APPLYING COORDINATE PRODUCTS
TO THE TOPOLOGICAL IDENTIFICATION OF NORMED SPACES

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Abstract. Using the $l^2$-products we find pre-Hilbert spaces that are absorbing sets for all Borelian classes of order $\alpha \geq 1$. We also show that the following spaces are homeomorphic to $\Sigma^\infty$, the countable product of the space $\Sigma = \{(x_n) \in R^\infty : (x_n) \text{ is bounded}\}$:

1. Introduction

We are interested in the topological classification of noncomplete normed linear spaces. The main tool in this area is the method of absorbing sets discovered and applied in the $\sigma$-compact case by Anderson and Bessaga and Pelczyński (see [2]). Absorbing sets which are not necessarily $\sigma$-closed in a considered copy $s$ of $l^2$ were developed by Bestvina and Mogilski [4]. A disadvantage of the approach presented in [4] was that two homeomorphic absorbing sets in $s$ might not have been relatively homeomorphic. The difficulty was overcome in [7] due to replacing the strong universality property by its relative version (see Theorem 2.2). We construct linear subspaces $F_\alpha$, $\alpha \geq 1$ (respectively, $G_\alpha$, $\alpha \geq 2$) of $l^2$ that are absorbing sets for the additive Borelian class $A_\alpha$ (respectively, the multiplicative Borelian class $\mathcal{M}_\alpha$) and such that the pair $(l^2, F_\alpha)$ (respectively, $(l^2, G_\alpha)$) is strongly $(\mathscr{A}, A_\alpha)$-universal (respectively, $(\mathscr{M}, \mathcal{M}_\alpha)$-universal). Applying Theorem 2.2, $(l^2, F_\alpha)$ and $(l^2, G_\alpha)$ are homeomorphic to $(s, \Lambda_\alpha)$ and $(s, \Omega_\alpha)$, respectively, where $\Lambda_\alpha$ and $\Omega_\alpha$ are absorbing sets in $s$ constructed in [4].

One may guess that $F_\alpha$ (respectively, $G_\alpha$) is the weak $l^2$-product $\sum l^2 H_n$ (respectively, the $l^2$-product $\prod l^2 H_n$) of pre-Hilbert spaces $H_n$ that contain a...
closed copy of $\Lambda_\alpha$ (respectively, $\Omega_\alpha$). The crucial step is to show that $\sum_{n \geq 1} H_n$ and $\prod_{n \geq 1} H_n$ are strongly $\mathcal{A}_\alpha$- and $\mathcal{M}_\alpha$-universal, respectively. Actually, we are able to verify the strong $(\mathcal{H}, \mathcal{L})$-universality property of an arbitrary normed coordinate product pair $(\prod C_n, \sum C_n)$ provided each element of $(\mathcal{H}, \mathcal{L})$ admits a relative closed embedding into every $(E_n, H_n)$ (see Proposition 3.1).

A version of 3.1 for cartesian products was earlier applied [11, 13, 12] in order to identify some function and sequence spaces that are homeomorphic to $\Omega_2 = \Sigma^\infty$. Applying 3.1 (and its variations), we show that several absolute $\mathcal{F}_n\mathcal{S}_\alpha$-spaces that underlie a "product" structure are homeomorphic to $\Omega_2$. In particular, we prove that every normed coordinate product $\prod C_n H_n$, $C$ being a Banach space, is homeomorphic to $\Omega_2$ provided each $H_n \in \mathcal{M}_2$ and infinitely many of the $H_n$'s are $Z_{\alpha}$-spaces. Another application concerns the function space $L^p = \cap_{p < q} L^p$ in the $L^q$-topology ($q < p$) and the sequence space $l^p = \cap_{p < q} l^p$ in the $l^q$-topology ($p < q$). We prove that $L^p$, $0 < q < p \leq \infty$, and $l^p$, $0 \leq p < q < \infty$, are homeomorphic to $\Omega_2$. Actually, we show that the pairs $(L^q, L^p)$, $(l^q, l^p)$, and $(s, \Omega_2)$ are homeomorphic. The fact that the space $L^p$ considered as a subspace of $L^0$ (of all measurable functions with the topology of convergence in measure) and the space $l^p$ as a subspace of $R^\infty$ are homeomorphic to $\Omega_2$ was previously obtained in [13]. Let us note that dealing with these different topologies on $L^p$ (same for $l^p$) the natural linear map $\Psi : L^0 \rightarrow (L^0)^{\infty}$ is employed. In the present paper $\Psi$ is considered as a linear isomorphism of $L^1$ onto $\prod L^1$ with the following key property:

$$\Psi(L^p) \cap \sum_{l^1} = \sum_{l^1} L^p.$$ 

In the last section we provide some examples of pre-Hilbert spaces with rather mysterious topological structure. They all are of the form

$$Y(A) \times F_\alpha \quad \text{and} \quad Y(A) \times G_\alpha,$$

where $Y(A)$ is the linear span of a linearly independent subset $A$ in $l^2$. In particular, we show that every projective class $P_n \setminus \bigcup_{k < n} P_k$, $n \geq 1$, contains uncountably many nonhomeomorphic pre-Hilbert spaces. The same is true for the class of spaces which are nonprojective. We observe that the argument of Henderson and Pelczyński [2] showing that there are uncountably many $\sigma$-compact pre-Hilbert spaces applies (after a minor change) to produce uncountably many nonhomeomorphic pre-Hilbert spaces in each class $\mathcal{A}_\alpha \setminus \mathcal{M}_\alpha$ and $\mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$ for $\alpha \geq 2$.

The results of §3 will be applied to construct absorbing sets for all projective classes in a forthcoming paper by the first named author.

The authors wish to note that J. Dijkstra and J. Mogilski have recently obtained the same results concerning $L^p$- and $l^p$-spaces [10].

Convention. All spaces considered are separable and metrizable. Maps are continuous functions.

2. Preliminaries

Let us recall that a closed subset $A$ of a space $X$ is a $Z$-set (respectively, a strong $Z$-set) if for every open cover $\mathcal{U}$ of $X$ there exists a $\mathcal{U}$-close to the
identity map \( f : X \to X \) such that \( f(X) \cap A = \emptyset \) (respectively, \( \text{cl}(f(X)) \cap A = \emptyset \)). A space which is a countable union of \( Z \)-sets is called a \( Z_\sigma \)-space. Note that every \( Z_\sigma \)-space is of the first category. In the case where \( X \) is an absolute neighborhood retract a closed set \( A \) is a \( Z \)-set iff given \( n \) every map of the \( n \)-dimensional cube \( I^n \) into \( X \) can be approximated by maps into \( X \setminus A \). Every not necessarily closed set \( A \) satisfying the above condition is called locally homotopy negligible in \( X \) (see [17]).

Fix a pair of spaces \((K, L)\), i.e., \( L \subseteq K \). We say that a pair of spaces \((X, Y)\) is strongly \((K, L)\)-universal if, for every closed subset \( D \) of \( K \), every map \( f : K \to X \) whose restriction to \( D \) is a \( Z \)-embedding (i.e., \( f|D \) is an embedding and \( f(D) \) is a \( Z \)-set in \( X \)) and satisfies the condition
\[
(f|D)^{-1}(Y) = D \cap L,
\]
and every open cover \( \mathcal{U} \) of \( X \), there exists a \( Z \)-embedding \( g : K \to X \) which is \( \mathcal{U} \)-close to \( f \) and satisfies the conditions
\[
g|D = f|D \quad \text{and} \quad g^{-1}(Y) = L.
\]

We find it convenient to formulate the following technical fact concerning the strong \((K, L)\)-universality (cf. [4, Proposition 2.2]).

**Proposition 2.1.** Let an absolute neighborhood retract \( X \), its subsets \( Y \subseteq Y' \), and a pair of spaces \((K, L)\) satisfy the following conditions:

(i) every \( Z \)-set in \( X \) is a strong \( Z \)-set,

(ii) \( X \setminus Y \) is locally homotopy negligible in \( X \),

(iii) \( Y' \) is locally homotopy negligible in \( X \),

(iv) given open subsets \( U \) of \( K \) and \( V \) of \( X \), a map \( f : K \to X \) with \( f(U) \subseteq V \cap Y \) and \( f(K \setminus U) \subseteq X \setminus V \), and an open cover \( \mathcal{V} \) of \( V \), there exists a closed embedding \( g : U \to V \) which is \( \mathcal{V} \)-close to \( f|U \) and satisfies \( g(U) \subseteq Y' \) and \( g^{-1}(V \cap Y) = L \cap U \).

Then for every \( Z \subseteq X \) with \( Z \cap Y' = Y \), the pair \((X, Z)\) is strongly \((K, L)\)-universal.

Before we give a proof of 2.1 we recall that \( f : K \to X \) is closed over a set \( A \subseteq X \) if for every \( a \in A \) and every neighborhood \( U \) of \( f^{-1}({a}) \) there exists a neighborhood \( V \) of \( a \) such that \( f^{-1}(V) \subseteq U \) (see [4]).

**Proof of 2.1.** Let \( D \) be a closed subset of \( K \) and let \( \overline{f} : K \to X \) be a map such that \( \overline{f}|D \) is a \( Z \)-embedding satisfying \( (\overline{f}|D)^{-1}(Z) = D \cap L \). Since \( \overline{f}(D) \) is a strong \( Z \)-set in \( X \) and \( X \setminus Y \) is locally homotopy negligible in \( X \), we can apply [4, Lemma 1.1; 17, Theorem 2.4] to approximate \( \overline{f} \) by \( f \) such that

1. \( \overline{f}|D = f|D \),

2. \( f \) is closed over \( \overline{f}(D) \),

3. \( f(K \setminus D) \subseteq Y \setminus f(D) \).

Set \( U = K \setminus D \) and \( V = X \setminus f(D) \). Let \( \mathcal{U} \) be an open cover of \( X \). Fix a metric \( d \) on \( X \) and choose an open cover \( \mathcal{V} \) of \( V \) which is inscribed in \( \mathcal{U} \) and such that

4. for every element \( W \) of \( \mathcal{V} \), \( \text{diam}(W) < \text{dist}(W, X \setminus V) \).

By our assumption, there exists a closed embedding \( g : U \to V \) which is \( \mathcal{V} \)-close to \( f|U \) and such that \( g^{-1}(V \cap Y) = U \cap L \) and \( g(U) \subseteq Y' \). By (4), \( g \)
can be continuously extended by \( \bar{f} = f \) over \( D \) to a one-to-one map which is \( \mathcal{U} \)-close to \( f \). Denote this extension also by \( g \). We have \( g^{-1}(V \cap Y) = U \cap L \) and consequently \( g(L) \subset Z \). Moreover, if \( g(x) \in Z \) and \( x \notin D \) then \( g(x) \in Z \cap Y' = Y \). This, together with \( (\bar{f}|D)^{-1}(Z) = D \cap L \), yields \( g^{-1}(Z) = L \). To show that \( g : K \to X \) is a closed embedding, let \( \{g(x_n)\}_{n=1}^{\infty} \) converge to \( y \in X \). If \( y \in \bar{f}(D) \) then, by (4), \( \{f(x_n)\}_{n=1}^{\infty} \) converges to \( y \) and consequently, by (2), \( \{x_n\}_{n=1}^{\infty} \) converges to \( f^{-1}(y) \). Otherwise, \( y \in V \) and \( \{x_n\}_{n=1}^{\infty} \) converges to \( g^{-1}(y) \). Since \( g(K) \subset \bar{f}(D) \cup Y' \), the union of a \( Z \)-set and a locally homotopy negligible set, \( g(K) \), is a \( Z \)-set in \( X \).

Let \( \mathcal{H} \) and \( \mathcal{L} \) be classes of spaces. We write \( (K, L) \in (\mathcal{H}, \mathcal{L}) \) provided \( K \in \mathcal{H} \) and \( L \in \mathcal{L} \). A pair of spaces \((X, Y)\) is said to be strongly \((\mathcal{H}, \mathcal{L})\)-universal if \((X, Y)\) is strongly \((K, L)\)-universal for every pair \((K, L) \in (\mathcal{H}, \mathcal{L})\). This concept was introduced in [6]. If the pair \((X, Y)\) is strongly \((K, L)\)-universal for every \( L \in \mathcal{L} \) then, according to [4], \( Y \) is strongly \( \mathcal{L} \)-universal.

In what follows, \( \mathcal{L} \) will satisfy the following conditions:

(a) if \( L \) and \( L' \) are homeomorphic and \( L \in \mathcal{L} \), then \( L' \in \mathcal{L} \),

(b) if a space \( L \) is a union of its two closed subspaces which belong to \( \mathcal{L} \), then \( L \in \mathcal{L} \),

(c) every closed subset of an element of \( \mathcal{L} \) belongs to \( \mathcal{L} \).

The following fact proved in [7] extends the uniqueness theorem for absorbing sets discovered by Anderson and Bessaga and Pelczyński (see [2]).

**Theorem 2.2** [7, Theorem 2.1]. Let \( X \) be a topological copy of \( l^2 \) and let \( Y_1 \) and \( Y_2 \) be two subsets of \( X \). Assume that both \( Y = Y_1 \) and \( Y_2 \) satisfy the following conditions:

(i) \( X \setminus Y \) is locally homotopy negligible in \( X \),

(ii) \( Y \) is a \( Z_\alpha \)-space,

(iii) \( Y \) is a countable union of closed sets that are elements of \( \mathcal{L} \),

(iv) \( (X, Y) \) is strongly \((\mathcal{H}, \mathcal{L})\)-universal, where \( \mathcal{H} \) is the class of completely metrizable spaces.

Then, for every open cover \( \mathcal{U} \) of \( X \), there exists a \( \mathcal{U} \)-close to the identity homeomorphism of \((X, Y_1)\) onto \((X, Y_2)\).

