solves (1) in the distribution sense. In contrast to the linear theory there exists a solution in $\mathscr{P}$ without an initial trace of $t = 0$. In fact, we have the following result.

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Theorem 1.1. *Let $D \subset \mathbf{R}^n$ be a bounded domain with a smooth boundary. Then there is a solution $\beta \geq 0$ of the porous medium equation such that $\beta \in \mathscr{P}$ and $u(x, t) \leq \beta(x, t)$ for all $u \in \mathscr{P}$ and all $(x, t) \in D \times (0, \infty)$.*

By removing this exceptional solution β one has the following theory for the initial value problem.

Theorem 1.2. *Let $D \subset \mathbf{R}^n$ be a bounded domain with a smooth boundary. If $u \in \mathscr{P}$, $u \neq \beta$ then*

$$\sup_{t>0} \int_D \delta(x) u(x, t) dx < \infty$$

where $\delta(x) = \operatorname{dist}(x, \partial D)$. For all $\eta \in C_0^\infty(\mathbf{R}^n)$ with $\eta = 0$ on ∂D the limit

$$\lim_{t \downarrow 0} \int_D u(x, t) \eta(x) dx = \Lambda(\eta, u) \tag{2}$$

exists. Furthermore, there are nonnegative measures ν and λ on D and ∂D respectively, such that

$$\int_D \delta(x) d\mu(x) < \infty, \quad \int_{\partial D} d\lambda < \infty \tag{3}$$

and

$$\Lambda(\eta, u) = \int_D \eta d\mu + \int_{\partial D} \frac{\partial \eta}{\partial n} d\lambda. \tag{4}$$

Here $\partial / \partial n$ denotes differentiation along the normal direction. Conversely, if μ and λ are two nonnegative measures satisfying (3), *then there is a unique $u \in \mathscr{P}$, $u \neq \beta$ satisfying* (4).

Before stating the next part of the theory we want to recall some facts from the potential theory. We refer the reader to e.g. Helms [6] for an account of this.

We will let G denote the Green function of D, i.e. G has the property that the solution of the problem

$$\Delta v = -f \text{ in } D, \quad v = 0 \text{ on } \partial D,$$

is given by the Green potential

$$v(x) = Gf(x) = \int_D G(x, y) f(y) dy.$$

For λ a nonnegative bounded measure on ∂D we denote by $\mathscr{H}\lambda$ the Poisson integral of λ, i.e. the unique nonnegative harmonic function in D that takes the boundary value λ in a weak sense.

A function $h: D \to (-\infty, \infty]$ is called superharmonic in a domain D if it is lower semicontinuous, not identically infinite and whenever $B = B(x, r) = \{\xi : |x - \xi| < r\} \subset D$ one has the inequality

$$h(x) \geq \frac{1}{|B|} \int_B h(\xi) d\xi,$$

where $|B| = \int_B d\xi$. The Riesz representation theorem states that h is nonnegative and superharmonic in D if and only if there are nonnegative measures μ and λ satisfying (3) such that $h = G\mu + \mathscr{H}\lambda$. Furthermore, the measures μ and λ are uniquely determined by h.

Theorem 1.3. *For $u \in \mathscr{P}$ denote the Green potential of u by*

$$w(x, t) = \int_D G(x, y)u(y, t)dy.$$

Then $\partial w/\partial t = -u^m \leq 0$, $\Delta w = -u$ and since w is nonincreasing in t the pointwise limit

$$h(x) = \lim_{t \downarrow 0} w(x, t) \in [0, +\infty]$$

exists for all $x \in D$. Now $u = \beta$ if and only if $h(x) = +\infty$ for all $x \in D$. If $u \in \mathscr{P} \backslash \{\beta\}$ then h is nonnegative and superharmonic in D. Conversely, given a nonnegative and superharmonic function h in D there is a unique solution $u \in \mathscr{P} \backslash \{\beta\}$ such that $h(x) = \lim_{t \downarrow 0} w(x, t)$. Furthermore, if h has the Riesz decomposition $h = G\mu + \mathscr{H}\lambda$ then u takes the initial value (μ, λ) in the sense of (4).

