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SUPERCUSPIDAL REPRESENTATIONS AND THE THETA CORRESPONDENCE. II: SL(2) AND THE ANISOTROPIC O(3)

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ABSTRACT. A parametrization is given of the local theta correspondence attached to the reductive dual pair $(\mathrm{SL}_2(F), \mathrm{O}(F))$ where F is a nonarchimedean local field of odd residual characteristic and O is the orthogonal group of a ternary quadratic form which is anisotropic over F. The parametrization is in terms of inducing data. Various lattice models of the oscillator representation are used.

In this paper we examine Howe's local theta correspondence for the reductive dual pair $(\mathrm{SL}_2(F), \mathrm{O}(F))$ where F is a nonarchimedean local field of odd residual characteristic and O is the orthogonal group of a quadratic form in three variables which is anisotropic over F. In particular we determine if a representation of $\mathrm{O}(F)$ occurs in the theta correspondence and determine the corresponding representation if that representation is supercuspidal. The determination is in terms of inducing data from compact open subgroups.

The correspondence we study here has been studied by Rallis and Schiffman [RS] and Waldspurger [W] among others. What is new here is that we provide an explicit parametrization of the correspondence. The two principal techniques used here are the lattice model of the oscillator (Weil) representation and the parametrization of supercuspidal representations via induction from compact open subgroups. These two techniques were also the primary techniques in our paper [M1] to which this paper is a sequel (also see [M2] where we treat the split case). In future sequels, we plan to further evidence the power of using these techniques in tandem to study the theta correspondence.

This paper is organized as follows. First, as this paper is a sequel we rely heavily on the notation, motivation, and results of [M1]. Assuming this material (while also providing references), the first section of this paper is devoted to a brief recounting of the theory of the irreducible admissible representations of $\mathrm{O}(F)$. Since this group is isomorphic to the direct product of the projectivization of the multiplicative group of the nonsplit quaternion algebra over F and $\mathbb{Z}/2\mathbb{Z}$ this is straightforward if not well known. We present it in a manner suitable for our parametrization of the correspondence. In §2 of the paper we explicitly parametrize the correspondence in terms of inducing data for the representations of $\mathrm{O}(F)$ and the supercuspidal representations of the nontrivial

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two-fold cover of $\mathrm{SL}_2(F)$. The key here is the appropriate choices of various lattice models and the methods and results of our paper [M1].

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1. Some parameters

In this section we establish notation and parametrize the admissible dual of the orthogonal group associated to an anisotropic ternary quadratic form over a p-adic field. Since most of this material is known or easily derived from the literature we will be quite brief in our discussion. For unexplained terminology or notation see [M1].

Let F be a nonarchimedean local field of residual characteristic p with p odd. Let $\mathscr{O} = \mathscr{O}_F$ be the ring of integers of F and let $\varpi = \varpi_F$ be a generator of the maximal ideal $P = P_F$ in $\mathscr{O}$. Let $k = k_F$ denote the residue class field $\mathscr{O}/P$ and let $q = q_F$ be the cardinality of k. Finally let $\nu(x) = \nu_F(x)$ denote the order of an element x in F and normalize the absolute value $|\ |$ on F so that $|\varpi_F| = q^{-1}$.

Let V_2 be a three-dimensional vector space equipped with a symmetric nondegenerate bilinear form $\langle\ ,\ \rangle_2$ which does not (nontrivially) represent zero. Then we may identify V_2 and $\langle\ ,\ \rangle_2$ with the (reduced) trace zero elements, D^0 say, in a nonsplit quaternion algebra, D say, over F equipped with the nondegenerate symmetric bilinear form defined by the (reduced) norm $N_{D/F}$ (see [D, p. 57]); let $\mathrm{tr}_{D/F}$ denote the trace map from D to F and let σ_D denote the involution on D such that $N_{D/F}(x) = x\sigma_D(x)$ and $\mathrm{tr}_{D/F}(x) = x + \sigma_D(x)$. Also let $\mathscr{O} = \mathscr{O}_D$ denote the ring of integers in D, let $P = P_D$ denote the prime ideal in $\mathscr{O}_D$ and set $k = k_D = \mathscr{O}_D/P_D$. Let $\nu_D(x)$ denote the order of an element x in D and normalize the absolute value $|\ |_D$ on D so that $|x|_D = q^{-2\nu_D(x)}$.

Now if E is a quadratic subfield of D, then we may realize D as the cyclic algebra $(E/F, \sigma, a)$ where σ is the nontrivial element of the Galois group $\Gamma(E/F)$ of E/F and a is an element of $F^\times$ which is not in the image of the norm map $N_{E/F}$ from E to F (see, e.g., [R, §31]). In particular given a generator x for E/F there exists an element in z in $D^\times$ such that $zxz^{-1} = \sigma(x)$ and $z^2 = a$. Note that if, in addition, $N_{E/F}(x)$ is not a square in $F^\times$, then $D^\times$ is generated by D^1, x, and z where D^1 is the subgroup of $D^\times$ consisting of elements of norm one.

Let σ_2 denote the involution of $A_F(D^0)$ ($= \mathrm{End}_F(D^0)$ as in [M1]) associated to $\langle\ ,\ \rangle_2$. If we let an element x in $D^\times$ act on D^0 via conjugation, then the associated element d_x say in $A_F(D^0)$ is in G_2—the isometry group of D^0 with respect to $\langle\ ,\ \rangle_2$. Then if we set $PD^1 = D^1/\{\pm 1\}$ we may identify PD^1 with a subgroup of G_2; in fact with $E = F[x]$ and z as above with $\mathrm{tr}\, x = 0$ we have $G_2 = (PD^1 \rtimes \langle d_x, d_z\rangle) \times \langle \sigma_D\rangle$ where $\langle\sigma_D\rangle$ denotes the subgroup of G_2 generated by σ_D (restricted to D^0 where it is just $-I$). Note that $\langle d_x, d_z\rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\langle\sigma_D\rangle \cong \mathbb{Z}/2\mathbb{Z}$. In what follows we will not distinguish between an element of D^1 and its image in PD^1 when considering G_2; the resultant ambiguity in notation will either not matter or be resolvable from context.

Now consider the Lie algebra $A(D^0)_-$ of G_2. For y in D^0, let ady denote the F-endomorphism of D defined by $x \mapsto yx - xy$. Then the restriction of ady to D^0 maps D^0 into D^0 and we may identify (as is well known or easily checked) D^0 with $A(D^0)_-$ via the map $y \mapsto ady$. Now let $A^*(D^0)$ denote the set of elements x in $A(D^0)$ such that $1+x$ is invertible; it is straightforward to check that $A^*(D^0)$ contains $A(D^0)_-$. Then recall that the Cayley transform is the well-defined map $c_A = c$ from $A^*(D^0)$ to itself defined by $c(x) = (1-x)(1+x)^{-1}$. Recall also that c maps $A(D^0)_-$ into G_2 injectively with inverse c itself.

