CHARACTERIZATION OF AUTOMORPHISMS ON THE BARRETT AND THE DIEDERICH-FORNAESS WORM DOMAINS

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Abstract. In this paper we show that every automorphism on either the Barrett or the Diederich-Fornaess worm domains is given by a rotation in $u$-variable. In particular, any automorphism on either one of these two domains can be extended smoothly up to the boundary.

I. Introduction

In several complex variables extending a biholomorphism or an automorphism smoothly up to the boundary is always a very important and fundamental problem which is closely related to the classification problem of domains in $\mathbb{C}^n$. The extension phenomenon in general is false as shown in Barrett [3] if the domains are sitting in some general complex manifolds. However, it is still widely believed that such extension phenomena should hold if the domains are contained in $\mathbb{C}^n$, namely, we conjecture the following two statements,

(1.1) Any biholomorphism between two smoothly bounded domains $D_1$ and $D_2$ in $\mathbb{C}^n$, $n \geq 2$, can be extended smoothly to a $CR$-diffeomorphism between $\overline{D_1}$ and $\overline{D_2}$,

and its weaker counterpart

(1.2) Any automorphism of a smoothly bounded domain $D$ in $\mathbb{C}^n$, $n \geq 2$, can be extended smoothly up to the boundary, i.e., $\text{Aut}(D) = \text{Aut}(\overline{D})$.

Indeed, it has been shown in Bell and Ligocka [6] that if condition $R$ holds on both $D_1$ and $D_2$, then (1.1) is valid. Here condition $R$ means that the Bergman projection associated with the domain $D$ maps $C^\infty(\overline{D})$ continuously into itself. Condition $R$ was shown to hold on a large class of (pseudoconvex or nonpseudoconvex) domains. But surprisingly Barrett constructed in [1] a smoothly bounded nonpseudoconvex domain $\Omega$ in $\mathbb{C}^2$ which fails to satisfy condition $R$.

On the other hand, Diederich and Fornaess in [8] constructed a smoothly bounded pseudoconvex domain $\Omega$, in $\mathbb{C}^2$ which possesses many pathological properties that include a nontrivial Nebenhulle and the nonexistence of a $C^3$
plurisubharmonic defining function for \( \Omega_r \). Very recently Barrett showed in [4] that the Bergman projection associated with \( \Omega_r \) does not preserve the Sobolev space \( W^k(\Omega_r) \) if \( k \in \mathbb{R} \) is large enough. It is still not clear whether condition \( R \) holds on \( \Omega_r \) or not.

In this article we want to show that despite these pathological properties found on \( \Omega \) and \( \Omega_r \), statement (1.2) is still valid on both \( \Omega \) and \( \Omega_r \). In fact we can prove more, namely,

**Main Theorem.** Any automorphism \( f \) of either the Barrett or the Diederich-Fornaess worm domains is given by a rotation in \( w \)-variable, i.e., \( f(z, w) = (z, e^{i\phi}w) \) for some constant \( \phi \in \mathbb{R} \). In particular, \( f \) can be extended smoothly up to the boundary.

We make a remark here that although \( \Omega \) does not satisfy condition \( R \), it still enjoys an a priori estimate on Sobolev space \( W^k(\Omega) \) as shown in Boas and Straube [7].

II. Proof on the Barrett's domain

We first recall the definition of \( \Omega \). The domain \( \Omega \) is a smoothly bounded domain defined in \( \mathbb{C}^2 \) as follows,

\[
\Omega = \{(z, w) \in \mathbb{C}^2 | 1 < |w| < 6, \ |z - c(|w|)| > r_1(|w|) \text{ and } |z| < r_2(|w|)\},
\]

where the functions \( r_1(|w|), r_2(|w|), \) and \( c(|w|) \) are chosen to meet the following conditions: Let \( k \) be a positive integer. Define

\[
r_1(x) = \begin{cases} 
3 - \sqrt{x - 1} & \text{for } x \text{ near } 1, \\
\text{decreasing} & \text{for } x \in [1, 2], \\
1 & \text{for } x \in [2, 5], \\
\text{increasing} & \text{for } x \in [5, 6], \\
3 - \sqrt{6 - x} & \text{for } x \text{ near } 6.
\end{cases}
\]

