RATIONAL ORBITS ON THREE-SYMMETRIC PRODUCTS OF ABELIAN VARIETIES

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Abstract. Let $A$ be an $n$-dimensional Abelian variety, $n \geq 2$; let $\text{CH}_0(A)$ be the group of zero-cycles of $A$, modulo rational equivalence; by regarding an effective, degree $k$, zero-cycle, as a point on $S^k(A)$ (the $k$-symmetric product of $A$), and by considering the associated rational equivalence class, we get a map $\gamma: S^k(A) \rightarrow \text{CH}_0(A)$, whose fibres are called $\gamma$-orbits.

For any $n \geq 2$, in this paper we determine the maximal dimension of the $\gamma$-orbits when $k = 2$ or $3$ (it is, respectively, 1 and 2), and the maximal dimension of families of $\gamma$-orbits; moreover, for generic $A$, we get some refinements and in particular we show that if $\dim(A) \geq 4$, $S^3(A)$ does not contain any $\gamma$-orbit; note that it implies that a generic Abelian four-fold does not contain any trigonal curve. We also show that our bounds are sharp by some examples.

The used technique is the following: we have considered some special families of Abelian varieties: $A_t = E_t \times B$ ($E_t$ is an elliptic curve with varying moduli) and we have constructed suitable projections between $S^k(A_t)$ and $S^k(B)$ which preserve the dimensions of the families of $\gamma$-orbits; then we have done induction on $n$. For $n = 2$ the proof is based upon the papers of Mumford and Roitman on this topic.

1. Introduction

Let $X$ be a $d$-dimensional smooth algebraic variety; a cycle $Z$ of codimension $r$ in $X$ is defined to be an element of the free Abelian group $\text{C}^r(X)$ generated by the irreducible subvarieties of codimension $r$ on $X$. We are interested in zero-cycles, i.e. when $r = d$. Two zero-cycles $Z_1$ and $Z_2$ of $X$ are rationally equivalent if there exists a cycle $Z$ on $X \times \mathbb{A}^1$, which intersects each fibre $X \times \{t\}$ in some points such that $Z_1$ and $Z_2$ are obtained respectively by intersecting $Z$ with the fibres $X \times \{0\}$ and $X \times \{1\}$. Note that this is in fact an equivalence relation and that the zero-cycles rationally equivalent to $0$ (the zero of $\text{C}^d(X)$) form a subgroup of $\text{C}^d(X)$, (see [H, R₁]).

We denote by $\text{CH}_0(X)$ the (Chow) group of zero-cycles on $X$, modulo rational equivalence. If $Z = \sum n_i P_i$ is a zero-cycle, where the $P_i$ are points of $X$, we define the degree of $Z$ to be $\sum n_i$. It is convenient to regard an effective...
zero-cycle $Z = \sum n_i P_i$, i.e., one where all the $n_i > 0$, as a point on the $k$th symmetric product $S^k(X)$ of $X$, where $k = \text{deg}(Z)$. Then by taking the associated rational equivalence class, we obtain a map $\gamma : S^k(X) \to \text{CH}_0(X)$; the fibres of this map will be called $\gamma$-orbits; the irreducible, connected, components of a $\gamma$-orbit will be called $\gamma$-components, ($\gamma$-curves if they have dimension 1, $\gamma$-surfaces if they have dimension 2, etc.).

Now let $A$ be an Abelian variety, if we consider the Albanese morphism $\alpha_k : S^k(A) \to \text{Alb}(S^k(A)) = A$ (i.e., $\alpha_k(x_1, x_2, \ldots, x_k) = x_1 + x_2 + \ldots + x_k$), we have that the fibres of $\alpha_k$ are all isomorphic and that every $\gamma$-orbit of $S^k(A)$ is contained in exactly one fibre of $\alpha_k$. Then, if we want to study the $\gamma$-orbits of $S^k(A)$, we have only to consider the $\gamma$-orbits contained in $K_k(A) = \ker(\alpha_k)$.

In [P] the author showed that for a generic Abelian variety $A$, with $\text{dim}(A) \geq 3$, its Kummer variety, $K(A)$, does not contain any rational curve. By remarking that $K(A)$ is $K_2(A)$ in the previous notations, you can think that in $S^2(A)$ there are no one-dimensional $\gamma$-orbits, (where “dimension” means: maximal dimension of the $\gamma$-components of the $\gamma$-orbit, see §3). In fact, as Clemens pointed out, the technique used in [P] is related to the famous Mumford’s paper [M] about the rational equivalence of zero cycles on a surface. So that, by those arguments, it is possible to show:

**Theorem (1.1).** Let $A$ be an Abelian variety, $\text{dim}(A) \geq 2$, then
(a) $S^2(A)$ does not contain any two-dimensional $\gamma$-orbit;
(b) if $A$ is generic and $\text{dim}(A) \geq 3$, $S^2(A)$ does not contain any one-dimensional $\gamma$-orbit.

The proof of (1.1) is essentially contained in [P]: you have only to change the words “rational curve” into “$\gamma$-curve”, (see also (7.1)).

In this paper we study the $\gamma$-orbits of $S^3(A), \text{dim}(A) \geq 2$, and we obtain the following results:

**Theorem (1.2).** Let $A$ be an Abelian variety, $\text{dim}(A) \geq 2$, then
(a) in $S^3(A)$ there are no $d$-dimensional $\gamma$-orbits with $d \geq 3$;
(b) in $K_3(A)$ there are no one-dimensional families of two-dimensional $\gamma$-orbits;
(c) if $\text{dim}(A) = 2$, in $K_3(A)$ there are no three-dimensional families of one-dimensional $\gamma$-orbits.

**Remark (1.3).** If $\text{dim}(A) = 2$, in $S^3(A)$ there are some two-dimensional $\gamma$-orbits and some two-dimensional families of one-dimensional $\gamma$-orbits, see Examples (5.2) and (5.3); so that (1.2) is sharp.

**Theorem (1.4).** Let $A$ be a generic Abelian variety, $\text{dim}(A) \geq 3$, then
(a) if $\text{dim}(A) = 3$, in $S^3(A)$ there are no two-dimensional $\gamma$-orbits;
(b) if $\text{dim}(A) = 3$, in $K_3(A)$ there are no two-dimensional families of one-dimensional $\gamma$-orbits;
(c) if $\text{dim}(A) \geq 4$, in $S^3(A)$ there are no one-dimensional $\gamma$-orbits.

The proof of (1.2), in §5, is based upon the results of Mumford and Roitman (see §3); but, to apply them, we have needed some linear algebra which we have condensed in §4.
To prove (1.4) we have considered some special families of Abelian varieties of this type: \( A_t = E_t \times B \) (where \( E \) is usually an elliptic curve with varying moduli), and we have used the projections between \( S^3(A_t) \) and \( S^3(B) \) which preserve the dimension of the families of \( \gamma \)-orbits, then we have applied (1.2) to \( S^3(B) \), (see §7).

Unfortunately we did not find an easy way to show that such projections do exist, not even when \( A \) is isogenous to a product of elliptic curves. So we were forced to prove the lemmas in §6; actually some proof could be shortened by using the De Franchis-Severi theorem (for curves and for surfaces, see [D-M]), but we have avoided this theorem, firstly since it is not strictly necessary, secondly since we hope to generalize our results to \( S^k(A), \ k \geq 4 \).

Our theorems have the following corollary, which solves the problem put at the end of [P]:

**Corollary (1.5).** Let \( A \) be a generic \( g \)-dimensional Abelian variety, \( g \geq 4 \). Then \( A \) is not a quotient of a Jacobian of a trigonal curve, in other words \( A \) does not contain trigonal curves.

