MOUNTAIN IMPASSE THEOREM AND
SPECTRUM OF SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. This paper studies a minimax problem for functionals in Hilbert space in the form of $G(u) = \frac{1}{2} \rho \|u\|^2 - g(u)$, where $g(u)$ is Fréchet differentiable with weakly continuous derivative. If $G$ has a "mountain pass geometry" it does not necessarily have a critical point. Such a case is called, in this paper, a "mountain impasse". This paper states that in a case of mountain impasse, there exists a sequence $u_j \in H$ such that

$$g'(u_j) = \rho_j u_j, \quad \rho_j \to \rho, \quad \|u_j\| \to \infty,$$

and $G(u_j)$ approximates the minimax value from above. If

$$\gamma(t) = \sup_{\|u\|^2 = t} g(u)$$

and

$$J_0 = \left( 2 \inf_{t_2 > t_1 > 0} \frac{\gamma(t_2) - \gamma(t_1)}{t_2 - t_1}, 2 \sup_{t_2 > t_1 > 0} \frac{\gamma(t_2) - \gamma(t_1)}{t_2 - t_1} \right),$$

then $g'(u) = \rho u$ has a nonzero solution $u$ for a dense subset of $\rho \in J_0$.

1. FORMULATION OF RESULTS

If $g$ is a $C^1$-functional on a Hilbert space and $u$ is a critical point of $g$ on a sphere, then $g'(u) = \rho u$, $\rho \in \mathbb{R}$. This approach to semilinear elliptic equations has been known for decades (cf. [1, 2]), but a question of the range of $\rho$ has remained open. A recent series of papers (cf. [7] and references therein) provides an answer that can be summarized as follows.

Let $H$ be an infinite dimensional Hilbert space and let $g : H \to \mathbb{R}$ be a $C^1$-map (with respect to Fréchet differentiation). Let $H_w$ be the space $H$ supplied with the weak topology. Assume that

(1.1) $g \in C(H_w \to \mathbb{R}),$

(1.2) $g' \in C(H_w \to H).$

(By continuity we always mean local contiuity without uniform bounds.)

In applications to semilinear elliptic problems condition (1.1)–(1.2) correspond to the subcritical growth of the right-hand side.

Consider the following function

(1.3) $\gamma(t) = \sup_{\|u\|^2 = t} g(u).$

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Theorem 1.1. Assume (1.1), (1.2). The function (1.3) is continuous, nondecreasing and possesses left- and right-hand derivatives \( \gamma'_-(t) \leq \gamma'_+(t) \). For every \( t > 0 \) such that \( \gamma'_-(t) \neq 0 \) (\( \gamma'_+(t) \neq 0 \)) there exist \( u_+ \in H \) (\( u_- \in H \)) such that

\[
\|u_\pm\|^2 = t, \quad g(u_\pm) = \gamma(t),
\]

and

\[
2\gamma'_\pm(t)u_\pm = g'(u_\pm).
\]

In other words, the spectrum \( \{\rho\} \) of the problem

\[
\rho u = g'(u), \quad u \in H \setminus \{0\},
\]

contains all the tangent slopes from the graph of \( 2\gamma(t) \). If \( \gamma(t) \) had a continuous derivative, then (1.6) would be solvable for all

\[
\rho \in J_0 = \left( \inf_{r>0} 2\gamma'_-(t), \sup_{r>0} 2\gamma'_+(t) \right).
\]

However, \( \gamma(t) \) is not necessarily differentiable (cf. [7]). On the other hand, [7] gives sufficient conditions for (1.6) to be solvable with \( \rho \in J_0 \). The argument of [7] involves a mountain pass lemma with an additional condition of the (PS)-type which we avoid here.

The main result of this paper is

Theorem 1.2. Let \( g \) satisfy (1.1), (1.2). Then for every \( \rho \in J_0 \) such that (1.6) is not solvable there is a sequence \( \rho_j \in J_0 \) and a sequence \( u_j \in H \), such that

\[
\rho_j > \rho, \quad \rho_j \to \rho, \quad \rho_j u_j = g'(u_j), \quad \|u_j\| \to \infty.
\]

Theorem 1.2 means that the set of eigenvalues of (1.6) is dense in \( J_0 \) and that a missing eigenvalue always can be approximated by a “blow up” sequence. Theorem 1.2 reflects a technical situation that might be called a mountain impasse. This is the case when a functional has a standard mountain pass geometry but no critical points. This situation is handled by Theorem 2.1 which is a refinement of Schechter's Mountain Pass Alternative [4]. Section 2 contains the proof of Theorem 2.1. Section 3 discusses applications to semilinear elliptic problems. The tangible benefit of Theorem 1.2 is not a mountain impasse itself (which to our best understanding was never observed in elliptic problems), but a relation between solvability and priori bounds, widely used before in the topological approach. We will discuss this in more detail at the end of §3.