Now recall that a lattice chain $L = \{L_i\}$ in D^0 is said to be self-dual if for each i the dual lattice $L_i^* = \{x \in D^0 | \langle x, l\rangle \in \mathscr{O} \ \forall l \in L\}$ to L_i is in L. Recall further that the only self-dual lattice chain in V_2 (up to equivalence) is the chain $L^D = \{L_i^D = P_D^i\}_{i\in\mathbb{Z}}$ so that the only σ_2-stable hereditary order in $A_F(D^0)$ is $\mathscr{A}(L^D)$ (see, for example [Mo2, §1]). Since the associated hereditary order is unique, we set $\mathscr{A}(D^0) = \mathscr{A}(L^D)$ and similarly define $\mathscr{P}(D^0)$, $\mathscr{P}^i(D^0)$, $\mathscr{P}^i_+(D^0)$, $\mathscr{P}^i_-(D^0)$, and $U^n(D^0)$ for n positive. With the identifications, we have $\mathscr{P}^i_-(D^0) = P^i_-$, where $P^i_- = P^i_D \cap D^0$, $U^n(D^0) = U^n_D$ for n a positive integer where $U^n_D = \{x \in D^1 | x-1 \text{ is in } P^n_-\}$ and $U^0(D^0) = U(D^0) = G_2$. Finally, we note that the *nonstandard* filtrations constructed by Morris [Mo2, §2] collapse in this case to the standard filtration by powers of the radical.

Lemma 1.1. *The Cayley transform yields a bijection between P^0_- and PD^1. Further, for n a positive integer this bijection takes P^n_- to U^n_D.*

Proof. If not well known, this is certainly straightforward.

Now let $D^* = D - \{-1\}$ and let $c_D \colon D^* \to D^*$ be the map defined by $y \mapsto (1-y)(1+y)^{-1}$. Then the following two lemmas are also straightforward.

Lemma 1.2. *With notation as above,*

(i) *c_D is a bijection onto D^* with inverse c_D itself.*

(ii) *The restriction of c_D to D^0 yields a bijection onto $D^* \cap D^1$.*

(iii) *For n a positive integer, the restriction of c_D to P^n_- yields a bijection onto U^n_D.*

Lemma 1.3. *Let y be an element of D^0. Then $c_A(y/2)^2 = c_D(y)$.*

We now turn to the representation theory of G_2. First, since G_2 is compact, its irreducible representations are finite dimensional, admissible, and supercuspidal. Second, since G_2 is $PD^\times \rtimes \langle\sigma_D\rangle$ a parametrization of its (admissible) dual can be easily derived from known parametrizations of the admissible dual of $D^\times$ (see, e.g., [C]) or derived from scratch. Thus we provide below, without proof, such a parametrization convenient for our purposes.

To begin, suppose that E is a nontrivial subfield (hence quadratic) of D. Then under our identifications we may view PE^1 as a subgroup of G_2 where E^1 denotes the set of x in E such that $N_{E/F}(x) = 1$ and $PE^1 = E^1/\{\pm 1\}$. Now set $\Lambda^2 = \Lambda^2(E) = \{\lambda^2 | \lambda \text{ a character of } E^1\}$. One can check that $(PE^1)^\frown \cong \Lambda^2$. Also let Λ^2_1 denote the subgroup of Λ^2 consisting of characters trivial on $PE^1 \cap U^1_E$ where, for a positive integer k, $U^k_E = \{u \in E | u-1 \in P^k_E\}$ and where U^k_E is viewed as a subgroup of PE^1. Note that the centralizer of PE^1 in G_2 is $(PE^1 \times \langle d_\beta\rangle) \times \langle\sigma_D\rangle$ where β is some generator of E/F of nonsquare norm.

Now suppose that α is a nonzero element in $E \cap D^0$ such that $\nu_d(\alpha) < 0$. (Note that α being nonzero and in D^0 implies that $F[\alpha] = E$ and, moreover, that α is in fact E/F-minimal.) Then, setting $n = -\nu_D(\alpha)$ and $m' =$

$[(n+2)/2]$, let Λ^2_α denote the set of λ in Λ^2 which agree with the character ψ_α of $U^{m'}(D^0)$ upon restriction to $U^{m'}(D^0) \cap PE^1 = U_E^{m'}$. Then to each λ in Λ^2_α we associate a character $\rho'(\alpha, \lambda)$ of $(PE^1)U^{m'}(D^0)$ in the obvious manner.

Let $m = [(n+1)/2]$. If n is odd, then $m = m'$ and E/F is ramified; in this case set $\rho(\alpha, \lambda) = \rho'(\alpha, \lambda)$. If n is even, then $m = m' - 1$ and E/F is unramified. If further m is even, set $\rho(\alpha, \lambda) = \rho'(\alpha, \lambda)$ as a representation of $(PE^1)U^m(D^0) = (PE^1)U^{m'}(D^0)$. Finally, if n is even while m is odd, let $\rho(\alpha, \lambda)$ denote the unique irreducible q-dimensional representation occurring in $\mathrm{Ind}((PE^1)U^m(D^0),\ (PE^1)U^{m'}(D^0);\ \rho'(\alpha, \lambda))$ with multiplicity one. We note that in this last case the induced representation decomposes as a sum of representations of the form $\rho(\alpha, \lambda')$ with λ' in Λ^2_α and $\lambda'\lambda^{-1}$ in Λ^2_1; indeed, all such representations occur and occur with multiplicity two with the exception of $\rho(\alpha, \lambda)$ itself which occurs with multiplicity one.

In all the above cases the representation

$$\pi(\alpha, \lambda) = \mathrm{Ind}(G_2, (PE^1)U^m(D^0); \rho(\alpha, \lambda))$$

decomposes as the sum of 4 distinct irreducible representations $\pi(\alpha, \lambda, \gamma_1, \gamma_2)$, where $\gamma_i = \pm 1$ and $\pi(\alpha, \lambda, \gamma_1, \gamma_2)$ corresponds to the subspace of the space of $\pi(\alpha, \lambda)$ where $\pi(\alpha, \lambda)(d_\beta)$ (β as above) acts by multiplication by γ_1 and $\pi(\alpha, \lambda)(\sigma_D)$ acts by multiplication by γ_2.