Every subset \( Y \) of \( X \) which is strongly \( \mathcal{L} \)-universal and fulfils (i)-(iii) is called an \( \mathcal{L} \)-absorbing set in \( X \). In [4], it was shown that two \( \mathcal{L} \)-absorbing sets in a copy of \( l^2 \) are homeomorphic. Theorem 2.2 may be rephrased in its weaker form as follows: two \( \mathcal{L} \)-absorbing sets in a copy of \( l^2 \) are relatively homeomorphic provided they are strongly \((\mathcal{H}, \mathcal{L})\)-universal.

### 3. Strong universality in products

Let \( C \) be a normed countable coordinate space (briefly, a normed coordinate space), i.e., \( C = (C, \| \cdot \|_C) \) is a normed linear space of real sequences such that

\[
\text{(c1)} \quad \text{for every bounded sequence } \lambda = (\lambda_n) \text{ and every } c = (c_n) \in C, \text{ we have } \lambda \cdot c = (\lambda_n c_n) \in C \text{ and } \|\lambda \cdot c\|_C \leq \|\lambda\|_\infty \|c\|_C, \text{ where } \|\lambda\|_\infty = \sup_{n \geq 1} |\lambda_n|,
\]

\[
\text{(c2)} \quad \text{for every } \varepsilon > 0 \text{ and every } (c_n) \in C \text{ there exists } k \text{ such that } \|\lambda, \ldots, 0, c_k, c_{k+1}, \ldots\|_C < \varepsilon,
\]
(c₃) each unit vector \( u_n = (δ^i_n) \) belongs to \( C \).

We took the notion of a normed coordinate space from [1] (see also [16]) where the following equivalent condition replaces \((c₃)\): \( C \) is contained in no hyperplane \( \{ (c^*_k) : c_k = 0 \} \), \( k \geq 1 \). (For examples of normed coordinate spaces, see [1].) Note that \( C \) contains all eventually zero sequences \( C_0 \). Later on we have to assume that \( C \setminus C_0 \neq \emptyset \). This, of course, is the case if \( C \) is a Banach space.

Let \( \{ (E_n, \| \cdot \|_n) \}_{n=1}^{\infty} \) be a sequence of normed linear spaces. We consider the linear spaces
\[
\prod_C E_n = \left\{ (y_n) \in \prod_{n=1}^{\infty} E_n : (\| y_n \|_n) \in C \right\}
\]
and
\[
\sum_C E_n = \left\{ (y_n) \in \prod_{n=1}^{\infty} E_n : y_n = 0 \text{ for almost all } n \right\}
\]
which are both equipped with the norm \( \| (y_n) \|_{C} = \| (\| y_n \|_n) \|_C \). These spaces are called, respectively, the normed coordinate product (of the \( E_n \)'s in the sense of \( C \)) and the weak normed coordinate product (briefly, \( C \)-product and weak \( C \)-product of the \( E_n \)'s). For \( y = (y_n) \in \prod_C E_n \) and \( k \geq 1 \), we write \( r_k(y) = (0, \ldots, y_k, y_{k+1}, \ldots) \), \( s_k(y) = y - r_k(y) \), and \( \pi_k(y) = y_k \). Identifying \( E_n \) with the natural subspace of \( \prod_C E_n \), we have
\[
\begin{align*}
(A) \quad & \| s_k(y) \| \leq \| y \|, \\
(B) \quad & \| \pi_k(y) \| \leq \| r_k(y) \| \leq \| y \|, \\
(C) \quad & \lim_k \| r_k(y) \| = 0,
\end{align*}
\]
for every \( k \geq 1 \) and \( y \in \prod_C E_n \).

We now give the main result of this section.

**Proposition 3.1.** Let \( \{ (E_n, H_n) \}_{n=1}^{\infty} \) be a sequence of pairs of nontrivial normed linear spaces with each \( H_n \) dense in \( E_n \) and let \( C \) be a normed coordinate space that contains an element with infinitely many nonzero terms. Fix a pair of spaces \( (K, L) \) and assume that for every \( n \geq 1 \) there exists a bounded closed embedding \( \psi_n : K \to E_n \) with \( \psi^{-1}(H_n) = L \). Then, for every \( Z \subseteq E = \prod_C E_n \) with \( Z \cap \sum_C E_n = \sum_C H_n \), the pair \( (E, Z) \) is strongly \((K, L)\)-universal.

We shall make use of the next two lemmas.

**Lemma 3.2.** There exists a homotopy
\[
\Phi = (\Phi_n) : (E \times [0, 1], \sum_C H_n \times [0, 1]) \to (E, \sum_C H_n)
\]
satisfying the following conditions:
\[
\begin{align*}
(i) \quad & \Phi(\cdot, 0) = \text{id}, \\
(ii) \quad & \text{if } n \geq \frac{1}{t} + 2, \text{ then } \Phi_n(y, t) = 0, \\
(iii) \quad & \text{if for some sequence } \{ (y(i), t_i) \}_{i=1}^{\infty} \subseteq E \times [0, 1] \text{ with } \lim t_i = 0 \text{ there exists } y \in E \text{ such that } \lim \Phi(y(i), t_i) = y, \text{ then } \lim y(i) = y.
\end{align*}
\]

**Lemma 3.3.** There exists a one-to-one map
\[
\phi : K \times (0, 1] \to E
\]
satisfying the following conditions:

(iv) $\varphi^{-1}(\sum C H_n) = L \times (0, 1]$,
(v) $|||\varphi(x, t)||| \leq t$ for all $(x, t) \in K \times (0, 1]$,
(vi) if $x \in K$ and $\frac{1}{n+1} < t \leq \frac{1}{n}$, then $\pi_{n+2}\varphi(x, t) \neq 0$ while $\pi_{k}\varphi(x, t) = 0$
for all $k < n$ and $k \geq n + 4$,
(vii) if the sequence $\{\varphi(x_i, t_i)\}_{i=1}^{\infty}$ converges in $E$, $\{(x_i, t_i)\}_{i=1}^{\infty} \subset K \times (0, 1]$,
and $\lim t_i = t_0 > 0$, then $\{x_i\}_{i=1}^{\infty}$ converges in $K$.

First, we derive Proposition 3.1 from Lemmas 3.2 and 3.3.

Proof of 3.1. We make use of 2.1 with the following data: $X = E$, $Y = \sum C H_n$, $Y' = \sum C E_n$, and $Z$. It is known that $Z$-sets in $E$ are strong $Z$-sets (see [5, Lemma 2.6; 13, Lemma 2.1]). It is also clear that $E \setminus \sum C H_n$ and $\sum C E_n$ are locally homotopy negligible in $E$ (see, e.g., [17]). Fix a map $\bar{f} : K \to E$ and open sets $U \subset K$ and $V \subset E$ such that $f = \bar{f}|U$ maps $U$ into $V \cap \sum C H_n$ and $\bar{f}(x) \notin V$ for all $x \notin U$. Let $\mathcal{V}$ be an open cover of $V$. Pick a map $\omega : V \to (0, 1]$ such that

(1) whenever $y \in V$ and $z \in E$ satisfy $|||y - z||| < 2\omega(y)$ then there exists an element $\mathcal{V}$ containing both $y$ and $z$.

Let $\Phi$ be a homotopy of 3.2. Pick a map $\varepsilon : E \to [0, 1]$ such that

(2) $\varepsilon^{-1}(\{0\}) = E \setminus V$,
(3) $|||\Phi(y, \varepsilon(y)) - y||| < \omega(y)$ for all $y \in V$,
(4) $\left(\frac{1}{\varepsilon(y)} + 4\right)^{-1} < \omega(y)$ for all $y \in V$.

Write $\varepsilon(x) = \varepsilon(f(x))$ and $\lambda(x) = \left(\frac{1}{\varepsilon(x)} + 4\right)^{-1}$. Pick a homotopy $\varphi$ from 3.3 and define $g : U \to E$ by the formula

$$g(x) = \Phi(f(x), \varepsilon(x)) + \varphi(x, \lambda(x)).$$

Applying (3)-(4) and (v), we get

$$|||f(x) - g(x)||| < \omega(f(x)) + \lambda(x) < 2\omega(f(x))$$
for every $x \in U$. The property (1) of $\omega$ assures that the range of $g$ is $V$ and that $g$ is $\mathcal{V}$-close to $f$. Clearly, $g$ takes values in $\sum C E_n$ and, by (iv),

$$g^{-1}(V \cap \sum C H_n) = L \cap U.$$ 

To finish the proof, it remains to show that $g : U \to V$ is a closed embedding. First we check that $g$ is one-to-one. If $\frac{1}{n+1} < \varepsilon(x) \leq \frac{1}{n}$ then, by (ii), $\Phi_p(f(x), \varepsilon(x)) = 0$ for all $p \geq n + 3$. Since $\frac{1}{n+3} < \lambda(x) \leq \frac{1}{n+4}$ we have, by (vi), $\varphi_{n+6}(x, \lambda(x)) \neq 0$ and $\varphi_k(x, \lambda(x)) = 0$ for $k \neq n+4, n+5, n+6, n+7$. Assume that $g(x') = g(x')$ and $\varepsilon(x') \leq \varepsilon(x)$. Letting $\frac{1}{n'+1} < \varepsilon(x') \leq \frac{1}{n'}$, we see that $n' \geq n$ and, by (vi), $\varphi_{n'+6}(x', \lambda(x')) \neq 0$. It follows that

$$\varphi_{n'+6}(x', \lambda(x')) = g_{n'+6}(x') = g_{n'+6}(x) = \varphi_{n'+6}(x, \lambda(x)) \neq 0;$$

hence, $n' = n$ or $n + 1$. Then, for every $p \geq n + 4$, we have $\Phi_p(f(x'), \varepsilon(x')) = \Phi_p(f(x), \varepsilon(x)) = 0$ and consequently $\varphi_p(x, \lambda(x)) = g_p(x) = g_p(x') = \varphi_p(x', \lambda(x'))$. Since, by (vi), $\varphi_p(x, \lambda(x)) = \varphi_p(x', \lambda(x')) = 0$ for all $p \leq n + 3$, we conclude that

$$\varphi(x, \lambda(x)) = \varphi(x', \lambda(x')).$$
The latter yields $x = x'$ because $\varphi$ is one-to-one. Now, suppose \( \{g(x_i)\}_{i=1}^{\infty} \) converges to $y = (y_n) \in V$ for some sequence \( \{x_i\}_{i=1}^{\infty} \subset U $. Write $\varepsilon_i = \varepsilon(x_i)$ and $\lambda_i = \lambda(x_i)$. We can assume that \( \{\varepsilon_i\}_{i=1}^{\infty} \) converges to $\varepsilon_0 \in [0, 1]$. If $\varepsilon_0 = 0$, then $\lim_{i \to \infty} \lambda_i = 0$. Using (v), we get $\lim_{i \to \infty} \Phi(f(x_i), \varepsilon_i) = y$. Then, by (iii), \( \{f(x_i)\}_{i=1}^{\infty} \) converges to $y$. By the continuity of $\varepsilon$, we get $\varepsilon(y) = 0$ which contradicts (2). Therefore, we can assume that $\varepsilon_0 > 0$. Let

$$\varepsilon_0 = s_0 \frac{1}{n} + (1 - s_0) \frac{1}{n+1}$$

for some $0 < s_0 \leq 1$. We can further assume that

$$\varepsilon_i = s_i \frac{1}{n} + (1 - s_i) \frac{1}{n+1}.$$

Then, we have $\Phi_{n+j}(x_i, \varepsilon_i) = 0$ for all $i$ and $j \geq 3$; and consequently the sequence \( \{\varphi_{n+j}(x_i, \lambda_i)\}_{i=1}^{\infty} = \{g_{n+j}(x_i)\}_{i=1}^{\infty} \), converges to $y_{n+j}$ for all $j \geq 3$. For $p \neq n+j$, we have, by (vi), $\varphi_p(x_i, \lambda_i) = 0$. It follows that \( \{\varphi(x_i, \lambda_i)\}_{i=1}^{\infty} \) converges in $E$. Since $\lim \lambda_i = \left( \frac{1}{k_i} + 4 \right)^{-1} > 0$, by (vii), the sequence \( \{x_i\}_{i=1}^{\infty} \) is convergent in $K$. If $\lim x_i = x \in K \setminus U$, then $\lim f(x_i) = f(x) \in E \setminus V$ and $\lim \varepsilon(x_i) = \varepsilon(f(x)) = 0$, contradicting (2). We have shown that $g$ is a closed embedding.

**Proof of 3.2.** Pick a vector $e_n \in H_n$ with $|||e_n||| = 1$. Define $\Phi : E \times [0, 1] \to E$ by $\Phi(y, 0) = y$,

$$\Phi \left( y, \frac{1}{n} \right) = (y_1, \ldots, y_{n-1}, 0, |||r_n(y)||| \cdot e_{n+1}, 0, 0, \ldots)$$

and

$$\Phi \left( y, s \frac{1}{n} + (1 - s) \frac{1}{n+1} \right) = s \Phi \left( y, \frac{1}{n} \right) + (1 - s) \Phi \left( y, \frac{1}{n+1} \right)$$

for every $n \geq 1$, $0 \leq s \leq 1$, and $y \in E$. It is clear that $\Phi$ transforms \( \sum_i H_n \times [0, 1] \) into \( \sum_i H_n \), is continuous on $E \times (0, 1]$, and satisfies (i) and (ii). The continuity of $\Phi$ at the points $(y, 0)$ will follow from the auxiliary estimations.

