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# ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF MÖBIUS EQUATIONS

XINGWANG XU

ABSTRACT. A generalization of the Schwarzian derivative to conformal mappings of Riemannian manifolds has naturally introduced the corresponding overdetermined differential equation which we call the Möbius equation. We are interested in study of the existence and uniqueness of the solution of the Möbius equation. Among other things, we show that, for a compact manifold, if Ricci curvature is nonpositive, for a complete noncompact manifold, if the scalar curvature is a positive constant, then the differential equation has only constant solutions. We also study the nonhomogeneous equation in an $n$-dimensional Euclidean space.

## 1. INTRODUCTION

Let $(M, g)$ be a Riemannian $n$-manifold with Riemannian metric $g$. Let $\nabla$ denote the Riemannian connection for $g$. For a smooth function $\phi: M \to \mathbf{R}$, we follow Osgood and Stowe [11] and define

$$B(\phi) = B_g(\phi) := \operatorname{Hess}(\phi) - d\phi \otimes d\phi - \frac{1}{n}\{\Delta\phi - \|\operatorname{grad}\phi\|^2\}g, \tag{1}$$

where $\operatorname{Hess}(\phi)$ is the Hessian of $\phi$. Thus, for any pair of vector fields $X, Y$ on $M$,

$$B(\phi)(X, Y) = X(Y\phi) - (\nabla_X Y)\phi - (X\phi)Y\phi - \frac{1}{n}\{\Delta\phi - \|\operatorname{grad}\phi\|^2\}\langle X, Y\rangle.$$

The operator $B(\phi)$ is a symmetric $(0, 2)$-tensor field and the final term has been chosen so that the trace, with respect to $g$, vanishes. We are interested in studying the differential equation $B(\phi) = h$ for a given symmetric trace-free $(0, 2)$ tensor field on a general Riemannian manifold. We will call an equation $B(\phi) = h$ a Möbius equation. This differential equation has its roots in differential geometry. Assume $g^*$ arises as the pullback of a Riemannian metric by a conformal transformation $f$ of $(M, g)$ onto $(M', g')$ and let $\phi = \log\|df\|$. Then $g^* = e^{2\phi}g$, and we define the *Schwarzian tensor* of $f$ to be

$$S(f) = S_g(f) := B_g(\phi).$$

If $f: (M, g) \to (M', g')$ is a conformal transformation with $S(f) = 0$, then we say that $f$ is a *Möbius transformation*. When $(M, g) = (M', g')$, the

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Möbius transformations form a group, denoted $\mathscr{M}\text{öb}(M)$, which is unchanged by a Möbius change of metric. The *homothety* group, $\text{Hty}(M)$, is the group of those conformal transformations $f$ of $(M, g)$ such that $\phi = \log\|df\|$ is constant. $\text{Hty}(M)$ is a subgroup of the conformal group $\text{Conf}(M)$ [11].

Observe from the above formula that if $f$ is an analytic function, $f'(z) \neq 0$, then $S(f) = 0$ if and only if $\mathbf{S}(f) = 0$ if and only if $f$ is a Möbius transformation in the usual sense. And it is well known that all of the Möbius transformations of $S^2$ are conformal mappings of $S^2$ into itself. By Theorem 3.2 [11] and the well-known Liouville theorem, it is also true for higher dimensional spheres and Euclidean spaces.

Note also that one of the classic theorems about Schwarzian derivative says that, for suitable transformation function $h$, $S(f) = h$ has a solution unique up to Möbius transformation on $S^2$. It is easy to see that for manifolds of dimension two, the differential system $B(\phi) = h$ is a determined system, if the dimension of the manifold is bigger than two, this system is overdetermined and as a differential equation, it does not have fixed type, so it is very difficult to handle. Our first step to deal with this problem is try to classify the solutions of $B(\phi) = 0$.

In the special case, when $h = 0$, $\phi$ is induced by conformal transformation, classifying the solutions of $B(\phi) = 0$ gives the classification of Möbius transformation on any complete Riemannian manifold. That is, by Osgood and Stowe's work, the existence of a nonconstant solution to the Möbius equation $B(\phi) = 0$ (actually, to a related linear equation) is directly tied to warped product structures. Then we are able to classify the complete, connected manifolds admitting a nonhomothetic Möbius metric and to obtain a list of the complete Einstein manifolds which admit a nonhomothetic conformal Einstein metric. In fact, Theorem 3.2 and Corollary 4.3 of [11] show us that if $M$ is Einstein manifold with $\dim M > 2$, then $\mathscr{M}\text{öb}(M) = \text{Conf}(M)$. Also if $M$ is a complete manifold and $\dim M > 1$, then one of the following is valid [11, Theorem 6.1 or §2],

1. $\mathscr{M}\text{öb}(M) = \text{Hty}(M)$,
2. $\mathscr{M}\text{öb}(M)$ contains $\text{Hty}(M)$ as a subgroup of index two, or
3. $M$ is isometric to a sphere.

The only instances of 2. are certain warped products $R \times_{f(t)} M$ or $\text{Sphere}(f)$ in which $f''/f$ is not constant.

All those results strongly suggest that $B(\phi) = 0$ only has constant solutions. It would be very interesting to see in fact that this indeed happens. In order to see this, first we look at the compact case. We have the following:

**Theorem 1.1.** *Let $M$ be an oriented compact Riemannian $n$-manifold without boundary with $n > 2$. If the Ricci curvature of $M$ is nonpositive, or the scalar curvature of $M$ is constant, then all the solutions of $B(\phi) = 0$ are constants unless $(M, g_0)$ is standard sphere.*

As an application of Theorem 1.1, we prove the following

**Theorem 1.2.** *Any compact, oriented $n$-dimensional Riemannian manifold without boundary with $n > 2$ has a complete Riemannian metric $g$ such that $\mathscr{M}\text{öb}(M, g) = \text{Hty}(M, g)$.*

For noncompact complete Riemannian manifolds, life is more complicated.

Both the homogeneous and the nonhomogeneous equations are hard to handle. First, we restrict our attention to warped product spaces as is suggested by Osgood and Stowe's work. We obtain that if $M = R \times_f M_0$ and $M$ has positive constant scalar curvature, then $B(\phi) = 0$ has only the constant solution. Then we prove the following.

**Theorem 1.3.** *Let $M$ be a complete noncompact Riemannian manifold without boundary. If the scalar curvature of $M$ is some positive constant, then $B(\phi) = 0$ has only the constant solutions.*

Finally we consider the nonhomogeneous case. If $M$ is an arbitrary complete Riemannian manifold, we do not know what we should expect. So, we confine our attention to $R^n$. We know that $B(\phi) = 0$ has a lot of nonconstant solutions and, for a suitable tensor field $h$ which is symmetric and trace-free, $B(\phi) = h$ has no solutions.

The paper is organized as follows. The next section will be used to simply recall some facts about the Möbius equation from the paper of Osgood and Stowe. Then in §§3 and 4, we are going to prove our first main result. As an application of this and Guo and Yau's result, we proved that any compact oriented three-dimensional manifold has a complete Riemannian metric such that the Möbius group of $M$ with respect to this metric is the same as the homothety group of $M$ with respect to the same metric. In §4, we will use conformal geometric methods to show the rest of our first main result. As a special case we proved that $f$ is a Möbius transformation on $M$ which has constant scalar curvature, the conformal factor $u$ is a constant function. Finally, combining this result and one of Aubin's theorems gives the proof of our second theorem.