\[
r_2(x) = \begin{cases} 
3 + \sqrt{x - 1} & \text{for } x \text{ near } 1, \\
\text{increasing} & \text{for } x \in [1, 2], \\
4 & \text{for } x \in [2, 5], \\
\text{decreasing} & \text{for } x \in [5, 6], \\
3 - \sqrt{6 - x} & \text{for } x \text{ near } 6.
\end{cases}
\]

and

\[
c(x) = \begin{cases} 
0 & \text{for } x \in [1, 2], \\
\text{decreasing} & \text{for } x \in [2, 3], \\
(x - 3)^{2k} - 1 & \text{for } x \text{ near } 3, \\
\text{increasing} & \text{for } x \in [3, 4], \\
-(x - 4)^{2k} + 1 & \text{for } x \text{ near } 4, \\
\text{decreasing} & \text{for } x \in [4, 5], \\
0 & \text{for } x \in [5, 6].
\end{cases}
\]

Then the following theorem that shows \( \Omega \) fails to satisfy condition \( R \) was proved in Barrett [1].
Theorem. $P(C_0^\infty(\Omega))$ is not contained in $L^p(\Omega)$ for $p \geq 2 + 1/k$, where $P$ is the Bergman projection associated with $\Omega$.

Now we proceed to prove our main theorem on this domain. Let $f = (f_1, f_2)$ be an automorphism of $\Omega$. We first show the following lemma. A similar statement was proved in Boas and Straube [7].

Lemma 2.1. Let $g$ be a bounded holomorphic function on $\Omega$. Then $g$ can be extended holomorphically to $D$, where

$$D = \{(z, w) \in \mathbb{C}^2 | |z| < r_2(|w|) \text{ and } 1 < |w| < 6\}.$$

Proof. By Laurent series expansion one can write

$$g(z, w) = \sum_{n=-\infty}^{\infty} a_n(z)w^n,$$

where

$$a_n(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{g(z, w)}{w^{n+1}} \, dw,$$

and $|w| = R$ is any circle contained in the slice corresponding to $z$. Then one can see easily from (2.3) that $a_n(z)$ is locally uniformly bounded and can be extended to a holomorphic function on $\Delta(0; 4)$; i.e.,

$$a_n(z) \in H(\Delta(0; 4)) \text{ for all } n \in \mathbb{Z}.$$

Next we show as in [7] that the series (2.2) in fact converges on $\{(z, w) \in \mathbb{C}^2 | |z| < 3 \text{ and } 1 < |w| < 6\}$ to a holomorphic function. Consider first the nonnegative indices, i.e.,

$$\sum_{n=0}^{\infty} a_n(z)w^n.$$

Put $u(z) = \lim_{n \to \infty} |a_n(z)|^{1/n}$, and let $u^*(z)$ be the upper semicontinuous regularization of $u(z)$, i.e.,

$$u^*(z) = \lim_{z' \to z} u(z').$$

Then by the fact that $a_n(z)$ is locally uniformly bounded, we see that $u^*(z)$ is subharmonic on the disk $\Delta(0; 4)$, and it is easy to see that $u^*(z) \leq \frac{1}{6}$ for $|z| = 3$. Hence by maximum principle we obtain that $u^*(z) \leq \frac{1}{6}$ for $|z| \leq 3$ and $u(z) \leq \frac{1}{6}$ for $|z| \leq 3$. It follows that the series (2.4) is holomorphic on $\{(z, w) \in \mathbb{C}^2 | |z| < 3 \text{ and } |w| < 6\}$. For the negative indices part we simply replace $w$ by $\frac{1}{w}$, then an analogous argument will go through as well. This completes the proof of the lemma.

It follows thus from Lemma 2.1 that $f_k(z, w) \in H(D)$ for $k = 1, 2$. We claim that in fact we have $f = (f_1, f_2) \in \text{Aut}(D)$.