Given $y = (y_n) \in E$, we have

$$|||y - \Phi \left( y, \frac{1}{n} \right)||| \leq |||y - s_n(y)||| + |||s_n(y) - \Phi \left( y, \frac{1}{n} \right)|||$$

$$= |||r_n(y)||| + |||r_n(y) \cdot e_{n+1}||| = 2 |||r_n(y)|||.$$

For $t = s \frac{1}{n} + (1 - s) \frac{1}{n+1}$, $0 \leq s \leq 1$, and $y \in E$, we have

1. $|||y - \Phi(y, t)||| = |||s \Phi(y, \frac{1}{n}) + (1 - s)(y - \Phi(y, \frac{1}{n+1}))|||$

$$\leq s |||y - \Phi(y, \frac{1}{n})||| + (1 - s)|||y - \Phi(y, \frac{1}{n+1})||| \leq 2s |||r_n(y)||| + 2(1 - s)|||r_{n+1}(y)||| \leq 2|||r_n(y)|||.$$

Let $\{(y(i), t_i)\}_{i=1}^{\infty}$ be a sequence of $E \times (0, 1]$ that is convergent to $(y, 0) \in E \times \{0\}$ and let

2. $t_i = s_i \frac{1}{n_i} + (1 - s_i) \frac{1}{n_i+1}$ for some $0 \leq s_i \leq 1$ and $n_i \to \infty$.

Using (1), we get

3. $|||y - \Phi(y(i), t_i)||| \leq |||y - y(i)||| + |||y(i) - \Phi(y(i), t_i)||| \leq |||y - y(i)||| + 2|||r_{n_i}(y(i))|||.$
On the other hand, applying (B), we obtain

\[ \| r_n(y(i)) \| \leq \| r_n(y') \| + \| r_n(y - y(i)) \| \leq \| r_n(y) \| + \| y - y(i) \|. \]

Combining (3) and (4), we get

\[ \| y - \Phi(y(i), t_i) \| \leq 3 \| y - y(i) \| + 2 \| r_n(y) \|. \]

The latter inequality together with (C) yields the continuity of \( \Phi \) at \( (y, 0) \).

Now, let \( \{ (y(i), t_i) \}_{i=1}^\infty \subset E \times (0, 1] \) be such that \( \lim_{i \to \infty} \Phi(y(i), t_i) = y \in E \) and \( \lim t_i = 0 \). Express \( t_i \) in the form of (2). We have

\[ \| y - y(i) \| \leq \| s_n(y) - s_n(y(i)) \| + \| r_n(y) \| + \| r_n(y(i)) \|. \]

We see that \( s_n(y(i)) = s_n(\Phi(y(i), t_i)) \). Therefore, after using (A), we get

\[ \| s_n(y) - s_n(y(i)) \| = \| s_n(y - \Phi(y(i), t_i)) \| \leq \| y - \Phi(y(i), t_i) \|. \]

This implies

\[ \| y - y(i) \| \leq \| y - \Phi(y(i), t_i) \| + \| r_n(y) \| + \| r_n(y(i)) \|. \]

Note that the first two terms tend to 0 if \( i \to \infty \). To show (iii), it remains to verify that the last term also tends to 0. It is clear that it is enough to consider the case where \( s_i \geq \frac{1}{2} \) for all \( i \) and the case where \( s_i < \frac{1}{2} \) for all \( i \). In the first case, we apply (B) to the \( (n_i + 1) \)-coordinate and get

\[ \| y - \Phi(y(i), t_i) \| \geq \| y_{n_i+1} - s_i \| r_n(y(i)) \| \cdot e_{n_i+1}. \]

Then, since \( \| e_{n_i+1} \| = 1 \), we estimate

\[
\| r_n(y(i)) \| \leq \frac{1}{s_i} (\| y - \Phi(y(i), t_i) \| + \| y_{n_i+1} \|)
\leq 2 (\| y - \Phi(y(i), t_i) \| + \| y_{n_i+1} \|).
\]

Finally, according to (B) and (C), \( \lim_{i \to \infty} r_n(y(i)) = 0 \). In the case where \( s_i < \frac{1}{2} \), we apply (B) to the \( n_i \)-coordinate and obtain

\[ \| y - \Phi(y(i), t_i) \| \geq \| y_{n_i} - (1 - s_i) y_{n_i}(i) \|
\]

and hence

\[ \| y_{n_i}(i) \| \leq \frac{1}{1 - s_i} (\| y - \Phi(y(i), t_i) \| + \| y_{n_i} \|)
\leq 2 (\| y - \Phi(y(i), t_i) \| + \| y_{n_i} \|).
\]

The same argument applied to the \( (n_i + 2) \)-coordinate yields

\[ \| y - \Phi(y(i), t_i) \| \geq \| y_{n_i+2} - (1 - s_i) r_{n_i+1}(y(i)) \| \cdot e_{n_i+2}. \]

As before, we get

\[ \| r_{n_i+1}(y(i)) \| \leq \frac{1}{1 - s_i} (\| y - \Phi(y(i), t_i) \| + \| y_{n_i+2} \|)
\leq 2 (\| y - \Phi(y(i), t_i) \| + \| y_{n_i+2} \|).
\]

The latter, in turn, implies

\[ \| r_n(y(i)) \| \leq \| y_{n_i}(i) \| + \| r_{n_i+1}(y(i)) \|
\leq 4 \| y - \Phi(y(i), t_i) \| + 2 (\| y_{n_i} \| + \| y_{n_i+2} \|).
\]

Finally, according to (B) and (C), the last two terms of the above inequality tend to 0 if \( i \to \infty \).
Note. Condition (iii) is equivalent to the fact that the map \((y, t) \to (\Phi(y, t), t)\) from \(E \times [0, 1]\) into \(E \times [0, 1]\) is closed over \(E \times \{0\}\).

Proof of 3.3. By our assumption there exists a closed embedding \(\psi_n : K \to E_n\) such that

\[\psi^{-1}(H_n) = L\quad\text{and}\quad |||\psi_n(x)||| \leq \frac{1}{2^n}.\]

Pick a vector \(e_n \in H_n\) with \(||e_n|| = \frac{1}{2^n}\). Define \(\varphi = (\varphi_p)\) as follows:

\[\varphi_p(x, \frac{1}{n}) = \begin{cases} 0 & \text{if } p \neq n, n + 2, \\
\psi_n(x) & \text{if } p = n, \\
e_{n+2} & \text{if } p = n + 2.\end{cases}\]

and

\[\varphi\left(x, s\frac{1}{n} + (1-s)\frac{1}{n+1}\right) = s\varphi\left(x, \frac{1}{n}\right) + (1-s)\varphi\left(x, \frac{1}{n+1}\right)\]

for \(n \geq 1\) and \(0 \leq s \leq 1\). It is clear that \(\varphi\) is continuous and satisfies (vi) and (iv). We have

\[||\varphi\left(x, \frac{1}{n}\right)|| \leq ||\psi_n(x)|| + ||e_{n+2}|| \leq \frac{1}{n}.\]

Consequently, we estimate

\[||\varphi(x, t)|| \leq s \left\|\varphi\left(x, \frac{1}{n}\right)\right\| + (1-s) \left\|\varphi\left(x, \frac{1}{n+1}\right)\right\|\]

\[\leq s\frac{1}{n} + (1-s)\frac{1}{n+1} = t.\]

To show that \(\varphi\) is one-to-one, let \(\varphi(x, t) = \varphi(x', t')\) for some \((x, t), (x', t') \in K \times (0, 1)\). If \(t = s\frac{1}{n} + (1-s)\frac{1}{n+1}\) with \(n \geq 1\) and \(0 \leq s < 1\) (respectively, \(t = 1\)), then the last nonvanishing coordinate of \(\varphi(x, t)\) is the \((n+3)\)-coordinate (respectively, the third coordinate) and it equals \((1-s)e_{n+3}\) (respectively, \(e_3\)). This shows that \(t = t'\). Clearly, we have \(\varphi_{n+1}(x, t) = (1-s)\psi_{n+1}(x)\) (respectively, \(\varphi_1(x, t) = \psi_1(x)\)). Since \(\psi_n\) (respectively, \(\psi_1\)) is an embedding, we get \(x = x'\).

Let \(\{(x_i, t_i)\}_{i=1}^{\infty}\) be a sequence of \(K \times (0, 1)\) such that \(\{(\varphi(x_i, t_i))\}_{i=1}^{\infty}\) converges in \(E\) and \(\lim t_i = t_0 > 0\). Assume that \(t_0 = s_0\frac{1}{n} + (1-s_0)\frac{1}{n+1}\) for some \(n \geq 1\) and \(0 < s_0 < 1\) (the case where \(t_0 = \frac{1}{n}, n \geq 1\), can be treated similarly). We may suppose that \(t_i = i\frac{1}{n} + (1-s_i)\frac{1}{n+1}\) for all \(i\), where \(0 < s_i < 1\) and \(\lim s_i = s_0\). Since \(\varphi_n(x_i, t_i) = s_i\psi_n(x_i)\), \(\psi_n(x_i)\) converges in \(E_n\). Finally, \(\{x_i\}_{i=1}^{\infty}\) converges in \(K\) because \(\psi_n\) is a closed embedding.

In §§4 and 5, we will employ the following variation of Proposition 3.1.

Proposition 3.4. Let \(\{(E_n, H_n)\}_{n=1}^{\infty}\) be a sequence of pairs of nontrivial normed linear spaces with each \(H_n\) dense in \(E_n\) and let \(C\) be a normed coordinate space. Fix a pair of spaces \((K, L)\). Assume there are pairwise disjoint infinite subsets \(N_1, N_2, \ldots\) of the set of integers \(N\) such that \(N_k \cap \{1, 2, \ldots, k-1\} = \emptyset\) and, writing

\[C_k = \{(c_p)_{p \in N_k} : \exists c \in C \forall p \in N_k \pi_p(c) = c_p\}\]
and identifying $C_k$ with the natural subspace of $C$, each $C_k$ contains an element with infinitely many nonzero terms and there exists a bounded closed embedding $\psi_k : K \to \prod C_k E_p$ with $\psi_k^{-1}(\prod C_k H_p) = L$ for $k \geq 1$. Then, the pair $(\prod C E_n, \prod C H_n)$ is strongly $(K, L)$-universal.

A proof of 3.4 will be omitted. Let us indicate that to get it one has to follow the proof of 3.1 and replace 3.3 by the lemma below.

**Lemma 3.5.** There exists a one-to-one map

$$\varphi : K \times (0, 1] \to \prod C E_n$$

satisfying conditions (v) and (vii) of 3.3 together with

(iv') $\varphi^{-1}(\prod C H_p) = L \times (0, 1]$,

(vi') given $n \geq 1$ there exists an integer $k_n > k_{n-1}$ ($k_1 \geq 1$) such that, if $x \in K$ and $\frac{1}{n+1} < t \leq \frac{1}{n}$ then $\pi_k \varphi(x, t) \neq 0$ while $\pi_k \varphi(x, t) = 0$ for all $k \in N \setminus (N_{n+1} \cup N_{n+2} \cup \{k_n, k_{n+1}\})$.

**Proof.** Pick $k_n \in N_k$ with $k_n > k_{n-1}$ ($k_1 \geq 1$). By our assumption, there exists a closed embedding $y_n : K \to \prod C E_p$, $n > 1$, such that

$$y_n^{-1}(\prod C H_p) = L \quad \text{and} \quad |||y_n(x)||| \leq \frac{1}{n}$$

for all $x \in K$. Pick a vector $e_{k_n} \in H_{k_n}$ with $||e_{k_n}|| = \frac{2}{n}$. Define $\varphi = (\varphi_k)$ as

$$\varphi_k \left(x, \frac{1}{n}\right) = \begin{cases} 0 & \text{if } k \in N \setminus (N_{n+1} \cup \{k_n\}), \\
 e_{k_n} & \text{if } k = k_n, \\
 \pi_k y_n(x) & \text{if } k \neq k_n,
\end{cases}$$

and

$$\varphi \left(x, s \frac{1}{n} + (1-s) \frac{1}{n+1}\right) = s \varphi \left(x, \frac{1}{n}\right) + (1-s) \varphi \left(x, \frac{1}{n+1}\right)$$

for $n \geq 1$ and $0 \leq s \leq 1$. Conditions (iv') and (vi') follow easily. To verify (v) and (vii), repeat a reasoning of the proof of 3.3.

The next result is a counterpart of Proposition 3.1 for cartesian products and can be viewed as a relative version of [4, Proposition 2.5]. We need to recall that by the weak product of $X_i$'s with the basepoints $*_i \in X_i$ we mean

$$W(X_i, *_i) = \left\{(x_i) \in \prod_{i=1}^\infty X_i : x_i = *_i \text{ for almost all } i\right\}$$

(endoeded with the subspace topology).

**Proposition 3.6.** Let $X_i$ be a noncompact absolute retract and let $Y_i$ be a subset of $X_i$ such that $X_i \setminus Y_i$ is locally homotopy negligible in $X_i$ for $i = 1, 2, \ldots$ Fix a pair of spaces $(K, L)$ and assume that for every $i > 1$ there exists a closed embedding

$$h_i : K \to X_i \text{ with } h_i^{-1}(Y_i) = L.$$ 

Then, for every choice of basepoints $*_i \in Y_i$ and every set $Z \subseteq X = \prod_{i=1}^\infty X_i$ with $Z \cap W(X_i, *_i) = W(Y_i, *_i)$, the pair $(X, Z)$ is strongly $(K, L)$-universal.

**Proof.** We apply Proposition 2.1 with the following data: $X$, $Y = W(Y_i, *_i)$, $Y' = W(X_i, *_i)$, and $Z$. It is clear that $X$ is an absolute retract and both $X \setminus Y$
and \( Y' \) are locally homotopy negligible in \( X \). By [13, Lemma 2.2] \( Z \)-sets are strong \( Z \)-sets in \( X \). Let \( \overline{f} : K \to X \) be such that \( f = (f_i) = \overline{f}|U \) maps \( U \) into \( V \cap Y \) and \( \overline{f}(x) \notin V \) for all \( x \notin U \), where \( U \subset K \) and \( V \subset X \) are open sets. Let \( \mathcal{V} \) be an open cover of \( V \). We pick a map \( \mu : X_i \times X_i \times [0, 1] \to X_i \) such that for all \( i \geq 1 \)

1. \( \mu_i(x, y, 0) = x \) and \( \mu_i(x, y, 1) = y \) for every \( x, y \in X_i \),
2. \( \mu_i(Y_i \times Y_i \times [0, 1]) \subset Y_i \).

