TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 336, Number 2, April 1993

GROUPS OF DUALITIES

GEORGI D. DIMOV AND WALTER THOLEN

ABSTRACT. For arbitrary categories $\mathscr{A}$ and $\mathscr{B}$, the "set" of isomorphism-classes of dualities between $\mathscr{A}$ and $\mathscr{B}$ carries a natural group structure. In case $\mathscr{A}$ and $\mathscr{B}$ admit faithful representable functors to **Set**, this structure can often be described quite concretely in terms of "schizophrenic objects" (in the sense of Johnstone's book on "Stone Spaces"). The general theory provided here allows for a concrete computation of that group in case $\mathscr{A} = \mathscr{B} = \mathscr{C}$ is the category of all compact and all discrete abelian groups: it is the uncountable group of algebraic automorphisms of the circle $\mathbb{R}/\mathbb{Z}$, modulo its subgroup $\mathbb{Z}_2$ of continuous automorphisms.

INTRODUCTION

There is such a big variety of articles, monographs and chapters in books on dualities that it seems doubtful that anything substantially new could be said on the topic, in particular in the context of ordinary **Set**-based categories which are considered in this paper. In fact, this paper lives very much on Lawvere's idea of describing dualities in terms of objects living in two categories, an idea which has inspired others to invent rather picturesque phrases: Isbell talks about objects keeping "summer and winter homes" (cf. Bergman [2]); H. Simmons suggested "schizophrenic object", a name used in Johnstone's book [14] and in this paper. However, a first characterization of contravariant adjunctions and dualities in terms of schizophrenic objects on a sufficiently abstract level only appeared in this paper's predecessor [6]. The objective of this paper is to present this characterization in a more complete fashion, to apply it in such a way that the group structure of the family of isomorphism classes of all dualities between two given categories becomes concretely accessible, and finally to use this description to prove a result announced in [6] (but probably already envisaged by Prodanov; there is a crucial remark in his unpublished paper [20]) on the group of all (Pontryagin-type) dualities of the category $\mathscr{C}$ of all compact and all discrete Abelian groups. This is, admittedly, a bad category, but one with an interesting group of dualities, and this fact in turn we see as an argument for developing a theory on dualities without restrictive assumptions on the categories involved.

In a short introductory section we describe the "group" $\mathrm{DUAL}(\mathscr{A}, \mathscr{B})/\cong$ we are interested in for abstract categories $\mathscr{A}$ and $\mathscr{B}$, before we then turn

Received by the editors January 28, 1991.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 18A40, 18D05, 22D35, 54B30, 54H10.
Partial support by NSERC is gratefully acknowledged.

to the setting kept throughout the paper, that $\mathscr{A}$ and $\mathscr{B}$ come equipped with **Set**-functors U and V respectively which need not even be right adjoints. In fact, without any conditions on these data, *natural* contravariant adjunctions $(S: \mathscr{B} \to \mathscr{A}, \; T: \mathscr{A} \to \mathscr{B})$, i.e. adjoint pairs (S, T) with $VT = \mathscr{A}(-, \widetilde{A})$, $US = \mathscr{B}(-, \widetilde{B})$ such that the units and co-units are given by U-initial and V-initial families (of evaluation maps) respectively, are described in terms of **Set**-bijections $\tau: U\widetilde{A} \to V\widetilde{B}$ (= schizophrenic objects), in such a way that it is clear how to construct adjunctions from a schizophrenic object (cf. Theorem 2.5). If U and V are conservative (= reflect isomorphisms), it is easy to detect the dualities amongst the natural adjunctions (cf. Theorem 2.9). Otherwise one has to use stronger conditions on $\widetilde{A}$ and $\widetilde{B}$ (see (A3), (B3) of 2.8, which, however, are still weaker than the condition used normally, namely that $\widetilde{A}$ and $\widetilde{B}$ be regular cogenerators of $\mathscr{A}$ and $\mathscr{B}$ respectively) in order to obtain not only a characterization of natural dualities but also that, in fact, every representable duality is isomorphic to a natural one (cf. Corollary 2.11).

In §3 we show that, for fixed $\widetilde{A} \in |\mathscr{A}|$, $\widetilde{B} \in |\mathscr{B}|$, the natural 2-categorical structure of the family of all $(\widetilde{A}, \widetilde{B})$-represented contravariant adjunctions or dualities between $\mathscr{A}$ and $\mathscr{B}$, provided there is at least one such duality, can be transferred to the schizophrenic objects characterizing them. The point of this transfer is that the composition of adjunctions becomes just the composition of bijective mappings (cf. Lemma 3.2), a fact that is easily conjectured but rather tedious to establish. From there it is easy to derive the main result of the paper concerning the group of isomorphism-classes of dualities between $\mathscr{A}$ and $\mathscr{B}$. The examples we have been considering in [6] will all just give groups of up to two elements. In this paper we therefore present only the example of all Pontryagin-type dualities on a category $\mathscr{C}$ for which $\mathrm{DUAL}(\mathscr{C}, \mathscr{C})/\cong$ is in fact an uncountable group.

Our list of references gives just a choice of "generic" articles. We would like to mention in particular the Lambek-Rattray paper [15] whose Propositions 2.5 and 2.6 come closest to our Theorems 2.5 and 2.9 as far as the general principle of comparing adjunctions and schizophrenic objects is concerned; the two propositions, however, require one of the two categories to be algebraic and are more involved with respect to this algebraic structure. Close relatives of these propositions, on different levels of generality, were previously published by Freyd [8], Pultr [21] and Isbell [13]. Further important general references to Stone-Gelfand-type dualities include [1, 3, 4, 12, 14, 17, 18, 19]. All categorical notions not defined in the text may be found in [11] or [16].

1. Contravariant adjunctions and dualities

1.1. A *contravariant adjunction* between two categories $\mathscr{A}$ and $\mathscr{B}$ consists of contravariant functors $T: \mathscr{A} \to \mathscr{B}$ and $S: \mathscr{B} \to \mathscr{A}$ such that

$$\mathscr{A}(A, SB) \cong \mathscr{B}(B, TA) \tag{1}$$

naturally in $A \in |\mathscr{A}|$ and $B \in |\mathscr{B}|$; we write $(S, T): \mathscr{A} \to \mathscr{B}$. There are units $\eta_B: B \to TSB$ and co-units $\varepsilon_A: A \to STA$ satisfying

$$T\varepsilon_A \cdot \eta_{TA} = 1_{TA} \quad \text{and} \quad S\eta_B \cdot \varepsilon_{SB} = 1_{SB} \tag{2}$$

naturally in $A \in |\mathscr{A}|$ and $B \in |\mathscr{B}|$. (S, T) is a *duality* if η and ε are functorial isomorphisms.