**Proof.** Let \( C \) be a trigonal curve such that there exists a surjective map

\[
f: J(C) \to A.
\]

By composing \( f \) with the Abel-Jacobi map, we get a nontrivial map \( C \to A \), hence we have a finite map: \( S^3(C) \to S^3(A) \); as \( C \) is trigonal we have another obvious map: \( \mathbb{P}^1 \to S^3(C) \to S^3(A) \); this gives rise to a rational curve in \( S^3(A) \), but it is not possible by (1.4)(c). \( \square \)

**Remark (1.6).** Obviously the Jacobian of a trigonal curve contains a trigonal curve: the curve itself; (1.5) shows that, among Abelian varieties, the Jacobians of genus 4 curves are special also under this point of view.

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2. **Notations and conventions**

- \( \oplus \) direct sum of vector spaces,
- \( \langle x_1, x_2, \ldots \rangle \) \( \mathbb{C} \)-vector space generated by \( x_1, x_2, \ldots \),
- variety by this term we mean a projective complex variety,
- \( n \)-fold \( n \)-dimensional variety (not necessarily smooth),
- surface two-fold,
- curve one-fold,
- generic by this word we mean: outside a countable union of proper analytic subvarieties,
- \( K_V \) canonical divisor of the variety \( V \) when it is smooth,
- \( V \times V \) Cartesian product of the variety \( V \) with itself,
- \( V^k \) \( k \)-Cartesian product of the variety \( V \),
- \( S^k(V) \) \( k \)-symmetric product of the variety \( V \),
- \( \mathbb{H}_n \) Siegel space of \( n \)-dimensional Abelian varieties.
3. RATIONAL EQUIVALENCE OF ZERO-CYCLES

In this paragraph we recall the results of Roitman and Mumford we need in the sequel.

**Proposition (3.1)** (see [R2]). Let $Z$ be a degree $k$ effective zero-cycle on a smooth variety $X$, then the $\gamma$-orbit of $X$ containing $Z$ is a countable union of closed subsets of $S^k(X)$; such a set is usually called c-closed.

We can define the dimension of a c-closed set as the maximal dimension of its irreducible components. In this way it is possible to define the dimension of the image: $\gamma(S^k(X)) \subseteq CH_0(X)$, even though it is not an algebraic variety, as

$$d_k = \dim(S^k(X)) - \min\{\text{dimension of a fibre of } \gamma\}.$$  

We say that $CH_0(X)$ is finite dimensional if the set of integers $d_k$ is bounded, otherwise we say that $CH_0(X)$ is infinite dimensional.

In [M] Mumford proved that if $X$ is a surface with geometric genus $p_g > 0$, then $CH_0(X)$ is infinite dimensional. In [R2] Roitman gave the following generalization:

**Theorem (3.2).** Let $X$ be a smooth variety; then there are integers $d(X)$ and $j(X) \geq 0$, and an integer $k_0$, such that for all $k \geq k_0$ we have $d_k = kd(X) + j(X)$. Moreover $d(X) \leq \dim(X)$, and $d(X) = 0$ if and only if $CH_0(X)$ is finite dimensional.

In [R1 and R2] Roitman proved the following:

**Theorem (3.3).** Let $X$ be a smooth variety, suppose that, for some positive integer $q$, there exists a nonzero global $q$-form $\omega$ on $X$. Then $\omega$ induces a $q$-form $\omega_k$ on $S^k(X)$ whose restriction to any $\gamma$-component of $S^k(X)$ is zero. Hence $d(X) \geq q$.

We recall that the $q$-form $\omega_k$ quoted in (3.3) is defined as follows: we consider $X^k$ and for any $i = 1, 2, \ldots, k$ we consider the natural projection onto the $i$th factor $p_i: X^k \rightarrow X$, now the $q$-form $\sum p_i^*\omega$ is well defined at the generic point of $S^k(X)$ because it is invariant under the action of the symmetric group; so we set $\omega_k = \sum p_i^*\omega$. In the same papers Roitman also shows the following:

**Theorem (3.4).** Let $f_1, f_2$ be two maps between a smooth variety $V$ and $S^k(X)$ such that $\forall v \in V \ f_1(v)$ is rationally equivalent to $f_2(v)$; let $\omega$ be a $q$-form defined on $X$; then $f_1^*(\omega_k) = f_2^*(\omega_k)$.

The previous theorem allows us to prove this corollary.

**Corollary (3.5).** Let $V$ be a smooth $n$-dimensional variety; let $f: V \rightarrow S^k(X)$ be a map; suppose that there exists a map $p: V \rightarrow B$, where $B$ is an $n - t$ dimensional variety, such that $\forall b \in B, \ f[p^{-1}(b)]$ is a $t$-dimensional $\gamma$-component of $S^k(X)$; let $\omega$ be a $q$-form defined on $X$. Then $f^*\omega_k = 0$ if $q > n - t$.

**Proof.** We can always choose a suitable subvariety $W$ of $V$ such that $p|_W$ is finite over $B$; let $V^* \be V \times_B W$ (fibre product). Let $p^*: V^* \rightarrow W$ and $\pi^*: V^* \rightarrow V$ the induced projections and $\sigma: W \rightarrow V^*$ be the canonical section.
of $p^*$; now we consider the maps $h, g: V^* \to S^k(X)$ such that $h(v) = f[\pi^*(v)]$ and $g(v) = h[\sigma(p^*(v))]$. Obviously $h(v)$ is rationally equivalent to $g(v)$ $\forall v \in V^*$, and therefore, by (3.4), $h^*\omega_k = g^*\omega_k$. But $g^*\omega_k = (p^*)^*\sigma^*h^*\omega_k$ and $\sigma^*h^*\omega_k = 0$ if $q > n - t$, as $\pi^*$ is finite on $V$, $f^*\omega_k = 0$. □

4. Some linear algebra

Let $V$ be $C^2$, and let $\{dz, dw\}$ be a basis for $V^*$. Let $L_2$ be the kernel of the map $\sigma: V \oplus V \oplus V \to V$ given by summation. Consider the following two-form on $L_2$:

\[
\begin{align*}
&[dz_1 \wedge dw_1 + dz_2 \wedge dw_2 + dz_3 \wedge dw_3]_{L_2} \\
&= [2dz_1 \wedge dw_1 + 2dz_2 \wedge dw_2 + dz_1 \wedge dw_2 + dz_2 \wedge dw_1]_{L_2} \\
&= [dz_1 \wedge d(2w_1 + w_2) + d(z_1 + 2z_2) \wedge dw_2]_{L_2}. 
\end{align*}
\]

As $(\cdot)$ has maximal rank on $L_2$, we have that any locally isotropic subspace of $V \oplus V \oplus V$ for $(\cdot)$, has dimension 2 at most. In fact there are such two-dimensional maximal subspaces, for instance: $\{(v, \rho v, \rho^2 v), \, v \in V, \, \rho \in C$ with $1 + \rho + \rho^2 = 0\}$.

Now let $W$ be $C^n$, $n \geq 2$, and let $L_n$ be the kernel of the map $\sigma: W \oplus W \oplus W \to W$ as before. Let $U$ be a linear subspace of $L_n$ such that for all projections $W \to V$, the induced map $L_n \to L_2$ sends $U$ into a totally isotropic subspace of $L_2$ for $(\cdot)$. Then dim$(U) \leq n$. In fact, for $n = 2$ this is true, for $n \geq 3$ we can proceed by induction on $n$: every projection $L_n \to L_{n-1}$ has kernel of dimension 2, so that dim$(U) \leq n + 1$; moreover if dim$(U) = n + 1$, the kernel of every projection $L_n \to L_{n-1}$ would lie in $U$, and this is not possible.