To construct \( \mu_i \), choose any map \( \lambda_i : X_i \times X_i \times [0, 1] \to X_i \) satisfying (1) and a homotopy \( (\varphi^i_t) : X_i \times [0, 1] \to X_i \) with \( \varphi^i_0 = \text{id}_{X_i} \) and \( \varphi^i_t(X_i) \subset Y_i \) for all \( t > 0 \) and define

\[
\mu_i(x, y, t) = \varphi^i_t(\lambda_i(x, y, t)).
\]

To produce \( \varphi^i_t \), use the fact that \( X_i \setminus Y_i \) is locally homotopy negligible in \( X_i \) and apply [17, Theorem 2.4]. The same property implies that \( Y_i \) is an absolute retract [17, Theorem 3.1]; moreover, since \( X_i \) is noncompact, \( Y_i \) is nontrivial. As a consequence, there exists an embedding \( \alpha_i : [0, 1] \to Y_i \) with

3. \( \alpha_i(0) = *_i \) for \( i \geq 1 \).

Fix \( t = s \frac{1}{n} + (1 - s) \frac{1}{n+1} \), \( n \geq 1 \) and \( 0 \leq s \leq 1 \), and let \( \Phi(x, t) = (y_i) \), where

4. \( y_i = f_i(x) \) for \( i \leq n \),
5. \( y_i = *_i \) for \( i = n + 6 \) and \( i > n + 9 \),
6. \( y_{n+1} = \mu_{n+1}(f_{n+1}(x), *_{n+1}, s) \),
7. \( y_{n+2} = \mu_{n+2}(*_n, h_{n+2}(x), s) \),
8. \( y_{n+i} = h_{n+i}(x) \) for \( i = 3 \) and \( 4 \),
9. \( y_{n+5} = \mu_{n+5}(h_{n+5}(x), *_{n+5}, s) \),
10. \( y_{n+7} = \alpha_{n+7}(s) \) and \( y_{n+8} = \alpha_{n+8}(1 - s) \).

Letting \( \Phi(x, 0) = f(x) \), we easily check that \( \Phi : K \times [0, 1] \to X \) is well defined and continuous. Notice that, by (2),

11. \( \Phi(K \times (0, 1]) \subset W(X_i, *_i) \),
12. \( \Phi^{-1}(W(Y_i, *_i)) \cap (K \times (0, 1]) = L \times (0, 1] \).

We claim that \( \Phi|K \times (0, 1] \) is one-to-one. In fact, let \( (x, t) \) and \( (x', t') \) be such that \( \Phi(x, t) = \Phi(x', t') \). If \( t = s \frac{1}{n} + (1 - s) \frac{1}{n+1} \), \( n \geq 1 \) and \( 0 \leq s \leq 1 \) (respectively, \( t = 1 \)), the last \( p \)th coordinate of \( \Phi(x, t) \), different from \( *_p \), occurs when \( p = n + 8 \) (respectively, \( p = 8 \)) and it is equal to \( \alpha_{n+8}(1 - s) \) (respectively, \( \alpha_8(1) \)). Since \( \alpha \) is an embedding, \( \Phi(x, t) \) determines \( t \) and hence \( t = t' \). According to (8), \( \Phi_{n+3}(x, t) = h_{n+3}(x) \) (respectively, \( \Phi_4(x, t) = h_4(x) \)). Therefore, \( h_{n+3}(x) = h_{n+3}(x') \) (respectively, \( h_4(x) = h_4(x') \)) and consequently we get \( x = x' \).

Choose a map \( e : X \to [0, 1] \) such that

13. \( e^{-1}([0]) = X \setminus V \),
14. whenever \( y \in V \) and \( y' \in X \) satisfy \( d(y, y') < e(y) \) then there is an element of \( \mathcal{V} \) containing both \( y \) and \( y' \),

where \( d \) is a metric on \( X = \prod_{i=1}^\infty X_i \) chosen so that \( d(y, y') < \frac{1}{n+1} \) if \( y \) and \( y' \) agree on the first \( n \) coordinates. By the choice of \( d \) and (4), we get

15. \( d(\Phi(x, e(f(x))), f(x)) < e(f(x)). \)

To see (15), observe that if \( \frac{1}{n+1} < e(f(x)) \leq \frac{1}{n} \), then \( d(\Phi(x, e(f(x))), f(x)) \leq \frac{1}{n+1}) \).
Define \( g : U \to X \) by \( g(x) = \Phi(x, \varepsilon(f(x))) \).

By (14) and (15), \( g \) is \( \mathcal{V}' \)-close to \( f \) and takes values in \( V \). In turn, (5) and (12) imply that \( g \) takes values in \( W(X_i, *_i) \) and satisfies \( g^{-1}(V \cap W(Y_i, *_i)) = L \cap U \).

It remains to verify that \( g : U \to V \) is a closed embedding. For some sequence \( \{x_k\}_{k=1}^{\infty} \subset U \) let \( \lim g(x_k) = y = (y_i) \in V \). We may assume that \( \{\varepsilon(f(x_k))\}_{k=1}^{\infty} \) converges to some \( \varepsilon_0 \in [0, 1] \). If \( \varepsilon_0 = 0 \) then, by (15), \( \lim f(x_k) = y \), contradicting the fact that \( \varepsilon(y) > 0 \). If \( \varepsilon_0 > 0 \), then we may assume that that \( \varepsilon(f(x_k)) \in (\frac{1}{n+1}, \frac{1}{n-1}) \) for some \( n \) and all \( k \). According to (8), \( g(x_{n+3}(x_k)) = h_{n+3}(x_k) \) and consequently \( \lim h_{n+3}(x_k) = y_{n+3} \). Since \( h_{n+3} \) is a closed embedding, \( \{x_k\}_{k=1}^{\infty} \) converges in \( K \). If \( \lim x_k = x \in K \setminus U \), then \( \lim f(x_k) = f(x) \notin V \) and, by (13), \( \lim \varepsilon(f(x_k)) = \varepsilon(f(x)) = 0 \), a contradiction.

**Note 3.7.** Our proof of Proposition 3.6 requires that at least infinitely many of the \( X_n \)'s are noncompact. Otherwise, it may happen that not all \( Z \)-sets are strong \( Z \)-sets in \( X \) (see [13]). However, the proof (after minor modifications) still works if one assumes that all the \( X_n \)'s are nontrivial local compacta.

4. **Borelian absorbing sets can be linearly represented in \( l^2 \)**

For every countable ordinal \( \alpha > 0 \), by \( \mathcal{A}_\alpha \) and \( \mathcal{M}_\alpha \) we denote the additive and multiplicative classes of all absolute Borelian sets of order \( \alpha \), respectively. To be more specific, \( \mathcal{M}_0 \) consists of all compacta, \( \mathcal{A}_1 \) consists of all \( \sigma \)-compact spaces, \( \mathcal{M}_1 = \mathcal{M} \) consists of all completely metrizable spaces, and \( \mathcal{M}_2 \) consists of all absolute \( F_{\sigma} \)-sets. By \( \mathcal{P}_n \), \( n \geq 1 \), we denote the class of all projective sets of order \( n \); \( \mathcal{P}_0 = \bigcup_\alpha \mathcal{A}_\alpha \). A set that does not belong to \( \bigcup_{n=1}^{\infty} \mathcal{P}_n \) is called nonprojective.

The aim of this section is to find a linear representation of an \( \mathcal{A}_\alpha \)-absorbing set \( F_\alpha \) (respectively, \( \mathcal{M}_\alpha \)-absorbing set \( G_\alpha \)) in \( l^2 \). To perform this, we will make use of \( \mathcal{A}_\alpha \)-absorbing sets \( \Lambda_\alpha \) and \( \mathcal{M}_\alpha \)-absorbing sets \( \Omega_\alpha \) constructed in copies \( s \) of \( l^2 \) in [4]. By the uniqueness theorem for absorbing sets [4], \( F_\alpha \) is homeomorphic to \( \Lambda_\alpha \) and \( G_\alpha \) is homeomorphic to \( \Omega_\alpha \). Actually, we show that the pairs \( (l^2, F_\alpha) \) and \( (s, \Lambda_\alpha), \alpha \geq 1 \) (respectively, \( (l^2, G_\alpha) \) and \( (s, \Omega_\alpha), \alpha \geq 2 \), are homeomorphic. The last is achieved by proving the strong \( (\mathcal{M}, \mathcal{A}_\alpha) \)- and \( (\mathcal{M}, \mathcal{M}_\alpha) \)-universality of suitable pairs. The multiplicative case of order \( \alpha = 1 \) differs from the others and is treated separately in [8]; we include a description of \( G_1 \) in our text in order to formulate the result in full generality.

We briefly recall the definition of \( \Lambda_\alpha \) and \( \Omega_\alpha \). Set \( \Lambda_1 = \Sigma \subset R^\infty = s \) and \( \Omega_1 = W(R^\infty, 0) \subset (R^\infty)^\infty = s \). Inductively, if \( \alpha = \beta + 1 \) let \( \Omega_\alpha = \Lambda^\infty_\beta \subset s^\infty_\beta = s \), where \( \Lambda_\beta \) is represented in \( s_\beta \). If \( \alpha \) is a limit ordinal let \( \Omega_\alpha = \bigcap_{0 < \beta < \alpha} \Lambda^\infty_\beta \subset \bigcap_{0 < \beta < \alpha} s^\infty_\beta = s \), where \( \Lambda_\beta \) is represented in \( s_\beta \). Finally, let \( \Lambda_\alpha = W(s_\alpha \setminus \Omega_\alpha, *) \subset s^\infty_\alpha = s \), where \( \Omega_\alpha \) is represented in \( s_\alpha \) and \( * \) is an arbitrary basepoint of \( s_\alpha \setminus \Omega_\alpha \). By the Kadec-Anderson theorem [2], the spaces \( s \) (in which \( \Lambda_\alpha \) and \( \Omega_\alpha \) are represented) are copies of \( l^2 \).

**Proposition 4.1** (cf. [4]). For every \( \alpha \geq 2 \), the pairs \( (s, \Lambda_\alpha) \) and \( (s, \Omega_\alpha) \) are strongly \( (\mathcal{M}, \mathcal{A}_\alpha) \) - and \( (\mathcal{M}, \mathcal{M}_\alpha) \)-universal, respectively. The pair \( (s, \Lambda_1) = (R^\infty, \Sigma) \) is strongly \( (\mathcal{M}, \mathcal{A}_1) \)-universal.
Proof. We only present a proof of the multiplicative case. We repeat the argument of [4, Lemma 6.3] and use Proposition 3.6.

To show that \((R^\infty, \Sigma)\) is strongly \((\mathcal{M}, \mathcal{A})\)-universal and fix a pair \((K, L) \in (\mathcal{M}, \mathcal{A})\). There exists a closed embedding \(h : K \to R^\infty\) with \(h^{-1}(\Sigma) = L\). (Take any closed embedding of \(K\) into \(R^\infty\) and compose it with a homeomorphism of \(R^\infty\) sending \(h(L) \cup \Sigma\) onto \(\Sigma\); see [2].) Now, by 3.6, \(((R^\infty)^\infty, W(\Sigma, 0))\) is strongly \((\mathcal{M}, \mathcal{A})\)-universal. Since the latter pair is homeomorphic to \((R^\infty, \Sigma)\) [2, p. 275], the strong \((\mathcal{M}, \mathcal{A})\)-universality of \((s, \Lambda_1)\) follows.

We assume that \(\alpha \geq 2\) and \(\alpha = \beta + 1\) (the case of a limit ordinal is analogous). Given \((K, L) \in (\mathcal{M}, \mathcal{A}_\alpha)\) there exists \(L_i \subset K, L_i \in \mathcal{A}_\beta\ (i \geq 1)\), such that \(L = \cap_{i=1}^\infty L_i\). By the inductive assumption, we find a closed embedding \(h_i : K \to s_\beta\) with \(h^{-1}(\Lambda_\beta) = L_i\). Writing \(h = (h_i)\), we see that \(h\) is a closed embedding of \(K\) into \(s_\infty = s\) with \(h^{-1}(\Omega_\alpha) = \cap_{i=1}^\infty L_i = L\). Finally, Proposition 3.6 yields the strong \((\mathcal{M}, \mathcal{A}_\alpha)\)-universality of \((s_\infty, \Lambda_\infty) = (s, \Omega_\alpha)\).

Theorem 4.2. For every \(\alpha > 1\), there exists a linear subspace \(F_\alpha\) of \(l^2\) which is an \(\mathcal{A}_\alpha\)-absorbing set and such that the pair \((l^2, F_\alpha)\) is strongly \((\mathcal{M}, \mathcal{A}_\alpha)\)-universal. In particular, \((l^2, F_\alpha)\) and \((s, \Lambda_\alpha)\) are homeomorphic.

Proof. Construction of \(F_\alpha\). Let \((A, B)\) be a copy of \((s, \Lambda_\alpha)\). Consider a closed embedding \(h\) of \(A\) onto a linearly independent subset of the unit sphere in \(l^2\) satisfying the following condition:

\[(*)\] for every \(a \in A\) and every closed subset \(F \subset A\) with \(a \not\in F\) there exists a continuous linear functional \(x^* : l^2 \to R\) such that \(x^*(h(F)) \subset \{0\}\) while \(x^*(h(a)) \neq 0\).