1.2. The contravariant adjunctions between $\mathscr{A}$ and $\mathscr{B}$ are the objects of a (meta-)category $\mathrm{ADJ}(\mathscr{A}, \mathscr{B})$ whose morphisms

$$(\delta, \gamma): (S, T) \to (S', T')$$

are given by natural transformations $\gamma: T \to T'$, $\delta: S \to S'$ with

$$\gamma S \cdot \eta = T'\delta \cdot \eta' \quad \text{and} \quad \delta T \cdot \varepsilon = S'\gamma \cdot \varepsilon'. \tag{3}$$

Of course, δ is uniquely determined by γ and (3) (and vice versa); hence $\mathrm{ADJ}(\mathscr{A}, \mathscr{B})$ is equivalent to the full subcategory of right adjoint functors in the functor category $\mathscr{B}^{\mathscr{A}^{\mathrm{op}}}$. $\mathrm{ADJ}(\mathscr{A}, \mathscr{B})$ contains the full subcategory $\mathrm{DUAL}(\mathscr{A}, \mathscr{B})$ of dualities between $\mathscr{A}$ and $\mathscr{B}$.

1.3. An *equivalence* of the categories $\mathscr{C}$ and $\mathscr{D}$ is a full and faithful (covariant) functor $E: \mathscr{C} \to \mathscr{D}$ which is surjective on objects up to isomorphisms; equivalently, there is a functor $F: \mathscr{D} \to \mathscr{C}$ such that $FE \cong \mathrm{Id}$ and $EF \cong \mathrm{Id}$. Let $\mathrm{EQU}(\mathscr{A})$ denote the (meta-)category of equivalences of $\mathscr{A}$ with itself, considered as a full subcategory of the functor category $\mathscr{A}^{\mathscr{A}}$. $\mathrm{EQU}(\mathscr{A})$ is actually a 2-category, with one 0-cell: since the composition of equivalences of $\mathscr{A}$ is again an equivalence of $\mathscr{A}$, one has a monoid structure. Modulo $\cong$, this is even a group structure since equivalences have "inverses up to $\cong$"; in other words: the quotient monoid $\mathrm{EQU}(\mathscr{A})/\cong$ is a group (disregarding size at this point).

1.4. Let $(S_0, T_0): \mathscr{A} \to \mathscr{B}$ be a duality. Then the categories $\mathrm{DUAL}(\mathscr{A}, \mathscr{B})$ and $\mathrm{EQU}(\mathscr{A})$ are equivalent; the assignments $\Psi: (S, T) \mapsto S_0T$, $(\delta, \gamma) \mapsto S_0\gamma$, define an equivalence $\Psi: \mathrm{DUAL}(\mathscr{A}, \mathscr{B}) \to \mathrm{EQU}(\mathscr{A})$. Via Ψ one may pull back the group structure of $\mathrm{EQU}(\mathscr{A})/\cong$ to the family $\mathrm{DUAL}(\mathscr{A}, \mathscr{B})/\cong$ of isomorphism-classes of dualities between $\mathscr{A}$ and $\mathscr{B}$. Explicitly, the multiplication in $\mathrm{DUAL}(\mathscr{A}, \mathscr{B})/\cong$ is given by

$$[S, T] \cdot [S', T'] = [S'T_0S, TS_0T'] \tag{4}$$

(brackets denote $\cong$-classes; (S_0, T_0) is the *given* duality), making

$$\Psi: \mathrm{DUAL}(\mathscr{A}, \mathscr{B})/\cong \to \mathrm{EQU}(\mathscr{A})/\cong$$

a group isomorphism. $[S_0, T_0]$ is neutral in $\mathrm{DUAL}(\mathscr{A}, \mathscr{B})/\cong$, and the inverse of $[S, T]$ is $[S_0TS_0, T_0ST_0]$. Note that the structure of the group $\mathrm{DUAL}(\mathscr{A}, \mathscr{B})/\cong$ does not depend on the choice of the duality (S_0, T_0); if we had started with (S_1, T_1) instead of (S_0, T_0) to obtain $\mathrm{DUAL}'(\mathscr{A}, \mathscr{B})/\cong$, then $[S, T] \mapsto [ST_0S_1, T_1S_0T]$ gives a group isomorphism $\mathrm{DUAL}(\mathscr{A},\mathscr{B})/\cong \to \mathrm{DUAL}'(\mathscr{A}, \mathscr{B})/\cong$.

1.5. *Remark.* Faith [7, p. 65] describes the group structure of $\mathrm{EQU}(\mathscr{A})/\cong$ vaguely, and says that an equivalence between two categories or, equivalently, a duality "is determined up to an autoequivalence", as is verified in 1.4 above. However, in proving his claim, Faith argues that for equivalences $H: \mathscr{C} \to \mathscr{C}$ and $T: \mathscr{A} \to \mathscr{C}$ one has $HT \cong T$, which is false even when $\mathscr{A} = \mathscr{C}$ and $T = \mathrm{Id}$: consider the nonidentical equivalence (in fact: isomorphism) $H: \mathscr{A} \to \mathscr{A}$ with $\mathscr{A}$ the category $A_1 \to A_0 \leftarrow A_2$ (with exactly three objects and two nonidentical morphisms); then $H \ncong \mathrm{Id}$.

2. Characterization of natural adjunctions and dualities

In this section, let $U: \mathscr{A} \to \mathbf{Set}$ and $V: \mathscr{B} \to \mathbf{Set}$ be arbitrary (covariant) functors.

2.1. A contravariant adjunction $(S, T): \mathscr{A} \to \mathscr{B}$ is called *strictly* $(\widetilde{A}, \widetilde{B})$-*represented*, with $\widetilde{A} \in |\mathscr{A}|$ and $\widetilde{B} \in |\mathscr{B}|$, if

$$VT = \mathscr{A}(-, \widetilde{A}) \quad \text{and} \quad US = \mathscr{B}(-, \widetilde{B}); \tag{5}$$

(S, T) is *strictly represented* if it is strictly $(\widetilde{A}, \widetilde{B})$-represented for suitable $\widetilde{A}, \widetilde{B}$. For such adjunctions, the units and co-units are essentially *evaluation maps*; more precisely, for $A \in |\mathscr{A}|$, $x \in UA$, and $B \in |\mathscr{B}|$, $y \in VB$, consider

$$\begin{aligned} \varphi_{A,x}&: \mathscr{A}(A, \widetilde{A}) \to U\widetilde{A}, \qquad s \mapsto (Us)(x), \\ \psi_{B,y}&: \mathscr{B}(B, \widetilde{B}) \to V\widetilde{B}, \qquad t \mapsto (Vt)(y), \end{aligned} \tag{6}$$

$$\begin{aligned} \tau = \tau_{S,T}&: U\widetilde{A} \to V\widetilde{B}, \qquad \tilde{x} \mapsto V[(U\varepsilon_{\widetilde{A}})(\tilde{x})](1_{\widetilde{A}}), \\ \sigma = \sigma_{S,T}&: U\widetilde{B} \to V\widetilde{A}, \qquad \tilde{y} \mapsto U[(V\eta_{\widetilde{B}})(\tilde{y})](1_{\widetilde{B}}). \end{aligned} \tag{7}$$

Then τ and σ are bijective, with $\sigma = \tau^{-1}$, and

$$V[(U\varepsilon_A)(x)] = \tau\varphi_{A,x}, \quad U[(V\eta_B)(y)] = \sigma\psi_{B,y} \tag{8}$$

(see [15, 6]). We call the strictly $(\widetilde{A}, \widetilde{B})$-represented adjunction (S, T) *natural* if, for every $A \in |\mathscr{A}|$ and $B \in |\mathscr{B}|$,

$((U\varepsilon_A)(x): TA \to \widetilde{B})_{x \in UA}$ is a V-initial family, and

$((V\eta_B)(y): SB \to \widetilde{A})_{y \in VB}$ is a U-initial family.