Note that dim$(U) = n$ is possible, for instance if $U = \{(w, \rho w, \rho^2 w), \, w \in W, \, \rho \in C$ with $1 + \rho + \rho^2 = 0\}$; we will see in (4.2) that it is the only possibility. Now we can prove the following:

**Proposition (4.1).** In the same notation as before, let $n = 3$, let $\{dz, dw, du\}$ be a basis for $W^*$; consider the following three-form:

\[
(\sim\sim) \quad dz_1 \wedge dw_1 \wedge du_1 + dz_2 \wedge dw_2 \wedge du_2 + dz_3 \wedge dw_3 \wedge du_3
\]

and suppose that $U$ is totally isotropic for $(\sim\sim)$. Then $\dim(U) \leq 2$.

**Proof.** By contradiction we suppose that $\dim(U) = 3$, then by projecting $W$ to $V$ three times along the respective axes we see that:

\[
U = \langle(a_1, 0, 0), (a_2, 0, 0), (a_3, 0, 0)\rangle + \langle(0, b_1, 0), (0, b_2, 0), (0, b_3, 0)\rangle
\]

\[
+ \langle(0, 0, c_1), (0, 0, c_2), (0, 0, c_3)\rangle
\]

with: $\sum a_i = \sum b_i = \sum c_i = 0$. So the vectors $a = (a_i), b = (b_i), c = (c_i)$ in $C^3$ lie in the plane $P$ defined by the equation: $\sum x_i = 0$. Since for all projections $W \to V$, the induced map $L_n \to L_2$ sends $U$ into a totally isotropic subspace of $L_2$ for $(\cdot)$, we have: $\sum a_i b_i = \sum b_i c_i = \sum c_i a_i = 0$.

Since the symmetric bilinear form on $C^3$ which has the identity associated matrix (with respect to the standard base) has rank 2 on $P$, we conclude from the above equations that either $a$, $b$ or $c$ is $0$, (this is impossible as we have supposed that $\dim(U) = 3$) or $a$, $b$, and $c$ are all multiples of the same vector.
w with \( \sum w_i = \sum (w_i)^2 = 0 \). So that w can be taken to be some permutation of \((1, \rho, \rho^2)\). Hence we can write: \( a = Aw, \ b = Bw \) and \( c = Cw \) for some nonzero complex numbers \( A, B, C \). But if we apply \((\sim)\) to these three vectors we have that the result is zero if and only if \( ABC = 0 \), contradiction!

By (4.1) it is very easy to prove the following:

**Proposition (4.2).** In the previous notation: let \( n \geq 4 \). Then \( U \subseteq \{(w, \rho w, \rho^2 w), w \in W, \rho \in \mathbb{C} \text{ with } 1 + \rho + \rho^2 = 0\} \) and if all projections of \( W \) into \( \mathbb{C}^3 \) send \( U \) into a totally isotropic subspace for \((\sim)\), we have that \( \dim(U) \leq 2 \).

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### 5. Proof of (1.2) and Some Examples

Let \( A \) be an \( n \)-dimensional Abelian variety. Firstly we want to recall some useful facts about \( S^k(A) \).

There is an action of the additive group \( A \) on the variety \( S^k(A) \): for every \( a \in A \) we have \( T_a : S^k(A) \rightarrow S^k(A) \) such that for every \((x_1, x_2, \ldots, x_k) \in S^k(A) \) \( T_a(x_1, x_2, \ldots, x_k) = (x_1 + a, x_2 + a, \ldots, x_k + a) \). For every \( a \in A \), \( T_a \) is an isomorphism of \( S^k(A) \) which we will call translation, by abuse of language.

If we consider the \( nk \)-dimensional Abelian variety \( A^k \), we have that there is a \((k!)\)-covering \( p : A^k \rightarrow S^k(A) \) which is obviously ramified on the points \((x_1, x_2, \ldots, x_k) \) of \( S^k(A) \) such that the \( x_i \) are not all distinct. Moreover there is another obvious \((k!)\)-covering \( \pi : A^{k-1} \rightarrow K_k(A) \) \((\text{the Kernel of the Albanese map, see } \S 1)\) such that \( \pi(x_1, x_2, \ldots, x_{k-1}) = (x_1, x_2, \ldots, x_{k-1}, -x_1 - x_2 - x_{k-1}) \). Remark that any \( d \)-dimensional \( \gamma \)-component in \( K_k(A) \) gives rise to a \( d \)-fold in \( A^{k-1} \) via \( \pi \).

Now we are able to prove (1.2); recall that, by the argument of \( \S 1 \), we have to study the \( \gamma \)-orbits contained in \( K_3(A) \).

**Proof of (1.2)(a).** Let \( V \) be the dual of the Lie algebra of \( A \), \( \dim(V) = \dim(A) = n \), and we recall that, for any Abelian variety \( A \), \( \forall q \geq 1, H^q(A) = \Lambda^q(V) \).

For any \( \omega \in \Lambda^q(V), q \geq 2 \), we consider the \( q \)-form \( \phi(\omega) \) induced by \( \omega \) on \( S^3(A) \) in the following way: \( \phi(\omega) = p_*(p_1^*\omega + p_2^*\omega + p_3^*\omega) \), where \( p : A^3 \rightarrow S^3(A) \)

and \( p_1, p_2, p_3 \) are the projections of \( A \times A \times A \) on \( A \).

The tangent space \( U \) at every smooth point of any \( \gamma \)-orbit of \( K_3(A) \) lies in \( L_n \) (see \( \S 4 \)); \( \phi(\omega) \) has to vanish on \( U \), by Theorem (3.3), for any \( \omega \in \Lambda^q(V) \), \( q = 2, 3, \ldots, n \); this means that the assumptions of (4.2) about the projections of \( U \) are satisfied. Hence \( \dim(U) \leq 2 \); therefore every \( \gamma \)-orbit has dimension 2 at most. \( \square \)

**Remark (5.1).** The previous proof is based on the fact that all the forms belonging to \( \phi(\Lambda^q(V)) \), \( q = 2, 3, \ldots, n \), have to vanish on the tangent spaces at the smooth points of any \( \gamma \)-component of \( K_3(A) \). So we can say that, if a \( d \)-fold, contained in \( K_3(A) \), has the same properties, then \( d \leq 2 \).
Proof of (1.2)(b). If there would be such a family \( \{ S_t \} \), \( t \in \mathbb{C} \), then in \( K_3(A) \) we would get a three-fold \( T \) which would be filled by two-dimensional \( \gamma \)-components. By using the same notations as in the proof of (1.2)(a), we have that, by Corollary (3.5), the forms belonging to \( \phi(A^q(V)) \), \( q = 2, 3, \ldots, n \), have to vanish on the tangent spaces at the smooth points of \( T \), but this implies that \( \dim(T) \leq 2 \) by Remark (5.1): contradiction! \( \square \)

Proof of (1.2)(c). If there would be a family \( \{ C_r \} \), \( r \in \mathbb{C}^3 \), of one-dimensional \( \gamma \)-orbits in \( K_3(A) \) then \( K_3(A) \) would be filled by one-dimensional \( \gamma \)-components and this is not possible by (3.2) and (3.3). \( \square \)

Now we prove, by some examples, that, when \( \dim(A) = 2 \), the one-dimensional \( \gamma \)-orbits can span a three-fold in \( S^3(A) \), and that there are two-dimensional \( \gamma \)-orbits.

Example (5.2). Let \( A \) be an Abelian surface; let \( C \) be a nonhyperelliptic genus 3 (smooth, irreducible) curve on \( A \). If we consider the divisor \( L \) supported by \( C \), we get \( L^2 = 4 \) by the genus formula, and \( h^0(L) = 2 \) by the Riemann-Roch and Kodaira vanishing theorems.

So \( C \) moves in a pencil \( \{ C_\mu \} \) which has four base points: \( A, B, C, D \). The adjunction formula yields: \( K_L = L|_L \); so that \( A + B + C + D \) is a canonical divisor on every curve \( C_\mu \) of the pencil.