Condition \((*)\) is taken from [3] where it was checked that the embedding described by Bessaga and Pelczyński [2, p. 193] fulfils \((*)\). Denote by \(H\) the linear span of \(h(B)\) in \(l^2\) and by \(\overline{H}\) the closure of \(H\). Since \(B\) is dense in \(A\), \(\overline{H}\) contains \(h(A)\) as a (closed) subset. Write \((E_n, H_n) = (H, H)\) and set

\[E = \prod_{l^2} E_n\] and \[F_\alpha = \sum_{l^2} H_n.\]

Clearly, \(E\) is isomorphic to \(l^2\) and \(F_\alpha\) is dense in \(E\).

According to 4.1 and the fact that \(\Lambda_\alpha\) is an \(\mathcal{A}_\alpha\)-absorbing set [4], it suffices to prove that the pair \((E, F_\alpha)\) fulfils the requirements (i)-(iv) of 2.2 for the class \(\mathcal{L} = \mathcal{A}_\alpha\). Condition (i) is a consequence of the fact that \(F_\alpha\) is linear and dense in \(E\) (see, e.g., [17, Remark 2.9]). Since each set \(A_k = \{(y_n) \in F_\alpha : y_i = 0\ for i \geq k + 1\}\) is a \(Z\)-set in \(F_\alpha\) and \(F_\alpha = \bigcup_{k=1}^\infty A_k\), \(F_\alpha\) is a \(Z_\sigma\)-space. Condition (iv) follows directly from 3.1 and 4.1 because \(A\) is closed in \(E_\alpha\). The remaining condition (iii) can be concluded from (ii) and the lemma below.

Lemma 4.3. The space \(H\) is in \(\mathcal{A}_\alpha\).

Proof. We shall make use of the cross-section argument due to Klee [2, p. 271]. The \(n\)-fold product \(B^n\) admits a \(\sigma\)-closed cross-section, i.e., there exists a subset \(F\) of \(B^n\) that is a countable union of closed sets \(F_k\) such that

1. if \((b_1, b_2, \ldots, b_n) \in F\) then \(b_i \neq b_j\ for i \neq j\),
2. whenever \(\{y_i\}_{i=1}^n\) are \(n\) distinct points of \(B\) then there exists exactly one permutation of \(y_1, y_2, \ldots, y_n\) that belongs to \(F\).
By (1) and (2) the linear combination map \( \chi \) given by

\[
((b_1, b_2, \ldots, b_n), (\lambda_1, \lambda_2, \ldots, \lambda_n)) \rightarrow \lambda_1 h(b_1) + \lambda_2 h(b_2) + \cdots + \lambda_n h(b_n)
\]
transforms in a one-to-one way the product \( F_k \times D_k^p \) onto \( N_k^p \subset H \), where

\[
D_k^p = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) : \frac{1}{p} \leq |\lambda_i| \leq p \text{ for all } i \right\}
\]
and \( p = 1, 2, \ldots \). Employing, as in [3, Lemma 3.3], the condition (*) one shows that \( x \in \prod_k x \) is a homeomorphism. It shows that each \( N_k^p \), and hence \( H^n = \bigcup_{k, p=1}^{\infty} N_k^p \), belongs to \( \mathcal{A}_\alpha \). Since \( H = \bigcup_{n=1}^{\infty} H^n \), we get \( H \in \mathcal{A}_\alpha \).

**Theorem 4.4.** For every \( \alpha \geq 1 \), there exists a linear subspace \( G_\alpha \) of \( l^2 \) which is an \( \mathcal{M}_\alpha \)-absorbing set and such that the pair \( (l^2, G_\alpha) \) is strongly \( (\mathcal{M}, \mathcal{M}_\alpha) \)-universal. In particular, \( (l^2, G_\alpha) \) and \( (s, \Omega_\alpha) \) are homeomorphic for \( \alpha \geq 2 \).

**Proof.** The case for \( \alpha = 1 \) differs from the others. In [8] it was shown that the space

\[
G_1 = \left\{ (x_n) \in l^2 : \sum_{n=1}^{\infty} |x_n| < \infty \text{ and } \sum_{n=1}^{\infty} x_n = 0 \right\}
\]
is an \( \mathcal{M} \)-absorbing set and, moreover, the pair \( (l^2, G_1) \) is strongly \( (\mathcal{M}, \mathcal{M}) \)-universal.

**Construction of \( G_\alpha \).** Let \( \omega \geq 2 \). If \( \alpha \) is a limit ordinal, then choose an increasing sequence of ordinals \( \{\beta_n\}_{n=1}^{\infty} \) convergent to \( \alpha \); otherwise, \( \alpha = \beta + 1 \) and let \( \beta_n = \beta \). Pick, by 4.2, a pair \( (E_n, F_n) = (l^2, F_{\beta_n}) \) which is \( (\mathcal{M}, \mathcal{A}_{\beta_n}) \)-universal. Set

\[
E = \prod_{l^2} E_n \text{ and } G_\alpha = \prod_{l^2} F_n.
\]
The space \( E \) is isomorphic to \( l^2 \) and \( G_\alpha \) is its linear dense subspace.

Using 4.1 and the fact that \( \Omega_\alpha \) is an \( \mathcal{M}_\alpha \)-absorbing set [4], it is enough to verify conditions (i)–(iv) of 2.2 for the pair \( (E, G_\alpha) \) and \( \mathcal{L} = \mathcal{M}_\alpha \). Condition (i) follows as in the proof of 4.2. A simple argument shows that \( G_\alpha \in \mathcal{M}_\alpha \). Since \( F_1 \) is a \( Z_0 \)-space, \( G_\alpha \) is also a \( Z_0 \)-space. It remains to verify that \( (E, G_\alpha) \) is strongly \( (\mathcal{M}, \mathcal{M}_\alpha) \)-universal. We will apply 3.4. Let \( N_1, N_2, \ldots \) be any decomposition of the set of integers \( N \) into pairwise disjoint infinite sets. Then, the space \( C_k \) defined in 3.4 is \( l^2(N_k) \), the space of all square summable sequences indexed by the integers of \( N_k \). As a result \( \prod_{l^2} E_p \prod_{l^2} H_p = \prod_{l^2} E_p \prod_{l^2} H_p \). By the choice of \( \beta_n \), it is clear that the following lemma will finish the proof of 4.4.

**Lemma 4.5.** For every \( (K, L) \in (\mathcal{M}, \mathcal{M}_\alpha) \) there exists a bounded closed embedding \( \psi : K \rightarrow E \) with \( \psi^{-1}(G_\alpha) = L \).

**Proof.** Let \( L = \bigcap_{k=1}^{\infty} L_k \), where \( L_k \subset K \), \( L_{k+1} \subset L_k \), and \( L_k \in \mathcal{A}_{\beta_{n_k}} \) for some \( n_k \) with \( n_{k+1} > n_k \). Write \( \beta(k) = \beta_{n_k} \). Since \( (E_{n_k}, F_{n_k}) \) is strongly \( (\mathcal{M}, \mathcal{A}_{\beta(k)}) \)-universal, there exists a closed embedding \( \psi_{n_k} : K \rightarrow E_{n_k} \) such that

1. \( \|\psi_{n_k}(x)\| \leq (\frac{1}{2})^{n_k} \) for all \( x \in K \),
2. \( \psi_{n_k}^{-1}(F_{n_k}) = L_k \).
Write \( \psi_k \equiv 0 \) for all \( n \neq n_k \) \((k \geq 1)\) and set \( \psi = (\psi_n) \). By (1), \( \psi \) is continuous and bounded. It is easy to see that \( \psi : K \to E \) is a closed embedding with \( \psi^{-1}(G) = \bigcap_{k=1}^{\infty} L_k = L \).

Remark 4.6. As pointed out in [8], the pair \((s, \Omega) = ((R^\infty)^\infty, W(R^\infty, 0))\) is not strongly \((\mathcal{M}, \mathcal{M})\)-universal. (If it were, then by 2.2, \((s, \Omega)\) and \((l^2, G_1)\) would be homeomorphic; consequently, \( G_1 \) would be \( \sigma \)-closed in \( l^2 \), which contradicts a result of [15].)

Remark 4.7. The spaces \( \Lambda_\alpha \) and \( \Omega_\alpha \) can be realized as linear subspaces in other normed coordinate products \( \prod E_n \). The only restriction is the condition (*)

Remark 4.8. The result of 4.1 can be readily generalized to the triple case. Representing \( \Lambda_\alpha \) \((\alpha \geq 1)\) and \( \Omega_\alpha \) \((\alpha \geq 2)\) in \( R^\infty \), we could consider the triples

\[
(R^\infty, R^\infty, \Lambda_\alpha) \quad \text{and} \quad (R^\infty, R^\infty, \Omega_\alpha),
\]

where \( R = [-\infty, +\infty] \). These triples are strongly \((\mathcal{M}_0, \mathcal{M}, \mathcal{A}_\alpha)\)- and \((\mathcal{M}_0, \mathcal{M}, \mathcal{A}_\alpha)\)-universal, respectively (with an obvious meaning of the triple strong universality).

In §6 we shall need the following fact concerning the complements of \( F_\alpha \) and \( G_\alpha \).

Corollary 4.9. The space \( l^2 \setminus F_\alpha \) (respectively, \( l^2 \setminus G_\alpha \)) has the following properties:

(i) \( l^2 \setminus F_\alpha \) (respectively, \( l^2 \setminus G_\alpha \)) is a Baire space,

(ii) \( l^2 \setminus F_\alpha \in \mathcal{M} \setminus \mathcal{A}_\alpha \) (respectively, \( l^2 \setminus G_\alpha \in \mathcal{A}_\alpha \setminus \mathcal{M} \)),

(iii) \( l^2 \setminus F_\alpha \) (respectively, \( l^2 \setminus G_\alpha \)) is homogeneous,

(iv) for every (closed) ball \( B \subset l^2 \), \( B \setminus F_\alpha \) (respectively, \( B \setminus G_\alpha \)) is an absolute retract.

Proof. We will only deal with the \( F_\alpha \)-case, the \( G_\alpha \)-case is analogous. Conditions (i) and (ii) follow from the fact that \( F_\alpha \) is of the first category and that \( F_\alpha \in \mathcal{A}_\alpha \setminus \mathcal{M} \). To show (iii), we produce a homeomorphism of \( l^2 \) that preserves \( F_\alpha \) and carries \( y \in l^2 \setminus F_\alpha \) onto \( y' \in l^2 \setminus F_\alpha \). Let \( h \) be any homeomorphism of \( l^2 \) with \( h(y) = y' \), e.g., \( h \) is the translation. Then, the pairs \((l^2, h(F_\alpha))\) and \((l^2, F_\alpha)\) are strongly \((\mathcal{M}, \mathcal{A}_\alpha)\)-universal. The proof of [7, Theorem 2.1] can be easily modified to achieve a homeomorphism \( k \) of \( l^2 \) that carries \( h(F_\alpha) \) onto \( F_\alpha \) and preserves \( \{y'\} \) (set \( X_0 = Y_0 = \{y'\} \subset X_1 \cap Y_1 \)). We see that \( k \circ h \) preserves \( F_\alpha \) and sends \( y \) onto \( y' \). Condition (iv) is a consequence of the fact that \( B \cap F_\alpha \) is locally homotopy negligible in \( B \) and [17, Theorem 3.1]. Assume \( 0 \in \text{int} B \) and pick \( y_0 \in B \setminus F_\alpha \). Since the homotopy \( f_t(y) = (1-t)y + ty_0 \) \((0 \leq t \leq 1)\) takes its values in \( B \setminus F_\alpha \) for \( t > 0 \) and \( f_0 = \text{id} \), the local homotopy negligibility of \( B \cap F_\alpha \) in \( B \) follows.

5. Application to \( F_{\sigma\delta} \)-spaces

In this section we identify various absolute \( F_{\sigma\delta} \)-sets carrying product structures to be homeomorphic to \( \Omega_\delta = \Sigma^\infty \). The spaces we deal with will be considered with both normed and cartesian product topologies. We start with a direct application of Proposition 3.4 to coordinate products of normed \( F_{\sigma\delta} \)-spaces. A counterpart of 5.1 for cartesian products was previously obtained in [13].
Theorem 5.1. Let $\prod C H_n$ be a normed coordinate product of absolute $F_\delta$-spaces $H_n$ in the sense of a Banach space $C$. Assume that infinitely many of the $H_n$'s are $Z_\sigma$-spaces. Then $\prod C H_n$ is homeomorphic to $\Omega_2$. Moreover, writing $E_n$ for the linear completion of $H_n$, the pairs $(\prod C E_n, \prod C H_n)$ and $(s, \Omega_2)$ are homeomorphic.

We shall make use of the two lemmas below. A proof of the first one is implicitly contained in [12, Lemma 5.4] and therefore it will be omitted.

Lemma 5.2. Let $X \in \mathcal{M}$ be an absolute retract and $Y \subset X$ be a $Z_\sigma$-space such that $X \setminus Y$ is locally homotopy negligible in $X$. Then, for every $L \in \mathcal{A}'$ of the Hilbert cube $I^\infty$, there exists a map $\varphi : I^\infty \to X$ with $\varphi^{-1}(Y) = L$.

Lemma 5.3. Let $(H_n, \| \cdot \|_{H_n})$ be a normed linear space that is noncompactly embedded into a Banach space $(E_n, \| \cdot \|_{E_n})$, i.e., $H_n \subseteq E_n$, $\| \cdot \|_{E_n} \leq \| \cdot \|_{H_n}$, and the $E_n$-closures of $H_n$-balls are noncompact. Then, for every coordinate Banach space $C$ and every $K \in \mathcal{M}$, there exists a bounded closed embedding $\psi : K \to \prod C E_n$ such that $\psi(K) \subseteq \prod C H_n$.

