FUNCTIONAL EQUATIONS SATISFIED
BY INTERTWINING OPERATORS OF REDUCTIVE GROUPS

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Abstract. This paper generalizes a recent work of Vogan and Wallach [VW] in which they derived a difference equation satisfied by intertwining operators of reductive groups. We show that, associated with each irreducible finite-dimensional representation, there is a functional equation relating intertwining operators. In this way, we obtain natural relations between intertwining operators for different series of induced representations.

0. Introduction

We use the convention of denoting a Lie group by a capital letter, and denoting its Lie algebra by the corresponding lower case German letter. A subscript \( \mathbb{C} \) denotes complexification.

Let \( G \) be a real reductive group, \( P = MAN \) a parabolic subgroup of \( G \) with a given Langlands decomposition [Kn], and \( \Phi(P, A) \) the set of positive restricted \( A \)-roots corresponding to \( N \). For \( \nu \in \mathfrak{a}_C^* \), let \( a^\nu \) be the character of \( A \): \( a^\nu = e^{\nu(\log a)} \), \( a \in A \). Let \( \sigma \) be an admissible representation of \( M \), and \( \mathcal{H}_\sigma \) the representation space. In the sequel, we shall require \( \sigma \) to have an infinitesimal character, whose definition is given in (2.1.5). We denote by \( I_{P, \sigma, \nu} = \text{Ind}_{MAN}^G(\sigma \otimes a^\nu \otimes 1) \) the space of \( C^\infty \) functions, \( f \), from \( G \) to \( \mathcal{H}_\sigma \) such that \( f(xman) = a^{-(\nu+\rho)}a^{-1}(m)f(x) \), where \( m \in M, a \in A, n \in N \), and \( \rho = \rho_P \), the half sum of the positive restricted \( A \)-roots counted with multiplicities. \( G \) acts on \( I_{P, \sigma, \nu} \) by left translation. \( I_{P, \sigma, \nu} \) is usually referred to as the generalized principal series.

Let \( \overline{P} = MAN \) be the opposite parabolic subgroup to \( P \). We also define a representation of \( G \) by left translation in \( I_{\overline{P}, \sigma, \nu} = \text{Ind}_{MAN}^G(\sigma \otimes a^\nu \otimes 1) \), the space of \( C^\infty \) functions, \( f \), from \( G \) to \( \mathcal{H}_\sigma \) such that \( f(xman) = a^{-(\nu-\rho)}a^{-1}(m)f(x) \). We recall the standard (but formal) intertwining operator:

\[
J(P; P, \sigma, \nu)f(x) = \int_{\mathcal{N}} f(x\bar{n})\,d\bar{n}, \quad f \in I_{P, \sigma, \nu}.
\]

It is well known (see [Kn]) that if \( \text{Re}(\nu, \alpha) \geq c = c_\sigma \) for \( \alpha \in \Phi(P, A) \) and \( c_\sigma \) some constant depending only on \( \sigma \), then the integral defining
$J(\overline{P} : P, \sigma, \nu)$ converges absolutely. Moreover, in the range of convergence, we have $J(\overline{P} : P, \sigma, \nu) : I_{P, \sigma, \nu} \rightarrow I_{\overline{P}, \sigma, \nu}$ and it intertwines the $G$-actions.

The problem of meromorphically continuing the operators $J(\overline{P} : P, \sigma, \nu)$ was resolved in the early seventies by purely analytic methods (see [KS1, KS2]). It was shown, among other things, that in the rank one case, one can in fact analytically continue these operators with respect to the parameter $\nu$, except at negative multiples of some value of $\nu$ where they have simple poles. In this sense, these intertwining operators behave typically like the classical gamma function.

Therefore, it seems tempting and natural to ask whether these intertwining operators indeed have functional equations like the classical gamma function. This was first shown to be so by a recent work of Vogan and Wallach [VW] in which they derived a difference equation satisfied by intertwining operators. The method they employed was by tensoring with a finite-dimensional spherical representation. Clearly, that is the most efficient way of establishing the meromorphic continuation.

The purpose of the present work is to generalize and, at the same time, to explain the Vogan-Wallach result by tensoring with arbitrary finite-dimensional representations. It is fair to say that our main contribution is to show how simple and general the result turns out to be.

Some words about the organization of this paper are in order. In §1 (§§1.1–1.2), some very general constructions from multilinear algebra are discussed. These constructions will enable us to define four maps $T, S, U, V$ and obtain most of their properties in a rather transparent way.

In §2 (§§2.1–2.8), two commuting diagrams are proved. The first commuting diagram (§2.4) relates $T, S$ with intertwining operators. Its proof relies solely on a simple property of some projection operator, which is discussed in §2.2. Our main result is contained in the second commuting diagram (§2.7), which relates $U, V$ with intertwining operators. Its proof uses a critical lemma due to Vogan, the first commuting diagram, and some general properties of $T, S, U, V$. What is important here is that the second commuting diagram gives us a functional equation connecting the intertwining operators $J(\mu, \nu)$ and $J(\mu + \mu_1, \nu + \nu_1)$, where $(\mu_1, \nu_1)$ is a regular dominant integral infinitesimal character coming from an irreducible finite-dimensional representation of $G$. Moreover, it can be used to express $J(\mu, \nu)$ in terms of $J(\mu + \mu_1, \nu + \nu_1)$, not the other way around as in the case of the first commuting diagram. Following [VW], the last section §2.8 gives some basic properties of the factor $r_\xi^G(\Lambda)$, which occurs in the second commuting diagram.

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1. SOME MULTILINEAR ALGEBRA

1.1. Generalities on dual and tensor product of vector spaces. Let $F$ be a finite-dimensional complex vector space, and $F^*$ its complex dual. In this section, all our maps are linear.
There is a natural map

\[(1.1.1) \quad \text{tr} : F \otimes F^* \to \mathbb{C} \]

specified by \( \text{tr}(v \otimes v^*) = v^*(v), \ v \in F, \ v^* \in F^*. \) This induces maps

\[(1.1.2) \quad U \otimes F \otimes F^* \xrightarrow{1_U \otimes \text{tr}} U, \]

\[\quad (1_U \otimes \text{tr})(u \otimes v \otimes v^*) = v^*(v)u, \quad u \in U. \]

Here, \( U \) is a complex vector space, possibly infinite dimensional.

We can also think of this as defining contraction maps

\[(1.1.3) \quad a \circ v^* : U \otimes F \to U, \]

\[\quad a \circ v^* = (1_U \otimes \text{tr})(a \otimes v^*), \quad a \in U \otimes F. \]

Thus, \( (u \otimes v) \circ v^* = v^*(v)u, \ u \in U. \)

\[(1.1.4) \quad \text{Remark.} \quad \text{For any dual bases} \ \{y_i, y_i^*\} \ \text{of} \ \ F \ \text{and} \ F^*, \ \text{we have} \]

\[\quad \sum_i (a \circ y_i^*) \otimes y_i = a, \quad a \in U \otimes F. \]

\[(1.1.5) \quad \text{Remark.} \quad \text{The contraction mapping has the following property:} \]

\[\quad (1_U \otimes A)(a) \circ v^* = a \circ A^*(v^*), \quad a \in U \otimes V, \ A \in \text{End}(F), \]

where \( A^* \in \text{End}(F^*) \) is the adjoint of \( A \).