The purpose of this note is to study the regularity properties of weak solutions of the porous medium equation in a cylinder. It was crucial for the above theory to deal with solutions that were a priori known to be continuous. It is therefore natural to ask whether weak solutions are continuous. In a companion paper to this one [6] we have established this for purely local solutions. The present work deals with weak solutions, which are zero in a weak sense on the lateral part of the boundary, and thus, the purely local theory does not apply.

Let as before $D \subset \mathbf{R}^n$ be a bounded domain with a smooth boundary. Let $\mathscr{M}$ denote the class of

$$\eta \in C_0^\infty(\mathbf{R}^n \times (0, \infty))$$

with the property that $\eta = 0$ on $\partial D \times (0, \infty)$. We will say that a nonnegative function u is a weak solution of (1) satisfying Dirichlet boundary conditions if

$$\iint_{D \times (a, b)} u^m dx\, dt < \infty$$

for all $0 < a < b < \infty$ and

$$\iint_\Omega \left(u \frac{\partial \psi}{\partial t} + u^m \Delta \psi \right) dx\, dt = 0$$

for all $\psi \in \mathscr{M}$. Here Ω denotes $D \times (0, \infty)$. We will denote the class of all such solutions by $\mathscr{P}_W$. Our main result is now that all weak solutions are continuous, i.e. $\mathscr{P}_W = \mathscr{P}$.

Theorem 1.4. *Let $D \subset \mathbf{R}^n$ be a bounded domain with a smooth boundary. Suppose $u \geq 0$ is a weak solution of the porous medium equation satisfying Dirichlet boundary conditions. Then there is a continuous function u^* on $\overline{D} \times (0, \infty)$ with $u = u^*$ a.e. in Ω, $u^* = 0$ on $\partial D \times (0, \infty)$.*

The idea for the proof of this result is to first show for each $T > 0$ there is a bounded measure μ_T on D such that for all $\psi \in \mathscr{M}$

$$\int_D \psi(x, T) d\mu_T = \iint_{t > T} \left(u \frac{\partial \psi}{\partial t} + u^m \Delta \psi \right) dx\, dt.$$

The second part of the proof consists of showing that $u = u_T$ a.e. in $\omega_T = \{(x, t): t > T\}$ where u_T is defined as the solution of the porous medium equation in ω_T with data μ_T at times $t = T$ constructed by Theorem 1.2.

2. Existence of trace

We will from now on assume that $D \subset \mathbf{R}^n$ is a bounded domain with a smooth boundary and that $u \geq 0$ is a weak solution of the porous medium equation (1) in $\Omega = D \times (0, \infty)$ with vanishing Dirichlet conditions.

Let $w = Gu$ be the Green potential of u in Ω. We then have that $w \geq 0$ with $\iint_{D\times(a,b)} w dx\, dt < \infty$ for all $0 < a < b < \infty$, furthermore

$$\iint_\Omega (w\Delta\eta + \eta u) dx\, dt = 0$$

for all $\eta \in \mathscr{M}$.

We next observe that $\partial w/\partial t = -u^m$ in the distribution sense. To see this let $\vartheta \in C_0^\infty(\Omega)$ and set $\eta = G\vartheta$. Then

$$\iint w\frac{\partial\vartheta}{\partial t} dx\, dt = -\iint w\Delta\frac{\partial\eta}{\partial t} dx\, dt = \iint u\frac{\partial\eta}{\partial t} dx\, dt - \iint u^m\Delta\eta dx\, dt$$
$$= \iint u^m\vartheta dx\, dt.$$

A particular consequence of this is that if $\tau > 0$ and if $\theta \in C_0^\infty(D)$, $\gamma \in C_0^\infty(\mathbf{R})$ with $\gamma(\tau) = 0$, then an approximation argument shows

$$\iint_{D\times(\tau,\infty)} \theta(x)(\gamma'(t)w(x, t) - \gamma(t)u(x, t)^m) dx\, dt = 0. \tag{5}$$