We now construct the remaining representations of G_2. If λ is in Λ^2_1, set $\pi(\lambda) = \mathrm{Ind}(G_2, (PE^1)U^1(D^0); \tilde{\lambda})$ where $\tilde{\lambda}$ is the character of $(PE^1)U^1(D^0)$ which is trivial on $U^1(D^0)$ and is λ on PE^1. Let $E = F[\beta]$ such that $N_{E/F}(\beta)$ is not a square. Then, if λ is not trivial, $\pi(\lambda)$ decomposes as the sum of four distinct irreducible two-dimensional representations $\pi(\lambda, \gamma_1, \gamma_2)$, where $\gamma_i = \pm 1$ and $\pi(\lambda, \gamma_1, \gamma_2)$ corresponds to the subspace of the space of $\pi(\lambda)$ where $\pi(\lambda)(d_\beta)$ acts by multiplication by γ_1 and $\pi(\lambda)(\sigma_D)$ acts by multiplication by γ_2. On the other hand, $\pi(1)$ decomposes as the sum of eight distinct one-dimensional representations $\pi^\pm(1, \gamma_1, \gamma_2)$ with γ_1 and γ_2 as above and $\pm$ referring to the action of d_ϖ in the obvious manner.

Proposition 1.4. *The representations* $\pi(\alpha, \lambda, \gamma_1, \gamma_2)$, $\pi(\lambda, \gamma_1, \gamma_2)$, *and* $\pi^\pm(1, \gamma_1, \gamma_2)$ *constructed above exhaust the admissible dual of* G_2. *These representations enjoy the following equivalences.*

(i) *A representation of the form* $\pi(\alpha, \lambda, \gamma_1, \gamma_2)$ *is never equivalent to a representation of the form* $\pi(\lambda', \gamma_1', \gamma_2')$ *or to a representation of the form* $\pi^\pm(1, \gamma_1', \gamma_2')$. *Likewise a representation of the form* $\pi(\lambda, \gamma_1, \gamma_2)$ *is never equivalent to a representation of the form* $\pi^\pm(1, \gamma_1', \gamma_2')$.

(ii) *Representations* $\pi(\alpha, \lambda, \gamma_1, \gamma_2)$ *and* $\pi(\alpha', \lambda', \gamma_1', \gamma_2')$ *are equivalent if and only if* $\gamma_1 = \gamma_1'$, $\gamma_2 = \gamma_2'$ *and there exists a* z *in* PD^1 *such that*

(a) $\alpha' - \alpha^z$ *is in* $p_D^{-[(n+1)/2]}$ *where* $n = \nu_D(\alpha)$;
(b) $F[\alpha]^z = F[\alpha']$;
(c) $\lambda^z = \lambda'$.

(iii) *Representations* $\pi(\lambda, \gamma_1, \gamma_2)$ *and* $\pi(\lambda', \gamma_1', \gamma_2')$ *are equivalent if and only if* $\gamma_1 = \gamma_1'$, $\gamma_2 = \gamma_2'$ *and there exists a* z *in* PD^1 *such that* $E = (E')^z$ *where* λ *is a character of* E *and* λ' *a character of* E' *such that* $\lambda = (\lambda')^z$ *or* $\lambda^{-1} = (\lambda')^z$.

(iv) *There are eight distinct representations of the form* $\pi^\pm(1, \gamma_1, \gamma_2)$.

Remark 1.5. (i) We note that a and b of (ii) above imply that we may parametrize a representation of the form $\pi(\alpha, \lambda, \gamma_1, \gamma_2)$ by $\pi(a, \lambda, \gamma_1, \gamma_2)$ where $N_D(\alpha) = -a$. Indeed an α may be recovered from a and λ as the unique element of $E \cap D^0$ (where λ is a quasicharacter of PE^1) of norm $-a$ (this determines α up to sign) such that ψ_α and λ agree on $U^{m'}(D^0)$ where $m' = [(-\nu_D(\alpha)+2)/2]$. Note further that a is a nonsquare and that any nonsquare of negative valuation can occur.

(ii) With notation as in the above proposition, we note for further reference that in the case n even m odd the representations occurring in the restriction of $\rho(\lambda, \alpha)$ to PE^1 are the characters λ' of PE_1 such that $\lambda'\lambda^{-1}$ is in Λ_1 and these representations occur with multiplicity two with the exception of λ which occurs with multiplicity one.

(iii) As in [RS], we say that a representation π of G_2 is *spherical* if it occurs in the decomposition of the natural action of G_2 on $L^2(D^0)$. Then one can verify directly (using a realization of D as an appropriate cyclic algebra) that if a representation of the form $\pi(\alpha, \lambda^2, \gamma_1, \gamma_2)$ or $\pi(\lambda^2, \gamma_1, \gamma_2)$ is spherical then $\gamma_2 = \lambda(-1)$ if $E = F[\alpha]/F$ is unramified and -1 is not a square in $F^\times$, and $\gamma_1 = \gamma_2 = \lambda(-1)$ otherwise. Similarly, one can show that if a representation of the form $\pi^\pm(1, \gamma_1, \gamma_2)$ is spherical and -1 is a square in $F^\times$ then $\gamma_1 = \gamma_2$, while if -1 is not a square in $F^\times$ then $\gamma_2 = 1$. Later, independent of [RS, W], we will show that these necessary conditions for a representation to be spherical are also sufficient. For the moment we call a representation satisfying the above necessary conditions *pseudospherical*.

(iv) We note that the four pseudospherical representations of the form $\pi^\pm(1, \gamma_1, \gamma_2)$ are the unique pseudospherical extensions of the characters $\psi_a \circ N$ of $PD^\times$ as a ranges over a set of equivalence classes for $F^\times/(F^\times)^2$.

2. The correspondence

We continue with the notation of [M1] and the previous section. In particular, G_2 is the isometry group of an anisotropic form $\langle\ ,\ \rangle_2$ in three variables. In keeping with the notation of [M1], we let V_1 be a two-dimensional F-vector space equipped with a nondegenerate skew-symmetric bilinear form $\langle\ ,\ \rangle_1$; let $G_1 \cong SL_2(F)$ denote the isometry group of V_1. Further, we set $W = \mathrm{Hom}(V_1, V_2)$ and equip W with the nondegenerate skew-symmetric bilinear form $\langle\ ,\ \rangle$ defined by $\langle w, w'\rangle = \mathrm{tr}\, w\lambda(w')$ (with λ as in [M1]). Let G denote the isometry group of $\langle\ ,\ \rangle$ and identify G_1 and G_2 with subgroups of G via their respective actions of premultiplications by inverses and postmultiplication. Now suppose χ is a (continuous) nontrivial additive character of F and let ω_χ denote the oscillator representation of $\widetilde{G}$ attached to χ where $\widetilde{G}$ is the unique nontrivial two-fold cover of G. Also for H a closed subgroup of G, let $\widetilde{H}$ denote the inverse image of H in G under the covering map. Note that $\widetilde{G}_1$ is the nontrivial two-fold cover of G_1 and $\widetilde{G}_2 \cong G_2 \times \{\pm 1\}$. Then, using the Schrödinger model of the oscillator representation, Rallis and Schiffman have shown that

$$\omega_\chi|_{\widetilde{G_1 \cdot G_2}} \cong \bigoplus_\rho \pi_\rho \otimes \rho,$$

where ρ runs over the set of spherical representations of G_2 (see Remark 1.5), the π_ρ are distinct and each π_ρ is an irreducible unitary representation of