Proof of the claim. Put

$$D_0 = \{(z, w) \in \mathbb{C}^2 | |z| < 4 \text{ and } 1 < |w| < 6\},$$

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and let $\tilde{D}$ be the envelope of holomorphy of $D$. We have that $\Omega \subseteq D \subseteq \tilde{D} \subseteq D_0$. Let $p = (z_0, w_0)$ be a point in $D - \Omega$. Consider the circle

$$C_p = \{(z, w_0) \in \Omega | |z| = 3\}.$$

By maximum modulus principle it is easy to see that $f(p) \in D_0$. We wish to show that $f(p) \in \tilde{D}$, and hence $f(D) \subseteq \tilde{D}$.

Suppose that $q = f(p) \notin \tilde{D}$ with $5 \leq |f_2(p)| \leq 6$. Set

$$E(\tilde{D}) = \tilde{D} \cup \{(z, w) \in C^2 | |z| < 4 \text{ and } |w| < 4\},$$

and

$$E_+(\tilde{D}) = \{(|z|, |w|) \in \mathbb{R}^2 | (z, w) \in E(\tilde{D})\}.$$ 

It is well known that the logarithmic image of $E_+(\tilde{D})$ is geometrically convex. Since the point $q^* = (\ln |f_1(p)|, \ln |f_2(p)|)$ is not in $\ln(E_+(\tilde{D}))$ and the rational number is dense in $\mathbb{R}$, one can find two positive integers $m$ and $n$ such that the straight line $L$,

$$L = \{(x, y) \in \mathbb{R}^2 | mx + ny = c_1 \text{ for some constant } c_1\},$$

go through $q^*$ and such that the whole set $\ln(|f(C_p)|_+)$ lies in the half plane defined by $\{(x, y) \in \mathbb{R}^2 | mx + ny - c_1 < 0\}$, where

$$|f(C_p)|_+ = \{(x, y) \in \mathbb{R}^2 | (x, y) = (|f_1(z, w_0)|, |f_2(z, w_0)|) \text{ for some } |z| = 3\}.$$

Now consider the entire holomorphic function $h(z, w) = e^{-c_1}z^m w^n$. We see that the restriction of $h \circ f$ to $\Delta_p$, where $\Delta_p = \{(z, w) \in D | |z| < 3 \text{ and } w = w_0\}$, will violate the maximum modulus principle. A similar argument via the mapping $w \mapsto \frac{1}{w}$ can be applied to show that no point of $D - \Omega$ can be mapped to $D_0 - \tilde{D}$ with $1 \leq |w| \leq 2$. This shows that $f(p) \in \tilde{D}$, thus we have $f(D) \subseteq \tilde{D}$.

Let $g$ be the inverse mapping of $f$. Since $g$ can be extended holomorphically to $\tilde{D}$, it is legitimate to consider $g \circ f: D \to C^2$. Then by identity theorem and the fact $g \circ f|_{\Omega} = \text{identity mapping}$, we get $g \circ f = \text{identity mapping}$ on $D$. Similarly $f \circ g$ is also the identity mapping on $D$. This shows that $f \in \text{Aut}(D)$, and the proof of the claim is now completed.

Next we observe that the domain $D$ is Reinhardt. Therefore $f$ can be extended holomorphically to a small open neighborhood of $\tilde{D}$. In particular, we have $f \in \text{Aut}(\tilde{D})$. For instance see Barrett [2]. However, we want to show more that $f'$ in fact is given by a rotation in $w$-variable. So we next characterize a Reinhardt hypersurface in $C^2$ that contains a Riemann surface in it. By a Reinhardt hypersurface we mean that the hypersurface is invariant under the rotations in all directions. The result might have some interest of itself. If $H$ is a Reinhardt hypersurface in $C^2$, we denote by $H_+$ the corresponding curve in $\mathbb{R}^2$. Then we have

**Lemma 2.5.** Let $H$ be a Reinhardt hypersurface in $C^2$ such that $H_+$ is decreasing in pr-space with $\rho = |w|$ and $r = |z|$. Then $H$ contains a Riemann surface near $p_0 \in H$ if and only if $H$ is either flat in one of the coordinates or $H_+$ is defined near $p_0$ by a hyperbola, namely,

$$H_+ = \{(\rho, r) \in \mathbb{R}^2 | r \rho^c = \text{constant, for some } c \geq 0\}.$$
Proof. Put \( z = (x, y) = re^{i\theta} \) and \( w = (u, v) = \rho e^{i\phi} \). Let the defining function for \( H_+ \) (hence for \( H \)) by \( \xi(\rho, r) \). Rewrite \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) in terms of polar coordinates, we obtain
\[
\frac{\partial}{\partial x} = \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} + \cos \theta \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial y} = \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial}{\partial r}.
\]