2. Mountain impasse theorem

Let

\[
G \in C^1(H \to \mathbb{R}), \quad G' \in C(H_w \to H_w)
\]

be a weakly lower semicontinuous functional with a mountain pass geometry, as follows. Let \( \delta > 0 \), \( t_0 > 0 \), \( e \in H \), \( \|e\|^2 > t_0 \) and assume that

\[
G(u) \geq 2\delta > 0 \text{ for } \|u\|^2 = t_0, \text{ while } G(0), G(e) \leq 0.
\]

We assume that \( G \) has no critical points:

\[
G'(u) \neq 0 \text{ when } G(u) \geq \delta.
\]
The following condition will also be required:

\[
\text{if } u_k \overset{w}{\to} u_0, \limsup (G'(u_k), u_k) \leq 0 \text{ and } \]

\[
G'(u_k) - (G'(u_k), u_k)u_k/\|u_k\|^2 \to 0, \text{ then } u_k \to u_0 \text{ in } H.
\] (2.4)

We should note that this condition of a weak (PS) type becomes a mere weak continuity condition when \( G \) is as in (3.2). Let

\[
S_t = \{ u \in H : \|u\|^2 = t \}, \quad B_t = \{ u \in H : \|u\|^2 \leq t \}.
\] (2.5)

For every \( t > \|\varepsilon\|^2 \) we define \( \Phi(t) \) as a collection of paths \( \varphi \in C([0, 1] \to B_t) \) such that

\[
\varphi(0) = 0, \quad \varphi(1) = \varepsilon.
\] (2.6)

Let

\[
\kappa(t) = \inf_{\varphi \in \Phi(t)} \max_{s \in [0, 1]} G(\varphi(s)).
\] (2.7)

From (2.2) it follows that

\[
\kappa(t) \geq 2\delta \quad \text{when } t > \|\varepsilon\|^2.
\] (2.8)

**Theorem 2.1.** Assume (2.1)–(2.4). There exist a sequence \( \alpha_j > 0 \), \( \alpha_j \to 0 \), and a sequence \( u_j \in H \setminus \{0\} \), \( \|u_j\| \to \infty \), such that

\[
G'(u_j) = -\alpha_j u_j,
\] (2.9)

\[
G(u_j) \geq \delta.
\] (2.10)

The proof of Theorem 2.1 will be given as a sequence of lemmas. Relations (2.1)–(2.4) are assumed throughout §2. The following statement can be found in [5].

**Lemma 2.2.** Let \( Z(u) \in C(B_t \to H) \) and \( Z(u) \neq 0 \) on \( B_t \setminus \{0\} \). Assume that there is a closed subset \( Q \) of \( B_t \setminus \{0\} \) and a \( \theta < 1 \) such that

\[
(Z(u), u) + \theta \|Z(u)\| \|u\| \geq 0, \quad u \in Q.
\] (2.11)

Then for each \( \alpha < (1 - \theta) \) there is a locally Lipschitz mapping \( Y(u) : B_t \setminus \{0\} \to H \) such that

\[
(Z(u), Y(u)) \geq \alpha \|Z(u)\|, \quad u \in B_t \setminus \{0\},
\] (2.12)

\[
(Y(u), u) > 0, \quad u \in Q,
\] (2.13)

and

\[
\|Y(u)\| \leq 1, \quad u \in B_t \setminus \{0\}.
\] (2.14)

**Lemma 2.3.** Assume that there is an \( \varepsilon > 0 \), such that

\[
G'(u) = \beta u
\] (2.15)

has no solution \( u \) when

\[
u \in H_\delta = \{ u \in H : G(u) \geq \delta \},
\] (2.16)

and \( \beta \in [-2\varepsilon, 0] \). Then for any \( t > \|\varepsilon\|^2 \) there exist a \( \theta < 1 \) such that

\[
(G'(u), u) + \theta \|G'(u)\| \|u\| \leq 0, \quad u \in H_\delta \cap B_t \Rightarrow (G'(u), u) \leq -\varepsilon \|u\|^2.
\] (2.17)
Proof. Assume the opposite, namely that there is a sequence \( u_j \in H_\delta \cap B_t \) and a sequence \( \theta_j \to 1 \), \( \theta_j < 1 \), such that

\[
(2.18) \quad (G'(u_j), u_j) + \theta_j \|G'(u_j)\| \|u_j\| \leq 0,
\]

\[
(2.19) \quad \beta(u_j) := (G'(u_j), u_j)/\|u_j\|^2 \in [-\varepsilon, 0].
\]

