CHARACTERIZATION OF AUTOMORPHISMS ON THE BARRETT AND THE DIEDERICH-FORNAESS WORM DOMAINS

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Abstract. In this paper we show that every automorphism on either the Barrett or the Diederich-Fornaess worm domains is given by a rotation in $w$-variable. In particular, any automorphism on either one of these two domains can be extended smoothly up to the boundary.

I. Introduction

In several complex variables extending a biholomorphism or an automorphism smoothly up to the boundary is always a very important and fundamental problem which is closely related to the classification problem of domains in $C^n$. The extension phenomenon in general is false as shown in Barrett [3] if the domains are sitting in some general complex manifolds. However, it is still widely believed that such extension phenomena should hold if the domains are contained in $C^n$, namely, we conjecture the following two statements,

(1.1) Any biholomorphism between two smoothly bounded domains $D_1$ and $D_2$ in $C^n$, $n \geq 2$, can be extended smoothly to a $CR$-diffeomorphism between $\overline{D}_1$ and $\overline{D}_2$, and its weaker counterpart

(1.2) Any automorphism of a smoothly bounded domain $D$ in $C^n$, $n \geq 2$, can be extended smoothly up to the boundary, i.e., $\text{Aut}(D) = \text{Aut}(\overline{D})$.

Indeed, it has been shown in Bell and Ligocka [6] that if condition $R$ holds on both $D_1$ and $D_2$, then (1.1) is valid. Here condition $R$ means that the Bergman projection associated with the domain $D$ maps $C^\infty(\overline{D})$ continuously into itself. Condition $R$ was shown to hold on a large class of (pseudoconvex or nonpseudoconvex) domains. But surprisingly Barrett constructed in [1] a smoothly bounded nonpseudoconvex domain $\Omega$ in $C^2$ which fails to satisfy condition $R$.

On the other hand, Diederich and Fornaess in [8] constructed a smoothly bounded pseudoconvex domain $\Omega_r$ in $C^2$ which possesses many pathological properties that include a nontrivial Nebenhulle and the nonexistence of a $C^3$
plurisubharmonic defining function for $\Omega_r$. Very recently Barrett showed in [4] that the Bergman projection associated with $\Omega_r$ does not preserve the Sobolev space $W^k(\Omega_r)$ if $k \in \mathbb{R}$ is large enough. It is still not clear whether condition $R$ holds on $\Omega_r$ or not.

In this article we want to show that despite these pathological properties found on $\Omega$ and $\Omega_r$, statement (1.2) is still valid on both $\Omega$ and $\Omega_r$. In fact we can prove more, namely,

**Main Theorem.** Any automorphism $f$ of either the Barrett or the Diederich-Fornaess worm domains is given by a rotation in $w$-variable, i.e., $f(z, w) = (z, e^{i\phi}w)$ for some constant $\phi \in \mathbb{R}$. In particular, $f$ can be extended smoothly up to the boundary.

We make a remark here that although $\Omega$ does not satisfy condition $R$, it still enjoys an a priori estimate on Sobolev space $W^k(\Omega)$ as shown in Boas and Straube [7].

**II. Proof on the Barrett's domain**

We first recall the definition of $\Omega$. The domain $\Omega$ is a smoothly bounded domain defined in $\mathbb{C}^2$ as follows,

$$\Omega = \{(z, w) \in \mathbb{C}^2 | 1 < |w| < 6, \ |z - c(|w|)| > r_1(|w|) \text{ and } |z| < r_2(|w|)\},$$

where the functions $r_1(|w|)$, $r_2(|w|)$, and $c(|w|)$ are chosen to meet the following conditions: Let $k$ be a positive integer. Define

$$r_1(x) = \begin{cases} 
3 - \sqrt{x - 1} & \text{for } x \text{ near } 1, \\
decreasing & \text{for } x \in [1, 2], \\
1 & \text{for } x \in [2, 5], \\
increasing & \text{for } x \in [5, 6], \\
3 - \sqrt{6 - x} & \text{for } x \text{ near } 6.
\end{cases}$$