The canonical model \( C'_\mu \) of \( C_\mu \) is a smooth plane quartic whose canonical series is cut by the lines, therefore the divisor of \( C'_\mu \) corresponding to \( A + B + C + D \) is cut on \( C'_\mu \) by a line.

Now we consider a point \( P_\mu \) on \( C_\mu \) and the linear series \( g^1_3 \) corresponding to the linear series \( g^1_3 \) cut on \( C'_\mu \) by the lines passing through the point corresponding to \( P_\mu \). So that for every \( \lambda \in \mathbb{P}^1 \) we have a divisor: \( P_\mu + Q_{\mu\lambda} + R_{\mu\lambda} + S_{\mu\lambda} \) on \( C_\mu \). We choose an Abel map \( \alpha_\mu: C_\mu \to J(C_\mu) \) such that \( \alpha_\mu(P_\mu) = 0 \), hence, by Abel theorem, \( \alpha_\mu(Q_{\mu\lambda} + R_{\mu\lambda} + S_{\mu\lambda}) = \tau_{P, \mu} \) is constant with respect to \( \lambda \).

The 3-ples: \( \alpha_\mu(Q_{\mu\lambda}), \alpha_\mu(R_{\mu\lambda}), \alpha_\mu(S_{\mu\lambda}) \) in \( J(C_\mu) \) gives rise to a rational curve in \( S^3[J(C_\mu)] \) as \( \lambda \) moves in \( \mathbb{P}^1 \).

We consider the following commutative diagram

\[
\begin{array}{ccc}
C_\mu & \xrightarrow{i_\mu} & A \\
\downarrow{\alpha_\mu} & & \\
J(C_\mu) & \xrightarrow{f_\mu} & J(C_\mu)
\end{array}
\]

in which \( i_\mu \) is the embedding of \( C_\mu \) in \( A \) and \( f_\mu \) is the homomorphism between Abelian varieties induced by \( \alpha_\mu \). By using \( f_\mu \) we get a rational curve in \( S^3(A) \); by translating this curve by \( f_\mu(\tau_{P, \mu}) \) we get a rational curve \( \gamma_{P, \mu} \) in \( K_3(A) \).

Now we let \( P \) vary on \( C_\mu \): for every point \( P \) we get a curve \( \gamma_{P, \mu} \) in \( K_3(A) \); these curves are all distinct because the used linear series \( g^1_3 \) on \( C'_\mu \) are distinct.

Now let \( P \) vary on \( C_\mu \) and let \( \mu \) vary in \( \mathbb{P}^1 \): for every couple \( P, \mu \) we get a curve \( \gamma_{P, \mu} \) in \( K_3(A) \); these curves are all distinct because they are made by points lying on different curves \( C_\mu \) of \( A \).
Obviously every curve $\gamma_{p,\mu}$ is contained in a $\gamma$-orbit of $K_3(A)$ and this example shows that in $K_3(A)$ there exist $\gamma$-orbits whose span is a three-fold.

**Example (5.3).** The previous example also shows that in $K_3(A)$ there exist some $\gamma$-orbits whose span is a surface. In fact for every curve $C_\mu$ of the previous example we can fix the point $A$, one of the base points of the pencil $\{C_\mu\}$, and for every $\mu \in \mathbb{P}^1$ we get a rational curve $\gamma_{A,\mu} = \gamma_\mu$ in $K_3(A)$.

In this case, by recalling the construction of the linear series $g_1$, we have that for every $\mu \in \mathbb{P}^1$ there exists a $\lambda \in \mathbb{P}^1$ such that $Q_{\mu\lambda} = B$, $R_{\mu\lambda} = C$, $S_{\mu\lambda} = D$. Therefore: $\alpha_\mu(B + C + D) = \tau_{A,\mu}$ and $f_\mu[\alpha_\mu(B + C + D)] = f_\mu(\tau_{A,\mu}) = i_{\mu}(B + C + D)$ is independent from $\mu$, hence the obtained curves in $S^3(A)$ belong to

$$\{(x, y, z) \in S^3(A) | x + y + z = i_{\mu}(B + C + D)\}$$

and all pass through the point: $(i_{\mu}(B), i_{\mu}(C), i_{\mu}(D))$ in $S^3(A)$.

So that the translated curves $\gamma_\mu$ in $K_3(A)$ all intersect between them. Therefore the curves $\gamma_\mu$ span a rational surface in $K_3(A)$ which is contained in a $\gamma$-orbit.

### 6. The lemmas

In this paragraph we prove some lemmas which will be useful in §7. We will need to study the projections of $d$-dimensional $\gamma$-components which are induced by natural projections between $K_3(V \times W)$ and $K_3(W)$, where $V$ and $W$ will be suitable Abelian varieties.

By the commutativity of the following diagram

$$
\begin{array}{ccc}
(V \times W) \times (V \times W) & \longrightarrow & W \times W \\
\downarrow \pi & & \downarrow \pi \\
K_3(V \times W) & \longrightarrow & K_3(W)
\end{array}
$$

we have to study the natural projections $(V \times W) \times (V \times W) \to W \times W$, this is the aim of the following two lemmas.

Let $X$ be a smooth irreducible $d$-fold and let $A$ be an $n$-dimensional Abelian variety; let $\sigma: X \to A \times A$ be a map, birational onto its image, such that $\sigma(X)$ generates $A \times A$. Assume that $A$ is isogenous to $D \times D \times B$ where $D$ and $B$ are Abelian varieties of dimension $q$ and $(n - 2q)$ respectively. We fix two "dual" isogenies $D \times D \times B \to A \to D \times D \times B$ such that their composition is the multiplication by an integer; in this way we get a map $f \circ \sigma: X \to B \times B$ by composing the natural projection $f$ with $\sigma$; let $Y$ be $f[\sigma(X)]$; assume that

$$\text{the natural projection } f: A \times A \to B \times B \text{ is such that } Y = f[\sigma(X)] \text{ is a } d \text{-dimensional subvariety of } B \times B.$$  

Now let $\nu_i: D \to D \times D \to A$ be the composition of an embedding of $D$ in $D \times D$ with the previously chosen isogeny; we can suppose that $i$ varies in a countable set, in fact among all embeddings $D \to D \times D$ there are the following morphisms of algebraic groups: $d \to (ad, bd)$ (for any $d \in D$ and for a fixed couple of coprime integers $a$, $b$). We set $B_i = [(D \times D)/\nu_i(D)] \times B$ and let $X_i$ be the image of $X$ under the composition of the natural projection $A \times A \to B_i \times B_i$ with $\sigma$. 

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In this situation we have the maps: \( q^*_i : H^1(X_i, \mathbb{Q}) \to H^1(X, \mathbb{Q}) \) and \( \sigma^*: H^1(A \times A, \mathbb{Q}) \to H^1(X, \mathbb{Q}) \); let \( \Lambda_i \) be the image of \( q^*_i \), then

**Lemma (6.2).** With the previous notations, there exists an index \( i \) at least (hence an embedding of \( D \) in \( D \times D \)) such that \( \Lambda_i \) contains the image of \( \sigma^* \).

**Proof.** Note that this proof actually shows more, i.e. \( \Lambda_i \) contains the image of \( H^1(A \times A, \mathbb{Q}) \) in \( H^1(X, \mathbb{Q}) \) save for a finite number of \( i \).