Let \( u_i \) be a renamed weakly convergent subsequence, and \( u_0 = w\)-lim \( u_j \). By (2.1) \( G'(u_j) \rightharpoonup G'(u_0) \) and therefore \( G'(u_j) \) is bounded in norm. Then (2.18) easily implies

\[
(2.20) \quad G'(u_j) - \beta(u_j)u_j \to 0.
\]

Then by (2.4) \( u_j \to u_0 \) in \( H_\delta \), \( G(u_0) \geq \delta \), and

\[
(2.21) \quad G'(u_0) = \beta(u_0)u_0, \quad u_0 \in H_\delta \cap B_t.
\]

By (2.19) \( \beta(u_0) \in [-\varepsilon, 0] \) which contradicts assumptions of the lemma. □

Lemma 2.4. There is a number \( r(t) > 0 \) and a number \( \mu > 0 \) independent of \( t \) such that

\[
(2.22) \quad \|G'(u)\| \geq r(t), \quad u \in H_\delta \cap B_t,
\]

\[
(2.23) \quad \|u\| \geq 2\mu, \quad u \in H_\delta \cap B_t.
\]

Proof. (1) Assume that (2.22) fails. Then there is a sequence \( u_j \rightharpoonup u_0 \in B_t \), such that \( G'(u_j) \to 0 \). Then by (2.4) \( u_j \to u_0 \) in \( H_\delta \), \( u_0 \in H_\delta \cap B_t \), and \( G'(u_0) = 0 \), which contradicts (2.2).

(2) Consider the lower bound of \( \|u\| \) on \( H_\delta \). If \( u_j \to 0 \) on \( H_\delta \), then \( G(u_j) \to G(0) \leq 0 \) by (2.2), which contradicts the assumption on \( u_j \). □

Lemma 2.5. Under assumptions of Lemma 2.3,

\[
(2.24) \quad D^+_t \kappa(t) \leq -\frac{1}{2} \varepsilon \mu^2 / t, \quad t > \|e\|^2,
\]

with \( \mu \) as in (2.23).

Proof. (1) Let us define the following sets:

\[
Q_0 = \{ u \in B_{2t} : |G(u) - \kappa(t)| \leq \delta / 2 \},
\]

\[
\tilde{Q}_0 = \{ u \in B_{2t} : |G(u) - \kappa(t)| \geq \delta \},
\]

\[
Q_1 = \left\{ u \in B_{2t} : \frac{(G'(u), u)}{\|G'(u)\| \|u\|} \leq -1 + \eta \right\},
\]

\[
Q_2 = \left\{ u \in B_{2t} : \frac{(G'(u)u)}{\|G'(u)\| \|u\|} \geq -1 + 2\eta \right\},
\]

where \( t > \|e\|^2 \), \( 0 < 2\eta < 1 - \theta \) with \( \theta \) as in Lemma 2.3 applied to the ball \( B_{2t} \). Let

\[
\chi_0(u) = d(u, \tilde{Q}_0)/(d(u, Q_0) + d(u, \tilde{Q}_0)),
\]

\[
\chi_1(u) = d(u, Q_2)/(d(u, Q_1) + d(u, Q_2)),
\]

\[
\chi_2(u) = 1 - \chi_1(u), \quad u \in H.
\]

The functions (2.26) are Lipschitz continuous, their range is \([0, 1]\), they equal one on \( Q_0, Q_1, Q_2 \), respectively, and vanish, respectively, on \( \tilde{Q}_0, Q_2 \) and \( Q_1 \).
We now wish to apply Lemma 2.2 for $B_{2t}$ with $Z = G'$, $Q = \text{supp } \chi_1 \cap \text{supp } \chi_0 = B_{2t}\setminus (\tilde{Q}_0 \cup Q_2)$ and $\theta$ as in Lemma 2.3. Consider the initial value problem

\begin{align}
\frac{d\sigma}{dh} &= \chi_0(\sigma)\chi_1(\sigma)\sigma - N\chi_0(\sigma)\chi_2(\sigma)Y(\sigma)/\|Y(\sigma)\|, \\
\sigma(h)|_{h=0} &= \varphi, \quad \varphi \in H, \quad N = 2et/(1 - 2\eta)r.
\end{align}

The right-hand side in (2.27) is locally Lipschitz continuous in $\sigma$ and thus the problem has a unique $C^1$-solution $\sigma$ defined for all $h$. Note that if $\varphi = 0$ or $\varphi = e$, then so is $\sigma$ for all $h$.