$$r_2(x) = \begin{cases} 
3 + \sqrt{x - 1} & \text{for } x \text{ near } 1, \\
increasing & \text{for } x \in [1, 2], \\
4 & \text{for } x \in [2, 5], \\
decreasing & \text{for } x \in [5, 6], \\
3 - \sqrt{6 - x} & \text{for } x \text{ near } 6.
\end{cases}$$

and

$$c(x) = \begin{cases} 
0 & \text{for } x \in [1, 2], \\
decreasing & \text{for } x \in [2, 3], \\
(x - 3)^{2k} - 1 & \text{for } x \text{ near } 3, \\
increasing & \text{for } x \in [3, 4], \\
-(x - 4)^{2k} + 1 & \text{for } x \text{ near } 4, \\
decreasing & \text{for } x \in [4, 5], \\
0 & \text{for } x \in [5, 6].
\end{cases}$$

Then the following theorem that shows $\Omega$ fails to satisfy condition $R$ was proved in Barrett [1].
Theorem. $P(C_0^\infty(\Omega))$ is not contained in $L^p(\Omega)$ for $p \geq 2 + \frac{1}{k}$, where $P$ is the Bergman projection associated with $\Omega$.

Now we proceed to prove our main theorem on this domain. Let $f = (f_1, f_2)$ be an automorphism of $\Omega$. We first show the following lemma. A similar statement was proved in Boas and Straube [7].

**Lemma 2.1.** Let $g$ be a bounded holomorphic function on $\Omega$. Then $g$ can be extended holomorphically to $D$, where

$$D = \{(z , w) \in \mathbb{C}^2 \mid |z| < r_2(|w|) \text{ and } 1 < |w| < 6\}.$$

**Proof.** By Laurent series expansion one can write

$$g(z , w) = \sum_{n=-\infty}^{\infty} a_n(z)w^n,$$

where

$$a_n(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{g(z , w)}{w^{n+1}} \, dw,$$

and $|w| = R$ is any circle contained in the slice corresponding to $z$. Then one can see easily from (2.3) that $a_n(z)$ is locally uniformly bounded and can be extended to a holomorphic function on $\Delta(0 ; 4)$; i.e.,

$$a_n(z) \in H(\Delta(0 ; 4)) \text{ for all } n \in \mathbb{Z}.$$

Next we show as in [7] that the series (2.2) in fact converges on $\{(z , w) \in \mathbb{C}^2 \mid |z| < 3 \text{ and } 1 < |w| < 6\}$ to a holomorphic function. Consider first the nonnegative indices, i.e.,

$$\sum_{n=0}^{\infty} a_n(z)w^n.$$

Put $u(z) = \lim_{n \to \infty} |a_n(z)|^{1/n}$, and let $u^*(z)$ be the upper semicontinuous regularization of $u(z)$, i.e.,

$$u^*(z) = \lim_{z' \to z} u(z').$$

Then by the fact that $a_n(z)$ is locally uniformly bounded, we see that $u^*(z)$ is subharmonic on the disk $\Delta(0 ; 4)$, and it is easy to see that $u^*(z) \leq \frac{1}{6}$ for $|z| = 3$. Hence by maximum principle we obtain that $u^*(z) \leq \frac{1}{6}$ for $|z| \leq 3$ and $u(z) \leq \frac{1}{6}$ for $|z| \leq 3$. It follows that the series (2.4) is holomorphic on $\{(z , w) \in \mathbb{C}^2 \mid |z| < 3 \text{ and } |w| < 6\}$. For the negative indices part we simply replace $w$ by $\frac{1}{w}$, then an analogous argument will go through as well. This completes the proof of the lemma.