The warped product space case will be treated in §5. In the final section we will cover the nonhomogeneous case on $R^n$. For the simplest possible $h$, we give a necessary and sufficient condition for $B(\phi) = h$ to have a solution.

Part of this paper is taken from my Ph.D. thesis at the University of Connecticut. I would like to take this opportunity to express my special thanks to my thesis advisor, William Abikoff.

## 2. The generalized Schwarzian derivatives and some of its properties

Let $M$ be a Riemannian manifold of dimension $\geq 2$ with metric $g = \langle\ ,\ \rangle$ and let $\nabla$ denote the Riemannian connection for $g$. For a smooth function $\varphi\colon M \to \mathbf{R}$, we define

$$B(\varphi) = B_g(\varphi) := \mathrm{Hess}(\varphi) - d\varphi \otimes d\varphi - \frac{1}{n}(\Delta\varphi - |\nabla\varphi|^2)g \tag{2}$$

where $\mathrm{Hess}(\varphi)$ is the Hessian of $\varphi$. Thus for vector fields $X$, $Y$ on $M$,

$$B(\varphi)(X, Y) = X(Y\varphi) - (\nabla_X Y)\varphi - x\varphi Y\varphi - \frac{1}{n}(\Delta\varphi - |\nabla\varphi|^2)\langle X, Y\rangle.$$

The operator $B(\varphi)$ is a symmetric $(0, 2)$-tensor, and the final term has been chosen to make the trace vanish with respect to $g$. We will always associate $B(\varphi)$ to a conformal metric $\hat{g} = e^{2\varphi}g$ and we call it the *Schwarzian tensor*. In particular, if $\hat{g}$ arises as the pull back of a Riemannian metric by a conformal

diffeomorphism, or immersion, $f\colon (M, g) \to (M', g')$ so that $\varphi = \log|\nabla f|$, we define the Schwarzian tensor to be $S(f) = S_g(f) = B_g(\varphi)$.

At first observation, we have

**Lemma 2.1.** *If $f : D \to D$ is a holomorphic function where $D$ is a domain in $\mathbf{S}^2$ and $f' \neq 0$, then*

$$S(f) = \begin{pmatrix} \operatorname{Re}\mathbf{S}(f) & -\operatorname{Im}\mathbf{S}(f) \\ -\operatorname{Im}\mathbf{S}(f) & -\operatorname{Re}\mathbf{S}(f) \end{pmatrix}.$$

Then we obtain

**Proposition 2.2.** *If $f$ is a holomorphic function, then $S(f) = 0$ and $f'(z) \neq 0$ if and only if $f$ is a Möbius transformation in the usual sense.*

After we carefully study the properties of the Möbius equation, we have

**Proposition 2.3.** *Let $\varphi, \sigma : M \to \mathbf{R}$ be smooth functions on $(M, g)$. Then*

$$B_g(\varphi + \sigma) = B_g(\varphi) + B_{\hat{g}}(\sigma)$$

*where $\hat{g} = e^{2\varphi} g$.*

**Definition 2.4.** *Möbius transformations* are conformal diffeomorphisms with vanishing Schwarzian derivative.

**Proposition 2.5.** *Composites and inverses of Möbius transformations are Möbius transformations. The Möbius transformations of a Riemannian manifold $(M, g)$ into itself form a group $\mathscr{M}\mathrm{ö}\mathrm{b}(M) = \mathscr{M}\mathrm{ö}\mathrm{b}(M, g)$ containing the homothety group $\mathrm{Hty}(M)$ and contained in the group $\mathrm{Conf}(M)$ of conformal transformations.*

**Theorem 2.6.** *Let $(M, g)$ be a Riemannian manifold of dimension $n \geq 3$ and let $g' = e^{2\varphi} g$ be a metric conformally related to $g$. Let $R = kI_g + R_\beta + C$ and $R' = k' I_{g'} + R_{\beta'} + C'$ be the decomposition of the Riemannian curvature tensors of $g$ and $g'$, respectively. Then*

(a) $k' = e^{2\varphi}(k - \frac{2}{n}\Delta\varphi - \frac{n-2}{2}|d\varphi|^2)$;

(b) $\beta' = \beta - B(\varphi)$;

(c) $C' = C$

*where $\Delta\varphi$, $|d\varphi|^2$, and $B(\varphi)$ are computed with respect to $g$.*

*Proof.* See [11]. □

**Corollary 2.7.** *Let $(M, g)$ be a connected Einstein manifold of dimension $n \geq 3$ and $g' = e^{2\varphi} g$. Then $(M, g')$ is Einstein if and only if $B(\varphi) = 0$.*

Now let us introduce *spherical warped products.* Let $f\colon \mathbf{R} \to \mathbf{R}$ be such that

(I) $f$ is odd and of period $2T > 0$,

(II) $f(t) > 0$ when $0 < t < T$,

(III) $f'(0) = -f'(T) = 1$.

Let $g$ be the Riemannian metric on $M = S^n$ which in the parametrization

$$(t, \theta) \mapsto (\cos(\pi t/T), \sin(\pi t/T) \cdot \theta), \qquad t \in R, \ \theta \in S^{n-1},$$

is given by $ds^2 = dt^2 + f(t)^2 |d\theta|^2$. This metric $g$ is smooth through the singularities $\pm(1, 0, \dots, 0)$ of the parametrization. This spherical warped product $(M, g) = \mathrm{Sphere}(f)$ has no new local features; near $\pm(1, 0, \dots, 0)$ it is a polar warped product, and elsewhere it is an ordinary one.

**Theorem 2.8.** *Let $M$ be a complete connected Riemannian manifold of dimension $n \geq 2$. Then either*

(i) $\mathrm{Hty}(M) = \mathscr{M}\mathrm{ob}(M)$; *or*
(ii) $\mathrm{Hty}(M)$ *has index two in* $\mathscr{M}\mathrm{ob}(M)$; *or*
(iii) $M$ *is a standard sphere.*

*The only instances of* (ii) *are certain warped products* $R \times_f P$ *or* $\mathrm{Sphere}(f)$ *in which* $f''/f$ *is not constant.*

*Proof.* See [11, Theorem 6.1]. □

One of the examples of the second kind of manifolds given in the Osgood-Stowe's paper is the following: Let $M = R \times_f M_0$ with a complete Riemannian metric $ds^2 = v^{-2}(dt^2 + f^2(t)\,ds_0^2)$, where $f(t) = 1/(1+t^2)$ and $v(t) = 1 + (1/2)\int_0^t f(s)\,ds$ is positive. Then the reflection about zero of the real line $R$ is a nonhomothetic Möbius transformation with respect to the metric $ds^2$.

## 3. Compact Riemannian manifolds with nonpositive Ricci curvature

We consider the Möbius equation

$$B(\varphi) = h \tag{3}$$

where $h$ is a given symmetric trace-free $(0,2)$ tensor field, and $\varphi$ is an unknown smooth real-valued function.