(Recall that a family $(f_i: A \to A_i)_{i \in I}$ in $\mathscr{A}$ is U-*initial* if, for any family $(g_i: C \to A_i)_{i \in I}$ in $\mathscr{A}$ and any map $h: UC \to UA$ with $Uf_i \cdot h = Ug_i$ $(i \in I)$, one has a unique $\mathscr{A}$-morphism $t: C \to A$ with $Ut = h$ and $f_i t = g_i$ $(i \in I)$.)

2.2. Triples (A, τ, B) with $A \in |\mathscr{A}|$ and $B \in |\mathscr{B}|$ and a bijection $\tau: UA \to VB$ are called $(\mathscr{A}, \mathscr{B})$-*schizophrenic objects*; they form the objects of a full subcategory $\mathrm{Sch}(\mathscr{A}, \mathscr{B})$ of the comma category $U \downarrow V$ (cf. [15]), thus a morphism $(s, t): (A, \tau, B) \to (A', \tau', B')$ in $\mathrm{Sch}(\mathscr{A}, \mathscr{B})$ is given by morphisms $s: A \to A'$ in $\mathscr{A}$ and $t: B \to B'$ in $\mathscr{B}$ such that

$$\begin{tikzcd} UA \arrow[r, "Us"] \arrow[d, "\tau"'] & UA' \arrow[d, "\tau'"] \\ VB \arrow[r, "Vt"] & VB' \end{tikzcd}$$

commutes. If U is *transportable* (that is: if for every bijection $h: UA \to X$ with $A \in |\mathscr{A}|$ there is an $\mathscr{A}$-isomorphism $f: A \to A'$ with $UA' = X$ and $Uf = h$), or if V is transportable, then $\mathrm{Sch}(\mathscr{A}, \mathscr{B})$ is obviously equivalent to its full subcategory $\mathrm{Sch}_{\mathrm{id}}(\mathscr{A}, \mathscr{B})$ of those schizophrenic objects (A, τ, B) for which $UA = VB$ and $\tau = \mathrm{id}$.

2.3. For a morphism $(\delta, \gamma) : (S, T) \to (S', T')$ in $\mathrm{ADJ}(\mathscr{A}, \mathscr{B})$ with (S, T) strictly (A, B)-represented and (S', T') strictly (A', B')-represented, the Yoneda Lemma yields uniquely determined morphisms $s: A \to A'$ in $\mathscr{A}$ and $t: B \to B'$ in $\mathscr{B}$ such that $V\gamma = \mathscr{A}(-, s)$ and $U\delta = \mathscr{B}(-, t)$; explicitly, $s = (V\gamma_A)(1_A)$ and $t = (U\delta_B)(1_B)$. Note that, if (δ, γ) is an isomorphism in $\mathrm{ADJ}(\mathscr{A}, \mathscr{B})$, then s and t are isomorphisms in $\mathscr{A}$ and $\mathscr{B}$ respectively. With $\tau = \tau_{S,T}$ and $\tau' = \tau_{S',T'}$ (see (7)) one easily checks that one obtains a morphism $(s, t) : (A, \tau, B) \to (A', \tau', B')$ in $\mathrm{Sch}(\mathscr{A}, \mathscr{B})$. Thus one has a functor

$$\Phi: \mathrm{ADJ}_{\mathrm{rep}}(\mathscr{A}, \mathscr{B}) \to \mathrm{Sch}(\mathscr{A}, \mathscr{B})$$

whose domain is the full subcategory of strictly represented adjunctions in $\mathrm{ADJ}(\mathscr{A}, \mathscr{B})$.

Without any conditions on U and V one has

Proposition. *The restriction of* Φ *to the full subcategory* $\mathrm{ADJ}_{\mathrm{nat}}(\mathscr{A}, \mathscr{B})$ *of natural contravariant adjunctions is full and faithful.*

Proof (sketch). Given any morphism

$$(s, t): \Phi(S, T) \to \Phi(S', T')$$

in $\mathrm{Sch}(\mathscr{A}, \mathscr{B})$, one uses V-initiality of $((U\varepsilon'_C)(x))_{x\in UC}$ to construct $\gamma_C: TC \to T'C$ for every $C \in |\mathscr{A}|$, and U-initiality of $((V\eta'_D)(y))_{y\in VD}$ to construct $\delta_D: SD \to S'D$ for every $D \in |\mathscr{B}|$. This way one obtains a unique pair (δ, γ) with $\Phi(\delta, \gamma) = (s, t)$ (cf. [6, 4.2]). □

Remarks. (1) If V is faithful, it suffices to envoke only the first initiality property to construct γ and obtain δ by adjunction. (2) If $U \cong \mathscr{A}(A_0, -)$ and $V \cong \mathscr{B}(B_0, -)$ are representable, then for any adjunction $(S, T): \mathscr{A} \to \mathscr{B}$ one has $US \cong \mathscr{B}(-, TA_0)$ and $VT \cong \mathscr{A}(-, SB_0)$. Moreover, if U and V are transportable, then there is a strictly (SB_0, TA_0)-representable adjunction (S', T') which is isomorphic to (S, T) in $\mathrm{ADJ}(\mathscr{A}, \mathscr{B})$. Therefore, $\mathrm{ADJ}_{\mathrm{rep}}(\mathscr{A}, \mathscr{B})$ is equivalent to $\mathrm{ADJ}(\mathscr{A}, \mathscr{B})$ for U, V representable and transportable.

2.4. For a natural adjunction (S, T) and for $(\widetilde{A}, \tau, \widetilde{B}) = \Phi(S, T)$, from 2.1 one has the following two properties:

(A1) for every $A \in |\mathscr{A}|$, there is a V-initial family $(e_{A,x}: TA \to \widetilde{B})_{x\in UA}$ with $Ve_{A,x} = \tau\varphi_{A,x}$, $x \in UA$;

(B1) for every $B \in |\mathscr{B}|$, there is a U-initial family $(d_{B,y}: SB \to \widetilde{A})_{y\in VB}$ with $Ud_{B,y} = \tau^{-1}\psi_{B,y}$, $y \in VB$.

Let $\mathrm{Sch}_1(\mathscr{A}, \mathscr{B})$ be the full subcategory of $\mathrm{Sch}(\mathscr{A}, \mathscr{B})$ containing the objects $(\widetilde{A}, \tau, \widetilde{B})$ that satisfy conditions (A1) and (B1). Recall that the functor U is called *amnestic* if every $\mathscr{A}$-isomorphism $f: A \to A'$ with $UA = UA'$ and $Uf = 1$ is an identity morphism itself.