It follows thus from Lemma 2.1 that $f_k(z , w) \in H(D)$ for $k = 1, 2$. We claim that in fact we have $f = (f_1, f_2) \in \text{Aut}(D)$.

**Proof of the claim.** Put

$$D_0 = \{(z , w) \in \mathbb{C}^2 \mid |z| < 4 \text{ and } 1 < |w| < 6\},$$
and let $\tilde{D}$ be the envelope of holomorphy of $D$. We have that $\Omega \subseteq D \subseteq \tilde{D} \subseteq D_0$. Let $p = (z_0, w_0)$ be a point in $D - \Omega$. Consider the circle

$$C_p = \{(z, w_0) \in \Omega \mid |z| = 3\}.$$ 

By maximum modulus principle it is easy to see that $f(p) \in D_0$. We wish to show that $f(p) \in \tilde{D}$, and hence $f(D) \subseteq \tilde{D}$.

Suppose that $q = f(p) \notin \tilde{D}$ with $5 \leq |f_2(p)| \leq 6$. Set

$$E(\tilde{D}) = \tilde{D} \cup \{(z, w) \in C^2 \mid |z| < 4 \text{ and } |w| < 4\},$$

and

$$E_+(\tilde{D}) = \{(z, w) \in \mathbb{R}^2 \mid (z, w) \in E(\tilde{D})\}.$$ 

It is well known that the logarithmic image of $E_+(\tilde{D})$ is geometrically convex. Since the point $q^* = (\ln|f_1(p)|, \ln|f_2(p)|)$ is not in $\ln(E_+(\tilde{D}))$ and the rational number is dense in $\mathbb{R}$, one can find two positive integers $m$ and $n$ such that the straight line

$$L = \{(x, y) \in \mathbb{R}^2 \mid mx + ny = c_1 \text{ for some constant } c_1\},$$

go through $q^*$ and such that the whole set $\ln(|f(C_p)|_+)$ lies in the half plane defined by $\{(x, y) \in \mathbb{R}^2 \mid mx + ny - c_1 < 0\}$, where

$$|f(C_p)|_+ = \{(x, y) \in \mathbb{R}^2 \mid (x, y) = (|f_1(z, w_0)|, |f_2(z, w_0)|) \text{ for some } |z| = 3\}.$$ 

Now consider the entire holomorphic function $h(z, w) = e^{-c_1 z^m w^n}$. We see that the restriction of $h \circ f$ to $\Delta_p$, where $\Delta_p = \{(z, w) \in D \mid |z| < 3 \text{ and } w = w_0\}$, will violate the maximum modulus principle. A similar argument via the mapping $w \mapsto \frac{1}{w}$ can be applied to show that no point of $D - \Omega$ can be mapped to $D_0 - \tilde{D}$ with $1 \leq |w| \leq 2$. This shows that $f(p) \in \tilde{D}$, thus we have $f(D) \subseteq \tilde{D}$.

Let $g$ be the inverse mapping of $f$. Since $g$ can be extended holomorphically to $\tilde{D}$, it is legitimate to consider $g \circ f : D \to \mathbb{C}^2$. Then by identity theorem and the fact $g \circ f|_{\Omega} = \text{identity mapping}$, we get $g \circ f = \text{identity mapping}$ on $D$. Similarly $f \circ g$ is also the identity mapping on $D$. This shows that $f \in \text{Aut}(D)$, and the proof of the claim is now completed.