For every \( i \) we have a diagram of equidimensional \( d \)-folds

\[
\begin{array}{ccc}
X & \longrightarrow & X_i \\
\downarrow & & \downarrow \\
Y & \longrightarrow & 
\end{array}
\]

(the map \( X_i \to Y \) is obtained by using the natural projection \( B_i \to B_i/[\{(D \times D)/\nu_i(D)\}] \) and by remarking that \( B_i/[\{(D \times D)/\nu_i(D)\}] \) is isogenous to \( B \). It follows that: \( K[Y] \subset K[X_i] \subset K[X] \) so that there are only a finite number of birational models for the \( X_i \). The maps in the following diagram are defined in the obvious way:

\[
\begin{array}{ccc}
H^1(B \times B, \mathbb{Q}) & \longrightarrow & H^1(B_i \times B_j, \mathbb{Q}) \\
\downarrow & & \downarrow \sigma^* \\
H^1(Y, \mathbb{Q}) & \longrightarrow & H^1(X_i, \mathbb{Q}) \\
\downarrow & & \downarrow \sigma^* \\
& & H^1(X, \mathbb{Q})
\end{array}
\]

and we remark that, as \( \sigma(X) \) generates \( A \times A \) and the natural projection \( A \times A \to B_i \times B_j \) is surjective, the map \( H^1(B_i \times B_j, \mathbb{Q}) \to H^1(X, \mathbb{Q}) \) is injective for any \( i \). Now if we choose two distinct, transverse, embeddings of \( D \) in \( D \times D \) for which the corresponding fields \( K[X_{i1}] \) and \( K[X_{i2}] \), contained in \( K[X] \), coincide, then we have that \( H^1(X_{i1}, \mathbb{Q}) \to H^1(X, \mathbb{Q}) \) and \( H^1(X_{i2}, \mathbb{Q}) \to H^1(X, \mathbb{Q}) \) are the same map; by the injectivity of \( H^1(B_{ij} \times B_{ij}, \mathbb{Q}) \to H^1(X, \mathbb{Q}) \), \( j = 1, 2 \), we have that \( \Lambda_{i1} = \Lambda_{i2} \) must contain the span of the images of \( H^1(B_{i1} \times B_{i1}, \mathbb{Q}) \) and \( H^1(B_{i2} \times B_{i2}, \mathbb{Q}) \) in \( H^1(X, \mathbb{Q}) \), hence \( \Lambda_{i1} = \Lambda_{i2} \) must contain the image of \( H^1(A \times A, \mathbb{Q}) \) in \( H^1(X, \mathbb{Q}) \). \( \Box \)

**Lemma (6.3).** With the same assumptions as in (6.2), we get the same thesis if we consider \( F^1H^1(-, \mathbb{C}) \), (in the sense of mixed Hodge structures, see [G]), instead of \( H^1(-, \mathbb{Q}) \).

**Remark (6.4).** Note that, if \( \text{dim}(X) = 1 \), (*) is always satisfied, (save, obviously, when \( A = E \times E \times B \), \( E \) elliptic curve, and \( X = E \)).

Now let \( \Delta \) be an analytic scheme \((0 \in \Delta)\), and \( h: A \to \Delta \) a proper fibration such that \( h^{-1}(t), \ t \in \Delta \), is an Abelian variety isogenous to \( D_t \times B \), \( B \) fixed, \( (h^{-1}(0)) \) isogenous to \( D_0 \times B \).

The infinitesimal variation of the Hodge structures induces the following map \( \phi: H^{1,0}(D_0) \to \text{Hom}(T_\Delta(0), H^{0,1}(D_0)) \), such that for any \( \mu \in H^{1,0}(D_0) \) and for any \( t \in T_\Delta(0) \), \( \phi(\mu)(t) \) is the derivative of \( \mu \) along \( t \). We have the following:
Lemma (6.5). With the previous assumptions, consider the commutative diagram
\[
\begin{array}{ccc}
Q_t: X_t & \longrightarrow & Z \\
\sigma_t & \downarrow & \downarrow \\
D_t \times B & \longrightarrow & B
\end{array}
\]
where \( X_t \) are varieties parametrized by \( t \), \( \sigma_t \) are maps birational onto their images, \( \sigma_t(X_t) \) generates \( D_t \times B \) for any \( t \), \( f_t \) is the natural projection, \( q_t \) is induced by \( f_t \), \( i \) is an inclusion and \( Z \) is fixed. Assume that \( \phi \) is injective; then
\[
\sigma_t^*[H^1,0(D_0)] \cap q_t^* F^1 H^1(Z) = 0 \in F^1 H^1(X_0).
\]
Proof. If \( \mu \) belongs to that intersection, \( \phi(\mu) = 0 \) as \( F^1 H^1(Z) \) is independent from \( t \); as \( \phi \) is injective we have \( \mu = 0 \). \( \square \)

Now let \( \Delta \) be an open set of \( \mathscr{H} \), \( (0 \in \Delta) \), we will call a \((\Delta, m, G)\)-situation (for \( \mathscr{H} \)) the following data:
(i) a bundle of Abelian varieties over \( \Delta: \mathbb{A} \times \Delta \mathbb{A} \times \cdots \times G \) \( (m \text{ times}) \) where \( \mathbb{A} \) is the tautological Abelian bundle over \( \Delta \) and \( G \) is a constant Abelian variety;
(ii) a family of \( d \)-dimensional varieties \( k: X \to \Delta \) over \( \Delta \);
(iii) a morphism of \( \Delta \) families \( \sigma: X \to \mathbb{A}^m \times G \), i.e. a commutative diagram as follows:
\[
\begin{array}{ccc}
X & \xrightarrow{\sigma} & \mathbb{A}^m \times G \\
\downarrow & & \downarrow \Delta \\
h & \xrightarrow{k} & \mathbb{A}^m \end{array}
\]
(we set \( X_t = k^{-1}(t) \) and \( h^{-1}(t) = (A_t)^m \times G \) for any \( t \in \Delta \));
(iv) the assumption that the image \( \sigma_t(X_t) \) generates \( (A_t)^m \times G \) as a group, for any \( t \in \Delta \).

We remark that, if conditions (i), (ii), (iii) are satisfied, the bundle of Abelian varieties generated by the images \( \sigma_t(X_t) \) must be isomorphic to \( \mathbb{A}^m \times G' \) where \( m' \leq m \) and \( G' \) is an Abelian subvariety of \( G \); so that, by changing the bundle, we always get a \((\Delta, m', G')\)-situation. With the above warning we can say that to have a \((\Delta, m, G)\)-situation is equivalent to have a \( d \)-dimensional variety in \( \mathbb{A}^m \times G \) where \( \mathbb{A} \) is generic in \( \Delta \); (i.e. for any \( t \in \Delta \) we have a \( d \)-fold \( X_t \) in \( (A_t)^m \times G \). Actually we usually will consider only the case: \( m = 2 \), \( G = 0 \), (hence \( h = h \times h \) for the sake of simplicity, from now on, this case will be simply called "\( \Delta \)-situation.")

Lemma (6.6). We suppose to be in a \( \Delta \)-situation; we choose \( \mathbb{A} \) isogenous to \( D \times D \times B \), (as in Lemma (6.2)), and for any linear embedding \( \nu_t: D \to D \times D \times B \) we fix an isogeny between \( \mathbb{A} \) and \( \nu_t(D) \times [(D \times D)/\nu_t(D)] \times B \).

Let \( \Delta_t = \{ t \in \Delta \mid \text{the fibre of } h \times \Delta h \text{ is } A_t \times A_t \text{ where } A_t \text{ is isogenous to } \nu_t(D) \times D_t \times B \} \); let \( \mathbb{A}_0 \) be isogenous to \( A \) by the isogeny induced by the previously fixed one. This defines an embedding \( \nu_t^*: \mathscr{H}_q \to \mathscr{H}_n \), such that \( \Delta_t = \Delta \cap [\nu_t^*(\mathscr{H}_q)] \); we set \( B_t = \nu_t(D) \times B \).