2.5 **Theorem.** $\mathrm{ADJ}_{\mathrm{nat}}(\mathscr{A}, \mathscr{B})$ *and* $\mathrm{Sch}_1(\mathscr{A}, \mathscr{B})$ *are equivalent categories; they are even isomorphic if* U *and* V *are amnestic.*

Proof (sketch). In order to show that the obvious restriction of Φ is an equivalence, one just needs to show that for every $(\widetilde{A}, \tau, \widetilde{B}) \in |\mathrm{Sch}_1(\mathscr{A}, \mathscr{B})|$ there is a natural adjunction (S, T) and $\Phi(S, T) = (\widetilde{A}, \tau, \widetilde{B})$. By (A1) of 2.4, T is already defined on objects; by V-initiality, for $f\colon A \to A'$ in $\mathscr{A}$, there is only one morphism Tf in $\mathscr{B}$ with $VTf = \mathscr{A}(f, \widetilde{A})$ such that

$$\begin{tikzcd} TA \arrow[r, "e_{A,x}"] & \widetilde{B} \\ TA' \arrow[u, "Tf"] \arrow[ur, "e_{A',(Uf)(x)}"'] & \end{tikzcd}$$

commutes for all $x \in UA$. The co-unit $\varepsilon_A\colon A \to STA$ is the morphism with $(U\varepsilon_A)(x) = e_{A,x}$ that makes

$$\begin{tikzcd} STA \arrow[r, "d_{TA,s}"] & \widetilde{A} \\ A \arrow[u, "\varepsilon_A"] \arrow[ur, "s"'] & \end{tikzcd}$$

commute for all $s \in \mathscr{A}(A, \widetilde{A})$. Denote the strictly $(\mathscr{A}, \mathscr{B})$-represented natural adjunction obtained this way by $\Psi(\widetilde{A}, \tau, \widetilde{B})$. Then $\Phi\Psi = \mathrm{Id}$ and $\Psi\Phi \cong \mathrm{Id}$. When U, V are amnestic, then the domains of the initial families used in (A1), (B1) are not only unique up to isomorphism, but unique, thus $\Psi\Phi = \mathrm{Id}$. $\square$

2.6. For fixed objects $\widetilde{A} \in |\mathscr{A}|$, $\widetilde{B} \in |\mathscr{B}|$ we may consider the categories $\mathrm{ADJ}_{\mathrm{nat}}^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B})$ of strictly $(\widetilde{A}, \widetilde{B})$-represented natural adjunctions and $\mathrm{Sch}_1^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B})$ of schizophrenic objects $(\widetilde{A}, \tau, \widetilde{B})$ that satisfy (A1) and (B1) of 2.4.

Corollary (cf. [6]). $\mathrm{ADJ}_{\mathrm{nat}}^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B})$ *and* $\mathrm{Sch}_1^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B})$ *are equivalent, even isomorphic if* U *and* V *are amnestic.* $\square$

2.7 *Remark.* For a schizophrenic object $(\widetilde{A}, \tau, \widetilde{B})$ that satisfies condition (A1) of 2.4 but not necessarily (B1), one can still construct the contravariant functor $T\colon \mathscr{A} \to \mathscr{B}$ with $VT = \mathscr{A}(-, \widetilde{A})$. Obviously, if V reflects limits, in particular if V is monadic, then T transforms colimits of $\mathscr{A}$ into limits of $\mathscr{B}$. Less obviously, the same conclusion can be drawn if U preserves colimits; more precisely, one can show that T transforms those colimits of $\mathscr{A}$ which are preserved by U, into limits of $\mathscr{B}$. Therefore, under suitable assumptions on V or U, and if $\mathscr{A}$ is compact (in the sense of Isbell, for instance if $\mathscr{A}$ is cocomplete and cowellpowered and has a generator), an adjoint S exists. However, that $US \cong \mathscr{B}(-, \widetilde{B})$ holds in general is not guaranteed.

2.8. For a *natural duality* (S, T), i.e. for a natural contravariant adjunction whose units η_B, $B \in |\mathscr{B}|$, and co-units ε_A, $A \in |\mathscr{A}|$, are all isomorphisms, translation of the necessary conditions that $U\varepsilon_A$ and $V\eta_B$ are isomorphisms in terms of $(\widetilde{A}, \tau, \widetilde{B}) = \Phi(S, T)$, with $e_{A,x} = (U\varepsilon_A)(x)$ and $d_{B,y} = (V\eta_B)(y)$, gives

(A2) for every $A \in |\mathscr{A}|$ and every $t \in \mathscr{B}(TA, \widetilde{B})$ there is a unique $x \in UA$ such that $e_{A,x} = t$ (that is essentially: every $t\colon TA \to \widetilde{B}$ in $\mathscr{B}$ is an evaluation map at a unique point $x \in UA$);

(B2) for every $B \in |\mathscr{B}|$ and every $s \in \mathscr{A}(SB, \widetilde{A})$ there is a unique $y \in VB$ such that $d_{B,y} = s$.

Condition (A2) means that the family $(e_{A,x})_{x \in UA}$ considered in (A1) is actually the hom-set $\mathscr{B}(TA, \widetilde{B})$; dually, $(d_{B,y})_{y \in VB}$ is $\mathscr{A}(SB, \widetilde{A})$. Hence, for a duality (S, T), conditions (A1), (B1), (A2), (B2) imply

(A3) for every $A \in |\mathscr{A}|$, the hom-set $\mathscr{A}(A, \widetilde{A})$ is U-initial;

(B3) for every $B \in |\mathscr{B}|$, the hom-set $\mathscr{B}(B, \widetilde{B})$ is V-initial.

Vice versa, let us consider a schizophrenic object $(\widetilde{A}, \tau, \widetilde{B})$ with (A1), (B1), (A2), (B2); then, by 2.5, we have a natural adjunction (S, T) with $\Phi(S, T) = (\widetilde{A}, \tau, \widetilde{B})$ and $U\varepsilon$, $V\eta$ isomorphisms. If U and V are *conservative*, i.e. reflect isomorphisms, then this suffices to conclude that (S, T) is a duality. For arbitrary U and V, one has to envoke conditions (A3), (B3) to construct inverses $\bar{\varepsilon}_A$ and $\bar{\eta}_B$ of ε_A and η_B respectively: $\bar{\varepsilon}_A$ is the arrow with $U\bar{\varepsilon}_A = (U\varepsilon_A)^{-1}$ that makes

$$\begin{tikzcd} A \arrow[r, "s"] & \widetilde{A} \\ STA \arrow[u, "\bar{\varepsilon}_A"] \arrow[ur, "d_{TA,s}"'] & \end{tikzcd}$$

commute for all $s \in \mathscr{A}(A, \widetilde{A})$.