$\widetilde{G}_1$ which is supercuspidal unless ρ is trivial (but is otherwise undetermined) and is L^2 if ρ is trivial [RS]. This is then a special case of the local theta correspondence. Now let ω_χ^∞ denote the restriction of ω_χ to smooth vectors so that we are in the more general setting for the theta correspondence (this is not necessary here but is used when both G_1 and G_2 are noncompact). If H is a closed subgroup of G_1 let $\mathscr{R}_\chi(\widetilde{H})$ denote the set of irreducible admissible representations of $\widetilde{H}$ which are quotients of the restriction of ω_χ^∞ to $\widetilde{H}$. Then in this section, independent of [RS and W], we will use various lattice models of the oscillator representation to determine the supercuspidal representations in $\mathscr{R}_\chi(\widetilde{G}_1)$ and $\mathscr{R}_\chi(G_2)$ and the correspondence afforded by $\mathscr{R}_\chi(\widetilde{G}_1\widetilde{G}_2)$. What is new here is that we explicitly parametrize the correspondence in terms of inducing data.

Remark 2.1. Before beginning the argument we note that it is easy to show that $\mathscr{R}_\chi(G_2)$ consists of the spherical representations (see, e.g., [RS]). The more difficult task is the determination of the correspondence.

To be specific as to the correspondence we first fix $\chi = \psi_\varpi$ where ψ is fixed as in [M1]. In particular, if k' is the subfield of $k = \mathscr{O}_F/P_F$ of cardinality P and $\psi_{k'}$ is the additive character of $\psi_{k'}$ such that $\psi_{k'}(1) = e^{2\pi i/p}$, then we require that ψ be an additive character of F that factors to the character ψ_k of k_F defined by $\psi_{k'} \circ \mathrm{Tr}_{k/k'}$ and we set $\psi_\varpi(a) = \psi(\varpi a)$.

Now let L_1 be a self-dual lattice in V_1 (with respect to χ and V_1) and let $L_2 = P_-^0$. Then $L = \mathrm{Hom}_{\mathscr{O}_F}(L_1, L_2)$ is not a self-dual lattice in W but we may identify L^* with $\mathrm{Hom}_{\mathscr{O}_F}(L_1, P_-^{-1})$ and we have $P_F L^* \subseteq L \subsetneq L^*$. Thus we may realize ω_χ in a non-self-dual lattice model associated to L; note that $\dim_{k_F}(L^*/L) = 4$.

For k an integer, set $L^k = \mathrm{Hom}_{\mathscr{O}_F}(L_1, P_-^k)$ and set $L_1^k = P_F^k L_1$. Further set $\mathscr{L} = \{L^k\}_{k\in\mathbb{Z}}$ and $\mathscr{L}_1 = \{L_1^k\}_{k\in\mathbb{Z}}$ and note that $\mathscr{L}$ and $\mathscr{L}_1$ are self-dual lattice chains in W and V_1 respectively. Finally set $\mathscr{A}_1 = \mathscr{A}(\mathscr{L}_1)$, $\mathscr{A}_2 = \mathscr{A}(D^0)$, and $\mathscr{A} = \mathscr{A}(\mathscr{L})$. Then one checks that for l a nonnegative integer $G_2 \cap U^l(\mathscr{A}) = U^l(\mathscr{A}_2)$ and $G_1 \cap U^l(\mathscr{A}) = U^{[(l+1)/2]}(\mathscr{A}_1)$.

Lemma 2.2. *With notation as above, let k be a positive integer and let Y_{2k} be the set of functions in Y (the space of ω_χ in our lattice model) supported on L^{-2k-1}. Then the following hold.*

(i) *$U^{2k+1}(\mathscr{A}_1)$ and $U^{4k+1}(\mathscr{A}_2)$ fix Y_{2k} pointwise.*

(ii) *If f is in Y_{2k} and h is in $U^k(\mathscr{A}_1)$ or $U^{2k}(\mathscr{A}_2)$, then*

$$\omega_\chi(h)f(w) = \rho_L(2c(h)w)\chi(\langle w, c(h)w\rangle)f(w),$$

where $c(h) = (1-h)(1+h)^{-1}$ is the Cayley transform.

(iii) *If f in Y_{2k} is supported on $-w + L^*$, then f transforms according to ψ_{b_1} and ψ_{b_2} under the actions of $U^{k+1}(\mathscr{A}_1)$ and $U^{2k+1}(\mathscr{A}_2)$ respectively where $b_1 = -\varpi\lambda(w)w/2$ and $b_2 = \varpi w\lambda(w)/2$. Moreover, b_1 is an element of $(\mathscr{P}_{1,-})^{-2k}$ and b_2 is an element of $(\mathscr{P}_{2,-})^{-4k}$.*

Proof. Argue as in the proof of [M1, Lemma 4.5] using $M = L^{2k}$.

Lemma 2.3. *With notation as above, under our identification of $A(D^0)_-$ with D^0, $N_D(b_2) = \det(b_1)/4$. Further, $\nu_D(b_2) = -4k$ if and only if the k-vector space homomorphism $\overline{w}\colon L_1^0/L_1^1 \to P_-^{-2k-1}/P_-^{-2k}$ induced by w has rank 2.*

Proof. One checks that if y in D^0 is nonzero, then ady has minimal polynomial $X(X^2+4N_D(y))$. On the other hand one can check that b_2 satisfies the equation $X(X^2+\det b_1)=0$ since b_1 satisfies $X^2+\det b_1=0$. Thus, if b_2 is nonzero, $N_D(b_2)=\det(b_1)/4$. If b_2 is zero, then, since $b_2=\varpi w\lambda(w)/2$, w cannot have rank two as an element of W but then $b_1=-\varpi\lambda(w)w/2$ must have rank 0 or 1 as an element of $A_F(V_1)$ so that $\det(b_1)=0$ whence the first part of the lemma.

Now if $\overline{w}$ has rank 2 then w maps L_1^0 onto P_-^{-2k-1} since

$$\dim_k P_-^{-2k-1}/P_-^{-2k}=2.$$

Then the equality $\langle wv_1, v_2\rangle_2=\langle v_1, \lambda(w)v_2\rangle_1$ for all v_1 in V_1 and v_2 in V_2 implies that b_1 maps L_1^0 onto L_1^{-2k} and thus $\nu_F(\det b_1)=-4k$ so that $\nu_D(b_2)=-4k$ also. If $\overline{w}$ does not have rank 2, then w does not map L_1^0 onto P_-^{-2k-1} and then b_1 maps L_1^0 properly into L_1^{-2k} whence the result.