For any \( t \in \Delta_t \), let \( f_{t,i}: A_t \times A_t \to B_i \times B_i \) be the natural projection; if we assume (*) for the natural projection \( f_{t,0}: A \times A \to B \times B \) and \( \sigma_0(X_0) \), we have
that, save a finite number of i at most, $f_{i,t}[\sigma_i(X_i)]$ is not a fixed subvariety of $B_i \times B_i$.

Proof. We proceed by contradiction: if (6.6) is false, then for any $i$, $f_{i,t}[\sigma_i(X_i)]$ is a fixed $d$-fold $X_i$ in $B_i$ for any $t$, and $X_i$ generates $B_i$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
q_{i,0} : X_0 & \longrightarrow & X_i \\
\downarrow \sigma_0 & & \downarrow i_i \\
\end{array}
$$

$$
\begin{array}{ccc}
f_{i,0} : (D \times D \times B)^2 & \longrightarrow & B_i \times B_i.
\end{array}
$$

Note that we can apply Lemma (6.5) because we are in a $\Delta$-situation, so we have that $\sigma_i^*[H^{1,0}(D^4)] \cap (q_{i,0})^*F_i^1H^1(X_i) = 0 \in F_i^1H^1(X_0)$ but, by Lemma (6.3), $(q_{i,0})^*F_i^1H^1(X_i)$ contains $\sigma_i^*[H^{1,0}(D^4)]$ except for a finite number of $i$, contradiction! $\square$

Lemma (6.7). We are supposed to be in a $\Delta$-situation; but now we choose $A$ isogenous to $D^m \times B$, and we consider the countable set of the linear embeddings $\nu_i : D^p \rightarrow D^m$ ($p \leq m$, positive integers, $D \in \mathcal{H}_q$, $B \in \mathcal{H}_{n-mq}$). For any embedding $\nu_i$ we fix an isogeny between $A$ and $\nu_i(D^p) \times [D^m/\nu_i(D^p)] \times B$; let $\Delta_i = \{t \in \Delta| \text{ the fibre of } h \times A h \text{ is } A_t \times A_t \text{ where } A_t \text{ is isogenous to } F_t \times [D^m/\nu_i(D^p)] \times B, F_t \in \mathcal{H}_{pq}\}$, $A_0$ is isogenous to $A$ as in the previous cases. This defines an embedding $\nu_i^* : \mathcal{H}_{pq} \rightarrow \mathcal{H}_n$ such that: $\Delta_i = \Delta \cap [\nu_i^*(\mathcal{H}_{pq})]$; we set: $B_i = [D^m/\nu_i(D^p)] \times B$.

For any $t \in \Delta_i$, let $f_{i,t} : A_t \times A_t \rightarrow B_i \times B_i$ be the natural projection; if we assume (*) for the natural projection $f_{i,0} : A \times A \rightarrow B \times B$ and $\sigma_0(X_0)$, we have that, save a finite number of $i$ at most, $f_{i,t}[\sigma_i(X_i)]$ is not a fixed subvariety of $B_i$.

Proof. See the proof of (6.6). $\square$

To apply the above lemmas we need condition (*); this is a crucial point: it allows us to avoid the use of the De Franchis-Severi theorem. When $X$ is of general type and $d = 1$ or 2, this theorem would assure the existence of a finite number of subfields $K[X_i]$ of $K[X]$ (see the proof of Lemma (6.2)), without the assumption that $f$ is generically finite, i.e., roughly speaking, without fixing a shield $Y = f[\sigma(X)]$.

We use the following remark: consider diagram (6.1): our natural projections between $(V \times W) \times (V \times W)$ and $W \times W$ are induced by natural projections between $K_3(V \times W)$ and $K_3(W)$, so that to verify (*) it suffices to verify the corresponding statement for projections between $K_3(V \times W)$ and $K_3(W)$, and vice versa. This explains the statements of the following other lemmas.

Lemma (6.8). Let $S$ be a $\gamma$-surface in $K_3(E \times E)$ where $E$ is a generic elliptic curve (in the sense of moduli); let $S'$ be the pullback of $S$ in $E^2 \times E^2$; let $E_{pq}$ be a fixed embedding of $E \times E$ in $E^2 \times E^2$ such that $E_{pq} = \{px, qx, py, qy\}$ where $(x, y) \in E \times E$ and $p, q$ are coprime integers. Then there exist infinitely many couples $(p, q)$ such that $E_{pq}$ intersects $S'$ properly. In these cases the natural projection $E^2 \times E^2 \rightarrow (E^2 \times E^2)/E_{pq}$ is generically finite on $S'$ (and the induced map $K_3(E \times E) \rightarrow K_3[(E \times E)/\{px, qy\}]$ is generically finite on $S$).
Proof. We will prove that there exists a couple \((p, q)\) at least, such that \(E_{pq}\) intersects \(S'\) properly, but, in fact, our proof will also show that the intersection is proper save for a finite number of couples.

We proceed by contradiction; we recall that if two surfaces in \(E^4\) does not intersect properly then, for every generic point of the first surface, there passes a translate of the second one which intersects the former one along a curve. In fact the intersection cycle of two surfaces in \(E^4\) depends only on their homology class, and the homology class is invariant under translations.

We fix a generic point \(P\) of \(S'\), if every \(E_{pq}\) does not intersect \(S'\) properly then, \(\forall p, q\), there exists a translate of \(E_{pq}\) passing through \(P\) and cutting \(S'\) along a curve; hence, by looking at the tangent spaces, we have that in the Lie algebra of \(E^4\) there are: a vector space generated by \((p, q, 0, 0)\) and \((0, 0, p, q)\), \(\forall p, q\), and the vector space \(\langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \rangle\) (corresponding to the tangent space to \(S'\) at \(P\)), such that the matrix:

\[
\begin{pmatrix}
p & 0 & a_1 & b_1 \\
q & 0 & a_2 & b_2 \\
0 & p & a_3 & b_3 \\
0 & q & a_4 & b_4
\end{pmatrix}
\]

is always singular. Now we show that, for generic \(E\), this situation is not possible.

As \((a_1, a_2, a_3, a_4)\) and \((b_1, b_2, b_3, b_4)\) are independent, it is possible to choose a base for the Lie algebra such that: \(a_1 = b_2 = 1\), \(b_1 = a_2 = 0\); otherwise is not possible that the previous matrix is singular \(\forall p, q\). Now it is easy to see that it is possible only if \(b_3 = a_4 = 0\) and \(b_4 = a_3 = \rho\), with \(\rho \in \mathbb{C}\). As \(S'\) is the pullback in \(E \times E \times E \times E\) of a \(\gamma\)-component \(S\) in \(K_3(E \times E)\) which is not contained in the branching locus of \(\pi\), the skew symmetric two-form \((\cdot, \cdot)\) considered in §4 has to vanish on the tangent space at the generic point \(P\) of \(S'\) by (3.3), hence: \(1 + \rho + \rho^2 = 0\) and \(\rho\) is a constant, independent from \(P\).