If we set $u = e^{-\varphi}$, then (3) reduces to the following second order linear equation (the linear equation associated to the Schwarzian derivative)

$$\mathrm{Hess}(u) - \frac{1}{n}\Delta u g = -uh. \tag{4}$$

Then we have the following

**Theorem 3.1.** *For $n \geq 3$, let $M$ be a compact, oriented, connected Riemannian n-manifold without boundary and assume that the Ricci curvature is nonpositive. Then $u > 0$ is a solution of the linearized Schwarzian equation* (4) *with $h = 0$ if and only if $u$ is a positive constant.*

*Proof.* By an easy computation, we have

$$\frac{1}{2}\Delta|\nabla u|^2 = \sum_{i,j} u_{ij}^2 + (\Delta u)_i u_i + R_{ij}u_i u_j. \tag{5}$$

Using the equation (4), we get that $\sum_{i,j} u_{ij}^2 = (\Delta u)^2/n$. Therefore the previous equation can be written as

$$\frac{1}{2}\Delta|\nabla u|^2 = \frac{1}{n}(\Delta u)^2 + (\Delta u)_i u_i + R_{ij}u_i u_j. \tag{6}$$

Integrating over $M$, notice that $M$ has no boundary, gives that

$$0 = \frac{1}{n}\int_M (\Delta u)^2 + \int_M (\Delta u)_i u_i + \int_M R_{ij}u_i u_j. \tag{7}$$

Now we do integration by parts on the second term. It will give the following

$$0 = \frac{1-n}{n}\int_M (\Delta u)^2 + \int_M R_{ij}u_i u_j. \tag{8}$$

Then since $n \geq 2$ and $R_{ij}u_iu_j \leq 0$, we have that $\Delta u = 0$. Now maximum principle gives the conclusion of the theorem. □

Let us return to the definition of the Schwarzian derivative for a moment. Suppose $f$ is a conformal self-mapping of a Riemannian manifold $M$, then

$$f^*g = e^{2\phi}g.$$

So

$$S(f) = B(\phi) = \mathrm{Hess}(\phi) - d\phi \otimes d\phi - \frac{1}{n}(\Delta\phi - |d\phi|^2)g. \tag{9}$$

Further if $S(f) = 0$, then we called $f$ a Möbius transformation, therefore, from the definition of Möbius transformation and the previous lemma, we have

**Corollary 3.2.** *Let $M$ be a compact connected Riemannian manifold with nonpositive Ricci curvature and having no boundary, then the Möbius group of $M$ and the Homothety group are the same.*

*Proof.* Suppose $f\colon (M, g) \to (M, g)$ is a self-conformal mapping. We write $f_*g = e^{2\phi}g$ and we further assume that $f$ is a Möbius transformation. By the definition of Möbius transformation, $B(\phi) = 0$. If we set $u = e^{-\phi}$, then $u$ satisfies

$$\mathrm{Hess}(u) = \frac{1}{n}(\Delta u)g.$$

By the previous theorem, we conclude that $u$ is a positive constant or $f$ is a homothety. Therefore $\mathscr{M}\mathrm{öb}(M) \subset \mathrm{Hty}(M)$. But by definition, the homothety group is a subgroup of the Möbius group. □

**Corollary 3.3.** *Let $T^n$ be the $n$-dimensional torus with the flat Riemannian metric. Then*

$$\mathscr{M}\mathrm{öb}(T^n) = \mathrm{Hty}(T^n).$$

**Corollary 3.4.** *Every 3-dimensional compact Riemannian manifold without boundary has a Riemannian metric $g$ such that $\mathscr{M}\mathrm{öb}(M, g) = \mathrm{Hty}(M, g)$.*

*Proof.* By the theorem of Gao and Yau [4], for every compact 3-manifold, there exists a Riemannian metric with nonpositive Ricci curvature. The corollary then follows from the previous theorem. □

*Remarks.* 1. In the next section, we will prove that this corollary can be generalized to higher dimensional manifolds.

2. Later in this paper we will see that, in all these arguments, compactness is a necessary assumption.

## 4. Compact Riemannian manifolds with constant scalar curvature

We take as definition of a conformal vector field on a Riemannian manifold $(M, g)$ that it is a vector field $X$ whose flow $(\xi_t)_{t\in\mathbf{R}}$ is made up of conformal transformations. Conformal vector fields form a Lie algebra $\mathscr{C}(M, G)$ where $G = [g]$ is the conformal class of $g$. For $M$ compact, $\mathscr{C}(M, G)$ is the Lie algebra of $\mathbf{C}(M, G) := \mathrm{Conf}(M, g)$ because the conformal group does not depend on the choice of metric.

Fix a background metric $g$ in the conformal class. Then, for any $\varphi \in \mathrm{Conf}(M, g)$, set

$$\begin{aligned} \varphi^*(g) &= e^{2u} g, && \text{for } n = 2; \\ \varphi^*(g) &= u^{4/(n-2)} g, && \text{for } n \geq 3. \end{aligned}$$

Using these two formulas, we define the map

$$\begin{aligned} \alpha_n : \mathrm{Conf}(M, g) &\mapsto \mathbf{C}^\infty(M) \\ \varphi &\mapsto \alpha_2(\varphi) = u, \quad \text{for } n = 2; \\ \varphi &\mapsto \alpha_n(\varphi) = u, \quad \text{for } n \geq 3. \end{aligned}$$

Then we have a natural action: $\mathrm{Conf}(M, g)$ acting on $G$ by

$$\begin{aligned} (\varphi, g') &= (\varphi, \exp(2v)g) \to \varphi^*(\exp(2v)g) \\ &= \exp[2(v \circ \varphi + \alpha_2(\varphi))]g, \quad \text{for } n = 2; \\ (\varphi, g') &= (\varphi, v^{4/(n-2)} g) \to \varphi^*(v^{4/(n-2)} g) \\ &= (v \circ \varphi \alpha_n(\varphi))^{4/(n-2)} g, \quad \text{for } n \geq 3. \end{aligned}$$

If we identify $g'$ with $v$, it will be convenient to denote this action by $\Box_n$, that is: $v \Box_2 \varphi = v \circ \varphi + \alpha_2(\varphi)$ and $v \Box_n \varphi = (v \circ \varphi)\alpha_n(\varphi)$ if $n \geq 3$.

The linearized action is now given by the following lemma.