Next we observe that the domain $D$ is Reinhardt. Therefore $f$ can be extended holomorphically to a small open neighborhood of $\tilde{D}$. In particular, we have $f \in \text{Aut}(\tilde{D})$. For instance see Barrett [2]. However, we want to show more that $f'$ in fact is given by a rotation in $w$-variable. So we next characterize a Reinhardt hypersurface in $\mathbb{C}^2$ that contains a Riemann surface in it. By a Reinhardt hypersurface we mean that the hypersurface is invariant under the rotations in all directions. The result might have some interest of itself. If $H$ is a Reinhardt hypersurface in $\mathbb{C}^2$, we denote by $H_+$ the corresponding curve in $\mathbb{R}^2$. Then we have

**Lemma 2.5.** Let $H$ be a Reinhardt hypersurface in $\mathbb{C}^2$ such that $H_+$ is decreasing in $pr$-space with $\rho = |w|$ and $r = |z|$. Then $H$ contains a Riemann surface near $p_0 \in H$ if and only if $H$ is either flat in one of the coordinates or $H_+$ is defined near $p_0$ by a hyperbola, namely,

$$H_+ = \{(\rho, r) \in \mathbb{R}^2 \mid r \rho^c = \text{constant, for some } c \geq 0\}.$$
Proof. Put $z = (x, y) = re^{i\theta}$ and $w = (u, v) = \rho e^{i\phi}$. Let the defining function for $H_+$ (hence for $H$) by $\xi(\rho, r)$. Rewrite $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in terms of polar coordinates, we obtain

$$\frac{\partial}{\partial x} = -\sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial y} = \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial}{\partial r}.$$ 

Next the tangential type-$(1, 0)$ vector field is generated by

$$L = \frac{\partial \xi}{\partial z} \frac{\partial}{\partial w} - \frac{\partial \xi}{\partial w} \frac{\partial}{\partial z} = \frac{1}{2} e^{-i\phi} \frac{\partial \xi}{\partial r} \frac{\partial}{\partial w} - \frac{1}{2} e^{-i\phi} \frac{\partial \xi}{\partial \rho} \frac{\partial}{\partial z}.$$ 

Suppose that $H$ contains a Riemann surface $\mathcal{R}$ near $p_0$ with $\frac{\partial \xi}{\partial r}(p_0) \neq 0$. Then one can choose $L$ to be

$$L = \frac{\partial}{\partial w} - e^{i(\phi-\theta)} \frac{\partial \xi}{\partial \rho} \frac{\partial}{\partial w} \left( \frac{\partial \xi}{\partial r} \right)^{-1} \frac{\partial}{\partial z}.$$ 

Put $g(z, w) = e^{i(\phi-\theta)} \frac{\partial \xi}{\partial \rho} \left( \frac{\partial \xi}{\partial r} \right)^{-1}$. Since $[L, \overline{L}]_{\mathcal{R}} \equiv 0 \mod (L \oplus \overline{L})$, we see that the restriction $g|_{\mathcal{R}}$ is holomorphic. Then consider

$$\frac{g w}{z} \big|_{\mathcal{R}} = \frac{\partial \xi/\partial \rho}{\partial \xi/\partial r} \cdot \frac{\rho}{r}.$$ 

It shows that the function $\frac{g w}{z} \big|_{\mathcal{R}}$ is holomorphic and real valued. Hence it must be a real constant function, namely,

$$\frac{g w}{z} \big|_{\mathcal{R}} \equiv c, \quad \text{for some } c \in \mathbb{R}.$$ 

Also locally one can express $r$ as a function of $\rho$, and the slope of $H_+$ near $p_0$ is given by

$$\frac{dr}{d\rho} = -\frac{\partial \xi/\partial \rho}{\partial \xi/\partial r} = -\frac{c}{\rho}.$$ 

Therefore by solving this first order differential equation we get $r \rho^c = e^{c_0}$, for some constant $c_0 \in \mathbb{R}$. Since $H_+$ is decreasing, the constant $c$ is nonnegative.

On the other hand, if $H_+$ is defined locally near some $p_0$ by $r \rho^c = c_1$ with $c, c_1 > 0$, then by direct computation we get

$$L = w \frac{\partial}{\partial w} - c z \frac{\partial}{\partial z}.$$ 

Hence we have $[L, \overline{L}] \equiv 0$. It follows that there exists a Riemann surface in $H$ near $p_0$. This completes the proof of the lemma.