Proof. Since $C$ is complete, there is $c = (c_n) \in C$ with all the $c_n$'s strictly positive. It is enough to construct a closed embedding $\psi = (\psi_n) : K \to \prod C E_n$ with $\psi_n(x) \in H_n$ and $\|\psi_n(x)\|_{H_n} \leq c_n$ for all $x \in K$ and $n \geq 1$. Fix a metric $d$ on the Hilbert cube $I^\infty = [0, 1]^\infty$. Embed $K$ into $I^\infty$ and write $I^\infty \setminus K = \bigcup_{n=1}^{\infty} F_n$, where each $F_n$ is a closed subset of $I^\infty$. Define $d_n(q) = \text{dist}_d(q, F_n)$, $q \in I^\infty$. We claim that there exists a closed embedding $\alpha_n : [0, \infty) \to E_n$ such that $\alpha_n(t) \in H_n$ and $\|\alpha_n(t)\|_{H_n} \leq 1$ for all $t \geq 0$ and $n \geq 1$. This follows from the noncompactness of the $E_n$-closure of the $H_n$-unit ball and Klee's result [14] that every noncompact closed convex subset $F$ of $E_n$ contains a copy of $[0, \infty)$. (The piecewise linear embedding $\alpha$ constructed by Klee can be improved to get the nodes of $\alpha$ contained in any dense linear subset of $F$.) Pick a vector $e_{2n} \in H_{2n}$ with $\|e_{2n}\|_{H_{2n}} = 1$. Define $\psi = (\psi_n)$ by

$$
\psi_{2n}(q) = c_{2n} q_n e_{2n} \quad \text{and} \quad \psi_{2n-1}(q) = c_{2n-1} \alpha_{2n-1}((d_n(q))^{-1})
$$

for $q = (q_n) \in K$. It is clear that $\psi : K \to \prod C E_n$ is one-to-one and $\|\psi_n(q)\|_{H_n} \leq c_n$, $n \geq 1$, $q \in K$. By [16, Lemma 1.4], $\psi$ is continuous. If $\{\psi(q(i))\}_{i=1}^{\infty}$ is convergent in $\prod C E_n$, then there exists $q_0 \in I^\infty$ with $\lim_{i \to \infty} q(i) = q_0$. Assume that $q_0 \in F_k$ for some $k$. Then $\lim_{i \to \infty} d_k(q(i)) = 0$, contradicting the fact that $\{(d_k(q(i)))^{-1}\}_{i=1}^{\infty}$ converges. The latter is a consequence of the facts that the sequence $\{c_{2n-1}((d_k(q(i)))^{-1})\}_{i=1}^{\infty}$ is convergent in $E_k$ and that $\alpha$ is a closed embedding.

Proof of 5.1. We show that the pair $(E, H) = (\prod C E_n, \prod C H_n)$ satisfies (i)-(iv) of 2.2 with $\mathcal{L} = \mathcal{M}_2$. By the Kadec-Anderson theorem [2], $E$ is a copy of $l^2$. A standard argument yields (i)-(iii). The strong $(\mathcal{M}, \mathcal{M}_2)$-universality of $(E, H)$ will be derived from Proposition 3.4. Fix a pair $(K, L) \in (\mathcal{M}, \mathcal{M}_2)$. Find pairwise disjoint infinite subsets $N_1, N_2, \ldots$ of $N$ so that $H_p$ is a $Z_\sigma$-space for every $p \in \bigcup_{k=1}^{\infty} N_k$. Let $C_k$ be the subspace of $C$ that corresponds to $N_k$ (see 3.4). Write $E^k = \prod C_k E_p$ and $H^k = \prod C_k H_p$. To fulfill the hypothesis of 3.4 we have to produce a bounded closed embedding $\psi_k : K \to E^k$ with $\psi_k^{-1}(E^k) = L$. To this end we split $C_k = C_k^1 \oplus C_k^2$ into two coordinate spaces and find a bounded closed embedding $\psi_k : K \to \prod_{C_k^1} E_p$ with $\psi_k(K) \subseteq \prod_{C_k^1} H_p$. 

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and a bounded map $\psi_k^2 : K \to \prod C_k E_p$ with $(\psi_k^2)^{-1}(\Pi_{C_k} H_p) = L$. Finally, letting $\psi_k = (\psi_k^1, \psi_k^2)$ we get a required embedding (we identify $\prod C_k E_p$ with $\Pi_{C_k} E_p \oplus \Pi_{C_k} E_p$).

To find $\psi_k^1$ we apply Lemma 5.3. In this case $H_p$ is a genuine subspace of an infinite-dimensional Banach space $E_p$; hence the balls in $E_p$ are noncompact. Let $c = (c_p) \in C_k^+$ be such that all $c_p$ are strictly positive. Embed $K$ into $I^\infty$ and represent $L = \cap_p L_p$ so that each $L_p$ is $\sigma$-compact and $\{L_p\}$ is descending. Use Lemma 5.2 with $X = B(c_p)$, the closed ball in $E_p$ centered at 0 with radius $c_p$, $Y = B(c_p) \cap H_p$, and $L = L_p$. We get maps $\varphi_p : I^\infty \to B(c_p)$ with $\varphi_p^{-1}(Y) = L_p$. Finally, we set $\psi_k^1(x) = (\varphi_p(x))$, $x \in K$. The continuity of $\psi_k^2$ follows from [16, Lemma 4.1]. To verify the hypothesis of 5.2, notice that $Y$ is convex and dense in $X$. Let $H_p = \bigcup_{m=1}^{\infty} A_m$, where each $A_m$ is a $Z$-set in $H_p$. Then $\text{int}(B(c_p)) \cap A_m$ is a $Z$-set in $\text{int}(B(c_p)) \cap H_p$. Since $\text{int}(B(c_p)) \cap H_p$ is convex and dense in $B(c_p) \cap H_p$, $B(c_p) \cap A_m$ is a $Z$-set in $B(c_p) \cap H_p$. It shows that $Y$ is a $Z_\sigma$-space.

A direct consequence of 5.1 is

**Note 5.4.** The simplest pre-Hilbert space representation of $\Omega_2$ is the space $\prod_{i=1}^2 l^2_i$, where $l^2_i = \{ (x_i) \in l^2 : x_i = 0$ for almost all $i \}$. Moreover, the pairs $(\prod_{i=1}^2 l^2_i, (\Omega_2 \times \Omega_2))$ are homeomorphic.

Consider the set $\prod C H_n = H$ as a subspace of the cartesian product $\prod_{n=1}^{\infty} E_n = E$. By the Kadec-Anderson theorem [2], $E$ is a copy of $l^2$. Easily, $E \cap H$ is locally homotopy negligible in $E$. We claim that $C$ is an $F_{\sigma\delta}$-subset of $R^\infty$. This is a consequence of the equality

$$C = \left\{ (x_n) \in R^\infty : \forall \varepsilon > 0 \exists k \forall m > k \left\| \sum_{i=k}^{m} x_i u_i \right\|_C \leq \varepsilon \right\}$$

($u_i$ is the $i$th unit vector). Consider the map $f(x) = (\|x_n\|)$, $x = (x_n) \in E$, and notice that $f^{-1}(C) = \prod C E_n$. This shows that $\prod C E_n \in \mathcal{M}_2$. Since $\prod C H_n = \prod C E_n \cap \prod_{n=1}^{\infty} H_n$, $H$ is an absolute $F_{\sigma\delta}$-set. Repeating (with obvious changes) the remaining part of the proof of 5.1, we get the following generalization of a result [13].

**Theorem 5.5.** Let $\{H_n\}_{n=1}^{\infty}$ be a sequence of normed linear spaces such that each $H_n$ is an absolute $F_{\sigma\delta}$-set and infinitely many of the $H_n$'s are $Z_\sigma$-spaces. Then, for every coordinate Banach space $C$, the space $\prod C H_n$ considered in the product topology, is homeomorphic to $\Omega_2$. Moreover, if $E_n$ is the linear completion of $H_n$ then the pairs $(\prod_{n=1}^{\infty} E_n, \prod C H_n)$ and $(s, \Omega_2)$ are homeomorphic.

**Remark 5.6.** The hypothesis that infinitely many of the $H_n$'s are $Z_\sigma$-spaces is essential. Consider the coordinate space $c_0 = \{ (x_i) \in R^\infty : \lim x_i = 0 \}$ with the $\| \cdot \|_\infty$-norm. Note that $c_0 \subset \bigcup_{k=1}^{\infty} B_\infty(k)$, where $B_\infty(k) = \{ x \in R^\infty : \|x\|_{\infty} \leq k \}$. This shows that $c_0$ is contained in a $\sigma$-compact subset of $R^\infty$. On the other hand $\Omega_2$ contains a copy of $R^\infty$ closed in $s$. This shows that $(R^\infty, \prod c_0 R)$ and $(s, \Omega_2)$ are not homeomorphic, contrary to the expectation expressed in [13]. Proposition 3.6 and Lemma 5.2 yield the strong $(\mathcal{M}_0, \mathcal{M}_2)$-universality of the pair $(R^\infty, c_0)$. 

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For the $c_0$-products we have the following generalization of [13, Theorem 4.2].

**Theorem 5.7.** Let, for $n \geq 1$, $X_n$ be a subset of a Banach space $(E_n, \| \cdot \|_n)$ with $0 \in X_n$ so that $\inf_{n \geq 1} \text{diam}(X_n) = \alpha > 0$. Assume that each $X_n$ is an absolute retract that is an absolute $F_{\sigma\delta}$-set. Then the space

$$X = \prod_{c_0} X_n = \left\{(x_n) \in \prod_{n=1}^{\infty} X_n : \|x_n\|_n \to 0\right\}$$

(endowed with the product topology) is homeomorphic to $\Omega_2$.

**Proof.** We will show that $X$ is an $\mathcal{M}_2$-absorbing set in some copy of $l^2$, i.e., we will verify conditions (i)–(iii) of 2.2 and the strong $\mathcal{M}_2$-universality of $X$. Then, the uniqueness theorem for absorbing sets [4] yields our assertion. Since $\prod_{n=1}^{\infty} X_n$ is an absolute retract and $\prod_{n=1}^{\infty} X_n \setminus X$ is locally homotopy negligible, $X$ is also an absolute retract [17, Theorem 3.1]. Decompose the set of integers $\mathbb{N}$ into pairwise disjoint infinite sets $N_1, N_2, \ldots$. Write $X' = \bigcup_{p \in \mathbb{N}} X_p$ and $Y' = \{(x^p) \in X': \|x^p\|_p \to 0\}$ and let $\Psi : \prod_{n=1}^{\infty} X_n \to \prod_{i=1}^{\infty} X'$ be the natural isomorphism. We have

1. $\Psi(X) \supset W(Y', 0)$.

Note that each $Y'$ is noncompact. (If it were compact then, because $Y'$ is dense in $X'$, we would get $X' = Y'$, contradicting the fact that $\alpha > 0$.) Now, each $Y'$ has a completion $Z'$ with $Z' \setminus Y'$ locally homotopy negligible in $Z'$ [17, Proposition 4.1]. Since $Y'$ is noncompact, we can assume that $Z'$ is also noncompact (if it were compact take $Z' \setminus \{\ast\}$, where $\ast \in Z' \setminus Y'$). Then the product $s = \prod_{i=1}^{\infty} Z'$ is a copy of $l^2$ [18, Theorem 5.1] with $s \setminus X$ locally homotopy negligible in $s$; this shows (i). An argument preceding Theorem 5.5 applies to show that $X$ is an absolute $F_{\sigma\delta}$-set. Writing $A_m = \{(x_n) \in X : \|x_j\|_j \leq \frac{\alpha}{2} \text{ for all } j \geq m + 1\}$, we see that $\bigcup_{m=1}^{\infty} A_m = X$ and that each $A_m$ is a $Z$-set in $X$. In proving the strong $\mathcal{M}_2$-universality of $X$ we employ 3.6 with $Y_i = Y'$, $K = L \in \mathcal{M}_2$, and $Z = \Psi(X)$. To produce a closed embedding of $L$ into $Y'$, we may assume that $Y' = X'$.

Write $B'((e)) = \{(x^p) \in X': \|x^p\|_p \leq e \text{ for all } p \in N_i\}$ and notice that

2. $\prod_{i=1}^{\infty} Y_i \cap B'((2^{-i}))$ is a closed subset of $\Psi(X)$.

By (2) and the fact that a countable product of $Z_{\sigma}$-spaces that are absolute retracts contains a closed copy of $\Omega_2$ [13, Corollary 2.5], it suffices to check that each $Y' \cap B'((2^{-i}))$ contains a closed $Z_{\sigma}$-space that is an absolute retract. Assuming $\alpha \geq 2^{-i}$, we choose in each $X_p$ an arc $T_p$ joining 0 with some $x_p$ with $\|x_p\|_p = 2^{-i}$. Write $T' = Y' \cap \prod_{p \in N_i} T_p$. Then $T'$ is a closed subset of $Y'$. The argument showing that $X$ is a $Z_{\sigma}$-space applies also to verify that $T'$ is a $Z_{\sigma}$-space. The proof is completed.

Let us note a relative version of [13, Corollary 2.7].

**Remark 5.8.** Let $X_n \in \mathcal{M}$ be a noncompact absolute retract and let $Y_n$ be a subset of $X_n$ such that $X_n \setminus Y_n$ is locally homotopy negligible in $X_n$, $n = 1, 2, \ldots$. If each $Y_n$ is an absolute $F_{\sigma\delta}$-set and infinitely many of the $Y_n$'s are $Z_{\sigma}$-spaces, then the pairs $(\prod_{n=1}^{\infty} X_n, \prod_{n=1}^{\infty} Y_n)$ and $(s, \Omega_2)$ are homeomorphic. Apply 2.2 together with 3.6. To produce a closed embedding $h : K \to \prod_{n=1}^{\infty} X_n$.
with \( h^{-1}(\prod_{n=1}^{\infty} Y_n) = L \) employ Lemma 5.2 and the fact that \( \prod_{n=1}^{\infty} X_n \) contains a closed copy of \([0, \infty)\) that lives in \( \prod_{n=1}^{\infty} Y_n \). Moreover, adopting Theorem 2.2 and Lemma 5.2 to the triple case one can get a homeomorphism of suitable triples (see [9]); 4.8.