**Lemma 4.1.** *If $(\xi_t)_{t \in \mathbf{R}}$ is the flow of a conformal vector field $X$ on $(M, g)$, then, for any smooth function $v$ ($v > 0$ if $n \geq 3$),*

$$\begin{aligned} \frac{d}{dt}(v \Box_2 \xi_t)|_{t=0} &= X \cdot v + \frac{1}{2} \operatorname{div}_g X, \\ \frac{d}{dt}(v \Box_n \xi_t)|_{t=0} &= X \cdot v + \frac{n-2}{2n}(\operatorname{div}_g X) v \quad \text{if } n \geq 3. \end{aligned} \tag{10}$$

*Proof.* Suppose $\varphi$ is a conformal transformation. First note that $\varphi^* g = (\det(\varphi))^2 g$ since $\varphi^* g(X, Y) = g(\varphi_* X, \varphi_* Y) = (\det(\varphi_*))^2 g(X, Y)$. Then we know, $\alpha_2(\varphi) = \log(\det(\varphi_*))^{1/2}$ and, if $n \geq 3$, $\alpha_n(\varphi) = (\det(\varphi_*))^{(n-2)/2n}$. Recall that if $A = A(t)$ is a diagonal matrix,

$$\frac{d}{dt}(\det(A(t)))|_{t=0} = \det A(0) \cdot \operatorname{trace}\left(A(0)^{-1} \frac{d}{dt}(A(t))|_{t=0}\right)$$

and the statement remains true if $A$ is any diagonalizable matrix. The map on the tangent bundle induced by any conformal transformation is diagonalizable in normal coordinates. Therefore if $(\xi_t)_{t \in \mathbf{R}}$ is the flow of a conformal vector field,

$$\begin{aligned} \frac{d}{dt}(\det((\xi_t)_*))|_{t=0} &= \det((\xi_t)_*)|_{t=0} \cdot \operatorname{trace}\left[((\xi_t)_*)^{-1} \frac{d}{dt}((\xi_t)_*)\right]\Bigg|_{t=0} \\ &= \operatorname{div}_g(X). \end{aligned} \tag{11}$$

Then we have

$$\frac{d}{dt}(v \Box_2 \xi_t)|_{t=0} = \frac{d}{dt}(v \circ \xi_t + \alpha_2(\xi_t))|_{t=0} = X \cdot v + \frac{1}{2} \operatorname{div}_g(X).$$

Using the product rule of differentiation and (11), we also obtain the following formula

$$\begin{aligned}\frac{d}{dt}(v\Box_n\xi_t)_{t=0} &= \frac{d}{dt}((v\circ\xi_t)\alpha_n(\xi_t))|_{t=0}\\ &= \frac{d}{dt}(v\circ\xi_t)\cdot\alpha_n(\xi_t)|_{t=0} + (v\circ\xi_t)\cdot\frac{d}{dt}(\alpha_n(\xi_t))|_{t=0}\\ &= X\cdot v + \frac{n-2}{2n}\operatorname{div}_g(X)v\end{aligned}$$

for $n \geq 3$. □

Let $g'$ and $g$ be two conformally related metrics. For the questions involving scalar curvature that we are going to consider, it is convenient (and classical) to write the conformal factor as before

$$\rho = \begin{cases} e^u & \text{for } n = 2, \\ u^{\frac{2}{n-2}} \quad \text{with } u > 0 & \text{for } n \geq 3. \end{cases}$$

If $s_g$ denotes the scalar curvature of the metric $g$, one then has the classical formulas (see [8] for details)

$$s_{g'} = \begin{cases} (-2\Delta_g u + s_g)e^{-2u} & \text{for } n = 2, \\ (-4\frac{n-1}{n-2}\Delta_g u + s_g u)u^{-\frac{n+2}{n-2}} & \text{for } n \geq 3. \end{cases}$$

We are thus led to introduce the family of quasilinear differential operators $F_n$ on the space $\mathbf{C}^\infty(M)$ of smooth functions defined as

$$F_2(u) = e^{-2u}(-2\Delta_g u + s_g),$$
$$F_n(u) = u^{-\frac{n+2}{n-2}}\left(-4\frac{n-1}{n-2}\Delta_g u + s_g u\right) \quad \text{for } n \geq 3.$$

For any function $U$ viewed as sitting in the tangent space $T_u(\mathbf{C}^\infty(M))$, the directional derivative $U\cdot F_n$ of $F_n$ in the direction $U$ is given by

$$(U\cdot F_2)(u) = 2e^{-2u}(-\Delta_g U - (-2\Delta_g u + s_g)U), \tag{12}$$

(13)

$$(U\cdot F_n)(u) = 4\frac{n-1}{n-2}u^{-\frac{n+2}{n-2}}\left(-\Delta_g U - \left(-\frac{n+2}{n-2}u^{-1}\Delta_g u + \frac{1}{n-1}s_g\right)U\right), \quad (n \geq 3).$$

**Lemma 4.2.** *The divergence of a conformal vector field $X$ satisfies the identity*

$$-\Delta_g(\operatorname{div}_g X) = \frac{1}{n-1}s_g \operatorname{div}_g X + \frac{n}{2(n-1)}X\cdot s_g. \tag{14}$$

*Proof*. Let $X$ be a conformal vector field and let $(\xi_t)_{t\in R}$ be a one-parameter conformal group generated by $X$. Then

$$F_n(1\Box_n\xi_t) = s_{\xi^*(g)} = F_n(1)\circ\xi_t = s_g\circ\xi_t$$

and $s_g = F_n(1)$.

Therefore, we have

$$\frac{\partial}{\partial t}(F_n(1\Box_n\xi_t))|_{t=0} = \left(\frac{\partial}{\partial t}(1\Box_n\xi_t)|_{t=0}\cdot F_n\right)(1) = X\cdot s_g \tag{15}$$

and by equation (12) and Lemma (4), we have

$$\frac{d}{dt}(1\Box_n\xi_t)|_{t=0} = \frac{n-2}{2n}\operatorname{div}_g X.$$

We know that (13)

$$\begin{aligned} U\cdot F_n(u) &= \frac{d}{dt}\bigg|_{t=0}(u+tU)^{-\frac{n+2}{n-2}}\left(-4\frac{n-1}{n-2}\Delta_g(u+tU) = s_g(u+tU)\right) \\ &= -\frac{n+2}{n-2}u^{-\frac{n+2}{n-2}-1}U\left(-4\frac{n-1}{n-2}\Delta_g u + s_g u\right) \\ &\quad + u^{-\frac{n+2}{n-2}}\left(-4\frac{n-1}{n-2}\Delta_g U + s_g U\right). \quad \Box \end{aligned}$$

Therefore we have

$$\begin{aligned} X\cdot s_g &= \frac{d}{dt}\bigg|_{t=0}(1\Box_n\xi_t)\cdot F_n(1) \\ &= -\frac{n+2}{n-2}s_g\left(\frac{d}{dt}\bigg|_{t=0}(1\Box_n\xi_t)\right) \\ &\quad -4\frac{n-1}{n-2}\Delta_g\left(\frac{d}{dt}\bigg|_{t=0}(1\Box_n\xi_t)\right) + s_g\left(\frac{d}{dt}\bigg|_{t=0}(1\Box_n\xi_t)\right) \\ &= -\frac{n+2}{n-2}s_g\frac{n-2}{2n}\operatorname{div}_g X - 4\frac{n-1}{2n}\Delta_g(\operatorname{div}_g X) + s_g\left(\frac{n-2}{2n}\operatorname{div}_g X\right) \\ &= -\frac{2}{n}s_g\operatorname{div}_g X - \frac{2(n-1)}{n}\Delta_g(\operatorname{div}_g X). \end{aligned}$$

So

$$-\frac{2(n-1)}{n}\Delta_g(\operatorname{div}_g X) = X\cdot s_g + \frac{2}{n}s_g\operatorname{div}_g X.$$

We have obtained

$$-\Delta_g(\operatorname{div}_g X) = \frac{n}{2(n-1)}X\cdot s_g + \frac{1}{n-1}s_g\operatorname{div}_g X$$

which is the desired result. □

*Remark.* 1. This formula is well known. This type of proof is suggested in [13], but not proved there. Other proofs may be found in [7, 5].