$\mathrm{DUAL}_{\mathrm{rep}}(\mathscr{A}, \mathscr{B})$ and $\mathrm{DUAL}_{\mathrm{nat}}(\mathscr{A}, \mathscr{B})$ denote the subcategories of dualities in $\mathrm{ADJ}_{\mathrm{rep}}(\mathscr{A}, \mathscr{B})$ and $\mathrm{ADJ}_{\mathrm{nat}}(\mathscr{A}, \mathscr{B})$ respectively, and by $\mathrm{Sch}_{1,2}(\mathscr{A}, \mathscr{B})$ and $\mathrm{Sch}_{1,2,3}(\mathscr{A}, \mathscr{B})$ we denote the subcategories of $\mathrm{Sch}_1(\mathscr{A}, \mathscr{B})$-objects that satisfy conditions (A2), (B2) and, in the case $\mathrm{Sch}_{1,2,3}(\mathscr{A}, \mathscr{B})$, in addition (A3), (B3). The above remarks sketch the proof of

2.9 **Theorem.** $\mathrm{DUAL}_{\mathrm{nat}}(\mathscr{A}, \mathscr{B})$ *is equivalent to* $\mathrm{Sch}_{1,2,3}(\mathscr{A}, \mathscr{B})$; *in case* U *and* V *are conservative, it is even equivalent to* $\mathrm{Sch}_{1,2}(\mathscr{A}, \mathscr{B})$. *These equivalences are isomorphisms of categories, if* U *and* V *are also amnestic.* □

Adapting the terminology of 2.6 to the context of 2.9 one obtains

2.10 **Corollary.** $\mathrm{DUAL}_{\mathrm{nat}}^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B})$ *is equivalent to* $\mathrm{Sch}_{1,2}^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B})$ *if* U *and* V *are conservative. "Equivalent" can be replaced by "isomorphic" if* U *and* V *are also amnestic.* □

2.11. As pointed out in 2.8, (A3), (B3) are necessary conditions for the existence of a natural strictly $(\widetilde{A}, \widetilde{B})$-represented duality between $\mathscr{A}$ and $\mathscr{B}$. Vice versa, considering a pair of objects $(\widetilde{A}, \widetilde{B})$ in $\mathscr{A} \times \mathscr{B}$ that satisfies (A3), (B3), *any* strictly $(\widetilde{A}, \widetilde{B})$-represented duality is natural. Hence one has

Corollary (cf. [6]). *For objects* $\widetilde{A}$, $\widetilde{B}$ *with* (A3), (B3), *one has that*

$$\mathrm{DUAL}_{\mathrm{rep}}^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B}) = \mathrm{DUAL}_{\mathrm{nat}}^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B})$$

is equivalent to $\mathrm{Sch}_{1,2}^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B})$, *even isomorphic if* U *and* V *are amnestic.*

2.12 *Remark.* Condition (A3) is certainly satisfied if $\widetilde{A}$ is a regular cogenerator of the complete category $\mathscr{A}$, and if U is faithful and preserves limits, in particular if U is a faithful representable. (Note that the existence of a strictly

$(\widetilde{A}, \widetilde{B})$-represented duality necessarily makes U and V representable, and also faithful if $\widetilde{A}$ and $\widetilde{B}$ are cogenerators.)

3. The main theorem

For simplicity, henceforth U and V are assumed to be faithful, transportable and amnestic. (The combination of the latter two properties means that, in the sense of Lambek and Rattray [15], U and V have unique transfer. Hence, in the case of U, any bijection $j\colon X \to UA'$ with $A' \in |\mathscr{A}|$ gives a uniquely determined $\mathscr{A}$-isomorphism $i\colon A \to A'$ with $Ui = j$.)

3.1. We return to the composition (4) of 1.4 which, however, will not be considered just for isomorphism classes of dualities or adjunctions. As before, we fix a duality $(S_0, T_0)\colon \mathscr{A} \to \mathscr{B}$, and then define 2-categories $\mathrm{ADJ}^{=}(\mathscr{A}, \mathscr{B})$ [and $\mathrm{DUAL}^{=}(\mathscr{A}, \mathscr{B})$] as follows: the only 0-cell is, in both cases, the given duality (S_0, T_0); 1-cells are all contravariant adjunctions [dualities] $(S, T)\colon \mathscr{A} \to \mathscr{B}$ with $US = US_0$ and $VT = VT_0$, and 2-cells are $\mathrm{ADJ}(\mathscr{A}, \mathscr{B})$-morphisms as described in 1.2. In order to define the composition of 1-cells (S, T), (S', T'), one has to find an isomorphic copy of the adjunction $(S'T_0S, TS_0T')$ that belongs to $\mathrm{ADJ}^{=}(\mathscr{A}, \mathscr{B})$: since V is transportable and amnestic, there is an endofunctor $F\colon \mathscr{B} \to \mathscr{B}$ and a functorial isomorphism $\mu\colon F \to TS_0$ with $VF = V$ and $V\mu = V\eta_0$ (with η_0 the unit of (S_0, T_0)), both uniquely determined; similarly, there are $E\colon \mathscr{A} \to \mathscr{A}$ and $\nu\colon E \to S'T_0$ with $UE = U$ and $U\nu = U\varepsilon_0$ (with ε_0 the co-unit of (S_0, T_0)); now put

$$(S, T)\cdot(S', T') = (ES, FT'). \tag{9}$$

This composition provides the 1-cells with a monoid structure and can be easily extended to all 2-cells, using the isomorphism

$$(\nu S, \mu T')\colon (ES, FT') \to (S'T_0S, TS_0T')$$

in $\mathrm{ADJ}(\mathscr{A}, \mathscr{B})$. We omit the details of checking that $\mathrm{ADJ}^{=}(\mathscr{A}, \mathscr{B})$ and $\mathrm{DUAL}^{=}(\mathscr{A}, \mathscr{B})$ become 2-categories this way. If (S_0, T_0) is strictly $(\widetilde{A}, \widetilde{B})$-represented, these 2-categories contain $\mathrm{ADJ}^{\widetilde{A},\widetilde{B}}_{\mathrm{rep}}(\mathscr{A}, \mathscr{B})$ and $\mathrm{DUAL}^{\widetilde{A},\widetilde{B}}_{\mathrm{rep}}(\mathscr{A}, \mathscr{B})$ as sub-2-categories respectively.

For (S_0, T_0), (S, T), (S', T') all strictly $(\widetilde{A}, \widetilde{B})$-represented with induced bijections τ_0, τ, τ' respectively, and with $\sigma_0 = \tau_0^{-1}$, one has the following crucial:

3.2 **Lemma.** *The schizophrenic object induced by the composition* $(S, T)\cdot(S', T')$ *is* $(\widetilde{A}, \tau\sigma_0\tau', \widetilde{B})$.

Proof. We must compute the schizophrenic object $(\widetilde{A}, \overline{\tau}, \widetilde{B})$ induced by the contravariant adjunction (9) whose co-unit $\overline{\varepsilon}$ is given by

$$\overline{\varepsilon} = \nu^{-1}SFT' \cdot S'T_0S\mu T' \cdot S'T_0\varepsilon S_0T' \cdot S'\eta_0^{-1}T' \cdot \varepsilon'.$$

For $\tilde{x} \in U\widetilde{A}$, $\overline{\tau}(\tilde{x})$ is given by formula (7), so that with $US = \mathscr{B}(-, \widetilde{B})$ and