Let us recall that by \( L^p[a, b] \) we denote the space of equivalence classes of Lebesgue measurable functions \( x : [a, b] \to \mathbb{R} \) with

\[
\|x\|_p = \left( \int_a^b |x(t)|^p \, dt \right)^{\min(1, \frac{1}{p})} < \infty
\]

with the topology induced by the \( F \)-norm \( \| \cdot \|_p \), \( 0 < p < \infty \). Write \( \widetilde{L}^p[a, b] = \bigcap_{p' < p} L_{p'}[a, b] \), \( 0 < p \leq \infty \), and by \( \widetilde{L}^q[a, b] \) denote the set \( \widetilde{L}^p[a, b] \) with the \( \| \cdot \|_q \)-topology, \( q < p \). Note that \( \widetilde{L}^q[a, b] \) is dense in \( L^q[a, b] \). We skip the symbol \([a, b]\) if \([a, b] = [0, 1]\).

**Theorem 5.9.** The pairs \((L^q, \widetilde{L}^p)\) and \((s, \Omega_2)\) are homeomorphic for \( 0 < q < p \leq \infty \).

**Proof.** Mazur's homeomorphism [2, p. 207] of \( L^1 \) onto \( L^q \) transforms \( \widetilde{L}^p \) onto \( \widetilde{L}^{pq} \). Therefore, it suffices to consider the case of \( q = 1 \) (and arbitrary \( p > 1 \)).

We write \( \widetilde{L}^p = \tilde{L}^p_q \). Since \( L^1 \) is a copy of \( l^2 \) [2], it is enough to verify conditions (i)-(iv) of 2.2. The local homotopy negligibility of \( L^1 \setminus \tilde{L}^p \) follows in a standard way. Note that each \( L^p \) is an \( F_\sigma \)-subspace of \( L^q \) for \( p > q \). (This is a consequence of the facts that \( L^p \) is an \( F_\sigma \)-subspace of \( L^0 \), the space of measurable functions with the convergence in measure topology (see [13]), and that the \( L^0 \)-topology is weaker than the \( \| \cdot \|_q \)-topology.) Select an increasing sequence \( \{p_n\}_{n=1}^\infty \subset (1, p) \) that converges to \( p \). Since \( \widetilde{L}^p = \bigcap_{n=1}^{\infty} L^{p_n} \), we get \( \widetilde{L}^p \in \mathcal{M}_2 \).

To prove that \( \widetilde{L}^p \) is a \( Z_\sigma \)-space, we choose \( 1 < p' \leq p \) and write

\[
B_{p'}(\varepsilon) = \{ x \in L^p : \|x\|_{p'} \leq \varepsilon \}.
\]

Since \( \widetilde{L}^p = \bigcup_{k=1}^{\infty} B_{p'}(k) \cap \widetilde{L}^p \) it suffices to check that each \( A = B_{p'}(k) \cap \widetilde{L}^p \) is a \( Z \)-set in \( L^p \). First of all, note that \( B_{p'}(k) \) is a \( Z \)-set in \( L^1 \) because it is a closed subset of a locally homotopy negligible set \( L^{p'} \) in \( L^1 \). Then, using the fact that \( L^1 \setminus \widetilde{L}^p \) is locally homotopy negligible in \( L^1 \), we infer that \( A \) is a \( Z \)-set in \( L^p \) (see [5, Lemma 2.6]).

We make use of 3.1 to verify the strong \((\mathcal{M}, \mathcal{M}_2)\)-universality of \((L^1, \widetilde{L}^p)\). The map \( \Psi \) given by

\[
\Psi(x) = (x[2^{-n}, 2^{-n+1}])_{n=1}^{\infty},
\]

\( x \in L^1 \), is a linear isomorphism of \( L^1 \) onto \( \prod_{n} L^1[2^{-n}, 2^{-n+1}] \). Writing \( Z = \Psi(\widetilde{L}^p) \), we have

\[
Z \cap \sum_{n} L^1[2^{-n}, 2^{-n+1}] = \sum_{n} \widetilde{L}^p[2^{-n}, 2^{-n+1}].
\]

Since the pair \((E_n, H_n) = (L^1[2^{-n}, 2^{-n+1}], \widetilde{L}^p[2^{-n}, 2^{-n+1}])\) is (naturally) isomorphic to \((L^1, \widetilde{L}^p)\), the lemma below verifies the hypothesis of 3.1 and thus finishes the proof of 5.9.
Lemma 5.10. Let \((K, L) \in (\mathcal{M}, \mathcal{M}_2)\). There exists a bounded closed embedding 
\(\psi : K \to L^1\) with \(\psi^{-1}(\tilde{L}^p) = L\).

Proof. We repeat a reasoning from the proof of 5.1. First, we find a bounded closed embedding 
\(\psi^1 : K \to L^1[\frac{1}{2}, 1]\) with \(\psi^1(K) \subset \tilde{L}^p\). Then, we produce a bounded map 
\(\psi^2 : K \to L^1[0, \frac{1}{2}]\) with \((\psi^2)^{-1}(\tilde{L}^p[0, \frac{1}{2}]) = L\). Finally, we let 
\(\psi = (\psi^1, \psi^2)\). To get \(\psi^1\), we apply 5.3 with \(H_n = L^p[\frac{1}{2}, 1]\), 
\(E_n = L^1[\frac{1}{2}, 1]\), and \(C = L^1\). It is clear that \(H_n\) is noncompactly embedded in 
\(E_n\). Consequently there exists a bounded closed embedding 
\(\psi^1 : K \to \prod L^1[\frac{1}{2}, 1] = L^1[\frac{1}{2}, 1]\) such that 
\(\psi^1(K) \subset \prod L^p[\frac{1}{2}, 1] \subset \prod L^p[\frac{1}{2}, 1] = L^p[\frac{1}{2}, 1] \subset \tilde{L}^p[\frac{1}{2}, 1]\). To obtain \(\psi^2\), embed \(K\) into \(L^\infty\) and represent 
\(K = \bigcup_{n=1}^\infty B_{B^p(2^{-n})} \bigcap B_{B^p(2^{-n})}\). We claim that each 
\(A = B_{B^p(2^{-n})} \cap \tilde{L}^p\) is a \(Z\)-set in \(Y\). Since \(B_{B^p(2^{-n})}\) is a \(Z\)-set in \(L^p\), it easily follows that 
\(B_{B^p(2^{-n})} \cap B_{B^p(2^{-n})}\) is a \(Z\)-set in \(B_{B^p(2^{-n})}\). Now the local homotopy negligibility of \(X \setminus Y\) in \(X\) implies, via [5, Lemma 2.6], that \(A\) is a \(Z\)-set in \(Y\). This finishes the proof.

Remark 5.11. One could likely elaborate an abstract scheme of identifying 
some normed coordinate products that are homeomorphic to \(\Omega_2\), as done for 
cartesian products in [13]. Due to replacing the convex structure by a suitable 
equiconnected structure on \(L^0([0, 1], G)\), the space of measurable \(G\)-valued functions on \([0, 1]\), it was proved in [13] that \(\tilde{L}^p([0, 1], G)\) (with the 
\(L^0\)-topology) is homeomorphic to \(\Omega_2\), provided \(G\) is a closed unbounded 
subset of a Banach space. Using 3.6 and 5.2, one can show that the pair 
\((L^0([0, 1], G), \tilde{L}^p([0, 1], G))\) is homeomorphic to\((s, \Omega_2)\) for \(0 < p \leq \infty\). 
To produce a closed embedding of \(R^\infty\) in \(L^0([0, 1], G)\) with values in 
\(B = \{ x \in L^1 : |x(t)| \leq \varepsilon \text{ almost everywhere } \}\), we use the argument of 5.3 and 
the fact that \(B \cap L^0([0, 1], G)\) is a copy of \(I^2\) [2]. It is likely that the pairs 
\((L^q([0, 1], G), \tilde{L}^q([0, 1], G))\) and \((s, \Omega_2)\) are also homeomorphic.

By \(l^p\) we denote the space of real-valued sequences \(x = (x_n)\) such that 
\[\|x\|_p = \left( \sum_{n=1}^\infty |x_n|^p \right)^{\min(1, \frac{1}{p})} < \infty\]
with the topology induced by the \(F\)-norm \(\| \cdot \|_p\), \(0 < p < \infty\). Write \(\tilde{l}^p = \)
\( \cap_{p' > p} l^{p'}, \quad 0 \leq p < \infty, \) and denote by \( \widetilde{l}^p_q \) the space \( \widetilde{l}^p \) with the \( \| \cdot \|_q \)-topology, \( q > p \). Note that \( \widetilde{l}^p_q \) is a dense linear subspace of \( l^q \).

**Theorem 5.12.** The pairs \((l^q, \widetilde{l}^p_q)\) and \((s, \Omega_2)\) are homeomorphic for \(0 < p < q < \infty\).

**Proof.** As in the proof of 5.9, we only need to check that \((l^1, \widetilde{l}^p)\), \(0 < p < 1\), fulfills conditions (i)-(iv) of 2.2; we write \( \widetilde{l}^p = \widetilde{l}^p_q \). A verification of (i) and (iii) is almost the same as in 5.9 and uses the observation that

\[
B_{p'}(e) = \{x \in l^{p'} : \|x\| < e\}
\]

is closed in the \( \| \cdot \|_{p'} \)-topology \((p > p')\). Also, every set \( B_{p'}(k) \cap \widetilde{l}^p \) is a Z-set in \( \widetilde{l}^p \), yielding (ii). To verify (iv) we make use of 3.1. Decompose \( N \) into pairwise disjoint infinite sets \( N_1, N_2, \ldots \). Consider the linear isomorphism \( \Phi : l^1 \rightarrow \prod_{n} l^1(N_n) \), where \( l^1(N_n) \) is an isomorphic copy of \( l^1 \) of sequences indexed by integers of \( N_n \), given by

\[
\Phi(x) = ((x_k)_{k \in N_1}, (x_k)_{k \in N_2}, \ldots),
\]

\( x \in l^1 \). Writing \( Z = \Phi(\widetilde{l}^p) \), we see that

\[
Z \cap \sum_{n} l^1(N_n) = \sum_{n} \widetilde{l}^p(N_n).
\]

The following lemma enables us to apply 3.1 and hence to finish the proof of 5.12.

**Lemma 5.13.** Let \((K, L) \in (M, M_2)\). There exists a bounded closed embedding \( \psi : K \rightarrow l^1 \) with \( \psi^{-1}(\widetilde{l}^p) = L \).

**Proof.** We follow the proof of 5.10. As in 5.10 we embed \( K \) into \( l^\infty \), represent \( L = \cap_{n=1} l^1(N_n) \), and pick a sequence \((p_n) \subseteq (p, 1)\) convergent to \( p \). A bounded closed embedding \( \psi^1 : K \rightarrow l^1(N_1) \) with \( \psi^1(K) \subseteq \widetilde{l}^p \) is obtained via Lemma 5.3. We take \( H_n = (l^p(N_1), \| \cdot \|_p), E_n = l^1(N_1), \) and \( C = l^1 \). (Formally, we are not eligible to apply 5.3 because \( H_n \) is not a normed space. This assumption was only used to construct the closed embedding of \([0, \infty)\). In our case the unit closed ball \( B \) in \( l^p(N_1) \) is homeomorphic, via Mazur's homeomorphism \([2, p. 207]\), to the closed unit ball in the Hilbert space which, in turn, is homeomorphic to \( R^\infty \). Therefore \( B \) being closed in \( l^1(N_1) \) admits a required embedding.) Hence, we get a bounded closed embedding \( \psi^1 : K \rightarrow \prod_{n} l^1(N_1) = l^1(N_1) \) with \( \psi^1(K) \subseteq \prod_{n} l^p \subseteq l^p \subset \widetilde{l}^p \). To produce \( \psi^2 \), we apply 5.2 to the pair \((X, Y) = (B_{p_n}(2^{-n}), B_{p_n}(2^{-n}) \cap \widetilde{l}^p)\) find maps \( \phi_n : K \rightarrow l^p(N_n), n \geq 2 \), with \( \phi_n^{-1}(\widetilde{l}^p(N_n)) = L_n \) and \( \|\phi_n(x)\|_{p_n} \leq 2^{-n} \) for all \( x \in K \). It is easy to see that the map \( \psi^2(x) = (\phi_n(x))_{n=2}^\infty \) satisfies \( (\psi^2)^{-1}(\widetilde{l}^p(N \setminus N_1)) = L \). We let \( \psi = (\psi^1, \psi^2) \).

Let us formulate a more specific result concerning \( \widetilde{l}^p \)-products whose proof is a modification of the proof of 5.7 (and therefore will be omitted).

**Theorem 5.14.** Let, for \( n \geq 1 \), \( X_n \) be a subset of a Banach space \((E_n, \| \cdot \|_n)\) with \( 0 \in X_n \) so that \( \inf_{n \geq 1} \text{diam}(X_n) = \alpha > 0 \). Assume that each \( X_n \) is an
absolute retract that is an absolute $F_{\omega \delta}$-set. Then the space

$$\tilde{l}^p(X_n) = \left\{ (x_n) \in \prod_{n=1}^{\infty} X_n : \forall p' > p, \sum_{n=1}^{\infty} \|x_n\|_{p'} < \infty \right\}$$

(as a subspace of $\prod_{n=1}^{\infty} X_n$) is homeomorphic to $\Omega_2$ for every $0 \leq p < \infty$.