2. This formula is also useful in studying scalar curvature problems (see [13] and the references there).

In order to complete the proof of our next result, we need the following theorem of Obata:

**Theorem 4.3.** *In order for a complete Riemannian manifold of dimension* $n \geq 2$ *to admit a nonconstant function* $\varphi$ *with* $\operatorname{Hess}(\varphi) = -c^2\varphi g$, *it is necessary and sufficient that the manifold be isometric with a sphere* $S^n(c)$ *of radius* $1/c$ *in the* $(n+1)$*-Euclidean space.*

*Proof.* See [10, Theorem A]. □

Now we are in a position to prove the following

**Theorem 4.4.** *Let $M$ be a compact connected oriented $n$-dimensional Riemannian manifold without boundary. If the scalar curvature of the Riemannian metric $g$ is constant, then homogeneous Möbius equation has a nonconstant positive solution if and only if $M$ is isometric to some sphere.*

*Proof*. Suppose $u > 0$ satisfies

$$\mathrm{Hess}(u) = \frac{\Delta_g u}{n} g. \tag{16}$$

Since $\nabla u$ is a conformal vector field on $M$, from (4), we have $\mathrm{div}(\nabla u)$ satisfies

$$-\Delta(\mathrm{div}\,\nabla u) = \frac{s_g}{n-1}(\mathrm{div}\,\nabla u) + \frac{n}{2(n-1)}\nabla u \cdot s_g = \frac{s_g}{n-1}\,\mathrm{div}\,\nabla u.$$

As we know that $\mathrm{div}\,\nabla u = -\Delta u$, the previous equation can be written as

$$\Delta\left(\Delta u + \frac{s_g}{n-1}u\right) = 0. \tag{17}$$

By the maximum principal, we have

$$\Delta u = -\frac{s_g}{n-1}u + C \tag{18}$$

for some constant $C$. Take the Hessian of both sides in the previous equation and use (16), to obtain

$$\mathrm{Hess}(\Delta u) = -\frac{s_g}{n-1}\,\mathrm{Hess}(u) = -\frac{s_g \Delta u}{n(n-1)} g. \tag{19}$$

*Case* I (if $\Delta u \neq 0$ and $s_g > 0$). In this case, by Obata's theorem (4.3), we have that $M$ is isometric to a sphere with radius $\sqrt{n(n-1)/s_g}$.

*Case* II (if $\Delta u \neq 0$ and $s_g = 0$ everywhere). Since $s_g = 0$, by equation (18), $\Delta u = 0 + C$. $M$ is compact and $u$ is a smooth function, therefore $C$ must be zero, that is, $\Delta u = 0$. Using Bochner's lemma, we know that $u$ has to be a constant, which contradicts our assumption. So this case cannot happen.

*Case* III (if $\Delta u \neq 0$ and $s_g < 0$). Since the function $u$ is a smooth function and $M$ is compact, $u$ must assume its maximum value at a point $p \in M$. At $p$, we have that $\Delta u \leq 0$. Using equation (18),

$$\frac{s_g}{n-1}u(p) \geq C.$$

But $s_g < 0$, so we have

$$u(p) \leq \frac{C(n-1)}{s_g}. \tag{20}$$

The same argument for the minimum value of the function $u$ gives us

$$u(q) \geq \frac{C(n-1)}{s_g}. \tag{21}$$

Compare equations (20) and (21), to see that $u$ must be a constant. This shows that this case cannot happen, either.

*Case* IV (if $\Delta u = 0$). It is easy to see that $u$ is a constant in this case. □

**Corollary 4.5.** *If $f''/f$ is not constant and the function $f(t)$ is symmetric with respect to $T/2$ where $2T$ is period of $f$, then the scalar curvature of* Sphere($f$) *cannot be constant.*

*Proof.* Osgood and Stowe [11] have shown that in this special case, we have $\mathscr{M}\text{öb}(\text{Sphere}(f)) \neq \text{Hty}(\text{Sphere}(f))$. □

**Corollary 4.6.** *Let $M$ be a compact connected oriented $n$-dimensional Riemannian manifold without boundary, if the scalar curvature of the Riemannian metric $g$ is constant, then either $\mathscr{M}\text{öb}(M) = \text{Hty}(M)$ or $M$ is isometric to some sphere.*

**Theorem 4.7.** *Any compact, oriented $n$-dimensional Riemannian manifold without boundary and $n > 2$ admits a complete Riemannian metric $g$ such that $\mathscr{M}\text{öb}(M, g) = \text{Hty}(M, g)$.*

*Proof.* This follows from Theorem 6.19 of [1] and the previous theorem. □

## 5. Complete Riemannian manifolds with constant scalar curvature

In this section, we really want to generalize the previous result to the complete noncompact case. First of all, we have the following easy example.

**Example.** On $R^n$, the homogeneous Möbius equation has a lot of nonconstant solutions. In fact, if $f_{a,b}(x) = a+b|x|^2 > 0$, $a > 0$, $b > 0$, then it is easy to see that $\Delta f_{a,b}(x) = 2bn$ and $(f_{a,b})_{ij} = 2b\delta_{ij}$. Therefore, $\text{Hess}(f_{a,b}) - \Delta f_{a,b}/nE = 0$ where $E$ is identity matrix. Of course, $f_{a,b}$ is not constant.

Therefore, we cannot expect similar results for the complete case without further assumptions. Since we work on complete Riemannian manifolds with constant scalar curvature, one assumption we can put on is nonzero scalar curvature. Euclidean space does not satisfy this assumption. But that is still not enough because we have

**Example.** On the standard hyperbolic space $H^n$, the Möbius equation has a nonconstant solution.

*Proof.* Here we take unit ball model with metric

$$ds^2 = \frac{1}{(1-|x|^2)^2} \sum_{i=1}^{n} dx_i^2$$

where $|x|^2 < 1$. If we set $u = 1/(1-|x|^2)$, then it is easy computation to show $u$ satisfies the Möbius equation. □

Next we turn to the positive scalar curvature case. In that case we have the following, the main result of this section.

**Theorem 5.1.** *If $M$ is a complete, connected, noncompact Riemannian manifold with positive constant scalar curvature, then the Möbius equation has only constant solutions.*

The proof will consist of several lemmas. In particular, Lemmas 5.4 and 5.5 give the proof of Theorem 5.1. First we examine warped product spaces.

Let $(M_0, ds_0^2)$ be a Riemannian manifold and let $\mathbf{R} \times_f M_0$ have metric $ds^2 = dx_0^2 + f(x_0)\, ds_0^2$. Assume

$$ds_0^2 = g_{ij}\, dx^i\, dx^j, \qquad i, j = 1, 2, \dots, n,$$

is complete. $ds^2$ is complete if the function $f(x_0)$ is positive on the real line $R$ and we will assume that $ds^2$ is complete.