It is interesting to note that the vector field $-2\text{Im}L$, where $L$ is given in $(2.6)$, is generated by the following $S^1$-action,

$$\Lambda: S^1 \times \mathcal{R} \to \mathcal{R}, \quad (\theta, (z, w)) \mapsto (e^{-ic\theta} z, e^{i\theta} w).$$

Now we go back to the automorphism $f$ of $D$. We see that $f$ will map biholomorphically a Riemann surface in the boundary onto another Riemann surface in the boundary. In particular, if we set

$$\mathcal{R}_{a, b, \theta} = \{(z, w) \in bD | z = 4e^{i\theta} \text{ and } a < |w| < b \text{ with } a \leq 2 \text{ and } b \geq 5\}.$$
to be the largest annulus sitting in the boundary with $|z| = 4$, and set
\[ C_{a, \theta} = \{(4e^{i\theta}, w) \in bD | w| = a\} \]
to be the inner boundary of $R_{a, b, \theta}$, and similarly let $C_{b, \theta}$ be the outer boundary of $R_{a, b, \theta}$. Then $f$ will map $R_{a, b, \theta}$ to a Riemann surface in the boundary. We claim that $C_{a, b, \theta}$ cannot be mapped to any Riemann surface contained in the boundary with $1 < |w| < 2$ or $5 < |w| < 6$. First it is not hard to see that there are only three different types of Riemann surfaces in these regions, they are

(i) $R_c \subseteq \{(z, w) \in bD | |z||w|^c = A$ for some constants $A$ and $c > 0$, and $\alpha < |w| < \beta$ with $5 < \alpha < \beta < 6\}$, or an equivalent counterpart in the region $1 < |w| < 2$.

(ii) $R_z = \{(z, w_0) \in bD | w_0 = pe^{i\phi}$ for some $p$ and $\phi$ with $5 < p < 6$ or $1 < p < 2$, and $s < |z| < t$ for some $3 < s < t < 4\}$.

(iii) $R_w = \{(z_0, w) \in bD | z_0 = re^{i\theta}$ for some $r$ and $\theta$ with $3 < r < 4$, and $s < |w| < t$ for some $1 < s < t < 2$ or $5 < s < t < 6\}$.

We can rule out $R_z$ and $R_w$ immediately by considering the ratio of the radii of the boundaries of these annuli. To knock out $R_c$ we first observe that if $f$ maps $R_{a, b, \theta}$ onto some $R_c$, then by continuity $f$ must map $c_{b, \theta}$ for any $\theta$ into exactly one of $\{(z, w) \in bD | |w| = \alpha\}$ or $\{(z, w) \in bD | |w| = \beta\}$. Suppose that $C_{b, \theta}$ is mapped to $\{(z, w) \in bD | |w| = \alpha\}$. Then by maximum modulus principal we see that $f$ will map $\{(z, w) \in \overline{D} | |z| \leq 4$ and $|w| = b\}$ biholomorphically onto
\[ \{(z, w) \in \overline{D} | |z| \leq A/\alpha^c$ and $|w| = \alpha\}. \]

In particular, $f$ must map a disk
\[ \Delta_w = \{(z, w) \in D | |z| < 4$ and $w = be^{i\phi}$ for some $\phi\} \]
biholomorphically onto another disk
\[ \Delta_{w'} = \{(z, w') \in D | |z| < A/\alpha^c$ and $w' = \alpha e^{i\phi'}$ for some $\phi'\}. \]

Since the restriction $f_{|\Omega}$ is an automorphism, we also have the following biholomorphic equivalence between two annuli induced by $f_{|\Omega}$, namely, $f_{|\Omega} : \Delta_w \cap \Omega \approx \Delta_{w'} \cap \Omega$. However, this cannot happen simply by examining the ratio of the radii of boundaries of these two annuli. Thus we have shown that $f(R_{a, b, \theta}) = R_{a, b, \eta(\theta)}$, for some real-valued function $\eta(\theta)$ that maps $S^1$ bijectively onto itself.