Lemma 2.4. *Let a be a nonsquare element of $F^\times$ such that $\nu_F(a)=-4k$ with k a positive integer. Then, a representation of the form, $\pi(a, \eta, \gamma_1, \gamma_2)$ occurs in $\mathscr{R}_\chi(G_2)$ if and only if it is pseudospherical.*

Proof. By Remark 1.5 it suffices to show that if the representation is pseudospherical then it occurs in $\mathscr{R}_\chi(G_2)$. Let w be an element of L^{-2k-1} such that $\overline{w}$ (as in the previous lemma) has rank 2. Then, by the previous lemma, $\nu_D(b_2)=-4k$. Let b be an element of $\mathscr{O}_F^\times$ such that $b^2N_D(b_2)=-a$ (note that $\operatorname{tr}(b_2)=0$ so $-N_D(b_2)$ is not a square) and let g be an element of $\mathscr{A}_1^\times$ such that $\det g=b$. Then one checks that for any x in X (the space of ρ_L) $y_{gw,x}$ transforms according to ψ_α under the action of $U^{2k+1}(\mathscr{A}_2)$ where α is an element of P_-^{-4k} such that $N_D(\alpha)=-a$. Without loss of generality (changing w if necessary) we may assume $g=1$. Then, by, for example, [M1, 2.12],

$$\omega_\chi(h)y_{w,x}=y_{w,\rho_L(2c(h)w)\chi(\langle w, c(h)w\rangle)x}$$

for h in $U^{2k}(\mathscr{A}_2)=(U^{2k}(\mathscr{A}_2)\cap PE^1)U^{2k+1}(\mathscr{A}_2)$. Now the images of the vectors $c(h)w$ in L^*/L lie in an isotropic (1-dimensional!) subspace so that we may realize ρ_L in a Schrödinger model (see [M1, 2.10]) where, for each h in $U^{2k}(\mathscr{A}_2)$, $\chi(\langle w, c(h)w\rangle)=1$ and $\rho_L(2c(h)w)$ acts by translation on x. Thus, changing x if necessary, we have that the $U^{2k}(\mathscr{A}_2)$-span of $y_{w,x}$ is isomorphic to $\operatorname{Ind}(U^{2k}(\mathscr{A}_2), U^{2k+1}(\mathscr{A}_2); \psi_\alpha)$. Now one checks that since $\overline{w}$ has rank 2 the stabilizer in $PD^\times$ of $-w+L^*$ in W/L^* is $U^{2k}(\mathscr{A}_2)$. Thus the $PD^\times$-span of $y_{w,x}$ is isomorphic to $\operatorname{Ind}(PD^\times, U^{2k}(\mathscr{A}_2); \psi_\alpha)$. But then since a spherical representation is uniquely determined by its restriction to $PD^\times$, the result follows.

Remark 2.5. That only spherical representations can occur in general can be proved using the lattice model. We will not prove this here since the Schrödinger model proof (as in [RS], see Remarks 2.1 and 1.5) is patently more simple. See the portion of the proof of Theorem 4.8 in [M1] concerning the nonoccurrence of the determinant representation of the unramified anisotropic O(2) for a lattice model argument similar in style.

Theorem 2.6. *Let $\pi(\mathscr{A}_1, \alpha, \eta)$ be a representation of $\widetilde{G}_1$ such that $\nu_{\mathscr{A}_1}(\alpha)$ is even. Then $\pi(\mathscr{A}_1, \alpha, \eta)$ occurs in $\mathscr{R}_\chi(\widetilde{G}_1)$ and pairs with $\pi(-\det(\alpha)/4, \eta^2, \gamma_1, \eta(-1))$ where $E=F[\alpha]$ has been identified with a subfield of D and*

$\gamma_1 = \eta(-1)$ if -1 is a square in $F^\times$ and $\gamma_1 = \eta(\mu)$ where μ is a fourth root of unity in E^1 if -1 is not a square in $F^\times$.

Proof. Since $\nu_{\mathscr{A}_1}(\alpha)$ is even (note that this implies that $F[\alpha]/F$ is unramified), by Lemma 2.3 there exists a w in W such that, for each x in X, $y_{w,x}$ transforms according to $\psi_{\alpha/2}$ under the action of $U^{2k+1}(\mathscr{A}_2)$ where $\nu_{\mathscr{A}_1}(\alpha) = -2k$. By Lemma 2.1 and Lemma 2.2, $y_{w,x}$ transforms according to some ψ_b under the action of $U^{k+1}(\mathscr{A}_1)$ where $\det(b) = \det\alpha$ and $\nu_F(\det\alpha) = -4k$. It follows that there exists a g in $\mathscr{A}_1^\times$ such that $b^g = \alpha$. Then, changing w if necessary (to wg^{-1}), we may assume $y_{w,x}$ transforms according to ψ_α under the action of $U^{k+1}(\mathscr{A}_1)$ and according to $\psi_{\alpha/2}$ under the action of $U^{2k+1}(\mathscr{A}_2)$.

For k a nonnegative integer set, $U^k(\mathscr{A}_{1,E}) = U^k(\mathscr{A}_1)\cap E^1$ and set $U(\mathscr{A}_{1,E}) = U^0(\mathscr{A}_{1,F}) \cap E^1$. Now let a be an element of E^1 such that either a is in $U^1(\mathscr{A}_{1,E})$ or a^2 is not in $U^1(\mathscr{A}_{1,E})$. Then there exists a b in F such that $a = c_1(b\alpha)$ where $c_1(x) = (1-x)(1+x)^{-1}$. Now using that $\alpha w/2 = -w\alpha$ one can check that $c_A(b\alpha/2)w = wc_1(b\alpha)$. Thus, by Lemma 1.3,