Let $(x_1, x_2, \dots, x_n)$ be a local coordinate on $M_0$, and $t = x_0$ a local coordinate on $\mathbf{R}$. Then

$$\hat{g}_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha = 0,\ \beta = 0, \\ 0 & \text{if } \alpha = 0 \text{ or } \beta = 0 \text{ but not both}, \\ f(t) g_{ij} & \text{if } \alpha = i > 0 \text{ and } \beta = j > 0. \end{cases}$$

Therefore

$$\hat{g}^{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha = 0,\ \beta = 0, \\ 0 & \text{if } \alpha = 0 \text{ or } \beta = 0, \\ f(t)^{-1} g^{ij} & \text{if } \alpha = i > 0 \text{ and } \beta = j > 0. \end{cases}$$

**Lemma 5.2.** *Let $M$ be $\mathbf{R} \times_{f(t)} M_0$ with metric as given above. Then the scalar curvature is*

$$R = \frac{(3n - n^2)}{4} \left[\frac{f'(t)}{f(t)}\right]^2 - n\frac{f''(t)}{f(t)} + \frac{R_0}{f(t)}. \tag{22}$$

*Proof.* This is a direct computation. We omit the details here. □

**Lemma 5.3.** *If $M = \mathbf{R} \times_{f(t)} M_0$ has constant scalar curvature, then $M_0$ also has constant scalar curvature.*

*Proof.* This is a direct consequence of the previous lemma. □

**Lemma 5.4.** *If $M$ is as in the previous lemma with positive constant scalar curvature and if the Möbius equation on $M$ has a nonconstant solution and $f(t) > 0$ everywhere on $R$, then $f(t)$ must be a constant.*

*Proof.* $M$ is $\mathbf{R} \times_{f(t)} M_0$ with metric as given above. Using the well-known formula

$$\Gamma_{ij}^k = 1/2 g^{kl} \left(\frac{\partial g_{lj}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right).$$

We have

$$\Gamma_{0i}^j = \frac{f'(t)}{2f(t)} \delta_i^j \quad \text{if } i, j \neq 0$$

and $\Gamma_{00}^i = 0$ for all $i$.

Now suppose $u > 0$ is a solution of the Möbius equation and $u$ is not a constant. That is, $u$ satisfies

$$\operatorname{Hess}(u) = \frac{\Delta u}{n+1} g. \tag{23}$$

By the above expression for the metric $g$, we have $u_{tt} = \Delta u/(n+1)$ and $u_{tx_i} = 0$. By the definition of the Laplacian, $\Delta u = u_{tt} + 1/f(t) g_0^{ij} \nabla_i \nabla_j u = u_{tt} + 1/f(t) \Delta_0 u$ where $\Delta_0$ is a Laplacian on $M_0$ with respect to the metric $g_0$. Therefore, we have

$$0 = u_{tx_i} = \frac{\partial^2 u}{\partial t \partial x_i} - \Gamma_{0i}^\beta \frac{\partial u}{\partial x_\beta}.$$

It follows that $\partial u/\partial t - [f'(t)/2f(t)]u$ is a function only depending on time $t$; call it $g(t)$. If we just consider $u$ as a function of $t$, then we can solve the first order ordinary differential equation. Specifically, we have $u = g_1(t) + g_2(t)v(x)$ where $g_1(t) = (f(t))^{1/2} \int g(t)/[f(t)]^{1/2}\,dt$ and $g_2(t) = [f(t)]^{1/2}$. Then, since

$$u_{tt} = \frac{\Delta u}{n+1},$$

we have

$$nu_{tt} = \frac{1}{f(t)}\Delta_0(u).$$

Therefore

$$n\left[\frac{\partial^2 g_1(t)}{(\partial t)^2}\right][f(t)]^{1/2} + n\left[\frac{\partial^2 [f(t)]^{1/2}}{(\partial t)^2}\right][f(t)]^{1/2}v(x) = \Delta_0 v(x). \tag{24}$$

*Case* A. If $(\partial^2[f(t)]^{1/2}/(\partial t)^2)[f(t)^{1/2}]$ is not a constant, then differentiating both sides of the above equation, we have $v(x)$ is a constant. Therefore $u$ only depends on time $t$. Then the equation (24) implies that $g_1(t) + Cg_2(t) = ct + d$. That is, $u(t) = ct + d$. And $u > 0$ implies that $c = 0$. So $u$ is a constant. It contradicts the assumption that $u$ is not a constant. So Case A cannot happen.

*Case* B. If $[\partial^2[f(t)]^{1/2}/(\partial t)^2](f(t))^{1/2}$ is a constant, then by simple computation, this says that

$$f''(t) - \frac{[f'(t)]^2}{2f(t)} = C. \tag{25}$$

Therefore the scalar curvature is

$$\begin{aligned} R &= \frac{3n-n^2}{4}\left[\frac{f'(t)}{f(t)}\right]^2 - n\frac{f''(t)}{f(t)} + \frac{R_0}{f(t)} \\ &= \frac{3n-n^2}{4}\left[\frac{f'(t)}{f(t)}\right]^2 - \frac{n}{f(t)}\left[\frac{[f'(t)]^2}{2f(t)} + C\right] + \frac{R_0}{f(t)} \\ &= \frac{3n-n^2-2n}{4}\left[\frac{f'(t)}{f(t)}\right]^2 + \frac{R_0 - Cn}{f(t)} \\ &= -\frac{n^2-n}{4}\left[\frac{f'(t)}{f(t)}\right]^2 + \frac{R_0 - Cn}{f(t)}. \end{aligned} \tag{26}$$

Since $R$ is a constant, differentiate both sides of the equation (26) with respect to time $t$ to obtain

$$\begin{aligned} 0 &= -\frac{n^2-n}{4}2\frac{f'(t)}{f(t)}\frac{f''(t)f(t) - [f'(t)]^2}{f^2(t)} - \frac{(R_0 - Cn)f'(t)}{f^2(t)} \\ &= \frac{f'(t)}{f(t)}\left[-\frac{n^2-n}{2}\frac{[(f'(t))^2/(2f(t)) + C]f(t) - [f'(t)]^2}{f^2(t)} - \frac{R_0 - Cn}{f(t)}\right] \\ &= -\frac{f'(t)}{f(t)}\left[\frac{n^2-n}{2}\frac{C}{f(t)} + \frac{R_0 - Cn}{f(t)} - \frac{n^2-n}{4}\frac{f'^2(t)}{f^2(t)}\right]. \end{aligned}$$

Since $f(t) > 0$, we have

$$f'(t)\left[\frac{(n^2-3n)C + 2R_0}{2f(t)} - \frac{n^2-n}{4}\frac{f'^2(t)}{f^2(t)}\right] = 0. \tag{27}$$

Hence by the continuity of $f(t)$, $f'(t)$, we have

*Case* 1. $f'(t) = 0$ everywhere on $R$; or

*Case* 2.

$$\frac{(n^2 - 3n)C + 2R_0}{2f(t)} - \frac{n^2 - n}{4}\left[\frac{f'(t)}{f(t)}\right]^2 = 0$$

on some open intervals of $R$.