Next we divide our arguments into two subcases.

Case 1. If $f$ maps $C_{b, \theta}$ to $C_{b, \eta(\theta)}$ for some $\theta$. Then by continuity $f$ will map $C_{b, \theta}$ to $C_{b, \eta(\theta)}$ for all $\theta \in [0, 2\pi]$. For each fixed $\theta \in [0, 2\pi]$, we see by reflection principle that $f_2$ can be extended to an entire function which preserves the modulus on $|w| = a$ and $|w| = b$. Hence $f_2$ must take the following form
\[ f_2(4e^{i\theta}, w) = e^{i\delta(\theta)} \cdot w, \]
for some real-valued function $\delta(\theta)$. Also we have
\[ f_1(4e^{i\theta}, w) = 4e^{i\eta(\theta)}, \]
independent of $w$ for $a < |w| < b$. 

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Then by the maximum modulus principle we have

$$|f_2(z, w)| = |w| \quad \text{for } |z| \leq 4 \quad \text{and} \quad a < |w| < b.$$  

This implies that $f$ will map $S_c$ biholomorphically onto $S_c$, where $S_c = \{(z, w) \in D \mid |z| \leq 4, \quad |w| = c \quad \text{with} \quad a < c < b\}$. Therefore, we conclude that $f$ must map a disk $\Delta_{c, \phi}$ onto another disk $\Delta_{c, \phi'}$ i.e.,

$$f: \Delta_{c, \phi} \sim \Delta_{c, \phi'},$$

where $\Delta_{c, \phi} = \{(z, w) \in \bar{D} \mid |z| \leq 4, \quad w = ce^{i\phi}\}$. Thus if we combine equations (2.7) and (2.9), we obtain for fixed $\phi$ that $f_2(4e^{i\theta}, ce^{i\phi}) = ce^{i\delta(\theta)} \cdot e^{i\phi} = ce^{i\phi'}$, for all $\theta \in [0, 2\pi]$. This implies that $\delta(\theta)$ is a constant function, namely, $\delta(\theta) = \phi_0$. Hence we obtain that $f_2(z, w) = e^{i\phi_0}w$.

Next equation (2.8) shows that the restriction $f_1|_{\Delta_{c, \phi}}$ of $f_1$ to every disk $\Delta_{c, \phi}$ with $a < c < b$ and all $\phi$ has the same boundary value. So we conclude that $f_1(z, w) = f_1(z)$ is independent of $w$. Then by the facts that $|f_1(4e^{i\theta})| = 4$ and $|f_1(3e^{i\theta})| = 3$, we get

$$f_1(z, w) = f_1(z) = e^{i\theta_0}z \quad \text{for some constant } \theta_0 \in \mathbb{R}.$$  

Since $f|_{\Omega}$ is an automorphism of $\Omega$, it is easy to see that $\theta_0 = 0$. This shows that

$$f = (f_1, f_2) = (z, e^{i\phi_0}w),$$

and the proof for Case 1 is now completed.

Finally we show that $f$ cannot map $C_{b, \theta}$ to $C_{a, \eta(\theta)}$. This will also complete the proof of our main theorem on the Barrett's domain.

**Case 2.** If $f$ maps $C_{b, \theta}$ to $C_{a, \eta(\theta)}$ for some $\theta$. Then again by continuity $f$ will map $C_{b, \theta}$ to $C_{a, \eta(\theta)}$ for all $\theta \in [0, 2\pi]$. Consider the map

$$g(z, w): D \to D',$$

$$(z, w) \mapsto \left(f_1(z, w), \frac{ab}{f_2(z, w)} \right),$$

where $D'$ is biholomorphic to $D$ via the map $(z, w) \mapsto (z, \frac{ab}{w})$.