Next the tangential type-(1, 0) vector field is generated by
\[
L = \frac{\partial \xi}{\partial z} \frac{\partial}{\partial w} - \frac{\partial \xi}{\partial w} \frac{\partial}{\partial z} = \frac{1}{2} e^{-i\theta} \frac{\partial \xi}{\partial r} \frac{\partial}{\partial w} - \frac{1}{2} e^{-i\phi} \frac{\partial \xi}{\partial \rho} \frac{\partial}{\partial z}.
\]

Suppose that \( H \) contains a Riemann surface \( \mathcal{R} \) near \( p_0 \) with \( \frac{\partial \xi}{\partial r}(p_0) \neq 0 \). Then one can choose \( L \) to be
\[
L = \frac{\partial}{\partial w} - e^{i(\theta-\phi)} \frac{\partial \xi}{\partial \rho} \left( \frac{\partial \xi}{\partial r} \right)^{-1} \frac{\partial}{\partial z}.
\]

Put \( g(z, w) = e^{i(\theta-\phi)} \cdot \frac{\partial \xi}{\partial \rho} \left( \frac{\partial \xi}{\partial r} \right)^{-1} \). Since \( [L, \overline{L}]_\mathcal{R} \equiv 0 \mod (L \oplus \overline{L}) \), we see that the restriction \( g|_\mathcal{R} \) is holomorphic. Then consider
\[
\frac{gw}{z} \bigg|_\mathcal{R} = \frac{\partial \xi / \partial \rho}{\partial \xi / \partial r} \cdot \frac{\rho}{r}.
\]

It shows that the function \( \frac{gw}{z} \bigg|_\mathcal{R} \) is holomorphic and real valued. Hence it must be a real constant function, namely,
\[
\frac{gw}{z} \bigg|_\mathcal{R} \equiv c, \quad \text{for some } c \in \mathbb{R}.
\]

Also locally one can express \( r \) as a function of \( \rho \), and the slope of \( H_+ \) near \( p_0 \) is given by
\[
\frac{dr}{d\rho} = \frac{-\partial \xi / \partial \rho}{\partial \xi / \partial r} = -\frac{c}{\rho}.
\]

Therefore by solving this first order differential equation we get \( r \rho^c = e^{c_0} \), for some constant \( c_0 \in \mathbb{R} \). Since \( H_+ \) is decreasing, the constant \( c \) is nonnegative.

On the other hand, if \( H_+ \) is defined locally near some \( p_0 \) by \( r \rho^c = c_1 \) with \( c, c_1 > 0 \), then by direct computation we get
\[
L = w \frac{\partial}{\partial w} - cz \frac{\partial}{\partial z}.
\]

Hence we have \( [L, \overline{L}] \equiv 0 \). It follows that there exists a Riemann surface in \( H \) near \( p_0 \). This completes the proof of the lemma.

It is interesting to note that the vector field \( -2i \Im L \), where \( L \) is given in (2.6), is generated by the following \( S^1 \)-action,
\[
\Lambda: S^1 \times \mathcal{R} \to \mathcal{R},
\]
\[
(\theta, (z, w)) \mapsto (e^{-i\theta} z, e^{i\theta} w).
\]

Now we go back to the automorphism \( f \) of \( D \). We see that \( f \) will map biholomorphically a Riemann surface in the boundary onto another Riemann surface in the boundary. In particular, if we set
\[
\mathcal{R}_{a, b, \theta} = \{ (z, w) \in bD | z = 4e^{i\theta} \text{ and } a < |w| < b \text{ with } a \leq 2 \text{ and } b \geq 5 \}.
\]
to be the largest annulus sitting in the boundary with \(|z| = 4\), and set

\[ C_{a, \theta} = \{(4e^{i\theta}, w) \in bD | |w| = a\} \]

to be the inner boundary of \( R_{a, b, \theta} \), and similarly let \( C_{b, \theta} \) be the outer boundary of \( R_{a, b, \theta} \). Then \( f \) will map \( R_{a, b, \theta} \) to a Riemann surface in the boundary. We claim that \( R_{a, b, \theta} \) cannot be mapped to any Riemann surface contained in the boundary with \( 1 < |w| < 2 \) or \( 5 < |w| < 6 \). First it is not hard to see that there are only three different types of Riemann surfaces in these regions, they are