(2) Let $\varphi_j \in \Phi(t)$ be a minimizing sequence for (2.7) and $\sigma_j$ be correspondent solutions of (2.27)-(2.28). Then for $t_1 \in (t, 2t)$,

\begin{align}
\kappa(t_1) &\leq \sup_{s \in [0, 1]} G(\sigma_j(h; s))
\end{align}

as long as

\begin{align}
\|\sigma_j(h; s)\|^2 < t_1 \quad \text{for all } s \in [0, 1].
\end{align}

Let us establish a bound on $h$ that implies (2.30). By (2.27)

\begin{align}
\frac{d}{dh}\|\sigma_j\|^2 &= 2\chi_0\chi_1\|\sigma_j\|^2 - 2\chi_0\chi_2 N(Y(\sigma_j), \sigma_j)\|Y(\sigma_j)\|^{-1} \leq 2\|\sigma_j\|^2.
\end{align}

Therefore,

\begin{align}
\|\sigma_j(h; s)\|^2 &\leq te^{2h}.
\end{align}

Consequently, assuming

\begin{align}
0 \leq h \leq \frac{1}{2} \ln(t_1/t)
\end{align}

we obtain (2.30). In the further course of the proof $h$ will be subject to additional bounds from above.

(3) Let us estimate the derivative of $G(\sigma)$.

\begin{align}
\frac{d}{dh}G(\sigma_j) &= \chi_0\chi_1(G'(\sigma_j), \sigma_j) - N\chi_0\chi_2(G'(\sigma_j), Y(\sigma_j))/\|Y(\sigma_j)\|.
\end{align}

By Lemma 2.2 as already applied,

\begin{align}
(G'(\sigma), Y(\sigma))/\|Y(\sigma)\| \geq (1 - 2\eta)\|G'(\sigma)\| \quad \text{when } \sigma \in \text{supp } \chi_0\chi_2.
\end{align}

By Lemma 2.3

\begin{align}
(G'(\sigma_j), \sigma_j) &\leq -e\|\sigma_j\|^2 \quad \text{when } \sigma \in \text{supp } \chi_0\chi_1.
\end{align}

Then (2.34) yields

\begin{align}
\frac{d}{dh}G(\sigma_j) &\leq -e\chi_0\chi_1\|\sigma_j\|^2 - (1 - 2\eta)N\chi_0\chi_2\|G'(\sigma_j)\| \\
&\leq -e\chi_0\chi_1\|\sigma_j\|^2 - (1 - 2\eta)N\chi_0\chi_2 r\|\sigma_j\|^2/2t \\
&\leq -e\chi_0(\chi_1 + \chi_2)\|\sigma_j\|^2 = -e\chi_0(\sigma_j)\|\sigma_j\|^2.
\end{align}

(4) Consider the following sets of $s \in [0, 1]$. Let

\begin{align}
I_1 = \{s \in [0, 1]: |G(\varphi(s)) - \kappa(t)| \geq \delta/2\}.
\end{align}
For \( j \) large enough the inequality in (2.38) holds only if \( G(\varphi_j(s)) \leq \kappa(t) - \delta/2 \), since \( \varphi_j \) is a minimizing sequence and \( \kappa(t) \) is approximated by the maximal values of \( G(\varphi_j(s)) \). By (2.37), \( G(\sigma_j(h; s)) \leq \kappa(t) - \delta/2 \) for \( s \in I_1 \). Now let \( I_2 \) be a subset of \([0, 1]\) \( \backslash I_1 \), such that \( \sigma_j(h; s) \in Q_0 \) for all \( h \in [0, h_1] \), \( h_1 := \frac{1}{2} \ln(t_1/t) \), and \( I_3 = [0, 1] \backslash (I_1 \cup I_2) \). On \( I_2 \) (2.37) implies

\[
\frac{d}{dh} G(\sigma_j) \leq -\varepsilon \|\sigma_j\|^2. 
\]