So one can repeat the above argument and obtain that

$$f_2(z, w) = \frac{ab}{w} e^{i\phi_0} \quad \text{for some constant } \phi_0 \in \mathbb{R}.$$  

Since $5 \leq ab \leq 12$, in order to preserve the boundaries at two ends, we must have $ab = 6$. We may also conclude that

$$f_1(z, w) = z.$$  

Next consider the point $p_0 = (\frac{1}{5}, 3)$. We see that $p_0 \in \Omega$. Since $f \in \text{Aut}(\Omega)$, we must have $f(p_0) \in \Omega$. However, equations (2.11) and (2.12) show that $f(p_0) = (\frac{1}{5}, 2e^{i\phi_0})$, and this point is clearly not in $\Omega$. This gives the desired contradiction.
Hence any automorphism on the Barrett’s domain is given by a rotation in $w$-variable.

III. Proof on the Diederich-Fornaess domains

We first recall briefly the definition of the Diederich-Fornaess domain here. Fix a smooth function $\lambda : \mathbb{R} \to \mathbb{R}$ satisfying

(a) $\lambda(x) = 0$ if $x \leq 0$,
(b) $\lambda(x) > 1$ if $x > 1$,
(c) $\lambda''(x) \geq 100\lambda'(x)$ for all $x$,
(d) $\lambda''(x) > 0$ if $x > 0$,
(e) $\lambda'(x) > 100$ if $\lambda(x) > \frac{1}{2}$.

Then for any $r > 1$ we define $\Omega_r = \{(z, w) \in \mathbb{C}^2|\rho_r(z, w) < 0\}$ where

$$\rho_r(z, w) = |z + i\lambda| z^2 - 1 + \lambda(1/|w|^2 - 1) + \lambda(|w|^2 - r^2).$$

**Theorem [8].** $\Omega_r$ is a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$. The boundary is strictly pseudoconvex everywhere except on the following annulus,

$$M_r = \{(z, w) \in \partial b_{\Omega_r}|z = 0 \text{ and } 1 < |w| < r\}.$$

Now let $f = (f_1, f_2)$ be an automorphism of $\Omega_r$. Then $f$ can be extended smoothly up to the boundary on $b_{\Omega_r} - M_r$. For instance, see Bell [5]. Therefore, if we consider the deleted torus

$$T_a = \{(z, w) \in b_{\Omega_r}|1 < |w| = a < r \text{ and } z \neq 0\},$$

we see that $\eta_r = \rho_r \circ f$ is a defining function for $T_a$, $1 < a < r$, namely, the equation $|f_1(z, w) + e^{i\ln|w|^2}| = 1$ defines $T_a$. This implies that $|f_2(z, w)|^2 = |w|^2 \cdot e^{2\pi k}$, for some fixed integer $k$. Hence by considering the points $(z, w) \in T_a$ with a close to either $1$ or $r$, we conclude that $k = 0$ and $|f_2(z, w)| = |w|$ for $(z, w) \in T_a$ with $1 < a < r$.

Next fix the constant $a$ with $1 < a < r$, and a point $z_0$ with $|z_0 + e^{i\ln|a|^2}| < 1$ such that $z_0$ lies in a small open neighborhood of $-2e^{i\ln|a|^2}$. Then we consider the annulus defined by

$$A_{z_0} = \{(z_0, w) \in \mathbb{C}^2 \cap \Omega_r$$

with the inner boundary $C_\alpha = \{(z_0, w) \in b_{\Omega_r}||w| = \alpha\}$ and the outer boundary $C_\beta$ such that $\alpha < a < \beta$. $A_{z_0}$ can be identified with $A = \{w \in \mathbb{C}|\alpha < |w| < \beta\}$. Hence via this identification we obtain that

$$f_2(z_0, C_\alpha) = C_\alpha \quad \text{and} \quad f_2(z_0, C_\beta) = C_\beta,$$

and $f_2(z_0, \cdot)$ can be extended to an entire function by reflection principle. Then by (3.1) we must have that