This may be checked by the formula

\[\quad (1_U \otimes A)(u \otimes v) \circ v^* = (u \otimes Av) \circ v^* = v^*(Av)u = (A^*v^*)(v)u = (u \otimes v) \circ A^*(v^*). \]

\[(1.1.6) \quad \text{Remark.} \quad \text{Let} \ F_1 \subseteq F \ \text{be a subspace. A map} \ P_1 \in \text{End}(F) \ \text{is called a} \]

\[\quad \text{projection operator from} \ F \ \text{onto} \ F_1 \ \text{if the image of} \ P_1 \ \text{is} \ F_1 \ \text{and} \ P_1|_{F_1} = 1_{F_1}. \]

By applying \( 1_U \otimes P_1 \) to the equation in Remark (1.1.4) and using Remark (1.1.5), we see

\[\quad \sum_i (a \circ P_1^*(y_i^*)) \otimes P_1(y_i) = a, \quad a \in U \otimes F_1. \]

Given a map

\[(1.1.7) \quad T : X \to U \otimes F, \]

there is an associated map

\[(1.1.8) \quad T^\dagger : X \otimes F^* \to U \]

defined by the following composition:

\[\quad X \otimes F^* \xrightarrow{T \otimes 1_{F^*}} U \otimes F \otimes F^* \xrightarrow{1_U \otimes \text{tr}} U. \]

Thus, \( T^\dagger(x \otimes v^*) = T(x) \circ v^*. \)

We can reconstruct \( T \) from \( T^\dagger \) as follows: Given \( S : X \otimes F^* \to U \), we can define \( S^\dagger : X \to U \otimes F \) via the diagram:

\[(1.1.9) \quad X \xrightarrow{j} X \otimes F^* \xrightarrow{S \otimes 1_F} U \otimes F, \]
where \( j(x) = \sum_i x \otimes y_i^* \otimes y_i \), with \( \{y_i\} \) any basis of \( F \) and \( \{y_i^*\} \) the dual basis of \( F^* \). Thus,

\[
S^\dagger(x) = \sum_i S(x \otimes y_i^*) \otimes y_i.
\]

(1.1.11) **Remark.** The expression \( \sum_i y_i^* \otimes y_i \in F^* \otimes F \) is independent of the choice of the dual bases \( \{y_i\} \) and \( \{y_i^*\} \). Under the well-known identification \( F^* \otimes F \cong \text{End}(F) \), \( \sum_i y_i^* \otimes y_i \in F^* \otimes F \) corresponds to the identity element of \( \text{End}(F) \). Here it is worthwhile to point out the following general principle:

For any bilinear map \( B(\cdot, \cdot) : F^* \times F \to W \), the expression \( \sum_i B(y_i^*, y_i) \in W \) is independent of the choice of the dual bases \( \{y_i\} \) and \( \{y_i^*\} \).

Now if we have \( T : X \to U \otimes F \), we then have \( T^\dagger : X \otimes F^* \to U \) and \( (T^\dagger)^\dagger : X \to U \otimes F \).

We compute

\[
(T^\dagger)^\dagger(x) = \sum_i T^\dagger(x \otimes y_i^*) \otimes y_i = \sum_i (T(x) \circ y_i^*) \otimes y_i = T(x)
\]

(by Remark (1.1.4)). Hence, \( (T^\dagger)^\dagger = T \).

1.2. **Generalities on induction, dual, tensor product of representations.** Let \( G \) be a group and \( H \) a subgroup. Let \( \rho \) be a finite-dimensional representation of \( G \), and \( \mu \) a representation of \( H \), possibly infinite-dimensional. Then

\[
\rho : G \to \text{Gl}(F), \quad \mu : H \to \text{Gl}(U).
\]

We define \( \lambda_\mu \) to be the representation of \( G \) induced from \( \mu \), i.e.,

\[
\lambda_\mu = \text{ind}_H^G \mu = \{f : G \to U | f(gh) = \mu^{-1}(h)f(g), \ g \in G, h \in H\}.
\]

\( G \) acts on \( \lambda_\mu \) by left translation.

Since \( \rho \) is a finite-dimensional representation, tensoring with \( \rho \) is a well-defined, purely algebraic operation.

The following proposition is quite routine, but because it is simple and yet very useful for our later purpose, we include a detailed proof.

(1.2.1) **Proposition.** The mapping \( \alpha : \lambda_\mu \otimes \rho \to \lambda_\mu \otimes \rho |_{\rho \in H} \) specified by

\[
\alpha(f \otimes v)(g) = f(g) \otimes \rho(g)^{-1}v, \quad f \in \lambda_\mu, \ v \in F, \ g \in G,
\]

is a \( G \)-isomorphism.

**Proof.** \( \alpha \) is a \( G \)-map, since

\[
\alpha \left( (\lambda_\mu \otimes \rho)(g') \left( \sum_i f_i \otimes v_i \right) \right)(g) = \alpha \left( \sum_i \lambda_\mu(g')f_i \otimes \rho(g')v_i \right)(g)
\]

\[
= \sum_i (\lambda_\mu(g')f_i)(g) \otimes \rho(g)^{-1}\rho(g')v_i
\]

\[
= \sum_i f_i(g^{-1}g') \otimes \rho((g^{-1}g')^{-1}v_i)
\]

\[
= \alpha \left( \sum_i f_i \otimes v_i \right)(g^{-1}g')
\]
and

\[
\alpha \left( \sum_i f_i \otimes v_i \right)(gh) = \sum_i f_i(gh) \otimes \rho((gh)^{-1})v_i \\
= \sum \mu(h)^{-1} f_i(g) \otimes \rho(h)^{-1} \rho(g^{-1})v_i \\
= (\mu \otimes \rho|_H)(h)^{-1} \left( \sum_i f_i(g) \otimes \rho(g)^{-1}v_i \right) \\
= (\mu \otimes \rho|_H)(h)^{-1} \left( \alpha \left( \sum_i f_i \otimes v_i \right)(g) \right).
\]

We construct an inverse of \( \alpha \). Let \( \phi \in \lambda_{\mu \otimes \rho|_H} \); then \( \phi(g) \in U \otimes F \). Given a basis \( \{y_i\} \) of \( F \) and a dual basis \( \{y_i^*\} \) of \( F^* \), we know that \( \{\rho(g)^{-1}y_i\} \) and \( \{\rho^*(g)^{-1}y_i^*\} \) are again dual bases of \( F \) and \( F^* \). Therefore, we can write

\[
\phi(g) = \sum_i (\phi(g) \circ \rho^*(g)^{-1}y_i^*) \otimes \rho(g)^{-1}y_i \quad \text{(see Remark (1.1.4))}.
\]

Let

\[
(1.2.2) \quad \beta \phi = \sum_i \phi_i \otimes y_i,
\]

where \( \phi_i(g) = \phi(g) \circ \rho^*(g)^{-1}y_i^* \).

We check

\[
(\alpha \beta)(\phi)(g) = \alpha \left( \sum_i \phi_i \otimes y_i \right)(g) = \sum_i \phi_i(g) \otimes \rho(g)^{-1}y_i \\
= \sum_i (\phi(g) \circ \rho^*(g)^{-1}y_i^*) \otimes \rho(g)^{-1}y_i = \phi(g).
\]

\[
(\beta \alpha)(f \otimes v) = \sum f_i \otimes v_i,
\]

where

\[
f_i(g) = \alpha(f \otimes v)(g) \circ \rho^*(g)^{-1}y_i^* = (f(g) \otimes \rho(g)^{-1}v) \circ \rho^*(g)^{-1}y_i^* \\
= f(g) \cdot (\rho^*(g)^{-1}y_i^*)(\rho(g)^{-1}v) = f(g) \cdot y_i^*(v),
\]

so

\[
(\beta \alpha)(f \otimes v) = \sum f \cdot y_i^*(v) \otimes y_i = f \otimes v.
\]

Now suppose \( \nu : H \rightarrow \text{Gl}(X) \) is another representation of \( H \), and suppose we have a \( H \)-intertwining map

\[
(1.2.3) \quad T_0 : X \rightarrow U \otimes F
\]

\[
\nu \rightarrow \mu \otimes \rho|_H.
\]

This is equivalent to having a \( H \)-intertwining map

\[
(1.2.4) \quad T_0^\dagger : X \otimes F^* \rightarrow U
\]

defined by \( T_0^\dagger(x \otimes v^*) = T_0(x) \circ v^* \).

The map \( T_0 \) induces a map

\[
(1.2.5) \quad (T_0)_G : \lambda_\nu \rightarrow \lambda_{\mu \otimes \rho|_H}
\]

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by simply composing function values:

\[(T_0)G(f)(g) = T_0(f(g)), \quad f \in \lambda_\nu.\]

Composing with \(\beta : \lambda_\mu \otimes \rho|_H \rightarrow \lambda_\mu \otimes \rho\) gives us a \(G\)-intertwining map

\[(1.2.6) \quad \beta(T_0)G : \lambda_\nu \xrightarrow{(T_0)G} \lambda_\mu \otimes \rho \xrightarrow{\beta} \lambda_\mu \otimes \rho\]

which is computed by the formula

\[(1.2.7) \quad (\beta(T_0)G)(f) = \sum_i f_i \otimes y_i,\]

where \(f_i(g) = T_0(f(g)) \circ \rho^*(g)^{-1} y_i^*, \{y_i\}\) is, as above, a basis for \(F\), and \(\{y_i^*\}\) is the dual basis of \(F^*\).