(i) \( R_c \subseteq \{(z, w) \in bD | |z||w|^c = A \) for some constants \( A \) and \( c > 0 \), and \( \alpha < |w| < \beta \) with \( 5 < \alpha < \beta < 6 \), or an equivalent counterpart in the region \( 1 < |w| < 2 \).

(ii) \( R_z = \{(z, w_0) \in bD | w_0 = p e^{i\phi} \) for some \( p \) and \( \phi \) with \( 5 < p < 6 \) or \( 1 < p < 2 \), and \( s < |z| < t \) for some \( 3 < s < t < 4 \).

(iii) \( R_w = \{(z_0, w) \in bD | z_0 = re^{i\theta} \) for some \( r \) and \( \theta \) with \( 3 < r < 4 \), and \( s < |w| < t \) for some \( 1 < s < t < 2 \) or \( 5 < s < t < 6 \).

We can rule out \( R_z \) and \( R_w \) immediately by considering the ratio of the radii of the boundaries of these annuli. To knock out \( R_c \) we first observe that if \( f \) maps \( R_{a, b, \theta} \) onto some \( R_c \), then by continuity \( f \) must map \( C_{a, \theta} \) for any \( \theta \) into exactly one of \( \{(z, w) \in bD | |w| = \alpha \} \) or \( \{(z, w) \in bD | |w| = \beta \} \).

Suppose that \( C_{b, \theta} \) is mapped to \( \{(z, w) \in bD | |w| = \alpha \} \). Then by maximum modulus principal we see that \( f \) will map \( \{(z, w) \in \overline{D} \} \) \( |z| \leq 4 \) and \( |w| = b \) biholomorphically onto

\[ \{(z, w) \in \overline{D} | |z| \leq A/\alpha^c \text{ and } |w| = \alpha \}. \]

In particular, \( f \) must map a disk

\[ \Delta_w = \{(z, w) \in D | |z| < 4 \text{ and } w = be^{i\phi} \text{ for some } \phi \} \]

biholomorphically onto another disk

\[ \Delta_{w'} = \{(z, w') \in D | |z| < A/\alpha^c \text{ and } w' = \alpha e^{i\phi'} \text{ for some } \phi' \}. \]

Since the restriction \( f|_\Omega \) is an automorphism, we also have the following biholomorphic equivalence between two annuli induced by \( f|_\Omega \), namely, \( f|_\Omega : \Delta_w \cap \Omega \sim \Delta_{w'} \cap \Omega \). However, this cannot happen simply by examining the ratio of the radii of boundaries of these two annuli. Thus we have shown that \( f(R_{a, b, \theta}) = R_{a, b, \eta(\theta)} \), for some real-valued function \( \eta(\theta) \) that maps \( S^1 \) bijectively onto itself.

Next we divide our arguments into two subcases.

Case 1. If \( f \) maps \( C_{b, \theta} \) to \( C_{b, \eta(\theta)} \) for some \( \theta \). Then by continuity \( f \) will map \( C_{b, \theta} \) to \( C_{b, \eta(\theta)} \) for all \( \theta \in [0, 2\pi] \). For each fixed \( \theta \in [0, 2\pi] \), we see by reflection principle that \( f_2 \) can be extended to an entire function which preserves the modulus on \( |w| = a \) and \( |w| = b \). Hence \( f_2 \) must take the following form

\[ f_2(4e^{i\theta}, w) = e^{i\delta(\theta)} \cdot w, \]

for some real-valued function \( \delta(\theta) \). Also we have

\[ f_1(4e^{i\theta}, w) = 4e^{i\eta(\theta)}, \]

independent of \( w \) for \( a < |w| < b \).
Then by the maximum modulus principle we have

\[ |f_2(z, w)| = |w| \quad \text{for } |z| \leq 4 \quad \text{and} \quad a < |w| < b. \]