We can therefore define the trace ν_τ of w at a time $\tau > 0$ by choosing a $\gamma \in C_0^\infty(\mathbf{R})$ with the property that $\gamma(\tau) = 1$

$$\int_D \theta d\nu_\tau = \iint_{D\times(\tau,\infty)} \theta(x)(\gamma(t)u(x, t)^m - \gamma'(t)w(x, t)) dx\, dt.$$

By (5) the measure ν_τ is well defined. We remark that $\nu_\tau \geq 0$. To see this for $\varepsilon > 0$ let $\gamma_\varepsilon \in C_0^\infty(\mathbf{R})$ be chosen so that $\gamma_\varepsilon'(t) \geq 0$ for $\tau - \varepsilon \leq t \leq \tau$, $\gamma_\varepsilon(t) = 0$ for $t \leq \tau - \varepsilon$ and $\gamma_\varepsilon(\tau) = 1$. From (5) it follows that if $\theta \geq 0$ then

$$-\int_d \theta d\nu_\tau = \iint_{D\times(\tau-\varepsilon,\tau)} \theta(x)(\gamma_\varepsilon(t)u(x, t)^m - \gamma_\varepsilon'(t)w(x, t)) dx\, dt \leq o(1) \text{ as } \varepsilon \downarrow 0.$$

It is also easy to see that $\int_D d\nu_\tau < \infty$ and that ν_τ is a weakly continuous function of τ.

We summarize the properties of ν_τ in the following lemma.

Lemma 2.1. *For all $\tau > 0$, w has a trace ν_τ. The trace ν_τ is a nonnegative measure with $\int d\nu_\tau < \infty$. Furthermore, ν_τ is a supersolution of the Laplace equation, i.e. $\Delta\nu_\tau \leq 0$ in the distribution sense. In particular, ν_τ is absolutely continuous.*

Proof. We need only verify that ν_τ is a supersolution. Let $\gamma \in C_0^\infty(\mathbf{R})$, $e_j \in C^\infty(\mathbf{R})$ have the properties that $\gamma(\tau) = 1$, $e_j' \geq 0$, $e_j(t) = 0$ for $t \leq \tau$ and $e_j(t) = 1$ for $t \geq \tau + \frac{1}{j}$. Letting $\theta \in C_0^\infty(D)$ with $\theta \geq 0$ we have that

$$\int_D \Delta\theta d\nu_\tau = \lim_{j\to\infty} \iint_\Omega \Delta\theta(x)e_j(t)(\gamma(t)u(x, t)^m - \gamma'(t)w(x, t)) dx\, dt.$$

So integrating by parts and using the equation gives

$$\int_D \Delta\theta d\nu_\tau = -\lim_{j\to\infty} \iint_\Omega \theta(x)e_j'(t)\gamma(t)u(x,t)dx\,dt \le 0$$

which completes the proof of the lemma.

3. Approximation procedure

We begin by defining a convenient approximation of the Green function. For $f \in C_0^\infty(D)$ let u_f denote the solution of the initial Dirichlet problem for the heat equation, i.e. u_f solves the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0 & \text{for } (x,t) \in D \times (0,\infty), \\ u(x,0) = f(x) & \text{for } x \in D, \\ u(x,t) = 0 & \text{for } x \in \partial D,\ t > 0. \end{cases}$$

We denote by $G_k f$ the operator defined by

$$G_k f = \int_{2^{-k}}^\infty u_f\,dt$$

and let $G_k(x,y)$ denote the corresponding kernel. Gf can be written as

$$Gf = \int_0^\infty u_f\,dt.$$

The following properties are easily verified:

$$G_k(x,y) = G_k(y,x), \quad \Delta G_k \le 0,$$
$$G_k(x,y) \uparrow G(x,y) \quad \text{as } k \uparrow \infty.$$