$U\nu = U\varepsilon_0$ one obtains

$$\begin{aligned}\overline{\tau}(\tilde{x}) &= V[(U\overline{\varepsilon}_{\widetilde{A}})(\tilde{x})](1_{\widetilde{A}})\\ &= V[(U\nu^{-1}_{SFT'\widetilde{A}} \cdot US'((\eta_0^{-1})_{T'\widetilde{A}} \cdot T_0\varepsilon_{S_0T'\widetilde{A}} \cdot T_0S\mu_{T'\widetilde{A}}) \cdot U\varepsilon'_{\widetilde{A}})(\tilde{x})](1_{\widetilde{A}})\\ &= V[(U(\varepsilon_0^{-1})_{SFT'\widetilde{A}} \cdot \mathscr{B}((\eta_0^{-1})_{T'\widetilde{A}} \cdot T_0\varepsilon_{S_0T'\widetilde{A}} \cdot T_0S\mu_{T'\widetilde{A}}, \widetilde{B}) \cdot U\varepsilon'_{\widetilde{A}})(\tilde{x})](1_{\widetilde{A}})\\ &= V[(U(\varepsilon_0^{-1})_{SFT'\widetilde{A}})(g)](1_{\widetilde{A}})\end{aligned}$$

with $g := (U\varepsilon'_{\widetilde{A}})(\tilde{x}) \cdot (\eta_0^{-1})_{T'\widetilde{A}} \cdot T_0\varepsilon_{S_0T'\widetilde{A}} \cdot T_0S\mu_{T'\widetilde{A}}$. Therefore, putting $h := (U(\varepsilon_0^{-1})_{SFT'\widetilde{A}})(g)$, we obtain $\overline{\tau}(\tilde{x}) = (Vh)(1_{\widetilde{A}})$. Hence, in order to show the desired formula, we must show

$$\tau\sigma_0\tau'(\tilde{x}) = (Vh)(1_{\widetilde{A}}). \tag{10}$$

An easy computation, envoking only formulas (5) and (7) and the naturality of η, gives that, for any $h \in \mathscr{B}(FT'\widetilde{A}, \widetilde{B})$,

$$\sigma((Vh)(1_{\widetilde{A}})) = (Us)(h)$$

with $s = (V\eta_{FT'\widetilde{A}})(1_{\widetilde{A}}) \in \mathscr{A}(SFT'\widetilde{A}, \widetilde{A}) = VT_0SFT'\widetilde{A}$. Therefore, in order to obtain (10), we need to show only

$$\tau'(\tilde{x}) = \tau_0((Us)(h)). \tag{11}$$

By definition of h, $g = (U(\varepsilon_0)_{SFT'\widetilde{A}})(h)$. Hence, by (8),

$$Vg = V[(U(\varepsilon_0)_{SFT'\widetilde{A}})(h)] = \tau_0\varphi_{SFT'\widetilde{A},h}.$$

Thus, with (6) and (5), for every $s \in \mathscr{A}(SFT'\widetilde{A}, \widetilde{A}) = VT_0SFT'\widetilde{A}$ we have

$$\begin{aligned}\tau_0((Us)(h)) &= (\tau_0\varphi_{SFT'\widetilde{A},h})(s) = (Vg)(s)\\ &= V[(U\varepsilon_{\widetilde{A}})(\tilde{x})](V(\eta_0^{-1})_{T'\widetilde{A}}[(VT_0(S\mu_{T'\widetilde{A}} \cdot \varepsilon_{S_0T'\widetilde{A}}))(s)])\\ &= V[(U\varepsilon'_{\widetilde{A}})(\tilde{x})](\tilde{s})\end{aligned}$$

with $\tilde{s} := V(\eta_0^{-1})_{T'\widetilde{A}}[s \cdot S\mu_{T'\widetilde{A}} \cdot \varepsilon_{S_0T'\widetilde{A}}]$. We shall show below

$$\tilde{s} = 1_{\widetilde{A}}. \tag{12}$$

This will then give us (11) as required, since

$$\tau_0((Us)(h)) = V[(U\varepsilon'_{\widetilde{A}})(\tilde{x})](1_{\widetilde{A}}) = \tau'(\tilde{x})$$

and therefore concludes the proof. Hence we are left with having to show (12). Indeed, for every $t \in \mathscr{B}(T'\widetilde{A}, \widetilde{B})$ we have, by repeated application of (6) and (8), and since $V\eta_0 = V\mu$,

$$\begin{aligned}U[(V((\eta_0)_{T'\widetilde{A}})(1_{\widetilde{A}})](t) &= \varphi_{S_0T'\widetilde{A},t}((V(\eta_0)_{T'\widetilde{A}})(1_{\widetilde{A}}))\\ &= \tau^{-1}\tau\varphi_{S_0T'\widetilde{A},t}((V\mu_{T'\widetilde{A}})(1_{\widetilde{A}}))\\ &= \tau^{-1}V[(U\varepsilon_{S_0T'\widetilde{A}})(t)](V\mu_{T'\widetilde{A}})(1_{\widetilde{A}})\\ &= \tau^{-1}\psi_{FT'\widetilde{A},1_{\widetilde{A}}}((U\varepsilon_{S_0T'\widetilde{A}})(t) \cdot \mu_{T'\widetilde{A}})\\ &= (\tau^{-1} \cdot \psi_{FT'\widetilde{A},1_{\widetilde{A}}} \cdot \mathscr{B}(\mu_{T'\widetilde{A}}, \widetilde{B}) \cdot U\varepsilon_{S_0T'\widetilde{A}})(t).\end{aligned}$$

Therefore, with (8) and (5) one obtains

$$\begin{aligned} U[(V((\eta_0)_{T'\widetilde{A}})(1_{\widetilde{A}})] &= U[(V\eta_{FT'\widetilde{A}})(1_{\widetilde{A}})] \cdot US\mu_{T'\widetilde{A}} \cdot U\varepsilon_{S_0T'\widetilde{A}} \\ &= U[(V\eta_{FT'\widetilde{A}})(1_{\widetilde{A}}) \cdot S\mu_{T'\widetilde{A}} \cdot \varepsilon_{S_0T'\widetilde{A}}]. \end{aligned}$$

Finally, since U is faithful, and by choice of s,

$$(V(\eta_0)_{T'\widetilde{A}})(1_{\widetilde{A}}) = s \cdot S\mu_{T'\widetilde{A}} \cdot \varepsilon_{S_0T'\widetilde{A}}$$

follows, hence $\tilde{s} = 1_{\widetilde{A}}$ by definition of s. □

3.3. We can use the functor Φ of 2.3 to transfer the 2-categorical structure of $\mathrm{DUAL}_{\mathrm{rep}}^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B})$ to $\mathrm{Sch}_{1,2}^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B})$, envoking 2.11. The point of 3.2 is that this structure has a simple description: the composition of 1-cells is just the composition of maps, i.e. writing τ for $(\widetilde{A}, \tau, \widetilde{B})$ one has $\tau \cdot \tau' = \tau\sigma_0\tau'$. In view of (4) of 1.4, this formula may not look impressive; however, a priori it is not at all clear that $\mathrm{Sch}_{1,2}^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B})$ is closed under this multiplication! We shall summarize the consequences of 2.9, 2.11, and 3.2 just for isomorphism-classes of dualities and schizophrenic objects (brackets denote, as in 1.4, $\cong$-classes):