This implies that \( f \) will map \( S_c \) biholomorphically onto \( S_c \), where \( S_c = \{(z, w) \in \mathbb{D} | |z| \leq 4, \ |w| = c \ \text{with} \ a < c < b \} \). Therefore, we conclude that \( f \) must map a disk \( \Delta_{c, \phi} \) onto another disk \( \Delta_{c, \phi'} \), i.e.,

\[
(2.9) \quad f: \Delta_{c, \phi} \sim \Delta_{c, \phi'},
\]

where \( \Delta_{c, \phi} = \{(z, w) \in \mathbb{D} | |z| \leq 4, \ w = ce^{i\phi} \} \). Thus if we combine equations (2.7) and (2.9), we obtain for fixed \( \phi \) that \( f_2(4e^{i\theta}, ce^{i\phi}) = ce^{i\delta(\theta)} \cdot e^{i\phi} = ce^{i\phi'} \), for all \( \theta \in [0, 2\pi] \). This implies that \( \delta(\theta) \) is a constant function, namely, \( \delta(\theta) = \phi_0 \). Hence we obtain that \( f_2(z, w) = e^{i\phi_0}w \).

Next equation (2.8) shows that the restriction \( f_1|_{\Delta_{c, \phi}} \) of \( f_1 \) to every disk \( \Delta_{c, \phi} \) with \( a < c < b \) and all \( \phi \) has the same boundary value. So we conclude that \( f_1(z, w) = f_1(z) \) is independent of \( w \). Then by the facts that \( |f_1(4e^{i\theta})| = 4 \) and \( |f_1(3e^{i\theta})| = 3 \), we get

\[
(2.11) \quad f_1(z, w) = f_1(z) = e^{i\theta_0}z \quad \text{for some constant } \theta_0 \in \mathbb{R}.
\]

Since \( f|_{\Omega} \) is an automorphism of \( \Omega \), it is easy to see that \( \theta_0 = 0 \). This shows that

\[
(2.10) \quad f = (f_1, f_2) = (z, e^{i\theta_0}w),
\]

and the proof for Case 1 is now completed.

Finally we show that \( f \) cannot map \( C_{b, \theta} \) to \( C_{a, \eta(\theta)} \). This will also complete the proof of our main theorem on the Barrett's domain.

**Case 2.** If \( f \) maps \( C_{b, \theta} \) to \( C_{a, \eta(\theta)} \) for some \( \theta \). Then again by continuity \( f \) will map \( C_{b, \theta} \) to \( C_{a, \eta(\theta)} \) for all \( \theta \in [0, 2\pi] \). Consider the map

\[
g(z, w): D \to D',
\]

\[
(z, w) \mapsto \left( f_1(z, w), \frac{ab}{f_2(z, w)} \right),
\]

where \( D' \) is biholomorphic to \( D \) via the map \( (z, w) \mapsto (z, \frac{ab}{w}) \).

So one can repeat the above argument and obtain that

\[
(2.11) \quad f_2(z, w) = \frac{ab}{w}e^{i\phi_0} \quad \text{for some constant } \phi_0 \in \mathbb{R}.
\]

Since \( 5 \leq ab \leq 12 \), in order to preserve the boundaries at two ends, we must have \( ab = 6 \). We may also conclude that

\[
(2.12) \quad f_1(z, w) = z.
\]

Next consider the point \( p_0 = (\frac{1}{2}, 3) \). We see that \( p_0 \in \Omega \). Since \( f \in \text{Aut}(\Omega) \), we must have \( f(p_0) \in \Omega \). However, equations (2.11) and (2.12) show that \( f(p_0) = (\frac{1}{2}, 2e^{i\phi_0}) \), and this point is clearly not in \( \Omega \). This gives the desired contradiction.
Hence any automorphism on the Barrett’s domain is given by a rotation in $w$-variable.

III. Proof on the Diederich-Fornaess domains

We first recall briefly the definition of the Diederich-Fornaess domain here. Fix a smooth function $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

(a) $\lambda(x) = 0$ if $x \leq 0$,
(b) $\lambda(x) > 1$ if $x > 1$,
(c) $\lambda''(x) \geq 100 \lambda'(x)$ for all $x$,
(d) $\lambda''(x) > 0$ if $x > 0$,
(e) $\lambda'(x) > 100$ if $\lambda(x) > \frac{1}{2}$.