The map \(T_0^\dagger : X \otimes V^* \rightarrow U\) induces a map

\[(1.2.8) \quad (T_0^\dagger)G : \lambda_\nu \otimes \rho^*|_H \rightarrow \lambda_\mu

again by composing function values:

\[(T_0^\dagger)G(\phi)(g) = (T_0^\dagger)(\phi(g)), \quad \phi \in \lambda_\nu \otimes \rho^*|_H.\]

Composing with \(\alpha : \lambda_\nu \otimes \rho^* \rightarrow \lambda_\nu \otimes \rho^*|_H\) gives us a \(G\)-intertwining map

\[(1.2.9) \quad (T_0^\dagger)G\alpha : \lambda_\nu \otimes \rho^* \xrightarrow{\alpha} \lambda_\nu \otimes \rho^*|_H \xrightarrow{(T_0^\dagger)G} \lambda_\mu,\]

which is computed by the formula

\[(1.2.10) \quad ((T_0^\dagger)G\alpha)(f \otimes \nu^*)(g) = T_0(f(g)) \circ \rho^*(g)^{-1} \nu^*, \quad f \in \lambda_\nu.\]

Let us compare (1.2.6) with (1.2.9). If \(f \in \lambda_\nu\), and \(\nu^* \in F^*\), then we compute

\[(\beta(T_0)G)^\dagger(f \otimes \nu^*)(g) = (\beta(T_0)G)(f \otimes \nu^*) = \left(\sum f_i \otimes y_i\right) \otimes \nu^* = \sum f_i \cdot \nu^*(y_i),\]

where

\[f_i(g) = T_0(f(g)) \circ \rho^*(g)^{-1} y_i^*.\]

Therefore,

\[((\beta(T_0)G)^\dagger)(f \otimes \nu^*)(g) = T_0(f(g)) \circ \sum \nu^*(y_i) \rho^*(g)^{-1} y_i^* = T_0(f(g)) \circ \rho^*(g)^{-1} \nu^*.\]

In other words,

\[(1.2.11) \quad (\beta(T_0)G)^\dagger = (T_0^\dagger)G\alpha.\]

Suppose we have a submodule

\[(1.2.12) \quad \rho_1 \subseteq \rho|_H, \quad F_1 \subseteq F.\]

Let \(P_1\) be a projection map from \(F\) onto \(F_1\) (not necessarily a \(H\)-map) (see Remark (1.1.6) for the definition of such a map). Suppose the image of \(T_0 \subseteq U \otimes F_1\). Then

\[T_0 = (1_U \otimes P_1)T_0.\]
Hence if $f \in \lambda_\nu$ and $v^* \in F^*$, we can compute

$$
(\beta(T_0 G)^\dag (f \otimes v^*)(g) = (T_0^\dag) G \alpha(f \otimes v^*)(g)
= T_0(f(g)) \circ \rho^*(g)^{-1} v^*
= (1 \otimes P_1) T_0(f(g)) \circ \rho^*(g)^{-1} v^*
= T_0(f(g)) \circ P_1^*(\rho^*(g)^{-1} v^*) \quad (\text{by Remark (1.1.5)}).
$$

Therefore, we have

$$
(1.2.14) \quad (\beta(T_0 G)(f) = \sum f_i \otimes y_i, \quad f \in \lambda_\nu,
$$

where $f_i = (\beta(T_0 G)^\dag (f \otimes y_i^*) = T_0(f(g)) \circ P_1^*(\rho^*(g)^{-1} y_i^*)$.

2. Natural relations between intertwining operator

2.1. Preliminaries. Let $g$ be a complex reductive Lie algebra. Choose a Borel subalgebra $b$ of $g$, and thus a Cartan subalgebra $h \subseteq b$.

(2.1.1) The irreducible finite-dimensional representations of $g$ can be parametrized by their highest weights with respect to $b$, which are dominant integral (see [Hu]).

(2.1.2) Let $U(g)$ be the universal enveloping algebra of $g$, and $Z(g)$ its center. We know from Harish-Chandra (see [VI], for example) that the homomorphisms of $Z(g)$ into $\mathbb{C}$ can be parametrized by elements of $h^*/W_G$, the set of Weyl group orbits in $h^*$. Let $C_G$ be the positive Weyl chamber specified by our choice of $b$. It is a fundamental domain for the action of $W_G$ on $h^*$. For $\lambda \in C_G \subseteq h^*$, the corresponding homomorphism of $Z(g)$ into $\mathbb{C}$ is denoted by $\chi_\lambda$.

We fix our parabolic subgroup $P = MAN$ as in the Introduction.

Choose a Borel subalgebra $b_0$ of $m_c$, and thus a Cartan subalgebra $h_0 \subseteq b_0$. Then $b = b_0 \oplus n_c$ is a Borel subalgebra of $g_c$, and $h = h_0 \oplus a_c$ is a Cartan subalgebra contained in $b$.

(2.1.3) We can apply (2.1.1) to the Lie algebras $m_c$ and $g_c$. Thus if $\mu_1 \in h_0^*$ is dominant integral with respect to $b_0$ and in the weight lattice, the unique irreducible finite-dimensional representation of $M$ with this highest weight is denoted by $\sigma_{\mu_1}$. Similarly if $(\mu_1, \nu_1) \in h^*$ is dominant integral with respect to $b$, $\mu_1 \in h_0^*$, $\nu_1 \in a_c^*$, the corresponding irreducible finite-dimensional representation of $G$ is denoted by $F_{\mu_1, \nu_1}$.

For a finite-dimensional representation $(\pi, F)$ of $G$, denote by

$$
F^N = \{v \in F| \pi(n)v = v \ \forall n \in N\}
$$

the space of $N$-fixed vectors. Similarly $F^\bar{N}$ denotes the space of $\bar{N}$-fixed vectors.

We have the following standard lemma [W1].

(2.1.4) **Lemma.** Let $(\pi, F)$ be an irreducible finite-dimensional representation of $G$, and $(\pi^*, F^*)$ the contragradient representation. Then $F^N$ and $(F^*)^\bar{N}$
are irreducible representations of $MA$. Moreover, as representations of $MA$, they are contragradient to each other.

Thus in the notation of this section,
\[ F_{\mu_1, \nu_1}^N \cong \sigma_{\mu_1} \otimes a^{\nu_1} \]
as $MA$-modules.

(2.1.5) Given an admissible representation $(\pi, V)$ of $G$, there is a resulting representation of $\mathcal{Z}(g_C)$ on $V_0$, the space of $K$-finite vectors, where $K$ is a maximal compact subgroup of $G$. If $\mathcal{Z}(g_C)$ acts by scalars on $V_0$, the corresponding homomorphism
\[ \chi: \mathcal{Z}(g_C) \to \mathbb{C}, \]
defined by
\[ z \cdot x = \chi(z)x, \quad x \in V_0, \; z \in \mathcal{Z}(g_C), \]
is called the infinitesimal character of $\pi$.

More generally, if for some integer $d > 0$,
\[ (z - \chi(z))^d x = 0 \quad \text{for all } x \in V_0, \; z \in \mathcal{Z}(g_C), \]
we say $\pi$ has the generalized infinitesimal character $\chi$.

If $\chi = \chi_{\lambda}$, $\lambda \in C_G$ (see (2.1.2)), we shall also say $\pi$ has the infinitesimal character $\lambda$ (resp. generalized infinitesimal character $\lambda$).

We quote the following result from [Kn]. Let $(\pi, V)$ be a Harish-Chandra module, i.e., a finitely generated admissible $G$-module. Then there exist linear functionals $\lambda_1, \ldots, \lambda_l$ on $h$, $\mathcal{Z}(g_C)$-invariant subspaces $V_1, \ldots, V_l$ of $V_0$, and an integer $d > 0$ such that
\begin{enumerate}
  \item $\lambda_1, \ldots, \lambda_l$ are mutually inequivalent under the Weyl group.
  \item $V_0 = V_1 \oplus \cdots \oplus V_l$.
  \item $(z - \chi_{\lambda_j}(z))^d$ acts as the zero operator in $V_j$ for all $z \in \mathcal{Z}(g_C)$.
\end{enumerate}

Thus, we have a canonically defined projection operator, denoted by $P_{\lambda}^G$, from the space of $K$-finite vectors of $V$ to the subspace with the generalized infinitesimal character $\lambda$.

Similarly, if $E$ is a Harish-Chandra module of $M$, we denote by $P_{\mu}^M$ the projection operator from the space of $K \cap M$ finite vectors of $E$ to the subspace with the generalized infinitesimal character $\mu \in C_M$, the positive Weyl chamber specified by our choice of $b_0$ in $m_C$.