$$aw = wa^2. \tag{2.6.1}$$

It follows that if y is a nonzero vector in the G_2-span of $y_{w,x}$ transforming according to $\pi(\alpha/2, \eta^2, \gamma_1, \eta(-1))$ as in the proof of the previous lemma, then y transforms to $\rho'(\mathscr{A}_1, \alpha, \eta)$ under the action of $U^1(\mathscr{A}_{1,E})U^{k+1}(\mathscr{A}_1)$ where $\rho'(\mathscr{A}_1, \alpha, \eta)$ denotes the restriction of the representation $\rho'(\mathscr{A}_1, \alpha, \eta)$ of $U(\mathscr{A}_{1,E})U^{k+1}(\mathscr{A}_1)$ (as defined in [M1, §1.3]) to $U^1(\mathscr{A}_{1,E})U^{k+1}(\mathscr{A}_1)$. Indeed, η^2 determines η up to the nontrivial real-valued character θ say of E^1 and θ is trivial on $U^1(\mathscr{A}_{1,E})$. Now using the representation theory of the Heisenberg group (for an entirely similar argument see the proof of [M1, Theorem 4.8(i)]) one can show that we may choose a nonzero y in the $U^1(\mathscr{A}_{1,E})U^k(\mathscr{A}_1)\cdot G_2$ span of $y_{w,x}$ such that under the action of G_2 y transforms according to $\pi(\alpha/2, \eta^2, \gamma_1, \eta(-1))$ and under the action of $U(\mathscr{A}_{1,E})U^k(\mathscr{A}_1)$ it transforms according to some extension $\tilde{\rho}$ say of $\rho_1(\mathscr{A}, \alpha, \eta)$ to $U(\mathscr{A}_{1,E})U^k(\mathscr{A}_1)$. Write $\tilde{\rho} = \rho(\mathscr{A}, \alpha, \eta')$ for some η' such that $(\eta')\eta^{-1}$ is in Λ_1. Now the proposition will follow from Frobenius reciprocity if we can show that $\eta = \eta'$. To this end, it suffices (see, e.g., [M1, 1.7]) to determine the characters and their multiplicities occurring in the restriction of the $U(\mathscr{A}_{1,E})U^k(\mathscr{A}_1)$-span of y to E^1. This however is a straightforward calculation involving (2.6.1), the properties of the lattice model (see, e.g., [M1, 2.11]), Remark 2.7 below, and Tanaka's determination [T, §9] of the theta correspondence for $(\mathrm{SL}_2(k_F), \mathrm{O}_2(k_F))$ where O_2 is the orthogonal group of an anisotropic binary quadratic form over k_F.

Remark 2.7. If not well known, the following is easily checked. In the notation of [M1], the representations occurring in the restriction to E^1 of the $(q-1)$-dimensional irreducible representation $\rho(\mathscr{A}_1, \eta)$ of $U(\mathscr{A}_1)$ (the inflation of the cuspidal representation of $\mathrm{SL}_2(k_F)$ associated to η) associated to a non-real-valued character η in Λ_1 are the characters μ in Λ_1 such that $\mu \neq \eta$ but $\mu(-1) = \eta(-1)$. Further, these characters occur with multiplicity two. Similarly, each of the non-real-valued characters in Λ_1 occurs and occurs with multiplicity one in the restriction of the $(q-1)/2$ dimensional irreducible cuspidal representations $\rho(\mathscr{A}_1, +)$ and $\rho(\mathscr{A}_1, -)$ to E^1 (notation also as in [M1]).

Theorem 2.8. *All pseudospherical representations of the form* $\pi(\eta, \gamma_1, \gamma_2)$, $\pi^+(1, \gamma_1, \gamma_2)$, *and* $\pi^-(1, -1, \gamma_2)$ *occur in* $\mathscr{R}_\chi(G_2)$. *These representations pair with representations of* $\widetilde{G}_1$ *as follows where* $\mathscr{A}_1$ *is the hereditary order in* $\mathrm{End}_F(V_1)$ *associated to the self-dual lattice* L_1 *as above and* E *is such that* $\mathscr{O}_E = L_1$.

(i) *If* η *is not real-valued, then* $\pi(\mathscr{A}_1, \eta)$ *pairs with* $\pi(\eta^2, \gamma_1, \eta(-1))$ *where* $\gamma_1 = \eta(-1)$ *if* -1 *is a square in* F *and* $\gamma_1 = \eta(\mu)$ *where* μ *is a fourth root of unity in* E^1 *if* -1 *is not a square in* F.

(ii) *The trivial representation* $\pi^+(1, 1, 1)$ *does not pair with a supercuspidal representation.*

(iii) *The unique nontrivial pseudospherical representations of the forms* $\pi^+(1, \gamma_1, \gamma_2)$ *and* $\pi^-(1, -1, \gamma_2)$ *pair with* $\pi(\mathscr{A}_1, \mathrm{sgn}(A))$ *and* $\pi(\mathscr{A}_1, -\mathrm{sgn}(A))$ *respectively where*

$$A = (-1)^{[(f+1)/2](p-1)/2}(-1)^{f+1}(-1)^{(p^2-1)f/8}$$

with $f = \log_p q$.

Proof. Consider the functions in Y supported on L^{-1}. Then a straightforward computation, Theorem 3.1 of [M1] and [T, §9] imply that all the pseudospherical representations listed do occur. Then (i) follows from a straightforward computation and [T, §9] while (iii) follows as in Theorem 4.8 and Remark 4.10 of [M1]. Finally, to show that (ii) holds it seems best to use a Schrödinger model. In particular, realize ω_χ in the Schrödinger model attached to a polarization (X, Y) say of W arising from a polarization (X_1, Y_1) say of V_1. Let f be the function in the Schwartz space on X supported at 0 and taking the value 1 there. Identifying X with D^0 the action of G_2 becomes linear and thus f is fixed by G_2. On the other hand let N be the unipotent radical of the ("upper") parabolic subgroup of G_1 associated to the polarization (X_1, Y_1) of V_1. Then N imbeds as a subgroup of $\widetilde{G}_1$ and fixes f. It follows then that the representation corresponding to the trivial representation could not be supercuspidal.

Remark 2.9. (i) The argument for (ii) above is standard. We only include it for completeness. For a determination of the corresponding representation (a discrete series representation) see [RS] or [Wa].

(ii) We note that (iii) also follows from Theorem 3.1 of [M1] and either [RS] or [Wa].

Now to detect other representations of G_2 attached to unramified tori and the corresponding representations of $\widetilde{G}_1$ we consider another lattice model. To be precise, let L_1 be a lattice in V_1 such that $L_1^* = P^{-1}L_1$ and let $L_2 = P_-^1$. Then $L = \mathrm{Hom}_{\mathscr{O}_F}(L_1, L_2)$ is not a self-dual lattice in W but we may identify L^* with $P^{-1}\mathrm{Hom}_{\mathscr{O}_F}(L_1, P_-^0)$ and we have $P_F L^* \subseteq L \subsetneq L^*$. We now realize ω_χ is a non-self-dual lattice model associated to L; note that $\dim_{k_F}(L^*/L) = 2$.

For k an integer, set $L^k = \mathrm{Hom}_{\mathscr{O}_F}(L_1, P_-^{k+1})$ and set $L_1^k = P_F^k L_1$; further set $\mathscr{L} = \{L^k\}_{k\in\mathbb{Z}}$ and $\mathscr{L}_1 = \{L_1^k\}_{k\in\mathbb{Z}}$ and note that $\mathscr{L}$ and $\mathscr{L}_1$ are self-dual lattice chains in W and V_1 respectively. Set $\mathscr{A}_1 = \mathscr{A}(\mathscr{L}_1)$, $\mathscr{A}_2 = \mathscr{A}(D^0)$, and $\mathscr{A} = \mathscr{A}(\mathscr{L})$. Then one checks that for l a nonnegative integer $G_2 \cap U^l(\mathscr{A}) = U^l(\mathscr{A}_2)$ and $G_1 \cap U^l(\mathscr{A}) = U^{[(l+1)/2]}(\mathscr{A}_1)$.