Then for any $r > 1$ we define $\Omega_r = \{(z, w) \in \mathbb{C}^2 | p_r(z, w) < 0\}$ where

$$p_r(z, w) = |z + i \lambda^{-1}(\ln |w|^2)|^2 - 1 + \lambda(1/|w|^2 - 1) + \lambda(|w|^2 - r^2).$$

**Theorem** [8]. $\Omega_r$ is a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$. The boundary is strictly pseudoconvex everywhere except on the following annulus,

$$M_r = \{(z, w) \in \partial \Omega_r | z = 0 \text{ and } 1 < |w| < r\}.

Now let $f = (f_1, f_2)$ be an automorphism of $\Omega_r$. Then $f$ can be extended smoothly up to the boundary on $\partial \Omega_r - M_r$. For instance, see Bell [5]. Therefore, if we consider the deleted torus

$$T_a = \{(z, w) \in b\Omega_r | 1 < |w| = a < r \text{ and } z \neq 0\},$$

we see that $\eta_r = \rho_r \circ f$ is a defining function for $T_a$, $1 < a < r$, namely, the equation $|f_1(z, w) + e^{i \ln |w|^2}| = 1$ defines $T_a$. This implies that $|f_2(z, w)|^2 = |w|^2 \cdot e^{2k\pi}$, for some fixed integer $k$. Hence by considering the points $(z, w) \in T_a$ with a close to either 1 or $r$, we conclude that $k = 0$ and $|f_2(z, w)| = |w|$ for $(z, w) \in T_a$ with $1 < a < r$.

Next fix the constant $a$ with $1 < a < r$, and a point $z_0$ with $|z_0 + e^{i \ln |a|^2}| < 1$ such that $z_0$ lies in a small open neighborhood of $-2e^{i \ln |a|^2}$. Then we consider the annulus defined by

$$A_{z_0} = \{(z_0, w) \in \mathbb{C}^2 \cap \Omega_r$$

with the inner boundary $C_\alpha = \{(z_0, w) \in b\Omega_r | |w| = \alpha\}$ and the outer boundary $C_\beta$ such that $\alpha < a < \beta$. $A_{z_0}$ can be identified with $A = \{w \in \mathbb{C} | \alpha < |w| < \beta\}$. Hence via this identification we obtain that

$$f_2(z_0, C_\alpha) = C_\alpha \text{ and } f_2(z_0, C_\beta) = C_\beta,$$

and $f_2(z_0, \cdot)$ can be extended to an entire function by reflection principle. Then by (3.1) we must have that

$$f_2(z, w) = e^{i \phi(z)} \cdot w$$

for some real-valued function $\phi(z)$. Since $f_2(z, w)$ is also holomorphic in $z$, we conclude that $\phi(z) = \phi_0$ for some constant $\phi_0 \in \mathbb{R}$, and

$$f_2(z, w) = e^{i \phi_0} \cdot w \text{ for } (z, w) \in \Omega_r.$$

Then we consider the open solid torus $\pi_a$ defined by

$$\pi_a = \{(z, w) \in \Omega_r | 1 < |w| = a < r \text{ and } |z + e^{i \ln |w|^2}| < 1\}. $$
Put
\[ \Delta_{a, \phi} = \{ (z, a e^{i\phi}) \in \Omega_r | |z + e^{i \ln |a|^2}| < 1 \} . \]

It follows that the restriction of \( f_1 \) to \( \Delta_{a, \phi_1} \) must map \( \Delta_{a, \phi_1} \) biholomorphically onto \( \Delta_{a, \phi_2} \) for some \( \phi_2 \). This implies that the restriction of \( f_1 \) to \( \Delta_{a, \phi_1} \) can be extended at least smoothly up to \( \Delta_{a, \phi_1} \). Since \( f_1(0, a e^{i\phi_1}) = 0 \), it follows that \( f_1(z, w) \) can be expressed via the automorphisms on the unit disk as

\[ f_1(z, w) = e^{i \ln |w|^2} \left( \frac{1 - b(w)e^{i \ln |w|^2}}{e^{i \ln |w|^2} - b(w)} \right) \left( \frac{z + e^{i \ln |w|^2} - b(w)}{1 - b(w)(z + e^{i \ln |w|^2})} \right) - e^{i \ln |w|^2}, \]