Setting

$$p_k(x,t) = \int_D G_k(x,y)u(y,t)dy$$

we observe that the inequalities $p_{k+1} \ge p_k$ and $\partial p_k/\partial t \le 0$ hold in the distribution sense. The first inequality follows since G_k is monotonically increasing in k. To see the second inequality pick $\eta \in C_0^\infty(D)$ with $\eta \ge 0$. Then

$$\iint \frac{\partial \eta}{\partial t} p_k\,dx\,dt = \iint \frac{\partial(G_k\eta)}{\partial t} u\,dx\,dt = -\iint \Delta(G_k\eta)u^m\,dx\,dt \ge 0.$$

Pick $\lambda \in C_0^\infty(R)$ satisfying $\lambda \ge 0$, $\int \lambda(t)dt = 1$, and $\lambda = 0$ outside the interval $(\frac{1}{2}, 1)$. Define

$$w_k(x,t) = \int p_k(x, t + s2^{-k})\lambda(s)ds.$$

Lemma 3.1. *The functions w_k are smooth and converge to w in the distribution sense. The following inequalities hold for $k = 1, 2, \ldots$:*

$$w_k \le w_{k+1}, \tag{6}$$

$$\frac{\partial w_k}{\partial t} \le 0, \tag{7}$$

$$\Delta w_k \le 0, \tag{8}$$

$$\frac{\partial w_k}{\partial t} \le \psi(\Delta w_k) \quad \textit{where } \psi(s) = \operatorname{sign}(s)|s|^m. \tag{9}$$

For every $\tau > 0$ *there is an* $f_\tau \in L^1(D)$, $f_\tau \geq 0$ *such that*

$$w_k(x, t) \leq Gf_\tau(x) \quad \text{for } t > \tau. \tag{10}$$

For all $0 < a < b < \infty$

$$\iint_{D\times(a,b)} \left| \frac{\partial w_k}{\partial t} - \frac{\partial w}{\partial t} \right| dx\, dt \to 0 \quad \text{as } k \to \infty. \tag{11}$$

Proof. The properties (6)–(8) are straightforward consequences of the corresponding properties for the functions p_k.

Let f_j denote an orthonormal basis for $L^2(D)$ of eigenfunctions for the Laplace operator, i.e.

$$\begin{cases} \Delta f_j = -\lambda_j f_j & \text{in } D, \\ f_j = 0 & \text{on } \partial D. \end{cases}$$

We define the operator H_t, $t > 0$, and its integral kernel $H_t(x, y)$ as the solution operator for the initial Dirichlet problem, i.e. for $f \in L^2(D)$, $f = \sum c_j f_j$

$$H_t f(x) = u_f(x, t) = \sum c_j e^{-\lambda_j t} f_j(x) = \int_D H_t(x, y) f(y) dy.$$

We next define the operator $\mathscr{P}_k$ for a function q on Ω by

$$\mathscr{P}_k q(x, t) = \iint H_{2^{-k}}(x, y)\lambda(s) q(y, t + s2^{-k}) dy\, ds.$$

With this notation $w_k = \mathscr{P}_k w$ and

$$\psi(\Delta w_k) - \frac{\partial w_k}{\partial t} = \mathscr{P}_k u^m - (\mathscr{P}_k u)^m \geq 0$$

by the fact that $\mathscr{P}_k$ is given by a nonnegative kernel with total integral less than or equal to 1 and Jensen's inequality. Since $\partial(w_k - w)/\partial t = u^m - \mathscr{P}_k u^m$ the property (11) follows from the fact that $H_{2^{-k}}$ is an approximation of the identity.

Since $p_k(x, t) = \int G_k(x, y) u(y, t) dy$ is a decreasing function of t and λ is supported on $[\frac{1}{2}, 1]$ it follows that if $t > \tau > 0$ then

$$w_k(x, t) = \int p_k(x, t + 2^{-k}s)\lambda(s) ds \leq G_k f_\tau \leq G f_\tau$$

where $f_\tau(x) = \frac{2}{\tau} \int_{\tau/2}^{\tau} u(x, s) ds$, which completes the proof of the lemma.