2.2. Two projection maps: $P_N$ and $P_{\overline{N}}$. From now on, we fix $F = F_{\mu_1, \nu_1}$, an irreducible finite-dimensional representation of $G$.

(2.2.1) Lemma. We have the following direct sum decompositions:
\begin{align*}
(2.2.2) \quad F &= F_N \oplus \overline{n}F, \\
(2.2.3) \quad F^* &= (F^*)_{\overline{N}} \oplus nF^* \quad \text{(as } MA\text{-modules}),
\end{align*}
where $n$ and $\overline{n}$ are Lie algebras of $N$ and $\overline{N}$, respectively.

Proof. Use $g_C = n + m + a + n$, the Poincare-Birkhoff-Witt theorem, and the irreducibility of $F$. We leave the details to the reader.
We denote by $P_N$ and $P_{\bar{N}}$, the projections to $F^N$ and $(F^*)^{\bar{N}}$, according to the decompositions (2.2.2) and (2.2.3), respectively. They are $MA$-homomorphisms.

We prove the following properties of $P_N$ and $P_{\bar{N}}$.

(2.2.4) **Proposition.** (i) $P_N(\bar{n}v) = P_N(v)$, $\bar{n} \in \overline{N}$, $v \in F$; $P_{\bar{N}}(nv^*) = P_{\bar{N}}(v^*)$, $n \in N$, $v^* \in F^*$.

(ii) $P_N$ and $P_{\bar{N}}$ are adjoint to each other as elements of $\text{End}(F)$ and $\text{End}(F^*)$.

**Proof.** (i) By definition, we have $P_N(Xv) = 0$, $X \in \overline{N}$, $v \in F$. From the well-known relationship between a Lie group and its Lie algebra, it follows that $P_N(\bar{n}v) = P_N(v)$, $\bar{n} \in \overline{N}$, $v \in F$.

(ii) Let $v = v_1 + v_2$, $v_1 \in F^N$, $v_2 \in \overline{n}F$,

$v^* = v_1^* + v_2^*$, $v_1^* \in (F^*)^{\overline{N}}$, $v_2^* \in nF^*$.

Since $v_1^* \in (F^*)^{\overline{N}}$, we have $v_1^*(Xv) = -(Xv_1^*)(v) = 0$ for $X \in \overline{N}$, $v \in F$. Therefore $v_1^*(v_2) = 0$, and $v_2^*(v_1) = 0$ by a similar computation. Hence,

$$(P_Nv^*)(v) = v_1^*(v) = v_1^*(v_1),$$

$$v^*(P_N(v)) = v^*(v_1) = v_1^*(v_1).$$

2.3. **Definitions of four maps $T, S, U, V$.** Let $F = F_{\mu_1\nu_1}$ be the irreducible representation of $G$ such that $F^N = \sigma_{\mu_1} \otimes a^{\nu_1}$ as representations of $MA$. Let $\sigma$ be an admissible representation of $M$ with an infinitesimal character $\mu \in C_M$ (see §2.1). We shall write $\sigma_{\mu_1}$ instead of $\sigma$ for the sake of notation.

Consider $I_{P,\mu,\nu} = \text{Ind}_{MAN}^G(\sigma_{\mu} \otimes a^\nu \otimes 1)$. Let

$$\sigma_{\mu+\nu_1}^{\text{def}} = P_{\mu+\nu_1}(\sigma_{\mu} \otimes \sigma_{\mu_1}).$$

We recall here that $P_{\mu+\nu_1}^M$ is the projection operator onto the generalized infinitesimal character $\mu + \mu_1$. We caution the reader that $\sigma_{\mu_1}$ has the infinitesimal character $\mu_1 + \rho_M$ instead of $\mu_1$, where $\rho_M$ is the half sum of the positive $h_0$ roots in $m_C$.

Let

$$I_{P,\mu+\nu_1,\nu,\nu_1}^{\text{def}} = \text{Ind}_{MAN}^G(\sigma_{\mu+\nu_1} \otimes a^{\nu+\nu_1} \otimes 1).$$

We shall define below four maps $T, S, U, V$, which fit into the following two diagrams:

(2.3.1)

$$\begin{array}{ccc}
I_{P,\mu,\nu} \otimes F_{\mu_1,\nu_1} & \overset{J(\overline{P}:P,\mu,\nu) \otimes I}{\longrightarrow} & I_{P,\mu,\nu} \otimes F_{\mu_1,\nu_1} \\
| \quad T | & | & | \\
I_{P,\mu+\nu_1,\nu_1} & \underset{J(\overline{P}:P,\mu+\nu_1,\nu_1)}{\longrightarrow} & I_{P,\mu+\nu_1,\nu_1} \\
\end{array}$$

(2.3.2)

$$\begin{array}{ccc}
I_{P,\mu,\nu} \otimes F_{\mu_1,\nu_1} & \overset{J(\overline{P}:P,\mu+\nu_1,\nu_1) \otimes I}{\longrightarrow} & I_{P,\mu+\nu_1,\nu_1} \otimes F_{\mu_1,\nu_1} \\
| \quad U | & | & | \\
I_{P,\mu,\nu} & \underset{J(\overline{P}:P,\mu,\nu)}{\longrightarrow} & I_{P,\mu,\nu} \\
\end{array}$$
The reader is referred to §§1.1 and 1.2 for various notation used in the following.

We have the natural inclusion:

\[ \sigma_{\mu+\nu_1} \hookrightarrow \sigma_\mu \otimes \sigma_{\mu_1}. \]

This will induce a natural inclusion of \( \sigma_{\mu+\nu_1} \otimes a^{\nu+v_1} \otimes 1 \) into

\[ (\sigma_\mu \otimes \sigma_{\mu_1}) \otimes a^{\nu} \otimes 1 \otimes (\sigma_{\mu_1} \otimes a^{\nu_1} \otimes 1) = (\sigma_\mu \otimes a^{\nu} \otimes 1) \otimes (F_{\mu_1, \nu_1})^N, \]

but \( (F_{\mu_1, \nu_1})^N \) is a \( MA \)-submodule of \( (F_{\mu_1, \nu_1})|_P \), so

\[ (2.3.3) \quad T_0 : \sigma_{\mu+\nu_1} \otimes a^{\nu+v_1} \otimes 1 \hookrightarrow (\sigma_\mu \otimes a^{\nu} \otimes 1) \otimes (F_{\mu_1, \nu_1})|_P \]

as \( P \)-modules.

Therefore, we have a natural \( G \)-map

\[ T = \beta(T_0)_G : \text{Ind}^G_P(\sigma_{\mu+\nu_1} \otimes a^{\nu+v_1} \otimes 1) \]

\[ (2.3.4) \quad \xrightarrow{(T_0)_G} \text{Ind}^G_P((\sigma_\mu \otimes a^{\nu} \otimes 1) \otimes (F_{\mu_1, \nu_1})|_P) \]

\[ \cong \text{Ind}^G_P(\sigma_\mu \otimes a^{\nu} \otimes 1) \otimes F_{\mu_1, \nu_1}. \]

Since the image of \( T_0 \subseteq (\sigma_\mu \otimes a^{\nu} \otimes 1) \otimes (F_{\mu_1, \nu_1})^N \), the above map \( T \) can be computed as follows:

\[ (2.3.5) \quad T(f) = \sum_i f_i \otimes y_i, \quad f \in I_{P, \mu+\nu_1}, \quad y_i \text{ dual bases of } F \quad \text{and} \quad \{y_i^*\} \text{ of } F^*. \]

where \( f_i(g) = f(g) \circ P_N(g^{-1}y_i^*) \), with \( \{y_i\} \) and \( \{y_i^*\} \) dual bases of \( F \) and \( F^* \) (see (1.2.14) and Proposition (2.2.4)).