3.4 **Theorem.** *Let $(S_0, T_0): \mathscr{A}, \to \mathscr{B}$ be a strictly $(\widetilde{A}, \widetilde{B})$-represented duality, with induced schizophrenic object $\tau_0: U\widetilde{A} \to V\widetilde{B}$ and $\sigma_0 = \tau_0^{-1}$, and let $\widetilde{A}$ and $\widetilde{B}$ satisfy (the cogenerator-like) conditions* (A3) *and* (B3) *of* 2.8. *Then there is an isomorphism of groups*

$$\mathrm{DUAL}_{\mathrm{rep}}^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B})/\cong \to \mathrm{Sch}_{1,2}^{\widetilde{A},\widetilde{B}}(\mathscr{A}, \mathscr{B})/\cong .$$

Here isomorphism-classes of dualities are composed as in 1.4(4), *and isomorphism-classes of bijections $\tau: U\widetilde{A} \to V\widetilde{B}$ satisfying* (A1), (B1), (A2), (B2) *of* 2.4, 2.8 *are composed by the rule* $[\tau] \cdot [\tau'] = [\tau\sigma_0\tau']$. □

4. Pontryagin-type dualities

Throughout this section, $\mathscr{C}$ is the category of all discrete and of all compact abelian groups. Its underlying **Set**-functor $U \cong \mathscr{C}(\mathbb{Z}, -): \mathscr{C} \to \mathbf{Set}$ is faithful, transportable, amnestic and representable; for a morphism φ in $\mathscr{C}$, we normally write again φ for $U\varphi$. Our objective is to describe the structure of the group $\mathrm{DUAL}(\mathscr{C}, \mathscr{C})/\cong$. The circle group $\mathbb{R}/\mathbb{Z}$ is denoted by $\mathbf{T}$.

4.1 **Lemma.** *Every duality $(S, T): \mathscr{C} \to \mathscr{C}$ is isomorphic to a strictly $(\mathbf{T}, \mathbf{T})$-represented duality whose induced schizophrenic object $\tau: \mathbf{T} \to \mathbf{T}$ is an (algebraic) isomorphism of abelian groups. In particular,* $\mathrm{DUAL}(\mathscr{C}, \mathscr{C})/\cong$ *is isomorphic to* $\mathrm{DUAL}_{\mathrm{rep}}^{\mathbf{T},\mathbf{T}}(\mathscr{C}, \mathscr{C})/\cong$.

Proof. The first assertion follows from Lemma 6 of [22], and from Lemmas 5, 1, 2 of [22] one obtains that S and T are additive functors. This fact easily gives that, for $A \in |\mathscr{C}|$, the underlying Abelian group of SA and TA is (isomorphic to) $\mathscr{C}(A, \mathbf{T})$ with its natural algebraic structure, and that the bijection τ is a homomorphism (cf. [6, 5.4]). □

The crucial property of $\mathbf{T}$ is that any algebraic homomorphism $h: A \to B$ of locally compact abelian groups is continuous as soon as $\chi h: A \to \mathbf{T}$ is continuous for every character (=cont. hom.) $\chi: B \to \mathbf{T}$, cf. [10, 23]. From here one easily derives that condition (A3)=(B3) is satisfied for $\widetilde{A} = \widetilde{B} = \mathbf{T}$ in $\mathscr{A} = \mathscr{B} = \mathscr{C}$. Hence we have with 3.4 and 4.1:

4.2 **Corollary.** $\mathrm{DUAL}(\mathscr{C}, \mathscr{C})/\cong$ *is isomorphic to* $\mathrm{Sch}_{1,2}^{\mathbf{T},\mathbf{T}}(\mathscr{C}, \mathscr{C})/\cong$. □

So we are left with the problem to describe $\mathrm{Sch}_{1,2}^{\mathbf{T},\mathbf{T}}(\mathscr{C}, \mathscr{C})/\cong$. We already know that every representative $\tau: \mathbf{T} \to \mathbf{T}$ of a $\cong$-class is an algebraic automorphism. The only continuous ones are id and $-\mathrm{id}$. Hence $\tau \cong \tau'$ holds in $\mathrm{Sch}_{1,2}^{\mathbf{T},\mathbf{T}}(\mathscr{C}, \mathscr{C})$ if and only if $\tau' = \tau$ or $\tau' = -\tau$. In other words: $\mathrm{Sch}_{1,2}^{\mathbf{T},\mathbf{T}}(\mathscr{C}, \mathscr{C})/\cong$ is a subgroup of $\mathrm{Aut}_{\mathbf{Grp}}(\mathbf{T})/\{\mathrm{id}, -\mathrm{id}\}$, with $\mathrm{Aut}_{\mathbf{Grp}}(\mathbf{T})$ the group of algebraic automorphisms of $\mathbf{T}$. (Note that, of course, we take the classical Pontryagin duality as our initially chosen (S_0, T_0) in 3.1 since then $\sigma_0 = \tau_0^{-1} = \mathrm{id}$, i.e. the formula of 3.3 becomes the composition formula of $\mathrm{Aut}_{\mathbf{Grp}}(\mathbf{T})$.)

4.3 **Theorem.** $\mathrm{DUAL}(\mathscr{C}, \mathscr{C})/\cong$ *is isomorphic to* $\mathrm{Aut}_{\mathbf{Grp}}(\mathbf{T})/\{\mathrm{id}, -\mathrm{id}\}$.

Proof. We just need to show that every group automorphism $\tau: \mathbf{T} \to \mathbf{T}$ satisfies conditions (A1), (B1), (A2), (B2), so that

$$\mathrm{Aut}_{\mathbf{Grp}}(\mathbf{T})/\{\mathrm{id}, -\mathrm{id}\} = \mathrm{Sch}_{1,2}^{\mathbf{T},\mathbf{T}}(\mathscr{C}, \mathscr{C})/\cong.$$

First, we can modify the natural topology of $\mathbf{T}$ to obtain a compact Abelian group $\mathbf{T}_\tau$ such that $\tau: \mathbf{T}_\tau \to \mathbf{T}$ is a homeomorphism. Now for $A \in |\mathscr{C}|$, define $T_\tau A \in |\mathscr{C}|$, as follows: for A compact, $T_\tau A$ is the group $\mathscr{C}(A, \mathbf{T})$ with the discrete topology; for A discrete, $T_\tau A$ is the group $\mathscr{C}(A, \mathbf{T}) = \mathbf{Grp}(A, \mathbf{T})$ with the subspace topology of the Tychonoff product $(\mathbf{T}_\tau)^A$. For every $x \in A$, one has the $\mathscr{C}$-morphism

$$e_x^\tau = (T_\tau A \xrightarrow{\pi_x} \mathbf{T}_\tau \xrightarrow{\tau} \mathbf{T})$$

with π_x the xth product projection. To show conditions (A1), (A2) (and symmetrically, (B1), (B2)) we must prove that $(e_x^\tau)_{x \in A}$ is a U-initial family, and that the map $x \mapsto e_x^\tau$ gives a bijection of the sets A and $\mathscr{C}(T_\tau A, \mathbf{T})$.