Remark 5.15. Assume that each $X_n \in \mathcal{M}$. We may ask whether $(\prod_{n=1}^{\infty} X_n, \tilde{l}^p(X_n))$ is homeomorphic to $(s, \Omega_2)$. This, in general, is not necessarily the case. The space $\tilde{l}^p(R) = \tilde{l}^p$ is contained in a $\sigma$-compact subset of $R^\infty$ (cf. Remark 5.6). Let us notice that the pair $(R^\infty, \tilde{l}^p)$ is strongly $(\mathcal{M}_0, \mathcal{M}_2)$-universal. The assertion of Theorem 2.2 holds if one replaces $\mathcal{M}$ by $\mathcal{M}_0$ and add in the hypothesis that both $Y_1$ and $Y_2$ are contained in $\sigma$-compact subsets of $X$. As a consequence, the pairs $(R^\infty, c_0)$ and $(R^\infty, \tilde{l}^p)$ are homeomorphic for $0 \leq p < \infty$. This shows that two $\mathcal{L}$-absorbing sets $Y_1$ and $Y_2$ can be relatively homeomorphic in a copy $X$ of $l^2$ while none of the pairs $(X, Y_1)$ and $(X, Y_2)$ are strongly $(\mathcal{M}, \mathcal{L})$-universal.

6. The spaces $F_\alpha$ and $G_\alpha$ as factors of exotic pre-Hilbert spaces

In this section we present some examples concerning the topological classification of pre-Hilbert spaces. Examples we deal with are of the form $Y(A) \times F_\alpha$ and $Y(A) \times G_\alpha$, where $Y(A) = \text{span}(A)$ and $A$ is a linearly independent subset of $l^2$.

Fix a linearly independent arc $T = [0, 1]$ in $l^2$ such that $Y(A)$ is dense in $l^2$ for every infinite set $A \subseteq T$ (see [2, p. 267]). Since $Y(A)$ is contained in a $\sigma$-compact subspace of $l^2$, $Y(A)$ is a $Z_\sigma$-space provided it is infinite-dimensional (i.e., $A$ is infinite).

Proposition 6.1. Let $A$ be any subset of $T$ and $\alpha \geq 1$. Then:

(a) $Y(A) \times F_\alpha$ (respectively, $Y(A) \times G_\alpha$) contains no closed copy of $l^2 \setminus G_{\alpha+1}$ (respectively, $l^2 \setminus F_{\alpha+1}$),

(b) $Y(A) \times F_\alpha$ (respectively, $Y(A) \times G_\alpha$) contains no closed copy of $l^2 \setminus F_\alpha$ (respectively, $l^2 \setminus G_\alpha$).

First, in full detail, we consider the following particular case of part (b) with $\alpha = 1$ (as $l^2 \setminus F_1$ is a copy of $l^2$, see [2]).

Lemma 6.2. For every subset $A$ of $T$, the space $Y(A) \times \Sigma$ contains no closed copy of $l^2$. In particular, $Y(A) \times \Sigma$ is homeomorphic neither to $\Lambda_\alpha$, $\alpha \geq 2$, nor to $\Omega_\alpha$, $\alpha \geq 1$.

Proof. We apply the cross-section argument described in 4.3. For $k$ and $p \geq 1$, we write

$$C^p_k = \left\{ (t_1, t_2, \ldots, t_k) \in T^k : t_1 \leq t_2 \leq \cdots \leq t_k, \|t_i - t_j\| \geq \frac{1}{p} \right\}$$

and

$$D^p_k = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_k) \in R^k : \frac{1}{p} \leq |\lambda_i| \leq p \text{ for all } i \right\}.$$  

(The union $\bigcup_{k,p=1}^{\infty} C^p_k$ is a particular $\sigma$-compact cross-section for $T^k$.) The map $\chi_k$ given by $\chi_k((t_1, t_2, \ldots, t_k), (\lambda_1, \lambda_2, \ldots, \lambda_k)) = \lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_k t_k$
is a homeomorphism of \( C_k^p \times D_k^p \) onto \( M_k^p \subset Y(T) \). Clearly,
\[
M_k^p \cap Y(A) = \chi_k((C_k^p \cap A^k) \times D_k^p) = N_k^p
\]
is a closed subset in \( Y(A) \). Since \( Y(A) = \{0\} \cup \bigcup_{k,p=1}^{\infty} N_k^p \), we get
\[
Y(A) \times \Sigma = (\{0\} \times \Sigma) \cup \bigcup_{k,p=1}^{\infty} (N_k^p \times \Sigma).
\]
Assume that \( X \) being a copy of \( l^2 \) is contained as a closed subset of \( Y(A) \times \Sigma \). Using a Baire category argument and the fact that no open set in \( l^2 \) is \( \sigma \)-compact we find indices \( k \) and \( p \) such that \( N_k^p \times \Sigma \) contains an open subset \( U \) of \( \Sigma \). It follows that a copy \( B \) of a closed ball in \( l^2 \) inscribed in \( U \) is closed in \( N_k^p \times \Sigma \). It is easy to see that there exists a connected set \( K \subset A^k \) such that
\[
B \subset \chi_k((K \cap C_k^p) \times D_k^p) \times \Sigma.
\]
Each connected subset \( K \) of \( A^k \) is of the form \( I_1 \times I_2 \times \cdots \times I_k \), where every \( I_j \) is a connected component of \( A \) (i.e., \( I_j \) is an interval). Hence, \( K \cap C_k^p \) is locally compact and so is \( \chi_k((K \cap C_k^p) \times D_k^p) \). Finally, \( B \), being a closed subset of a \( \sigma \)-compact space \( \chi_k((K \cap C_k^p) \times D_k^p) \times \Sigma \), is itself \( \sigma \)-compact, a contradiction.

**Proof of 6.1.** Assume \( Y(A) \times F_\alpha \) contains a closed copy \( X \) of \( l^2 \setminus G_{\alpha+1} \). Using the notation of the proof of 6.2, we have
\[
Y(A) \times F_\alpha = (\{0\} \times F_\alpha) \cup \bigcup_{k,p=1}^{\infty} (N_k^p \times F_\alpha).
\]
Since \( X \) is a Baire space (see 4.9), there exist \( k \) and \( p \) and a closed set \( P \subset N_k^p \times F_\alpha \) such that \( P \) has nonempty interior in \( X \) and \( P \) is a copy of \( B \setminus G_{\alpha+1} \), for some closed ball in \( l^2 \). According to 4.9, \( P \) is connected. As in the proof of 6.2, we get
\[
P \subset \chi_k((K \cap C_k^p) \times D_k^p) \times F_\alpha,
\]
where \( K \cap C_k^p \) is locally compact. Now, it follows that \( \chi_k((K \cap C_k^p) \times D_k^p) \times F_\alpha \in \mathcal{A}_\alpha \); consequently \( P \in \mathcal{A}_\alpha \). Since \( X \) is homogeneous (see 4.9) and the interior of \( P \) in \( X \) is nonempty, \( X \) is locally in the class \( \mathcal{A}_\alpha \). The latter yields \( X \in \mathcal{A}_\alpha \), contradicting \( G_{\alpha+1} \in \mathcal{A}_\alpha \setminus \mathcal{A}_{\alpha+1} \).

All the remaining cases can be proved in the same way. (A minor change is needed for \( G_1 \); namely, \( G_1 \) must be represented as a countable union of complete metrizable spaces.)

**Corollary 6.3.** For every subset \( A \) of \( T \) the spaces \( Y(A) \times F_\alpha \) and \( Y(A) \times G_\alpha \), \( \alpha \geq 1 \), are topologically distinct.

**Proof.** By 6.1, \( Y(A) \times F_\alpha \) contains no closed copy of \( l^2 \setminus F_\beta \in \mathcal{M}_\beta \). Since \( \mathcal{M}_\alpha \subset \mathcal{A}_\beta \cap \mathcal{A}_\beta \) for \( \beta > \alpha \), the spaces \( Y(A) \times F_\beta \) and \( Y(A) \times G_\beta \) do contain such a copy. Also \( Y(A) \times G_\alpha \) contains a closed copy of \( l^2 \setminus F_\alpha \). As a consequence, we conclude that \( Y(A) \times F_\alpha \) is not homeomorphic to \( Y(A) \times F_\beta \) for \( \alpha \neq \beta \) and that \( Y(A) \times F_\alpha \) is not homeomorphic to \( Y(A) \times G_\beta \) for \( \beta \geq \alpha \). Analogously, we prove that \( Y(A) \times G_\alpha \) is not homeomorphic to \( Y(A) \times G_\beta \) for \( \beta \neq \alpha \) and that \( Y(A) \times G_\beta \) is not homeomorphic to \( Y(A) \times F_\alpha \) for \( \beta \leq \alpha \).

The same argument applies in the following
Corollary 6.4. For every subset $A$ of $T$, the spaces $Y(A) \times F_{\alpha}$ and $Y(A) \times G_{\alpha}$ are homeomorphic neither to $F_{\beta}$ nor to $G_{\beta}$ for $\beta \neq \alpha$.

Corollary 6.5. We have:

(a) if $A \in \mathcal{A}_{\alpha} \setminus \mathcal{M}_{\alpha}$, then the spaces $F_{\alpha}$, $Y(A) \times F_{\beta}$, and $Y(A) \times G_{\beta}$ belong to $\mathcal{A}_{\alpha} \setminus \mathcal{M}_{\alpha}$ and are topologically distinct for $\beta < \alpha$,

(b) if $A \in \mathcal{M}_{\alpha} \setminus \mathcal{A}_{\alpha}$, then the spaces $G_{\alpha}$, $Y(A) \times G_{\beta}$, and $Y(A) \times F_{\beta}$ belong to $\mathcal{M}_{\alpha} \setminus \mathcal{A}_{\alpha}$ and are topologically distinct for $\beta < \alpha$,

(c) if $A \in \mathbb{P}_{n} \setminus \bigcup_{k<n} \mathcal{P}_{k}$, then the spaces $Y(A) \times F_{\alpha}$ and $Y(A) \times G_{\alpha}$ belong to $\mathbb{P}_{n} \setminus \bigcup_{k\leq n} \mathcal{P}_{k}$ and are topologically distinct,

(d) if $A \notin \bigcup_{n=1}^{\infty} \mathbb{P}_{n}$, then the spaces $Y(A) \times F_{\alpha}$ and $Y(A) \times G_{\alpha}$ do not belong to $\bigcup_{n=1}^{\infty} \mathbb{P}_{n}$ and are topologically distinct.

Corollary 6.5 is a direct consequence of 6.3 and 6.4 and the following fact, which seems to be well known; however we could not find it formulated in such a generality in literature.

Lemma 6.6. We have:

(a) if $A \in \mathcal{A}_{\alpha} \setminus \mathcal{M}_{\alpha}$, then $Y(A) \in \mathcal{A}_{\alpha} \setminus \mathcal{M}_{\alpha}$, $\alpha \geq 1$,

(b) if $A \in \mathcal{M}_{\alpha} \setminus \mathcal{A}_{\alpha}$, then $Y(A) \in \mathcal{M}_{\alpha} \setminus \mathcal{A}_{\alpha}$, $\alpha \geq 2$,

(c) if $A \in \mathbb{P}_{n} \setminus \bigcup_{k<n} \mathcal{P}_{k}$, then $Y(A) \in \mathbb{P}_{n} \setminus \bigcup_{k\leq n} \mathcal{P}_{k}$, $n \geq 1$,

(d) if $A \notin \bigcup_{n=1}^{\infty} \mathbb{P}_{n}$, then $Y(A) \notin \bigcup_{n=1}^{\infty} \mathbb{P}_{n}$.

Proof. Since $A$ is closed in $Y(A)$, $A \notin \mathcal{L}$ implies $Y(A) \notin \mathcal{L}$ provided $\mathcal{L}$ is closed with respect to closed subsets. Therefore, it suffices to show that $A \in \mathcal{L}$ implies $Y(A) \in \mathcal{L}$, where $\mathcal{L} = \mathcal{A}_{\alpha}, \mathcal{M}_{\alpha}$ and $\bigcup_{k<n} \mathcal{P}_{k}$. The case $\mathcal{A}_{\alpha}$ and $\bigcup_{k<n} \mathcal{P}_{k}$ is a result of Klee [2, p. 272]. Let $A \in \mathcal{M}_{\alpha}$ and $\alpha \geq 2$. Represent $A = \bigcap_{n=1}^{\infty} A_{n}$, $A_{n} \in \mathcal{P}_{\beta_{n}}$ for $\beta_{n} < \alpha$, and employ $Y(A_{n}) \in \mathcal{P}_{\beta_{n}}$ to conclude that $Y(A) = \bigcap_{n=1}^{\infty} Y(A_{n}) \in \mathcal{A}_{\alpha}$.

Remark 6.7. Corollary 6.5(a) and (b) (see also 6.2) provide a negative answer to the question of whether a pre-Hilbert space that contains a Hilbert cube and is of the exact Borelian class of order $\alpha$ must be homeomorphic to either $F_{\alpha}$ or $G_{\alpha}$, $\alpha \geq 2$. The answer to this question is “yes” for $\mathcal{A}$.

Remark 6.8. From 6.5(c) it follows that each class $\mathbb{P}_{n} \setminus \bigcup_{k<n} \mathcal{P}_{k}$ contains uncountably many topologically distinct pre-Hilbert spaces that are $Z_{\alpha}$-spaces.

Remark 6.9. Each class $\mathcal{A}_{\alpha} \setminus \mathcal{M}_{\alpha}$, $\alpha \geq 1$, and $\mathcal{M}_{\alpha} \setminus \mathcal{A}_{\alpha}$, $\alpha \geq 2$, contains uncountably many topologically distinct pre-Hilbert spaces that are $Z_{\alpha}$-spaces. To show this, take $A \in \mathcal{A}_{\alpha} \setminus \mathcal{M}_{\alpha}$ (respectively, $A \in \mathcal{M}_{\alpha} \setminus \mathcal{A}_{\alpha}$) and repeat Henderson and Pelczyński’s argument to the spaces $Y(A) \times X$, $X \in \mathcal{H}$, where $\mathcal{H}$ is that of [2, p. 282].

References


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