Lemma 2.10. *With notation as above, let k be a positive integer and let Y_{2k} be the set of functions in Y (the space of ω_χ in our lattice model) supported on L^{-2k}. Then the following hold.*

(i) *$U^{2k}(\mathscr{A}_1)$ and $U^{4k-1}(\mathscr{A}_2)$ fix Y_{2k} pointwise.*

(ii) *If f is in Y_{2k} and h is in $U^k(\mathscr{A}_1)$ or $U^{2k-1}(\mathscr{A}_2)$ then*

$$\omega_\chi(h)f(w) = \rho_L(2c(h)w)\chi(\langle w\,,\, c(h)w\rangle)f(w),$$

where $c(h) = (1-h)(1+h)^{-1}$ is the Cayley transform.

(iii) *If f in Y_{2k} is supported on $-w + L^*$ then f transforms according to ψ_{b_1} and ψ_{b_2} under the actions of $U^k(\mathscr{A}_1)$ and $U^{2k}(\mathscr{A}_2)$ respectively where $b_1 = -\varpi\lambda(w)w/2$ and $b_2 = \varpi w\lambda(w)/2$. Moreover b_1 is an element of $(\mathscr{P}_{1,-})^{-2k+1}$ and b_2 is an element of $(\mathscr{P}_{2,-})^{-4k+2}$.*

Proof. Argue similarly to the proof of Lemma 2.2 with $M = L^{2k-1}$.

Lemma 2.11. *With notation as above, under our identification of $A(D^0)_-$ with D^0, $N_D(b_2) = \det(b_1)/4$. Further, $\nu_D(b_2) = -4k+2$ if and only if the k-vector space homomorphism $\overline{w}\colon L_1^0/L_1^1 \to P_-^{-2k+1}/P_-^{-2k+2}$ induced by w has rank 2.*

Proof. The proof of this lemma is similar to that of Lemma 2.3.

Proposition 2.12. *Let $\pi(\mathscr{A}_1, \alpha, \eta)$ be an element of $\widetilde{G}_1$ such that $E = F[\alpha]/F$ is unramified and $\nu_{\mathscr{A}_1}(\alpha)$ is odd. Then $\pi(\mathscr{A}_1, \alpha, \eta)$ occurs in $\mathscr{R}_\chi(G_1)$.*

Proof. With notation as above, let $\nu_{\mathscr{A}_1}(\alpha) = -2k+1$ with k a positive integer and let w be an element of L^{-2k} such that $\overline{w}$ has rank 2. Then by the previous lemmas $\nu_D(b_1) = -2k+1$ and $-\det(b_1)$ is nonsquare. Now let b be an element of $\mathscr{O}_F^\times$ such that $b^2\det(b_1) = \det(\alpha)$ and let g be an element of $\mathscr{A}_1^\times$ such that $\det g = b$. Then one checks that for any x in X (the space of ρ_L) $y_{gw,x}$ transforms according to $\psi_{\alpha'}$ under the action of $U^k(\mathscr{A}_1)$ where $\alpha' = \alpha^h$ for some h in $U(\mathscr{A}_1)$. Then one checks that $y_{hgw,x}$ transforms according to ψ_α under the action of $U^k(\mathscr{A}_1)$. Without loss of generality assume $hg = 1$. Finally since $\overline{w}$ has rank 2 the stabilizer in G_1 of $-w+L^*$ in W/L^* is $U^{2k}(\mathscr{A}_1)$ (recall that $G_1 \cap U^{2k-1}(\mathscr{A}) = U^{2k}(\mathscr{A}_1)$) so that the lemma now follows from Frobenius reciprocity.

Theorem 2.13. *With notation as above, let $\pi(\mathscr{A}_1, \alpha, \eta)$ be a representation of $\widetilde{G}_1$ such that $F[\alpha]/F$ is unramified and $\nu_{\mathscr{A}_1}(\alpha)$ is odd. Then $\pi(\mathscr{A}_1, \alpha, \eta)$ occurs in $\mathscr{R}_\chi(\widetilde{G}_1)$ and pairs with $\pi(-\det(\alpha)/4,\ \eta^2,\ \gamma_1,\ \eta(-1))$ where $E = F[\alpha]$ is identified with a subfield of D and $\gamma_1 = \eta(-1)$ is a square in F and $\gamma_1 = \eta(\mu)$ where μ is a fourth root of unity in E^1 if -1 is not a square in F.*

Proof. Since $\nu_{\mathscr{A}_1}(\alpha)$ is odd, by Lemma 2.12 there exists a w in W such that for each x in X, $y_{w,x}$ transforms according to ψ_α under the action of $U^k(\mathscr{A}_1)$ where $\nu_{\mathscr{A}_1}(\alpha) = -2k+1$. By Lemma 2.10 and Lemma 2.11, $y_{w,x}$ transforms according to some ψ_b under the action of $U^{2k}(A_2)$, where $N_D(b) = \det(\alpha)/4$. Using b to identify E with a subfield of D we may assume $y_{w,x}$ transforms according to ψ_α under the action of $U^k(\mathscr{A}_1)$ and $\psi_{\alpha/2}$ under the action of $U^{2k}(\mathscr{A}_2)$ for any x in X. Arguing as in the proof of Lemma 2.6 one can show that if y is a nonzero vector in the $U(\mathscr{A}_1)$-span of $y_{w,x}$ transforming according to $\rho(\mathscr{A}_1, \alpha, \eta)$ then y transforms according to $\rho_1'(\alpha/2, \eta^2)$ under the action of $U^1(\mathscr{A}_{2,E})U^{2k}(\mathscr{A}_2)$ where $\rho_1'(\alpha/2, \eta^2)$

denotes the restriction of the representation $\rho'(\alpha/2, \eta^2)$ of $U(\mathscr{A}_{2,E})U^{2k}(\mathscr{A}_2)$ to $U^1(\mathscr{A}_{2,E})U^{2k}(\mathscr{A}_2)$. Then one checks that, for x appropriately chosen, the $U^{2k-1}(\mathscr{A}_2)$ span of $y_{w,x}$ is isomorphic to the restriction of $\rho(\alpha/2, \sigma^2)$ to $U^{2k-1}(\mathscr{A}_2)$ for any σ (recall that $U^{2k-1}(\mathscr{A}_2) \cap PE^1 = U^{2k}(\mathscr{A}_2) \cap PE^1$). It follows from the representation theory of the Heisenberg group that we may choose a nonzero y in the $U(\mathscr{A}_1)U^1(\mathscr{A}_{2,E})U^{2k-1}(\mathscr{A}_2)$ span of $y_{w,x}$ such that, under the action of $U(\mathscr{A}_1)$, y transforms according to $\rho(\mathscr{A}_1, \alpha, \eta)$ and, under the action of $U^1(\mathscr{A}_{2,E})U^{2k-1}(\mathscr{A}_2)$, it transforms according to some extension $\tilde{\rho}$ say of $\rho_1(\alpha/2, \eta^2)$ to $U(\mathscr{A}_{2,E})U^{2k}(\mathscr{A}_2)$. Now the remainder of the argument is similar to the final portion of the proof of Proposition 2.6.