From the lemma follows that for all $(x, t) \in \Omega$ $\lim_{k\to\infty} w_k(x, t)$ exists (it may possibly equal $+\infty$). From now on we redefine w to equal this limit.

Lemma 3.2. *For every* τ *the trace* $d\nu_\tau = w(x, \tau) dx$ *and* $w(x, \tau) = G\mu_\tau(x)$ *for a nonnegative measure* μ_τ *with* $\int d\mu_\tau < \infty$. *Furthermore*

$$\lim_{t \downarrow \tau} w(x, t) = w(x, \tau).$$

Proof. Pick $\tau > 0$, $\theta \in C_0^\infty(D)$ and $\gamma \in C_0^\infty(0, \infty)$ with $\gamma(\tau) = 1$. Then

$$\int_D \theta w_k(x, \tau) = -\iint_{D\times(\tau,\infty)} \theta(x) \left(\gamma(t) \frac{\partial w_k(x, t)}{\partial t} + \gamma'(t) w_k(x, t) \right) dx\, dt.$$

By Lemma (3.1) $\partial w_k/\partial t \to \partial w/\partial t$ in $L^1(D) \times \operatorname{supp}(\gamma)$ and $w_k \uparrow w$ as $k \to \infty$ which shows $d\nu_\tau = w(x, \tau) dx$.

It also follows from Lemma (3.1) that $w(x, \tau)$ is superharmonic with $w(x, \tau) \leq Gf_\tau$, $0 \leq f_\tau \in L^1(D)$, since w is the increasing limit of the superharmonic functions w_k. We recall that a superharmonic function is a Green potential in D if and only if the largest harmonic minorant is identically zero, see Helms [6]. Since $w(x, \tau)$ is bounded by a potential we therefore have that $w(x, \tau) = G\mu_\tau(x)$ for some measure $\mu_\tau \geq 0$. To see that μ_τ has finite mass we notice that if $h_j = G\sigma_j$, $\sigma_j \geq 0$, is an increasing sequence of Green potentials with σ_j having a compact support in D and $\lim_{j\to\infty} h_j(x) = 1$ for all $x \in D$ then

$$\int h_j \, d\mu_\tau = \int w(x, \tau) d\sigma_j \leq \int Gf_\tau \, d\sigma_j = \int h_j f_\tau dx \leq \int f_\tau \, dx.$$

Letting $j \to \infty$ shows that $\int d\mu_\tau < \infty$. We remark that if D_j is a sequence of domains with $\overline{D}_j \subset D_{j+1}$ and $\bigcup D_j = D$ then h_j can be defined by $h_j = 1$ in $\overline{D}_j$ and $h_j = 0$ on ∂D with $\Delta h_j = 0$ in $D \backslash D_j$.

Since the functions w_k are decreasing functions of t it follows that w has the same property and hence the pointwise limit

$$h_\tau(x) = \lim_{t \downarrow \tau} w(x, t)$$

exists for all $x \in D$ and is superharmonic. Pick $\eta \in C_0^\infty(\mathbf{R}^n)$, $\eta \geq 0$, $\int \eta dx = 1$ and set for $\xi \in \mathbf{R}^n$, $\varepsilon > 0$ $\eta_{\xi,\varepsilon}(x) = \varepsilon^{-n}\eta(\frac{\xi - x}{\varepsilon})$. It now follows from the weak continuity of the trace of w that

$$\int \eta_{\xi,\varepsilon}(x) w(x, \tau) dx = \int \eta_{\xi,\varepsilon}(x) h_\tau(x) dx \tag{12}$$

for all ξ in D. We now recall that for all superharmonic functions h in D all points are Lebesgue points, i.e. for all $\xi \in D$

$$\lim_{\varepsilon \downarrow 0} \int \eta_{\xi,\varepsilon}(x) h(x) dx = h(x).$$

Therefore by taking the limit as $\varepsilon \downarrow 0$ in (12) shows that $h_\tau(\xi) = w(\xi, \tau)$ for all $\xi \in D$ which completes the proof of the lemma.