By (2.31)

\[
\frac{d}{dh} \|\sigma_j\|^2 \geq -2N\|\sigma_j\|
\]

and consequently,

\[
\|\sigma_j\| \geq \|\varphi_j\| - Nh, \quad s \in I_2.
\]

Since \( \varphi_j(s) \in H_\delta \) when \( s \in I_2 \), Lemma 2.4 implies

\[
\|\sigma_j\| \geq 2\mu - Nh
\]

and assuming

\[
h \leq \mu/N,
\]

one has

\[
\|\sigma_j\| \geq \mu \quad \text{for} \ s \in I_2,
\]

and therefore,

\[
\frac{d}{dh} G(\sigma_j) \leq -\varepsilon \mu^2, \quad s \in I_2.
\]

Finally, if \( s \in I_3 \), let \( h_0 \in [0, h_1] \) be a maximal \( h \), such that \( \sigma_j(h; s) \in Q_0 \) for \( h \in [0, h_0] \). Then

\[
G(\sigma_j(h; s)) \leq G(\sigma_j(h_0, s)) = \kappa(t) - \delta/2, \quad s \in I_3.
\]

Combining (2.38), (2.45), and (2.44), one has from (2.29) that

\[
\kappa(t_1) \leq \max_{s \in [0, 1]} G(\sigma_j(h; s))
\]

\[
\leq \max \left\{ \kappa(t) - \delta/2, \max_{s \in I_1} G(\varphi_j) - \varepsilon \mu^2 h \right\}
\]

\[
\leq \max \{ \kappa(t) - \delta/2, m_j - \varepsilon \mu^2 h \},
\]

where \( m_j = \max_{s \in [0, 1]} G(\varphi_j) \), \( m_j \to \kappa(t) \). With \( j \) large enough and an additional upper bound on \( h \)

\[
h < \delta / 6\varepsilon \mu^2,
\]

one has

\[
\kappa(t_1) \leq \kappa(t) - \varepsilon \mu^2 h.
\]

Relation (2.49) is valid only as far as \( h \) satisfies restrictions (2.48), (2.43), and (2.33). Three of them can be reduced to (2.33) when

\[
t_1 < \min\{2t, te^{2h_2}, te^{2h_3}\}, \quad \text{where} \ h_2 = \frac{\mu}{N} \text{ and } h_3 = \frac{\delta}{6\varepsilon \mu^2}.
\]
Then (2.49) with $h = \frac{1}{2} \ln(t_{1}/t)$ immediately implies (2.24).

**Proof of Theorem 2.1.** It is already proved in [4] that for any $t > \|e\|^2$ there is $u \in H_{\delta} \cap B_t$ and $\alpha > 0$ such that

\begin{equation}
G'(u) = -\alpha u.
\end{equation}

Assume that there are no sequences $\alpha_j > 0$, $u_j \in H_{\delta}$ satisfying (2.51) and such that $\alpha_j \to 0$. Then the conditions of Lemma 2.3 are satisfied with some $\varepsilon > 0$ and by Lemma 2.5

\begin{equation}
\limsup_{t \to \infty} \kappa(t) \to -\infty,
\end{equation}

which contradicts (2.8). Therefore $\{\alpha_j\}$ necessarily has a subsequence with zero limit. Assume now that $\{u_j\}$ has a bounded subsequence. Then there is a weakly convergent renamed subsequence $u_j \rightharpoonup u_0$ and

\begin{equation}
G'(u_j) \to 0.
\end{equation}

Then by (2.4) $u_j \to u_0$, $G(u_0) \geq \delta$, and $G'(u_0) = 0$, which contradicts (2.3). Thus $\|u_j\| \to \infty$.

\section{Applications to Elliptic Problems}

We wish to show now that Theorem 2.1 implies Theorem 1.2. Our argument is somewhat repetitious of [7] and we omit details.

1. Let

\begin{equation}
\rho_0 \in \mathcal{J}_0,
\end{equation}

and

\begin{equation}
G(u) = \frac{1}{2} \rho_0 \|u\|^2 - g(u) + c, \quad c \leq g(0).
\end{equation}

It is easy to see that if $g$ satisfies (1.1) and (1.2), then $G$ satisfies (2.1).

2. Let us verify (2.2). Let

\begin{equation}
\Gamma(t) = \frac{1}{2} \rho_0 t - \gamma(t) + c.
\end{equation}

If the function $\Gamma(t)$ has a point of a local minimum on $(0, \infty)$, then $G$ has a nonzero critical point (cf. [6] or [7]), which contradicts the assumptions. If the function $\Gamma(t)$ is monotone, then by Theorem 1.2 all the derivatives of $\gamma(t)$ must be either greater or smaller than $\frac{1}{2} \rho_0$ which contradicts (3.1).