Proposition 2.14. *The representation $\pi^-(1, 1, 1)$ occurs in $\mathscr{R}_\chi(G_2)$ and pairs with $\pi(\mathscr{A}_1, -\operatorname{sgn} A)$ where A as before is $(-1)^{[(f+1)/2](p-1)/2}(-1)^{f+1}(-1)^{(p^2-1)f/8}$ with $f = \log_p(q)$.*

Proof. Consider the functions in Y supported on L^{-1}. Then the result follows from a straightforward computation and Theorem 3.3 and Remark 3.4 of [M1] (it also follows from Theorem 3.1 of [M1] and either [RS] or [Wa]).

We now turn to those representations of G_2 arising from ramified extensions. To deal with these representations we realize ω_χ in a self-dual lattice model as follows. Let L_1 be a self-dual lattice in V_1 and let L_1' be a lattice in V_1 such that $(L_1')^* = P^{-1}L_1'$ and $PL_1 \subseteq L_1' \subseteq L_1$. Now set $M_1^i = \operatorname{Hom}(L_1, P_-^{i-1})$, $M_2^i = \operatorname{Hom}(L_1', P_-^i)$, and $L^i = M_1^i \cap M_2^i$. Then one can check that $\mathscr{L} = \{L^k\}$ is a self-dual lattice chain in W of period two such that $(L^k)^* = L^{-k}$. One can also check that if we set $M^i = L_1^{i/2}$ if i is even and $M_i = (L_1')^{(i-1)/2}$ if i is odd then $\mathscr{M} = \{M^i\}$ is a self-dual lattice chain in V_1 of period two such that $(M^i)^* = M^{-i}$. Set $\mathscr{A}_1 = \mathscr{A}(\mathscr{M})$, $\mathscr{A}_2 = \mathscr{A}(D^0)$, and $\mathscr{A} = \mathscr{A}(\mathscr{L})$. Then one checks that for l a nonnegative integer $G_i \cap U^l(\mathscr{A}) = U^l(\mathscr{A}_i)$. In what follows we realize ω_χ in the self-dual lattice model attached to L^0.

Lemma 2.15. *With notation as above let k be a positive integer and let Y_k be the set of functions in Y (the space of ω_χ in our lattice model) supported on L^{-k}. Then the following hold.*

(i) *$U^{2k}(\mathscr{A}_i)$, for $i = 1$ or 2, fixes Y_k pointwise.*

(ii) *For w in L^{-k}, y_w transforms according to ψ_{b_1} and ψ_{b_2} under the actions of $U^k(\mathscr{A}_1)$ and $U^k(\mathscr{A}_2)$ respectively where $b_1 = -\varpi\lambda(w)w/2$ and $b_2 = \varpi w\lambda(w)/2$. Moreover, b_1 is an element of $(\mathscr{P}_{1,-})^{-2k+1}$ and b_2 is an element of $(\mathscr{P}_{2,-})^{-2k+1}$.*

Proof. Argue as in the proof of Lemma 4.3 of [M1].

Lemma 2.16. *With notation as above, under our identification of $A(D^0)_-$ with D^0, $N_D(b_2) = \det(b_1)/4$. Further $\nu_D(b_2) = -2k+1$ if and only if the images w_1 and w_2 of w in L^{-k}/M_i^{-k+1} for $i = 1, 2$ respectively are both nonzero.*

Proof. This is similar to the proof of Lemma 2.3.

Proposition 2.17. *Let a be a nonsquare element of $F^\times$ such that $\nu_{\mathscr{A}}(a) = -2k+1$ with k a positive integer. Then a representation of the form $\pi(a, \eta, \gamma_1, \gamma_2)$ occurs in $\mathscr{R}_\chi(G_2)$ if and only if it is pseudospherical.*

Proof. Let w be an element of L^{-k} such that w_1 and w_2 as in the previous lemma are nonzero. Then, by the previous lemma, $\nu_D(b_2) = -2k+1$. Now suppose that there exists b in $\mathscr{O}_F^\times$ such that $b^2 N_D(b_2) = -a$. Then letting g be an element of $A_1^\times$ such that $\det g = b$ one checks that y_{gw} transforms according to ψ_α under the action of $U^k(\mathscr{A}_2)$ where α is an element of P_-^{-2k+1} such that $N_D(\alpha) = -a$. If $-N_D(b_2)/a$ is not a square then changing w_1 if k is even and w_2 if k is odd and a similar argument allows us to find such g, w, and α. In either case, we may assume without loss of generality that we have an element w of L^{-k} such that y_w transforms according to ψ_α where α is an element of P_-^{-2k+1} such that $N_D(\alpha) = -a$. Then one checks that the stabilizer in G_2 of $-w+L$ in W/L is $U^k(\mathscr{A}_2)$. The result then follows from Frobenius reciprocity.

Theorem 2.18. *Let $\pi(\mathscr{A}_1, \alpha, \eta)$ be a representation of G_1 such that $F[\alpha]/F$ is ramified. Then $\pi(\mathscr{A}_1, \alpha, \eta)$ occurs in $\mathscr{R}_\chi(\widetilde{G}_1)$ if and only if there exists a w in $L^{(\nu(\alpha)-1)/2}$ such that $\varpi_F \lambda(w) w = -2\alpha$. If $\pi(\mathscr{A}_1, \alpha, \eta)$ occurs in $\mathscr{R}_\chi(\widetilde{G}_1)$ then it pairs with $\pi(-\det(\alpha)/4, \eta^2, \eta(-1), \eta(-1))$.*

Proof. The proof of this result is similar to the proof of Theorem 2.6 only simpler since we are using a self-dual model.

Remark 2.19. (i) As a consequence of Proposition 2.5, Theorem 2.6, Theorem 2.8, Theorem 2.13, Proposition 2.17, and Theorem 2.18 we have that all pseudospherical representations of G_2 occur in $\mathscr{R}_\chi(G_2)$ and are thus spherical. We also have explicit pairings for these representations.

(ii) It is a straightforward exercise to determine the effect of changing χ. See for example Remark 2.4(iii) of [M1].

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