4. Main result

For $\tau > 0$ we define $u^*_{k,\tau}$ as the strong solution of the porous medium equation (1) in $D \times (\tau, \infty)$ that satisfies $u^*_{k,\tau} = 0$ on $\partial D \times (\tau, \infty)$ and $u^*_{k,\tau}(x, \tau) = u_k(x, \tau)$, $x \in D$, where $u_k = -\Delta w_k$. Set $w^*_{k,\tau} = Gu^*_{k,\tau}$. Since by Theorem 1.3 $w^*_{k,\tau}$ is decreasing in t we have for $(x, t) \in D \times (\tau, \infty)$ that

$$w^*_{k,\tau}(x, t) \leq w_k(x, \tau) \leq w(x, \tau).$$

Letting $\psi(s) = \operatorname{sign}(s)|s|^m$ we have that w_k is smooth and solves the inequality $\partial w_k / \partial t \leq \psi(\Delta w_k)$ and $\partial w^*_{k,\tau}/\partial t = \psi(\Delta w^*_{k,\tau})$. It now follows from the maximum principle (for a proof see Pierre [7] or Dahlberg and Kenig [5]) that

$$w_k \leq w^*_{k,\tau} \quad \text{in } D \times (\tau, \infty). \tag{13}$$

(The proof of the maximum principle in the above papers is only carried out for solutions but it also carries over to the present case of a smooth subsolution.)

Letting β be the extremal solution of the porous medium equation as described in Theorems (1.2) and (1.3) we have $u^*_{k,\tau}(x,\tau) \le \beta(x, t-\tau)$, $(x,t) \in D \times (\tau, \infty)$. The family $\{u^*_{k,\tau}\}_{k=1}^{\infty}$ is therefore uniformly bounded and hence equicontinuous (see Sacks [8]) on $\overline{D} \times [a, \infty)$ for every $a > \tau$. A subsequence will therefore converge uniformly to a strong solution u^*_τ of the porous medium equation on $\overline{D} \times [a, \infty)$ for each $a > \tau > 0$. Since $\{w^*_{k,\tau}\}_{k=1}^{\infty}$ is a monotonically increasing sequence of functions (again by the maximum principle explained above) it follows that the limit w^*_τ satisfies

$$\lim_{k\to\infty} w^*_{k,\tau} = Gu^*_\tau = w^*_\tau .$$

In particular we have that $w(x,t) \le w^*_\tau(x,t)$ for $(x,t) \in D \times (\tau, \infty)$. Setting $M(t) = \sup\{w(x,t) : x \in D\}$ and observing that $\tau > 0$ is arbitrary we see that $M(t) < \infty$ for all $t > 0$. Also, $w^*_\tau(x,t) \le M(t)$ for $t > \tau$.

We denote by $H_0(D)$ the Sobolev space that is the Hilbert space completion of $C_0^\infty(D)$ equipped with the norm $\|\theta\| = \int |\operatorname{grad}\theta|^2 dx$. We recall that if μ is a nonnegative measure on D then $G\mu \in H_0(D)$ if and only if $\mathscr{E}(\mu) = \int G\mu d\mu < \infty$ with the energy $\mathscr{E}(\mu) = \|G\mu\|^2$.

Since $w(x,t) = G\mu_t$ with μ_t a nonnegative measure with $\int d\mu_t < \infty$ we have that $\mathscr{E}(\mu_t) < \infty$ for all $t > 0$. We need the following lemma in order to study the continuity of $w(\cdot, t)$ in the H_0-norm.