Let \( U \) be the following natural \( G \)-map:

\[ U = (T_0)_G^\dagger : \text{Ind}^G_P(\sigma_{\mu+\nu_1} \otimes a^{\nu+v_1} \otimes 1) \otimes F_{\mu+\nu_1}^* \]

\[ (2.3.6) \quad \xrightarrow{\alpha} \text{Ind}^G_P((\sigma_{\mu+\nu_1} \otimes a^{\nu+v_1} \otimes 1) \otimes F_{\mu_1, \nu_1}^*) \]

\[ \xrightarrow{(T_0)_G} \text{Ind}^G_P(\sigma_\mu \otimes a^{\nu} \otimes 1). \]

We know by (1.2.11) that \( U = T_1 \), i.e.,

\[ (2.3.7) \quad U(f \otimes v^*) = T(f) \circ v^*. \]

By (1.2.13), we have

\[ (2.3.8) \quad U(f \otimes v^*)(g) = f(g) \circ P_N(g^{-1}v^*), \quad f \in I_{P, \mu+\nu_1}, \quad g \in I_{P, \nu_1}. \]

Also, we have the following natural epimorphism:

\[ \sigma_\mu \otimes \sigma_{\mu_1} \twoheadrightarrow \sigma_{\mu+\mu_1}, \]

\[ u \otimes u_1 \to P^M_{\mu+\mu_1}(u \otimes u_1), \quad u \in \sigma_\mu, \quad u_1 \in \sigma_{\mu_1}. \]

This will induce a \( MA \)-map

\[ \sigma_\mu \otimes a^{\nu} \otimes (F_{\mu_1, \nu_1})^N \twoheadrightarrow \sigma_{\mu+\mu_1} \otimes a^{\nu+v_1}, \]

where \( (F_{\mu_1, \nu_1})^N \) is a \( MA \)-quotient of \( F_{\mu_1, \nu_1} \) under the map

\[ P_N : F_{\mu_1, \nu_1} \twoheadrightarrow (F_{\mu_1, \nu_1})^N. \]
So we have
\[ \sigma_\mu \otimes a^\nu \otimes F_{\mu_1, \nu_1} \to \sigma_\mu \otimes a^\nu \otimes (F_{\mu_1, \nu_1})^N \to \sigma_{\mu+\mu_1} \otimes a^\nu a^{\nu_1}. \]

Since the map \( P_N \) has the property \( P_N(\bar{n}v) = P_N(v) \) for \( \bar{n} \in \bar{N} \) (Proposition (2.2.4)), the composition of the above two epimorphisms yields an epimorphism
\[ (2.3.9) \quad S_0 : (\sigma_\mu \otimes a^\nu \otimes 1) \otimes F_{\mu_1, \nu_1} \bar{p} \to \sigma_{\mu+\mu_1} \otimes a^\nu a^{\nu_1} \otimes 1 \]
as \( \bar{P} = MA\bar{N} \)-modules.

Therefore, we have a natural \( G \)-map
\[ \begin{align*}
S &= (S_0)_G : \text{Ind}^G_F(\sigma_\mu \otimes a^\nu \otimes 1) \otimes F_{\mu_1, \nu_1} \\
(2.3.10) &\cong \text{Ind}^G_F(\sigma_{\mu+\mu_1} \otimes a^{\nu+\nu_1} \otimes 1). 
\end{align*} \]

It can be computed as follows:
\[ (2.3.11) \quad S(h \otimes v)(g) = P^{M}_{\mu+\mu_1}(h(g) \otimes P_N(g^{-1}v)), \quad h \in I_{\bar{P}, \mu, \nu}, \]

Let \( V \) be the following natural \( G \)-map,
\[ \begin{align*}
V &= \beta(S_0^1)_G : \text{Ind}^G_F(\sigma_\mu \otimes a^\nu \otimes 1) \\
(2.3.12) &\cong \text{Ind}^G_F(\sigma_{\mu+\mu_1} \otimes a^{\nu+\nu_1} \otimes 1) \otimes F_{\mu_1, \nu_1}. 
\end{align*} \]

By (1.2.11), \( S = V^\dagger \). Therefore,
\[ (2.3.13) \quad V(h) = \sum S(h \otimes y_i) \otimes y_i^*, \]
or explicitly, by (2.3.11)
\[ (2.3.14) \quad V(h) = \sum h_i \otimes y_i^*, \quad h \in I_{\bar{P}, \mu, \nu}, \]
where \( h_i(g) = P^{M}_{\mu+\mu_1}(h(g) \otimes P_N(g^{-1}y_i)). \)

(2.3.15) Remark. When \( \mu_1 = 0 \), i.e., \( M \) acts trivially on \( F_N \), by a theorem of Helgason [He], \( F_N \) is one dimensional and \( F \) is spherical in the sense that it has a \( K \)-fixed vector, where \( K \) is a maximal compact subgroup of \( G \), as before. The above definitions of \( T, S, U, V \) reduce to the formulas of \( T, S, U, V \) given in [VW].

2.4. First commuting diagram. Choose a constant \( c \) such that, if \( \text{Re}(\nu, \alpha) \geq c \) for \( \alpha \in \Phi(P, A) \), then both integrals defining \( J(\bar{P} : P, \mu, \nu) \) and \( J(\bar{P} : P, \mu+\mu_1, \nu+\nu_1) \) converge absolutely (see the Introduction).

(2.4.1) Proposition. If \( \text{Re}(\nu, \alpha) \geq c \) for \( \alpha \in \Phi(P, A) \), then the following diagram commutes:
\[ \begin{align*}
I_{P, \mu, \nu} \otimes F_{\mu_1, \nu_1} &\xrightarrow{J(\bar{P} : P, \mu, \nu) \otimes I} I_{\bar{P}, \mu, \nu} \otimes F_{\mu_1, \nu_1} \\
I_{P, \mu+\mu_1, \nu+\nu_1} &\xrightarrow{J(\bar{P} : P, \mu+\mu_1, \nu+\nu_1)} I_{\bar{P}, \mu+\mu_1, \nu+\nu_1} \\
\end{align*} \]
Proof. Let \( \{y_i\} \) and \( \{y_i^*\} \) be dual bases of \( F_{\mu_1, \nu_1} \) and \( F_{\mu, \nu}^* \), as before. Set \( \tilde{f}_i(g) = f(g) \circ P_{\tilde{N}}(g^{-1}y_i^*) \) for \( f \in I_{P, \mu+\mu_1, \nu+\nu_1} \). Then, by (2.3.5),

\[
T(f) = \sum_i f_i \otimes y_i.
\]

Thus,

\[
S(J(\tilde{P} : P, \mu, \nu) \otimes I)Tf = \sum_i S(J(\tilde{P} : P, \mu, \nu) \otimes I)(f_i \otimes y_i)
\]

\[
= \sum_i S(J(\tilde{P} : P, \mu, \nu)f_i \otimes y_i).
\]

By the formula of \( S \) in (2.3.11), we have

\[
S(J(\tilde{P} : P, \mu, \nu)f_i \otimes y_i)(g) = P_{\mu+\mu_1}^M((J(\tilde{P} : P, \mu, \nu)f_i)(g) \otimes P_N(g^{-1}y_i))
\]

\[
= P_{\mu+\mu_1}^M \left( \int_{\tilde{N}} f_i(g\tilde{n}) \, d\tilde{n} \otimes P_N(g^{-1}y_i) \right)
\]

\[
= P_{\mu+\mu_1}^M \left( \int_{\tilde{N}} (f(g\tilde{n}) \circ P_{\tilde{N}}((g\tilde{n})^{-1}y_i^*)) \otimes P_N(g^{-1}y_i) \, d\tilde{n} \right).
\]

Since

\[
P_N(g^{-1}y_i) = P_N((g\tilde{n})^{-1}g^{-1}y_i) = P_N((g\tilde{n})^{-1}y_i), \quad \text{(by Proposition (2.2.4))},
\]

the expression inside the integral (2.4.3) is equal to

\[
(f(g\tilde{n}) \circ P_{\tilde{N}}((g\tilde{n})^{-1}y_i^*)) \otimes P_N((g\tilde{n})^{-1}y_i).
\]

Now

\[
\sum_i (f(g\tilde{n}) \circ P_{\tilde{N}}((g\tilde{n})^{-1}y_i^*)) \otimes P_N((g\tilde{n})^{-1}y_i)
\]

\[
= \sum_i (f(g\tilde{n}) \circ P_{\tilde{N}}(y_i^*)) \otimes P_N(y_i) = f(g\tilde{n})
\]

by Remark (1.1.6), since \( f(g\tilde{n}) \in \sigma_{\mu+\mu_1} = P_{\mu+\mu_1}^M(\sigma_\mu \otimes \sigma_{\mu_1}) \subseteq \sigma_\mu \otimes \sigma_{\mu_1} \), and the underlying space of \( \sigma_{\mu_1} \) is \( F^N \).

Therefore,

\[
S(J(\tilde{P} : P, \mu, \nu) \otimes I)Tf(g) = P_{\mu+\mu_1}^M \left( \int_{\tilde{N}} f(g\tilde{n}) \, d\tilde{n} \right) = \int_{\tilde{N}} f(g\tilde{n}) \, d\tilde{n}
\]

\[
= J(\tilde{P} : P, \mu+\mu_1, \nu+\nu_1)f(g),
\]

which is the desired commutativity.