for some real analytic function \( b(w) \) satisfying \( |b(w)| < 1 \) for \( 1 < |w| < r \). Equation (3.3) shows that there exists a small number \( \epsilon > 0 \) such that \( f_1(z, w) \) is real analytic on \( \Delta(0; \epsilon) \times A_\delta \), where \( A_\delta = \{ w \in C | 1 + \delta < |w| < r - \delta \} \) for some small \( \delta > 0 \). This in turn implies that \( f_1(z, w) \) is holomorphic on \( \Delta(0; \epsilon) \times A_\delta \). Therefore, one can write

\[ f_1(z, w) = \sum_{k=1}^{\infty} a_k(w)z^k, \]

with \( a_k(w) \in H(A_\delta) \) for all \( k \geq 1 \). By direct computation we get

\[ a_1(w) = \frac{\partial f_1}{\partial z}(0, w) = \frac{1 - |b(w)|^2}{|1 - b(w)e^{i \ln |w|^2}|^2}. \]

It shows that \( a_1(w) \) is a positive real constant, i.e., \( a_1(w) = c > 0 \). Next the computation of \( a_2(w) \) shows that

\[ a_2(w) = \frac{1}{2} \frac{\partial^2 f_1}{\partial z^2}(0, w) = c \cdot \frac{\frac{\overline{b(w)}}{1 - b(w)e^{i \ln |w|^2}}}{1 - b(w)e^{i \ln |w|^2}}. \]

We claim that \( a_2(w) = 0 \). Set

\[ g(w) = \frac{a_2(w)}{c} = \frac{\overline{b(w)}}{1 - b(w)e^{i \ln |w|^2}} \in H(A_\delta), \]

we have

\[ c = \frac{1 - |b(w)|^2}{|1 - b(w)e^{i \ln |w|^2}|^2} = |1 + g(w)e^{i \ln |w|^2}|^2 - |g(w)|^2 = 1 + 2 \text{Re}(g(w)e^{i \ln |w|^2}). \]

Therefore, one can write

\[ g(w)e^{i \ln |w|^2} = c_0 + iI(w), \]

with \( c_0 = \frac{1}{2}(c - 1) \) and \( I(w) \) is a smooth real-valued function on \( A_\delta \). Hence we obtain

\[ g(w) = c_0 e^{-i \ln |w|^2} + iI(w)e^{-i \ln |w|^2} \in H(A_\delta). \]

Locally one can multiply equation (3.5) by \( e^{2i \ln w} \) to get a new well-defined holomorphic function, and get

\[ g(w) e^{2i \ln w} = c_0 e^{-2 \text{Arg } w} + iI(w)e^{-2 \text{Arg } w}. \]
The real part of \( g(w)e^{2i\ln w} \) is a harmonic function. So let \( w = u + iv \), by direct computation we get
\[
\Delta_w(c_0e^{-2\text{Arg}w}) = c_0\Delta_w(e^{-2\tan^{-1} v/u}) = \frac{4c_0}{u^2 + v^2}e^{-2\tan^{-1} v/u} \equiv 0.
\]
It follows that \( c_0 = 0 \), and hence \( c = 1 \). This reduces (3.5) to
\[
(3.7) \quad -ig(w) = I(w)e^{-i\ln|w|^2} \in H(A_\delta).
\]
Then repeat the same argument, we see that
\[
-ig(w)e^{2i\ln w} = I(w)e^{-2\text{Arg}w} = c_1,
\]
where \( c_1 \) is a global constant. Hence
\[
I(w) = c_1e^{2\text{Arg}w}
\]
is a well-defined function on \( A_\delta \). It forces \( c_1 = 0 \). This shows \( g(w) \equiv 0 \), and the proof of our claim is now completed.

It follows then from (3.4) that we have \( b(w) \equiv 0 \) on \( A_\delta \), and equation (3.3) can be simplified to
\[
(3.8) \quad f_1(z, w) = z \quad \text{on} \quad \Omega_r.
\]
Our main theorem now follows from (3.2) and (3.8). So we are done.

References


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