But for Case 2, using equation (26), it is not hard to see that on such an interval, $f(t)$ is a constant. Of course, Case 1 implies that $f(t)$ is a constant. Therefore the proof of Lemma 5.4 is complete. □

**Lemma 5.5.** *If $M$ is a complete noncompact Riemannian manifold and the Möbius equation has a nonconstant solution, then $M = R \times_{f(t)} M_0$ where $f(t) > 0$, $f(t)$ is not a constant and $M_0$ is complete.*

*Proof*. This has been done in §5 of B. Osgood and D. Stowe [11]. Since here we only consider complete noncompact Riemannian manifold with positive constant scalar curvature, this kind of manifold cannot be a simply connected Riemannian manifold with constant section curvature. It also cannot be Sphere($f$). □

## 6. The inhomogeneous Möbius equation on Euclidean space

In this section, we prove the following

**Proposition 6.1.** *On $\mathbf{R}^3$, let $h = \mathrm{diag}(b_1, b_2, b_3)$ with the constants $b_i$ satisfying $b_1 + b_2 + b_3 = 0$, and $h \neq 0$. Then equation* (4) *has a positive solution if and only if one of the following conditions hold*:

(i) $2b_1 + b_2 = 0$, $b_2 < 0$;
(ii) $2b_2 + b_3 = 0$, $b_3 < 0$;
(iii) $2b_3 + b_1 = 0$, $b_1 < 0$.

*Proof*. In order to prove that this condition is sufficient, we only need a positive function $u$ which satisfies equation (4) with given $h$. By symmetry, it is enough to give a function in one case. For example, for (iii), we can take

$$u = -\frac{1}{b_1}\exp[(-3b_1/2)^{1/2}x_1].$$

Then, it is easy to see that this function satisfies equation (4) with $h$.

Now we start to prove that this condition is also necessary. First of all, from equation (4), we have

$$\frac{\partial^2 u}{\partial x_j^2} + b_j u = \frac{1}{3}\Delta u \tag{28}$$

and

$$\frac{\partial^2 u}{\partial x_k \partial x_j} = 0 \tag{29}$$

if $k \neq j$.

Then in order to complete the proof clearly and quickly, we need to prove the following

**Lemma 6.2.** *Suppose* (4) *has a positive solution with* $h = \operatorname{diag}(b_1, b_2, b_3)$ *and* $2b_i + b_j = 0$ *where* $i \neq j$ *and* $h \neq 0$, *then* $b_j < 0$.

*Proof* (*of lemma*). From equations (28) and (29), we have

$$\frac{\partial^3 u}{(\partial x_j)^2 \partial x_k} + \frac{\partial(b_j u)}{\partial x_k} = \frac{1}{3}\frac{\partial^3 u}{(\partial x_k)^3}.$$

Now we assume that $i, j, k$ are not equal to each other. The above equation tells us

$$b_j u = \frac{1}{3}\frac{\partial^2 u}{(\partial x_i)^2} + f(x_j, x_k)$$

and

$$b_j u = \frac{1}{3}\frac{\partial^2 u}{(\partial x_k)^2} + g(x_i, x_j).$$

Using equation (29), we have

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial^2 (b_j u)}{\partial x_j \partial x_k} = 0$$

and

$$\frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{\partial^2 (b_j u)}{\partial x_i \partial x_j} = 0.$$

Therefore, we are able to express the functions $f, g$ as $f(x_j, x_k) = f_1(x_j) + f_2(x_k)$ and $g(x_i, x_j) = g_1(x_i) + g_2(x_j)$ where $f_2$ and $g_1$ have no constant terms. Then we have

$$b_j u = \frac{1}{3}\frac{\partial^2 u}{(\partial x_i)^2} + f_1(x_j) + f_2(x_k) \tag{30}$$

and

$$b_j u = \frac{1}{3}\frac{\partial^2 u}{(\partial x_k)^2} + g_1(x_i) + g_2(x_j). \tag{31}$$

From equation (28), we obtain

$$\frac{\partial^2 u}{(\partial x_j)^2} + b_j u = \frac{1}{3}\frac{\partial^2 u}{(\partial x_j)^2} + b_j u - f_1(x_j) - f_2(x_k) + b_j u - g_1(x_i) - g_2(x_j).$$

So, we have

$$\begin{aligned} \frac{2}{3}\frac{\partial^2 u}{(\partial x_j)^2} - b_j u &= -[f_1(x_j) + f_2(x_k) + g_1(x_i) + g_2(x_j)], \\ \frac{2}{3}\frac{\partial^2 u}{(\partial x_i)^2} - 2b_j u &= -2f_1(x_j) - 2f_2(x_k) \end{aligned} \tag{32}$$

and

$$\frac{2}{3}\frac{\partial^2 u}{(\partial x_k)^2} - 2b_j u = -2g_1(x_i) - 2g_2(x_j).$$

Add these three equations together, to get

$$\frac{2}{3}\Delta u - 5b_j u = -3[f_1(x_j) + f_2(x_k) + g_1(x_i) + g_2(x_j)]. \tag{33}$$

Therefore, by equation (32),

$$\begin{aligned}\frac{\partial^2 u}{(\partial x_i)^2} + b_i u &= 3b_j u + b_i u - 3f_1(x_j) - 3f_2(x_k)\\ &= (3b_j + b_i)u - 3[f_1(x_j) + f_2(x_k)],\end{aligned}$$

and, on the other hand, by equation (28) and equation (33)

$$\frac{\partial^2 u}{(\partial x_i)^2} + b_i u = \frac{5}{2}b_j u - \frac{3}{2}[f_1(x_j) + f_2(x_k) + g_1(x_i) + g_2(x_j)].$$

Therefore, we have

$$\left(\frac{1}{2}b_j + b_i\right)u = \frac{3}{2}[f_1(x_j) + f_2(x_k) - g_1(x_i) - g_2(x_j)].$$

Since $2b_i + b_j = 0$ and $f_2$, $g_1$ have no constant terms, $f_1(x_j) - g_2(x_j) = 0$ and $f_2(x_k) = 0$; and $g_1(x_i) = 0$.

Then equation (31) and equation (30) become

$$b_j u = \frac{1}{3}\frac{\partial^2 u}{(\partial x_k)^2} + g_2(x_j) \tag{34}$$

and

$$b_j u = \frac{1}{3}\frac{\partial^2 u}{(\partial x_i)^2} + f_1(x_j) \tag{35}$$

respectively.

Using equation (34) and equation (29), we have $\frac{\partial(b_j u)}{\partial x_i} = 0$.