2.5. Vogan's Lemma. The following lemma is due to Vogan [V1].

(2.5.1) Lemma. Let \( \mathfrak{g} \) be a complex reductive Lie algebra, \( I \subseteq \mathcal{U}(\mathfrak{g}) \) any two-sided ideal, and let \( \mathcal{T}_I \) denote the category of \( \mathcal{U}(\mathfrak{g}) \)-modules \( M \) such that \( I \cdot M = 0 \). Let \( \phi \) be any natural construction which associates to each object \( X \) in \( \mathcal{T}_I \) the following \( \mathcal{U}(\mathfrak{g}) \)-module map:

\[
\phi_X : X \to X.
\]
Then there is a unique element \( \overline{Z} \in \mathcal{Z}(g)/I \cap \mathcal{Z}(g) \) such that, for all \( X \in \mathcal{F}_I \),
\[
\phi_X(x) = \overline{Z} \cdot x \quad (x \in X).
\]
Here \( Z \) is a representative of \( \overline{Z} \) in \( \mathcal{Z}(g) \). In particular, if \( I \cap \mathcal{Z}(g) \) is maximal in \( \mathcal{Z}(g) \), then there is a constant \( c \in \mathbb{C} \) such that
\[
\phi_X(x) = cx, \quad x \in X.
\]

(2.5.2) Remark. A formula for \( Z \) is also given in [V1], namely, \( Z = \phi_{N_I}(\pi(1)) \), where \( N_I = \mathbb{Z}(g)/I \) and \( \pi \) is the natural map from \( \mathbb{Z}(g) \) to \( N_I \).

From the above lemma, it follows

(2.5.3) Proposition. Let \( G \) be a real reductive Lie group, \( \mathcal{F}_\lambda \) the category of \( \mathbb{U}(\mathfrak{g}_\mathbb{C}) \)-modules having the infinitesimal character \( \lambda \in \mathcal{C}_G \) (\( \mathcal{C}_G \) is a fixed positive Weyl chamber as in §2.1), and \( F_\lambda \) the finite-dimensional irreducible representation of \( G \) with the highest weight \( \Lambda \). Let \( X \in \mathcal{F}_\lambda \) and define \( \phi : X \to X \) by
\[
\phi(x) = \sum_i P_{\lambda + \Lambda}^G(x \otimes y_i) \circ y_i^*,
\]
where \( \{y_i\} \) and \( \{y_i^*\} \) are any dual bases of \( F_\lambda \) and \( F_\lambda^* \), and \( \circ \) is the contraction between \( F_\lambda \) and \( F_\lambda^* \). Then there exists a scalar \( r_{\lambda, \Lambda}^G(\lambda) \) depending on \( \lambda \) and \( \Lambda \) such that \( \phi(x) = r_{\lambda, \Lambda}^G(\lambda)x \forall x \in X \).

(2.5.4) Remark. Similarly, define
\[
\psi(x) = \sum_i P_{\lambda + \Lambda}^G(x \otimes v_i^*) \circ v_i, \quad x \in X \in \mathcal{F}_{\lambda + \Lambda}.
\]
Then there exists a scalar \( R_{\lambda + \Lambda}^G(-\Lambda) \) depending on \( \lambda \) and \( \Lambda \) such that \( \psi(x) = R_{\lambda + \Lambda}^G(-\Lambda)x \forall x \in X \).

2.6. Properties of \( T, S, U, V \). We refer the reader to §2.3 for the definitions of \( T, S, U, V \). We first give a simple lemma about the induction functor.

(2.6.1) Lemma. \( \text{Ind}_{MAN}^G \) and \( \text{Ind}_{MAN}^G \) are exact functors.

Proof. As an induction functor, it is always left exact. Since \( G = K\text{MAN} \), restricting \( \text{Ind}_{MAN}^G \) to \( K \) gives an isomorphism.

Since
\[
\text{Ind}_{MAN}^G|K \cong \text{Ind}_{MAN}^K.
\]
and \( \text{Ind}_{MAN}^K \) is right exact by the "unitary trick," we see that \( \text{Ind}_{MAN}^G \) is also right exact.

We observe that
(i) \( T_0 \) is injective.
(ii) \( S_0 \) is surjective.

The exactness of the two induction functors implies

(2.6.2) Proposition. \( T \) is injective and \( S \) is surjective.

Assume \( r_{\mu}^M(\mu_1) \neq 0 \), and let \( \{w_j\} \) and \( \{w_j^*\} \) be any dual bases of \( \sigma_{\mu_1} \) and \( \sigma_{\mu_1}^* \). Thus, by Proposition (2.5.3), we have
\[
\frac{1}{r_{\mu}^M(\mu_1)} \sum_j P_{\mu + \mu_1}^M(u \otimes w_j) \circ w_j^* = u, \quad u \in \sigma_{\mu}.
\]
The above clearly implies

(iii) $\text{span}_{v^* \in \sigma_{\mu}^*} \{ T_0(\sigma_{\mu+\mu_1}) \circ v^* \} = \text{span}_{v^* \in \sigma_{\mu}^*} \{ (\sigma_{\mu+\mu_1}) \circ v^* \} = \sigma_{\mu}$.

(iv) If $u \in \sigma_{\mu}$ has the property that $S_0(u \otimes u_1) = P_{\mu+\mu_1}(u \otimes u_1) = 0 \forall u_1 \in \sigma_{\mu_1}$, then $u = 0$.

Again the exactness of the two induction functors implies

\[
(2.6.3) \quad \text{span}_{v^* \in F_{\mu+\mu_1}} \{ T(I_{\mu+\mu_1}, \nu+\nu_1) \circ v^* \} = I_{\mu+\mu_1, \nu},
\]

and if $S(h \otimes v) = 0 \forall v \in F$, then $h = 0$.

Since

\[
U(f \otimes v^*) = T(f) \circ v^* \quad \text{(by (2.3.7))},
\]

\[
V(h) = \sum S(h \otimes v_i) \otimes v_i^* \quad \text{(by (2.3.13))},
\]

we immediately have

\[
(2.6.4) \quad \text{Proposition.} \quad \text{Assume } P_{\mu}^M(\mu_1) \neq 0. \text{ Then } U \text{ is surjective and } V \text{ is injective.}
\]

The proofs of the following two propositions can be copied almost word for word from [VW]. We give some details for the sake of completeness.

\[
(2.6.5) \quad \text{Proposition.} \quad \text{There exists a nonzero complex valued polynomial } \phi_1 \text{ on } a_C^* \text{ such that, if } \phi_1(\nu) \neq 0, \text{ then}
\]

\[
T : I_{\mu+\mu_1, \nu+\nu_1} \rightarrow P_{\mu+\mu_1, \nu+\nu_1}(I_{\mu, \nu} \otimes F_{\mu_1, \nu_1})
\]

and

\[
S : P_{\mu+\mu_1, \nu+\nu_1}(I_{\mu, \nu} \otimes F_{\mu_1, \nu_1}) \rightarrow I_{\mu+\mu_1, \nu+\nu_1}
\]

are linear bijections.

\textbf{Proof.} Let $F = F_1 \supset F_2 \supset \cdots \supset F_r \supset F_{r+1} = (0)$ be a Jordan-Holder series for $F = F_{\mu_1, \nu_1}$ as a $P$-module. We may assume that $F_r$ is a module of the form $a_\nu \otimes 1$ as a $P$-module. If $V$ is an $(m, K \cap M)$-module and if $\lambda \in a_C^*$, we denote by $V_\lambda$ the $(p, K \cap M)$-module $V$ with $n$ acting trivially, $m$ acting as it did on $V$, and $a$ acting by $\lambda$. Then each $F_i/F_{i+1}$ is of the form $(V_i)_\lambda$, with $V_i$ an irreducible finite-dimensional $(m, K \cap M)$-module. Thus $I_{\mu, \nu} \otimes F$ has a composition series $I_{\mu, \nu} \otimes F = M_1 \supset M_2 \supset \cdots \supset M_r \supset M_{r+1} = (0)$ with $M_i/M_{i+1} \cong I_{\mu, \nu} \otimes (V_i)_{\nu_1}$. Now each $\sigma_\nu \otimes V_i$ has a direct sum decomposition into $(m, K \cap M)$-submodules $V_{ij}$ each having a different (meaning inequivalent under the Weyl group) infinitesimal character (see §2.1). Thus, $I_{\mu, \nu} \otimes F$ has a composition series with intermediate quotients $I_{\mu, \nu} \otimes (V_i)_{\nu_1}$. Here $\lambda_i$ is a weight of the action of $a$ on $F$ and if $\lambda_i = \nu_1$, then $i = r$, and there is only one $j$ with $V_{ij} \cong \sigma_{\mu+\mu_1}$ (cf. the definition of $\sigma_{\mu+\mu_1}$ in §2.3). Let $\mu_{ij}$ be the infinitesimal character parameter for $V_{ij}$. Then