The remaining possibility is that $\Gamma(t)$ has a global maximum at $t = t_0 \in (0, \infty)$. Then (2.2) is satisfied with the following choices. Let $t_1 > t_0$ and let $e$ be an element of a maximizing sequence for $g$ on $S_{t_1}$, such that

\begin{equation}
\frac{1}{2} \rho_0 t_1 - g(e) + c < M + c,
\end{equation}

where

\begin{equation}
M = \frac{1}{2} \rho_0 t_0 - \gamma(t_0).
\end{equation}

Finally, set

\begin{equation}
\delta = \frac{1}{2} \min(M - \frac{1}{2} \rho_0 \|e\|^2 + g(e), M + g(0))
\end{equation}

and

\begin{equation}
c = 2\delta - M.
\end{equation}
If one assumes that (1.6) has no solution with \( p = p_0 \), then \( G \) satisfies (2.3). Now note that (2.4) follows from (1.2) and one can apply Theorem 2.1. \( \square \)

We discuss here only the applications to the problem

\[
-\rho \Delta u = f(u), \quad u \in H_0^1(\Omega) \setminus \{0\},
\]

where \( \Omega \subset \mathbb{R}^n \), \( n \geq 3 \), is an open bounded set, \( f : \mathbb{R} \to \mathbb{R} \) is continuous and subcritical at infinity, i.e.,

\[
f(s) = o(|s|^{(n+2)/(n-2)}) \quad \text{as} \quad s \to \infty.
\]

Let

\[
F(s) = \int_0^s f(\sigma) \, d\sigma,
\]

\[
g(u) = \int_\Omega F(u) \, dx.
\]

It is well known that \( g \) satisfies (1.1) and (1.2) on \( H_0^1(\Omega) \). Equation (3.8) is an equation for a critical point of \( g \) on

\[
S_t = \left\{ u \in H_0^1(\Omega) : \int_\Omega |
abla u|^2 = t \right\}.
\]

Theorem 1.2 then implies that (3.8) is solvable for a dense subset of \( \rho \in J_0 \). The interval \( J_0 \) is defined here as an open interval between the lower and upper bound of the slopes on the graph of

\[
2\gamma(t) = \sup_{u \in S_t} 2 \int_\Omega F(u) \, dx.
\]

One can reverse Theorem 1.2 and state the solvability of (1.6) for all \( \rho \in J_0 \) with an additional condition of an a priori bound on a Hilbert norm of \( u \).

**Theorem 3.1.** Assume (1.1), (1.2), and (1.8). If there is a \( \nu > 0 \) and a \( c > 0 \), such that for any \( u \) satisfying (1.6) with \( |\rho - \rho_0| < \nu \),

\[
||u|| \leq c,
\]

then (1.6) has a solution with \( \rho = \rho_0 \).

This statement is an elementary corollary of Theorem 1.2.

There are several important results which establish a priori bounds in \( L^\infty \) motivated by the topological approach to (3.8) (cf. [3] and references therein). The following statement uses an argument from [3].

**Corollary 3.2.** Let \( n \geq 3 \), \( \Omega \) be starshaped, \( f \geq 0 \), and let

\[
\frac{F(s)}{s^{\sigma}} \quad \text{be a decreasing function near} \quad s = +\infty \quad \text{with some} \quad \sigma < \frac{2n}{n-2}.
\]

Then the problem (3.8) satisfies the condition (3.14) and consequently is solvable for all \( \rho \in J_0 \).

**Proof.** Since \( \Omega \) is starshaped, a well-known Pohozaev-Rellich identity (resulting from multiplication of (3.8) by \( (x \cdot \nabla)u \)) provides

\[
\rho ||u||^2 \leq \frac{2n}{n-2} \int F(u).
\]
At the same time, multiplication of (3.8) by \( u \) gives

\begin{equation}
\rho \| u \|^2 = \int f(u)u.
\end{equation}

From (3.15) and since the problem might have only positive solutions, one has

\begin{equation}
\int F(u) \leq \sigma \int f(u)u + c, \quad c \in \mathbb{R}.
\end{equation}

Then (3.16)–(3.18) immediately provide (3.14). \( \square \)

This corollary includes cases of \( f \) with sub- (or super-)linear behavior both at 0 and at \( \infty \).

**References**


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