Lemma 4.1. *Let $\mu, \mu_1, \mu_2, \ldots$ be nonnegative measures on D with $G(\mu_j) \le G(\mu_{j+1}) \le G(\mu)$ and $\mathscr{E}(\mu) < \infty$. Assume also that $\lim_{j\to\infty} G\mu_j(x) = G\mu(x)$ for all $x \in D$. Then $\mathscr{E}\mu_j \uparrow \mathscr{E}(\mu)$ and $\|G\mu_j - G\mu\| \to 0$ as $j \to \infty$.*

Proof. Using Fubini's theorem we see that

$$\begin{aligned}\mathscr{E}(\mu_j) &= \int G\mu_j d\mu_j \le \int G\mu_{j+1} d\mu_j = \int G\mu_j d\mu_{j+1} \\ &\le \int G\mu_{j+1} d\mu_{j+1} \le \mathscr{E}(\mu_{j+1}).\end{aligned}$$

Hence $\mathscr{E}(\mu_j) \le \mathscr{E}(\mu)$. If $j \ge k$ then

$$\mathscr{E}(\mu_j) \ge \int G\mu_k d\mu_j = \int G\mu_j d\mu_k$$

so by the monotone convergence theorem

$$\lim_{j\to\infty} \mathscr{E}(\mu_j) \ge \int G\mu d\mu_k = \int G\mu_k d\mu$$

so letting $k \to \infty$ and again using the monotone convergence theorem shows that $\lim_{j\to\infty} \mathscr{E}(\mu_j) = \mathscr{E}(\mu)$ and that

$$\|\mu_j - \mu\|^2 = \mathscr{E}(\mu) + \mathscr{E}(\mu_j) - 2\int G\mu_j d\mu \to 0$$

as $j \to \infty$ which yields the lemma.

From Lemma (4.1) it follows for all $\tau > 0$ that

$$\begin{aligned}\lim_{t\downarrow\tau} \|w(\cdot,t) - w(\cdot,\tau)\| &= \lim_{t\downarrow\tau} \|w^*_\tau(\cdot,t) - w(\cdot,\tau)\| \\ &= \lim_{k\to\infty} \|w_k(\cdot,\tau) - w(\cdot,\tau)\| = 0.\end{aligned}$$

Set $e_k(t) = \|w_k(\cdot, t)\|^2$. Since w_k is smooth we find by differentiating e_k that if $0 < \tau < T$ then there is a constant $C = C(\tau, T)$ such that

$$\iint_{D\times(\tau, T)} (\mathscr{P}_k u)(\mathscr{P}_k u^m) dx\, dt = e_k(\tau) - e_k(T) \leq C.$$

Since $\mathscr{P}_k^u$ and $\mathscr{P}_k u^m$ converge pointwise a.e. to u and u^m respectively, we find by Fatou's theorem that $\iint_{D\times(\tau, T)} u^{m+1} dx\, dt < \infty$ and hence

$$\iint_{D\times(\tau, T)} (|\mathscr{P}_k u - u|^{m+1} + |\mathscr{P}_k u^m - u^m|^{(m+1)/m}) dx\, dt \to 0$$

as $k \to \infty$. Differentiating the energy of $w_k(\cdot, t) - w_\tau^*(\cdot, t)$ we see for $0 < \tau < T$ that

$$\|w_k(\cdot, \tau) - w_\tau^*(\cdot, \tau)\|^2 = \|w_k(\cdot, T) - w_\tau^*(\cdot, T)\|^2 + \iint_{D\times(\tau, T)} (\mathscr{P}_k u - u_\tau^*)(\mathscr{P}_k u^m - (u_\tau^*)^m) dx\, dt.$$

Letting $k \to \infty$ we see that

$$\|w(\cdot, T) - w_\tau^*(\cdot, T)\|^2 + \iint_{D\times(\tau, T)} (u - u_\tau^*)(u^m - (u_\tau^*)^m) dx\, dt = 0.$$

Hence $u = u_\tau^*$ a.e. on $D \times (\tau, \infty)$ which yields Theorem 1.4.

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