\[
\chi(\mu_{ij}, \nu+\lambda_i)(C) - \chi(\mu_{ij}, \nu+\nu_1)(C)
\]

\[
= ((\mu_{ij}, \nu+\lambda_i), (\mu_{ij}, \nu+\lambda_i))
\]

\[
- ((\mu_1 + \mu, \nu+\nu_1), (\mu_1 + \mu, \nu+\nu_1))
\]

\[
= (\mu_{ij}, \mu_{ij}) - (\mu_1 + \mu, \mu_1 + \mu_1) + (\lambda_i, \lambda_i)
\]

\[
- (\nu_1, \nu_1) + 2(\nu, \lambda_i - \nu_1) = \phi_{ij}(\nu),
\]
where $C$ is the second order Casimir operator of $G$. It is obvious that $\phi_{ij}(v)$ is a nonzero polynomial of $v$ for $i < r$.

Set $\phi_1 = \prod_{i<r,j} \phi_{ij}$. Then if $\phi_1(v) \neq 0$, the only $I_{P_{ij}, \nu_i, \nu_i + \lambda_i}$ which can have the infinitesimal character $(\mu + \mu_1, \nu + \nu_1)$ is the one with $i = r$ and $V_{ij} \cong \sigma_{\mu + \mu_1}$, and so $P_{\mu + \mu_1, \nu + \nu_1}^G(I_{P_{ij}, \mu, \nu} \otimes F) \cong I_{P_{ij}, \mu + \mu_1, \nu + \nu_1}$. Similarly, if $\phi_1(v) \neq 0$, then $P_{\mu + \mu_1, \nu + \nu_1}^G(I_{F_{ij}, \sigma, \nu} \otimes F) \cong I_{P_{ij}, \mu + \mu_1, \nu + \nu_1}$. Since $T$ is injective and $S$ is surjective, the proposition now follows.

In view of Proposition (2.6.4), a similar argument as in above gives the following

(2.6.6) **Proposition.** Assume $r_{\mu}^M(\mu_1) \neq 0$. Then there exists a nonzero complex valued polynomial $\phi_2$ on $\mathfrak{a}_C$ such that, if $\phi_2(v) \neq 0$, then

$$U : P_{\mu, \nu}(I_{P_{ij}, \mu + \mu_1, \nu + \nu_1} \otimes F_{\mu_1, \nu_1}) \rightarrow I_{P_{ij}, \mu, \nu}$$

and

$$V : I_{F_{ij}, \sigma, \nu} \rightarrow P_{\mu, \nu}^G(I_{F_{ij}, \sigma, \nu} \otimes F_{\mu_1, \nu_1})$$

are linear bijections.

(2.6.7) **Remark.** $r_{\mu}^M(\mu_1) \neq 0$ if $\mu$ is a dominant integral regular infinitesimal character (see §2.8).

2.7. **Second commuting diagram: functional equations for the intertwining operators.** Again choose a constant $c$ such that, if $\text{Re}(\nu, \alpha) \geq c$ for $\alpha \in \Phi(P, A)$, then both integrals defining $J(P : P, \mu, \nu)$ and $J(P : P, \mu + \mu_1, \nu + \nu_1)$ converge absolutely.

Assume $r_{\mu}^M(\mu_1) \neq 0$. Our main result is the following

(2.7.1) **Theorem.** If $\phi_1(v) \neq 0$, $\phi_2(v) \neq 0$ (see §2.6 for their definitions), $\text{Re}(\nu, \alpha) \geq c$ for $\alpha \in \Phi(P, A)$, then the following diagram is commutative.

(2.7.2)

$$
\begin{array}{ccc}
I_{P_{ij}, \mu, \nu} & \rightarrow & I_{F_{ij}, \mu + \mu_1, \nu + \nu_1} \\
\downarrow U & & \downarrow \text{J(\mu, \nu)} \\
I_{P_{ij}, \mu, \nu} & \rightarrow & I_{F_{ij}, \mu + \mu_1, \nu + \nu_1}
\end{array}
$$

**Proof.** Let $\{y_i\}$ and $\{y_i^*\}$ be dual bases of $F_{\mu_1, \nu_1}$ and $F_{\mu_1, \nu_1}^*$, as before, and let $f = \sum_i f_i \otimes y_i^*$ be any element in $P_{\mu, \nu}^G(I_{P_{ij}, \mu + \mu_1, \nu + \nu_1} \otimes F_{\mu_1, \nu_1})$. Then

$$(J(\mu + \mu_1, \nu + \nu_1) \otimes I) \left( \sum_i f_i \otimes y_i^* \right) \in P_{\mu, \nu}^G(I_{P_{ij}, \mu + \mu_1, \nu + \nu_1} \otimes F_{\mu_1, \nu_1}).$$

Since $r_{\mu}^M(\mu_1) \neq 0$ and $\phi_2(v) \neq 0$, $V$ is a linear isomorphism by Proposition (2.6.6).

Therefore, there is a unique $h \in I_{F_{ij}, \mu, \nu}$ such that

(2.7.3)

$$
(J(\mu + \mu_1, \nu + \nu_1) \otimes I) \left( \sum_i f_i \otimes y_i^* \right) = \sum_i J(\mu + \mu_1, \nu + \nu_1) f_i \otimes y_i^* = V(h).
$$
By (2.3.13),
\[ V(h) = \sum_i S(h \otimes y_i) \otimes y_i^*, \]
so
\[ \sum_i J(\mu + \mu_1, \nu + \nu_1) f_i \otimes y_i^* = \sum_i S(h \otimes y_i) \otimes y_i^*, \]
and hence
\[ J(\mu + \mu_1, \nu + \nu_1) f_i = S(h \otimes y_i), \quad \text{for each } i. \]

By the first commuting diagram (2.4.2), namely,
\[ S(J(\mu , \nu) \otimes I) T = J(\mu + \mu_1, \nu + \nu_1), \]
we obtain
\[ S(J(\mu , \nu) \otimes I) T f_i = J(\mu + \mu_1, \nu + \nu_1) f_i = S(h \otimes y_i) \]
\[ = S[P_{\mu+\mu_1, \nu+\nu_1}^G (h \otimes y_i)]. \]

Since \( \phi_1(\nu) \neq 0 \), \( S \) is a linear isomorphism by Proposition (2.6.5), and so
\[ (J(\mu , \nu) \otimes I) T f_i = P_{\mu+\mu_1, \nu+\nu_1}^G (h \otimes y_i). \]
Thus, by contracting with \( y_i^* \) and summing over \( i \), we obtain by Proposition (2.5.3)
\[ \sum_i ((J(\mu , \nu) \otimes I) T f_i) \circ y_i^* = \sum_i P_{\mu+\mu_1, \nu+\nu_1}^G (h \otimes y_i) \circ y_i^* = r_{\mu, \nu}^G (\mu_1, \nu_1) h. \]

The left-hand side is equal to
\[ \sum_i J(\mu , \nu) (T f_i \circ y_i^*) = \sum_i J(\mu , \nu) U (f_i \otimes y_i^*) \quad \text{(by (2.3.7))} \]
\[ = J(\mu , \nu) U \left( \sum_i f_i \otimes y_i^* \right). \]
Thus, we obtain
\[ J(\mu , \nu) U \left( \sum_i f_i \otimes y_i^* \right) = r_{\mu, \nu}^G (\mu_1, \nu_1) h. \]
(2.7.4)

So,
\[ r_{\mu, \nu}^G (\mu_1, \nu_1) (J(\mu + \mu_1, \nu + \nu_1) \otimes I) \left( \sum_i f_i \otimes y_i^* \right) \]
\[ = r_{\mu, \nu}^G (\mu_1, \nu_1) V(h) \quad \text{(by (2.7.3))} \]
\[ = V J(\mu , \nu) U \left( \sum_i f_i \otimes y_i^* \right) \quad \text{(by (2.7.4))}, \]
which is exactly the commutativity of the diagram (2.7.2).
(2.7.5) Remark. In [VW], a similar commutative diagram is proved with $F$ an irreducible finite-dimensional spherical representation. Our proof is different from theirs.