By equation (35) $b_j u = f_1(x_j) = g_2(x_j)$. So using equation (28),

$$\frac{\partial^2 f_1(x_j)}{(\partial x_j)^2} + b_j f_1(x_j) = \frac{1}{3}\frac{\partial^2 f_1(x_j)}{(\partial x_j)^2}.$$

That is,

$$\frac{2}{3}\frac{d^2 f_1(x_j)}{(dx_j)^2} + b_j f_1(x_j) = 0. \tag{36}$$

Equation (36) has a positive solution everywhere if and only if $b_j < 0$. Because when $b_j > 0$, the general solutions of equation (36) have the form

$$f_1(x_j) = c_1 \sin\left[\left(\frac{3}{2}b_j\right)^{1/2} x_j\right] + c_2 \cos\left[\left(\frac{3}{2}b_j\right)^{1/2} x_j\right].$$

This function cannot be positive everywhere. Even though we can add a constant to $f_1(x_j)$, in order that the function is a solution of equation (4), this constant must be zero since $b_j \neq 0$. This completes the proof of the lemma. □

*Proof of the proposition* (*continued*). Now let us fix $j = 1$ in equation (28). Using equation (29), we have

$$\frac{\partial(b_1 u)}{\partial x_2} = \frac{1}{3}\frac{\partial(\Delta u)}{\partial x_2} = \frac{\partial^3 u}{3(\partial x_2)^3}$$

or

$$b_1 u = \frac{\partial^2 u}{3(\partial x_2)^2} + f(x_1, x_3). \tag{37}$$

Similarly, we have

$$b_1 u = \frac{\partial^2 u}{3(\partial x_3)^2} + g(x_1, x_2) \tag{38}$$

where $f(x_1, x_3)$ and $g(x_1, x_2)$ are arbitrary functions which only depend on $x_1, x_3$ and $x_1, x_2$ respectively. From equations (37) and (38), we have

$$f(x_1, x_3) = f_1(x_1) + f_2(x_3), \qquad g(x_1, x_2) = g_1(x_1) + g_2(x_2)$$

where, in addition, we are able to assume that $f_2$ and $g_2$ do not have any constant terms. Therefore,

$$b_1 u = \frac{\partial^2 u}{3(\partial x_2)^2} + f_1(x_1) + f_2(x_3) \tag{39}$$

and

$$b_1 u = \frac{\partial^2 u}{3(\partial x_3)^2} + g_1(x_1) + g_2(x_2). \tag{40}$$

Using equations (28), (39) and (40)

$$\frac{\partial^2 u}{(\partial x_1)^2} + b_1 u = \frac{\Delta u}{3} = \frac{\partial^2 u}{3(\partial x_1)^2} + b_1 u - f_1(x_1) - f_2(x_3) + b_1 u - g_1(x_1) - g_2(x_2).$$

That is

$$\frac{2\partial^2 u}{3(\partial x_1)^2} - b_1 u = -[f_1(x_1) + f_2(x_3) + g_1(x_1) + g_2(x_2)]. \tag{41}$$

Equations (39) and (40) are equivalent to

$$\frac{2\partial^2 u}{3(\partial x_2)^2} - 2b_1 u = -2[f_1(x_1) + f_2(x_3)] \tag{42}$$

and

$$\frac{2\partial^2 u}{3(\partial x_3)^2} - 2b_1 u = -2[g_1(x_1) + g_2(x_2)]. \tag{43}$$

Add (41), (42), (43) together to obtain

$$\Delta u - \frac{15 b_1 u}{2} = -\frac{9}{2}[f_1(x_1) + f_2(x_3) + g_1(x_1) + g_2(x_2)]. \tag{44}$$

On the one hand, from (39)

$$\begin{aligned} \frac{\partial^2 u}{(\partial x_2)^2} + b_2 u &= 3b_1 u + b_2 u - 3[f_1(x_1) + f_2(x_3)] \\ &= (3b_1 + b_2)u - 3[f_1(x_1) + f_2(x_3)]. \end{aligned}$$

On the other hand, from (28) and (44)

$$\frac{\partial^2 u}{(\partial x_2)^2} + b_2 u = \frac{5b_1 u}{2} - \frac{3}{2}[f_1(x_1) + f_2(x_3) + g_1(x_1) + g_2(x_2)]. \tag{46}$$

From equations (45), (46)

$$(b_1 + 2b_2)u = 3[f_1(x_1) + f_2(x_3) - g_1(x_1) - g_2(x_2)]. \tag{47}$$

Similarly, we have

$$(b_1 + 2b_3)u = 3[g_1(x_1) + g_2(x_2) - f_1(x_1) - f_2(x_3)]. \tag{48}$$

*Case* I. $b_1 + 2b_3 = 0$. In this case, we can invoke the previous lemma to see that $b_1 < 0$.

*Case* II. $b_1 + 2b_3 \neq 0$. In this case, we have $b_1 + 2b_2 \neq 0$, since $b_1 + b_2 + b_3 = 0$. Therefore, by equation (47)

$$u = \frac{3[f_1(x_1) + f_2(x_3) - g_1(x_1) - g_2(x_2)]}{b_1 + 2b_2}.$$

For this function $u$, use equation (28) with $j = 1$ to obtain

$$\begin{aligned} &2[(f_1)''(x_1) - (g_1)''(x_1)] + 3b_1[f_1(x_1) - g_1(x_1)] \\ &\quad - (f_2)''(x_3) + 3b_1 f_2(x_3) + (g_2)''(x_2) - 3b_1 g_2(x_2) = 0. \end{aligned}$$

Thus we can assume that

$$2[(g_1)''(x_1) - (f_1)''(x_1)] + 3b_1[g_1(x_1) - f_1(x_1)] = c_1$$

and

$$3b_1 g_2(x_2) = (g_2)''(x_2) + c_2 \tag{49}$$

and

$$3b_1 f_2(x_3) = (f_2)''(x_3) + c_3 \tag{50}$$

where $c_1 - c_2 + c_3 = 0$ and $c_1$, $c_2$, $c_3$ are constants.

Using equation (28) with $j = 2$, we have

$$2(g_2)''(x_2) + 3b_2 g_2(x_2) = c_4. \tag{51}$$

Then equations (49) and (51) show us that

$$(2b_1 + b_2)g_2(x_2) = \frac{c_4 + 2c_2}{3} \tag{52}$$

which implies that $g_2 = \text{constant}$. By assumption, the constant must be zero.

Using the same argument on $f_2$, we get $f_2 = 0$.

Therefore,

$$u = \frac{3[f_1(x_1) - g_1(x_1)]}{b_1 + 2b_2}.$$

From (28) with $j = 2$ and $j = 3$ respectively, we have

$$\frac{3b_2[f_1(x_1) - g_1(x_1)]}{b_1 + 2b_2} = \frac{[f_1(x_1) - g_1(x_1)]''}{b_1 + 2b_2}$$

and

$$\frac{3b_3[f_1(x_1) - g_1(x_1)]}{b_1 + 2b_2} = \frac{[f_1(x_1) - g_1(x_1)]''}{b_1 + 2b_2}$$

Then either $b_2 = b_3$ or $g_1(x_1) = f_1(x_1)$. The first case cannot happen, for otherwise, $b_1 + 2b_3 = 0$ which contradicts the hypothesis that $b_1 + 2b_3 \neq 0$. The second case also cannot happen since it would imply that $u = 0$. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CALIFORNIA 90089-1113

*Current address*: Department of Mathematics, National University of Singapore, 10 Kent Ridge Road, Singapore 0511

*E-mail address*: matxuxw@nuscc.nus.sg