Case 1: *A is discrete.* Products and subspaces are, by definition, formed in such a way that $(\pi_x)_{x \in A}$ is U-initial, hence $(e_x^\tau)_{x \in A}$ is U-initial as well since $\tau: \mathbf{T}_\tau \to \mathbf{T}$ is a homeomorphism. To show (A2) we first consider the case $\tau = \mathrm{id}$; that is the case of the classical Pontryagin duality $P: \mathscr{C} \to \mathscr{C}$, with $PA = T_{\mathrm{id}} A$, for which one knows that every character $\chi \in \mathscr{C}(PA, \mathbf{T})$ is of the form $\chi = \pi_x$ $(= e_x^{\mathrm{id}})$ for a unique point $x \in A$. But PA and $T_\tau A$ are homeomorphic under the restriction of the homeomorphism $\tau^A: (\mathbf{T}_\tau)^A \to \mathbf{T}^A$, so the assertion (A2) follows from the corresponding classical result.

Case 2: *A is compact.* In this case both PA and $T_\tau A$ provide $\mathscr{C}(A, \mathbf{T})$ with the discrete topology, hence they are equal. Classically one knows that $(\pi_x: PA \to \mathbf{T})_{x \in A}$ is equal to $\mathscr{C}(PA, \mathbf{T})$ and U-initial (by 3.4). But $\mathscr{C}(PA, \mathbf{T}) = \mathscr{C}(T_\tau A, \mathbf{T}_\tau)$, by discreteness, hence $(e_x^\tau)_{x \in A} = (\tau \pi_x)_{x \in A}$ is $\tau \cdot \mathscr{C}(T_\tau A, \mathbf{T}_\tau) = \mathscr{C}(T_\tau A, \mathbf{T})$. □

Note that the isomorphism $PA \cong T_\tau A$ which exists for all $A \in |\mathscr{C}|$, is *not* natural in A.

4.4 **Corollary.** *There are uncountably many nonisomorphic dualities of $\mathscr{C}$ with itself.*

Proof. It suffices to show that the group $\operatorname{Aut}\mathbf{T} = \operatorname{Aut}_{\mathbf{Grp}}(\mathbf{T})$ is not countable. In fact,

$$\mathbf{T} \cong \bigoplus_{p \text{ prime}} Z(p^\infty) \oplus \bigoplus_{\alpha<\mathfrak{c}} \mathbb{Q}_\alpha$$

(with $Z(p^\infty)$ the p-Sylow subgroup of $\mathbf{T}$, $\mathbb{Q}_\alpha = \mathbb{Q}$ and $\mathfrak{c}$ the cardinal number of the continuum), see [9]. Clearly, each $\operatorname{Aut} Z(p^\infty)$ is a subgroup of $\operatorname{Aut}\mathbf{T}$, and it is isomorphic to the units of the ring of p-adic integers, of which there are $\mathfrak{c}$-many (cf. [9]). □

4.5 *Remarks.* (1) There are actually uncountably many dualities of $\mathscr{C}$ with itself of type (T, T). For this it suffices to find uncountably many involutions in $\operatorname{Aut}_{\mathbf{Grp}}(\mathbf{T})$. But, using the presentation of $\mathbf{T}$ as a direct sum as in 4.4, for each $\alpha < \mathfrak{c}$ we can find an automorphism τ_α of $\mathbf{T}$ with $\tau_\alpha|_{\mathbb{Q}_\alpha} = -\mathrm{id}$ and τ_α identical on all other summands. Since $\tau_\alpha^2 = \mathrm{id}$, the proof is complete.

(2) Note that 4.4 no longer holds if $\mathscr{C}$ is replaced by the category $\mathscr{A}$ of all locally compact Abelian groups: the Pontryagin-van Kampen duality of $\mathscr{A}$ with itself is, up to isomorphism, the only duality of $\mathscr{A}$ with itself (cf. [22, 6]). Generalizations of these results appear in [5].

References

1. B. Banaschewski, *More on compact Hausdorff spaces and finitary duality*, Canad. J. Math. **36** (1984) 1113–1118.
2. G. M. Bergman, *On the scarcity of contravariant left adjunctions*, Algebra Universalis **24** (1987), 169–185.
3. E. Binz, *Continuous convergence on* $C(X)$, Lecture Notes in Math., vol. 469, Springer-Verlag, 1975.
4. B. J. Day, *Onto Gelfand transformations*, Cahiers Topologie Géom. Différentielle Catégoriques **28** (1987), 227–234.
5. D. Dikranjan and A. Orsatti, *On an unpublished manuscript of Ivan Prodanov concerning locally compact modules and their dualities*, Comm. Algebra **17** (1989), 2739–2771.
6. G. D. Dimov and W. Tholen, *A characterization of representable dualities*, Proc. Internat. Conf. on Categorical Topology (Prague, 1988), World Scientific, Singapore, 1989, pp. 336–357.
7. C. Faith, *Algebra.* I. *Rings, modules and categories*, corrected reprint, Springer-Verlag, 1981.
8. P. J. Freyd, *Algebra valued functors in general and tensor products in particular*, Colloq. Math. **14** (1966), 89–106.
9. L. Fuchs, *Infinite Abelian groups*, vols. I and II, Academic Press, London, 1970.
10. I. Glicksberg, *Uniform boundedness for groups*, Canad. J. Math. **14** (1962), 209–276.
11. H. Herrlich and G. E. Strecker, *Category theory*, 2nd ed., Heldermann Verlag, Berlin, 1979.
12. K. H. Hofmann and K. Keimel, *A generalized character theory of partially ordered sets and lattices*, Mem. Amer. Math. Soc. No. 122 (1972).
13. J. R. Isbell, *General functorial semantics*, I, Amer. J. Math. **94** (1972), 535–596.
14. P. T. Johnstone, *Stone spaces*, Cambridge Univ. Press, Cambridge, 1982.
15. J. Lambek and B. A. Rattray, *A general Stone-Gelfand duality*, Trans. Amer. Math. Soc. **248** (1979), 1–35.
16. S. Mac Lane, *Categories for the working mathematician*, Springer-Verlag, 1971.

17. L. D. Nel, *Topological universes and smooth Gelfand-Naimark duality*, Contemp. Math., vol. 30, Amer. Math. Soc., Providence, R. I., 1984, pp. 244–276.

18. V. V. Pashenkov, *Homogeneous and non-homogeneous duality*, Uspekhi Mat. Nauk **42:5** (1987), 79–99; English transl. Russian Math. Surveys **42:5** (1987), 95–121.

19. H.-E. Porst and M. B. Wischnewsky, *Every topological category is convenient for Gelfand duality*, Manuscripta Math. **25** (1978), 169–204.

20. I. Prodanov, *An axiomatic characterization of the Pontryagin duality*, unpublished manuscript (Sofia, 1984).

21. A Pultr, *The right adjoints into the category of relational systems*, Lecture Notes in Math., vol. 137, Springer-Verlag, 1970.

22. D. W. Roeder, *Functorial characterizations of Pontryagin duality*, Trans. Amer. Math. Soc. **154** (1971), 151–175.

23. N. Varopoulos, *Studies in harmonic analysis*, Proc. Cambridge Philos. Soc. **60** (1964), 465–516.

INSTITUTE OF MATHEMATICS, BULGARIAN ACADEMY OF SCIENCES, 1090 SOFIA, BULGARIA

DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, NORTH YORK, ONTARIO, M3J 1P3, CANADA

E-mail address: tholen@vm1.yorku.ca