(2.7.6) Remark. Since each representation present in diagram (2.7.2) is generically irreducible, one knows a priori that this diagram is commutative up to a multiple generically (Schur's Lemma).

To get this multiple, one forms $\sum_i V J(\mu, \nu) U P_{\mu, \nu}^G (f \otimes y_i^*) \circ y_i$ and computes as follows:

$$\sum_i V J(\mu, \nu) U P_{\mu, \nu}^G (f \otimes y_i^*) \circ y_i$$

$$= \sum_i V J(\mu, \nu) U (f \otimes y_i^*) \circ y_i$$

$$= \sum_{i,j} S(J(\mu, \nu) U (f \otimes y_i^*) \otimes y_j^*) \otimes y_i \circ y_i \text{ (by (2.3.13))}$$

$$= \sum_i S(J(\mu, \nu) \otimes I) U (f \otimes y_i^*) \otimes y_i$$

$$= \sum_i S(J(\mu, \nu) \otimes I) T f \text{ (since } U = T^\dagger, \text{ see 2.3.7)}$$

$$= J(\mu + \mu_1, \nu + \nu_1) f \text{ (by the commutativity of diagram (2.4.2)).}$$

Thus,

$$R_{\mu + \mu_1, \nu + \nu_1}^G (-\mu_1, -\nu_1) \sum_i V J(\mu, \nu) U P_{\mu, \nu}^G (f \otimes y_i^*) \circ y_i$$

$$= R_{\mu + \mu_1, \nu + \nu_1}^G (-\mu_1, -\nu_1) J(\mu + \mu_1, \nu + \nu_1) f$$

$$= \sum_i P_{\mu, \nu}^G (J(\mu + \mu_1, \nu + \nu_1)f \otimes y_i^*) \circ y_i \text{ (by Remark (2.5.4)).}$$

Combining the above identity with the commutativity of diagram (2.7.2), we have

(2.7.7) Corollary. If $r_{\mu \mu}^M (\mu_1) \neq 0$, $\phi_1(\nu) \neq 0$, $\phi_2(\nu) \neq 0$, and $\Re(\nu, \alpha) \geq c$ for $\alpha \in \Phi(P, A)$, then

$$r_{\mu, \nu}^G (\mu_1, \nu_1) R_{\mu + \mu_1, \nu + \nu_1}^G (-\mu_1, -\nu_1) = 1.$$
Let \( \{w_j\} \) and \( \{w_j^*\} \) be dual bases of \( F_{\lambda-\rho_G} \) and \( F_{\lambda-\rho_G}^* \). In particular, the above identity gives

\[
\sum_i P^{G}_{\lambda+\Lambda}(w_j \otimes y_i) \circ y_i^* = r^G_\lambda(\Lambda)w_j \quad \forall j.
\]

Hence by contracting with \( w_j^* \) and summing over \( j \), we obtain

\[
(2.8.2) \quad \sum_j \sum_i P^{G}_{\lambda+\Lambda}(w_j \otimes y_i) \circ y_i^* \circ w_j^* = r^G_\lambda(\Lambda) \sum_j w_j \circ w_j^* = r^G_\lambda(\Lambda) \dim F_{\lambda-\rho_G}.
\]

Since the left-hand side is independent of the choice of dual bases \( \{w_j \otimes y_i\} \) and \( \{w_j^* \otimes y_i^*\} \) of \( F_{\lambda-\rho_G} \otimes F_\Lambda \) and \( (F_{\lambda-\rho_G} \otimes F_\Lambda)^* \), we have

\[
(2.8.2) = \text{trace of } P^{G}_{\lambda+\Lambda} = \dim P^{G}_{\lambda+\Lambda}(F_{\lambda-\rho_G} \otimes F_\Lambda) = \dim F_{\lambda+\Lambda-\rho_G}.
\]

Thus,

\[
r^G_\lambda(\Lambda) = \frac{\dim F_{\lambda+\Lambda-\rho_G}}{\dim F_{\lambda-\rho_G}} = \frac{\prod_{\alpha>0}(\lambda + \Lambda, \alpha)/\prod_{\alpha>0}(\rho_G, \alpha)}{\prod_{\alpha>0}(\lambda, \alpha)\prod_{\alpha>0}(\rho_G, \alpha)} = \frac{\prod_{\alpha>0}(\lambda + \Lambda, \alpha)}{\prod_{\alpha>0}(\lambda, \alpha)}
\]

by the famous Weyl dimension formula.

Let \( \mathcal{G}(\Lambda) \) be the region of \( \lambda \) such that if \( \tau \) is a \( \mathfrak{h} \)-weight of \( F_\lambda \) and \( \tau \neq \Lambda \), then \( \chi_{\lambda+\tau}(C) \neq \chi_{\lambda+\Lambda}(C) \). Recall here that \( C \) is the second order Casimir operator and \( \chi_{\lambda+\tau} \) is the infinitesimal character associated to \( \lambda + \tau \in \mathfrak{h}^* \) via the Harish-Chandra isomorphism (see §2.1). Thus \( \mathcal{G}(\Lambda) \) is the complement in \( \mathfrak{h}^* \) of a finite number of hyperplanes.

The proof of the following proposition is essentially given in [VW].

(2.8.3) **Proposition.** \( r_\lambda(\Lambda) \) is a rational function in \( \mathcal{G}(\Lambda) \).

**Proof.** Let \( N_\lambda = \mathcal{U}(\mathfrak{g})/I_\lambda \), where \( I_\lambda \) is the two-sided ideal of \( \mathcal{U}(\mathfrak{g}_C) \) generated by \( z - \chi_\lambda(z), \, z \in \mathcal{U}(\mathfrak{g}_C) \). Then

\[
N_\lambda \otimes F_\Lambda = \bigoplus_{\tau \in \pi(F_\Lambda)} P^G_{\lambda+\tau}(N_\lambda \otimes F_\Lambda),
\]

where \( \pi(F_\Lambda) \) is the set of \( \mathfrak{h} \)-weights of \( F_\Lambda \) (see [Ko]).

Let \( r(\tau) \) be such that

\[
(C - \chi_{\lambda+\tau}(C))^{r(\tau)} P^G_{\lambda+\tau}(N_\lambda \otimes F_\Lambda) = 0.
\]

Set

\[
U_\lambda = \prod_{\tau \in \pi(F) - \{\Lambda\}} (C - \chi_{\lambda+\tau}(C))^{r(\tau)},
\]

\[
\tilde{Z}_\lambda = \chi_{\lambda+\Lambda}(U_\lambda)^{-1} U_\lambda \quad \text{(it makes sense since } \chi_{\lambda+\tau}(C) \neq \chi_{\lambda+\Lambda}(C), \text{ for } \tau \neq \Lambda).\]

Then the projection of \( N_\lambda \otimes F_\Lambda \) onto \( P^G_{\lambda+\Lambda}(N_\lambda \otimes F_\Lambda) \) is given by the action of \( \tilde{Z}_\lambda \) on \( N_\lambda \otimes F_\Lambda \).

Let

\[
Z_\lambda = \sum \tilde{Z}_\lambda(1 \otimes y_i) \circ y_i^*.
\]

Then \( r_\lambda(\Lambda) = \chi_\lambda(Z_\lambda) \) by Remark (2.5.2). It is clear that \( r_\lambda(\Lambda) \) is rational in \( \lambda \).

Let \( \Phi(\mathfrak{b}, \mathfrak{h}) \) be the set of positive \( \mathfrak{h} \)-roots in \( \mathfrak{g}_C \) specified by our choice of Borel subalgebra \( \mathfrak{b} \). The above proposition clearly implies

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Corollary. There exists a $c > 0$ such that $r_\lambda(\Lambda)$ is a rational function of $\lambda$ in the region \( \{ \lambda | \Re(\lambda, \alpha) \geq c \text{ for } \alpha \in \Phi(b, h) \} \).

Combining Corollary (2.8.4) with Proposition (2.8.1), and observing that a nonzero rational function cannot have "half a lattice" as its zeros, we obtain

Proposition.

\[
 r_\lambda(\Lambda) = \frac{\prod_{\alpha > 0}(\lambda + \Lambda, \alpha)}{\prod_{\alpha > 0}(\lambda, \alpha)}
\]

if $\Re(\lambda, \alpha) \geq c$ for $\alpha \in \Phi(b, h)$.

References


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