This means that the only surfaces in \(E^4\) which does not intersect properly \(E_{pq}\) \(\forall p, q\), are, up to translations, those Abelian surfaces \(S'\) which are the embeddings of \(E \times E\) in \(E^4\) such that \(S' = \{x, y, \rho x, \rho y\}\) where \((x, y) \in E \times E\) and \(\rho \in \mathbb{C}\) with \(1 + \rho + \rho^2 = 0\); but this implies that \(E\) has an endomorphism: \(x \rightarrow \rho x\ \forall x \in E\), with \(1 + \rho + \rho^2 = 0\), and this is not possible for generic \(E\). \(\square\)

Lemma (6.9). Let \(S\) be a \(\gamma\)-surface in \(K_3(E \times E \times E)\) where \(E\) is a generic elliptic curve; let \(S'\) be the pullback of \(S\) in \(E^3 \times E^3\); let \(E(p, q, r, p', q', r')\) be a fixed embedding of \(E^2 \times E^2\) in \(E^3 \times E^3\) such that \(E(p, q, r, p', q', r') = \{px + p'y, qx + q'y, rx + r'y, pz + p'w, qz + q'w, rz + r'w\}\) where \((x, y, z, w) \in E^2 \times E^2\), and \((p, q, r), (p', q', r')\) are triple of coprime integers, and such that the following matrix has rank 2:

\[
\begin{pmatrix}
p & q & r \\
p' & q' & r'
\end{pmatrix}
\]

Then there exist infinitely many choices \((p, q, r, p', q', r')\) such that \(E(p, q, r, p', q', r')\) intersects \(S'\) properly. In these cases the natural projection

\[
E^3 \times E^3 \rightarrow (E^3 \times E^3)/E(p, q, r, p', q', r')
\]
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is generically finite on \( S' \),

\[
(K_3(E \times E \times E) \to K_3[(E \times E \times E)/\{px + p'y, qx + q'y, rx + r'y\}]
\]
is generically finite on \( S \).

Proof. We can proceed as in the proof of Lemma (6.8). □

Lemma (6.10). Let \( E \) be a generic elliptic curve and let \( T \) be a three-fold in \( K_3(E^3) \) which is filled by a two-dimensional family of \( \gamma \)-curves; let \( T' \) be the pullback of \( T \) in \( E^3 \times E^3 \); let \( E_{pqr} \) be a fixed embedding of \( E \times E \) in \( E^3 \times E^3 \) such that \( E_{pqr} = \{px, qx, rx, py, qy, ry\} \) where \( (x, y) \in E \times E \) and \( p, q, r \) are coprime integers. Then there exist infinitely many triples \( (p, q, r) \) such that \( E_{pqr} \) does not intersect \( T' \) or intersects \( T' \) in a finite number of points. In these cases the natural projection \( E^3 \times E^3 \to (E^3 \times E^3)/E_{pqr} \) is generically finite on \( T' \) (and the induced map \( K_3(E^3) \to K_3[E^3/\{px, qx, rx\}] \) is generically finite on \( T \)).

Proof. By arguing as in Lemma (6.8) we get that the only three-folds in \( E^3 \times E^3 \) which does not intersect properly \( E_{pqr} \) are, up to translations, those Abelian three-folds \( T' \) which are the embeddings of \( E \times E \times E \) in \( E^3 \times E^3 \) such that \( T' = \{x, y, z, sx, sy, sz\} \) where \( (x, y, z) \in E \times E \times E \) and \( s \in \mathbb{C} \) with \( s(s + 1) = 0 \).

This would imply that, in \( K_3(E^3) \), \( T \) would be given by the unordered triples: \( \{P, sP, -(s + 1)P\} \), where \( s = 0 \) or \( s = -1 \) and \( P \in E^3 \); in any case we could define an embedding \( \lambda: T \to K_2(E^3) \) such that

\[
\lambda(\{P, sP, -(s + 1)P\}) = \{P, -P\};
\]

\( \lambda(T) \) would be a three-fold filled out by \( \gamma \)-curves; but this is not possible by (1.1)(b): recall that \( E \) is generic and the locus of nonsimple Abelian three-folds is dense in \( \mathscr{H}_3 \).

7. Proof of (1.4)

For the sake of simplicity, in every \( \Delta \)-situation considered in §7 we will identify \( X_t \) with \( \sigma_t(X_t) \).

Proof of (1.4)(a). We proceed by contradiction: we assume that for any three-dimensional Abelian variety \( A \), \( S^3(A) \), and therefore \( K_3(A) \), contains a \( \gamma \)-surface; by their pullback via \( \pi \) we have a surface in any \( A^2 \), so we are in a \( \Delta \)-situation. Then we can construct a fibration \( \eta: h \times \Delta: A \times \Delta \to \Delta \subset \mathcal{H}_3 \) as in §6. We want to apply Lemma (6.6) with \( D = B = E \), \( E \) generic elliptic curve. To have (*) we use Lemma (6.9): we can fix an Abelian variety \( A \) isogenous to \( E \times E \times E \), such that, when we project the \( \gamma \)-surface \( X \) contained in \( K_3(E \times E \times E) \) into \( K_3(E) \) (the last factor), by the natural projection, we obtain another \( \gamma \)-surface \( Y \). This means that the natural projection \( f: A \times A \to B \times B \) satisfies (*)

Now let \( E_{pq} \) be the image in \( E \times E \) of the embedding \( \nu_{pq} \) of \( E \) such that \( \nu_{pq}(x) = (px, qx) \) \( \forall x \in E \), \( (p, q) \) is a couple of coprime integers. We fix an isogeny between \( A \) and \( E_{pq} \times B_{pq} \) where \( B_{pq} = [(E \times E)/E_{pq}] \times E \). Let \( \Delta_{pq} = \Delta \cap [\nu_{pq}^*(\mathcal{H}_1)] \) the open subset of \( \Delta \) such that the fibre over \( t \in \Delta_{pq} \) is
AtA where A is isogenous to EtxBpq (A0 isogenous to A by the previously fixed isogeny) and El is an elliptic curve whose moduli depend on t.

Let φt be the natural projection between K3(At) and K3(Bpq), by our assumption there is a γ-surface Xt in every At and X0 = X. For small t, we can assume that Yt = φt(Xt) is a γ-surface of K3(Bpq); in fact Y0 = φ0(X0) = φ0(X) is a surface in K3(Bpq) because X projects into a surface in K3(E).

By Lemma (6.6), we can choose (p, q) such that \{Yt\} is a one-dimensional family of γ-surfaces of K3(Bpq) (i.e. the union of the Yt span a three-fold in K3(Bpq)); but dim(Bpq) = 2 and this is a contradiction with (1.2)(b). □

Remark (7.1). Here we want to give a short outline of the proof of (1.1)(b) when dim(A) = 3. Firstly we need (1.1)(a) for dimension 2: this is just an application of (3.2) and (3.3): if (1.1)(a) were false, for the generic point of S2(A) would pass a positive dimensional γ-orbit, but then d2 would be strictly less than 4.

Now we proceed by contradiction: we assume that for the generic Abelian three-fold A, S2(A), and therefore K2(A) (which is the Kummer variety K(A) of A), contains a γ-curve. By their pullback via π we get a curve in any A; by using these we can build a family of curves that gives rise to a Δ-situation. By arguing as in the proof of (1.4)(a) we can choose a suitable projection from K2(A) = K(A) onto K(E×E), where A is isogenous to EtxExE, E generic elliptic curve, in such a way that the images of our curves cover K(E×E). Since the image of a γ-orbit is a γ-orbit, we get a contradiction with (1.1)(a).

Proof of (1.4)(b). We proceed by contradiction: we assume that for any three-dimensional Abelian variety S3(A), and therefore K3(A) contains a three-fold filled by γ-curves: by their pull-back via π we have a three-fold in any A2. So we are in a Δ-situation and we can construct a fibration h×₄₂ h: A×₄₂ A → Δ ⊂ ℝ3 as in §6. Pay attention: now we proceed in a very similar way to the proof of (1.4)(a), but we cannot use Lemma (6.6) in that manner.

We fix an Abelian variety A isogenous to E × E × E, E generic elliptic curve. Let E_{pqr} be the image in E × E × E of the embedding ν_{pqr} of E such that ν_{pqr}(x) = (px, qx, rx) ∀x ∈ E, (p, q, r) is a triple of coprime integers; let F_{pqr} be (E × E × E)/E_{pqr}, we fix an isogeny between A and E_{pqr} × F_{pqr}.

By Lemma (6.10) we can assume that, when we project the three-fold T, filled by γ-curves, contained in K3(E × E × E), into K3(F_{pqr}), by the natural projection, we obtain another three-fold T# with the same property.

Let Δ_{pqr} = Δ ∩ [ν_{pqr}⁻¹(Δ)] the open subset of Δ such that the fibre over t ∈ Δ_{pqr} is A_t × A_t where A_t is isogenous to E_t × F_{pqr} (A_0 isogenous to A) and E_t is an elliptic curve whose moduli depend on t.

Let φ_t be the natural projection between K3(A_t) and K3(F_{pqr}), by our assumption there is a three-fold T_t, filled by γ-curves, in every A_t and T_0 = T. Moreover φ_0(T_0) = φ_0(T) = T# is a three-fold in K3(F_{pqr}) by the previous remarks. Therefore, by choosing a smaller disk, we can assume that T_t# = φ_t(T_t) is three-fold in K3(F_{pqr}).

We can use Lemma (6.6) (and Remark (6.4)), to assure that there exist triples (p, q, r) (for instance with r = 0) such that every one-dimensional family \{C_t\} of γ-curves of K3(A_t) projects into another similar family of K3(F_{pqr}). We choose one of these triples.
Now we consider two cases: if $T_{f}^{a}$ is a variable three-fold in $K_{3}(F_{pqr})$, by the previous condition, we would get a three-dimensional family of $\gamma$-curves in $K_{3}(F_{pqr})$, but $\dim(F_{pqr}) = 2$ and this is forbidden by (1.2)(c).

If $T_{f}^{a} = T_{0}^{a}$ is a fixed three-fold in $K_{3}(F_{pqr})$ then, by the previous condition, infinitely many $\gamma$-components pass through any point of $T_{0}^{a}$, hence we would have a one-dimensional family of $\gamma$-surfaces in $K_{3}(F_{pqr})$ at least, and this is not possible by (1.2)(b).

Proof of (1.4)(c). Firstly we assume that $\dim(A) = 4$ and we proceed by contradiction: we assume that for any four-dimensional Abelian variety $A$, $S^{3}(A)$, and therefore $K_{3}(A)$ contains a $\gamma$-curve; by their pullback via $\pi$ we have a curve in any $A^{2}$. So we have a $\Delta$-situation and then we can construct a fibration $h \times_{\Delta} h : A \times_{\Delta} A \to \Delta \subset H_{4}$ as in §6.

We want to use Lemma (6.7) with $D = B = E$, $E$ generic elliptic curve, $p = 2$, $m = 3$; note that (*) is satisfied by Remark (6.4).

We fix an Abelian variety $A$ isogenous to $E \times E \times E \times E$. For any embedding $\nu_{i} : E^{2} \to E^{3}$ let $\Delta_{i} = \Delta \cap [\nu_{i}^{*}(H)]$ the open subset of $\Delta$ such that the fibre over $r \in \Delta_{i}$ is $A_{i} \times A_{i}$ where $A_{i}$ is isogenous to $D_{i} \times [E^{3}/\nu_{i}(E^{2})] \times E$ ($A_{0}$ isogenous to $A$ as usual), $D_{i}$ is an Abelian surface and $D_{0}$ is isogenous to $E \times E$; in this case we set: $B_{i} = [E^{3}/\nu_{i}(E^{2})] \times E$. By our assumptions there exists a one-dimensional $\gamma$-component $C_{i}$ in any $K_{3}(A_{i})$.

Now we use Lemma (6.7) and we have that, for all $i$ except a finite number, when we project $K_{3}(A_{i})$ into $K_{3}(B_{i})$, by the natural projection, we get that every one-dimensional family of $\gamma$-curves projects into a one-dimensional family of $\gamma$-curves.

So we get a three-dimensional family of $\gamma$-curves in $K_{3}(B_{i})$; they cannot cover all $K_{3}(B_{i})$ by Theorem (3.2) (recall that $B_{i}$ is isogenous to $E^{2}$); they cannot cover a three-fold, otherwise this three-fold would be filled by $\gamma$-surfaces and this is not possible by (1.2)(b); so the only possibility is the following: they all project in a fixed surface $S$ in $K_{3}(B_{i})$, which is a $\gamma$-component.

Note that, by Lemma (6.8), we can suppose that $S$ project into a fixed surface $S_{\sim}$ when we project $K_{3}(B_{i}) = K_{3}([E^{3}/\nu_{i}(E^{2})] \times E)$ into $K_{3}(E)$ by the natural projection on the last factor, hence $S_{\sim}$ is $K_{3}(E)$.

Now we choose $D_{i} = E_{\sigma} \times E_{s}$ ($\sigma$, $s$ belonging to the moduli space of elliptic curves) and generic embeddings $\nu_{i}$ in infinitely many different ways; for any choice, by using all the previous arguments, we get

—a $\gamma$-curve $C_{\sigma,s}$ in any $K_{3}(E_{\sigma} \times E_{s} \times E \times E)$,
—a surface $S_{i}$ in $K_{3}(E_{\sigma} \times E \times E)$, ($S_{i}$ covered by $\gamma$-curves),
—a fixed surface $S$ in $K_{3}(E \times E)$ into which all $S_{i}$ project,
—a fixed surface $S_{\sim}$ in $K_{3}(E)$ into which $S$ projects,

(we always use natural projections).

We want to prove that this is a contradiction to Lemma (6.6). These facts create a situation which is very similar to a $\Delta$-situation, (see §6): actually, in this case, we have only a one-dimensional family of Abelian varieties: $A_{s} \times A_{s} = (E_{s} \times E \times E) \times (E_{s} \times E \times E)$ and a surface in every $A_{s} \times A_{s}$ which is the pullback, via $\pi$, of the surface $S_{i}$ contained in $K_{3}(E_{s} \times E \times E)$. So we have a fibration defined only over an open set $\Delta_{s}' \subset H_{s}$ and the surfaces $\{S_{s}\}$ are the fibres over $\Delta_{s}'$. However, by looking at the proof of (6.6), it is obvious that it is true even in this case.
But, in our case, we also have that, if we choose $A$ isogenous to $E \times E \times E$, there are infinitely many embeddings $\mu_{pq}: E \rightarrow E^2$ $(\forall x \in E \quad \mu_{pq}(x) = (px, qx), \quad p, q \text{ coprime integers, the } \mu_{pq} \text{ are induced by the } \nu_1)$ such that, when we choose a family $\{A_s \times A_s\}, \ s$ belonging to a suitable open set $\Delta_{pq} \subset \mathcal{H}_1$, (depending on $\Delta'_1$), such that $\forall s \in \Delta_{pq}, \ A_s$ is isogenous to $E_s \times \mu_{pq}(E) \times E$ (as usual $A_0$ is isogenous to $[E^2/\mu_{pq}(E)] \times \mu_{pq}(E) \times E$, isogenous to $A$), then

- $K_3(A_s)$ contains a surface $S_s$ for any $s$,
- $S_0$ projects into a surface $S'$ in $K_3(E)$ by the natural projection on the last factor, (hence condition (*) is satisfied),
- all surfaces $S_s$ project into a fixed surface $S$ in $K_3[\mu_{pq}(E) \times E]$.

This is a contradiction to Lemma (6.6)!

Now we assume that $\dim(A) = n \geq 5$ and we proceed by induction on $n$. Suppose that for any $n$-dimensional Abelian variety $A$, $S^3(A)$, and therefore $K_3(A)$ contains a $\gamma$-curve; then it is true for those Abelian $n$-folds which are isogenous to $E \times B$ where $E$ is a generic elliptic curve and $B$ is a generic Abelian $(n - 1)$-fold. It is easy to see that, by choosing a suitable isogeny, we also get a $\gamma$-curve in $K_3(B)$, and this is a contradiction to our induction hypothesis. $\square$

**References**


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