$$f_2(z, w) = e^{i\phi(z)} \cdot w$$

for some real-valued function $\phi(z)$. Since $f_2(z, w)$ is also holomorphic in $z$, we conclude that $\phi(z) = \phi_0$ for some constant $\phi_0 \in \mathbb{R}$, and

$$f_2(z, w) = e^{i\phi_0} \cdot w \quad \text{for } (z, w) \in \Omega_r.$$

Then we consider the open solid torus $\pi_a$ defined by

$$\pi_a = \{(z, w) \in \Omega_r|1 < |w| = a < r \text{ and } |z + e^{i\ln|w|^2}| < 1\}.$$
Put
\[ \Delta_{a, \phi} = \{ (z, ae^{i\phi}) \in \Omega_r | |z + e^{\ln|a|^2}| < 1 \} . \]

It follows that the restriction of \( f_1 \) to \( \Delta_{a, \phi_1} \) must map \( \Delta_{a, \phi_2} \) biholomorphically onto \( \Delta_{a, \phi_2} \) for some \( \phi_2 \). This implies that the restriction of \( f_1 \) to \( \Delta_{a, \phi_1} \) can be extended at least smoothly up to \( \Delta_{a, \phi_1} \). Since \( f_1(0, ae^{i\phi_1}) = 0 \), it follows that \( f_1(z, w) \) can be expressed via the automorphisms on the unit disk as
\[ f_1(z, w) = e^{\ln|w|^2} \left( \frac{1 - \overline{b(w)}e^{\ln|w|^2}}{e^{\ln|w|^2} - b(w)} \right) \left( \frac{z + e^{\ln|w|^2} - b(w)}{1 - \overline{b(w)}(z + e^{\ln|w|^2})} \right) - e^{\ln|w|^2}, \]

for some real analytic function \( b(w) \) satisfying \( |b(w)| < 1 \) for \( 1 < |w| < r \).

Equation (3.3) shows that there exists a small number \( \epsilon > 0 \) such that \( f_1(z, w) \) is real analytic on \( \Delta(0; \epsilon) \times A_\delta \), where \( A_\delta = \{ w \in \mathbb{C} | 1 + \delta < |w| < r - \delta \text{ for some small } \delta > 0 \} \). This in turn implies that \( f_1(z, w) \) is holomorphic on \( \Delta(0; \epsilon) \times A_\delta \). Therefore, one can write
\[ f_1(z, w) = \sum_{k=1}^{\infty} a_k(w) z^k, \]

with \( a_k(w) \in H(A_\delta) \) for all \( k \geq 1 \). By direct computation we get
\[ a_1(w) = \frac{\partial f_1}{\partial z}(0, w) = \frac{1 - |b(w)|^2}{|1 - \overline{b(w)}e^{\ln|w|^2}|^2}. \]

It shows that \( a_1(w) \) is a positive real constant, i.e., \( a_1(w) = c > 0 \). Next the computation of \( a_2(w) \) shows that
\[ a_2(w) = \frac{1}{2} \frac{\partial^2 f_1}{\partial z^2}(0, w) = c \cdot \frac{\overline{b(w)}}{1 - \overline{b(w)}e^{\ln|w|^2}}. \]

We claim that \( a_2(w) \equiv 0 \). Set
\[ g(w) = \frac{a_2(w)}{c} = \frac{\overline{b(w)}}{1 - \overline{b(w)}e^{\ln|w|^2}} \in H(A_\delta), \]

we have
\[ c = \frac{1 - |b(w)|^2}{|1 - \overline{b(w)}e^{\ln|w|^2}|^2} = |1 + g(w)e^{\ln|w|^2}|^2 - |g(w)|^2 = 1 + 2 \text{Re}(g(w)e^{\ln|w|^2}). \]

Therefore, one can write
\[ g(w)e^{\ln|w|^2} = c_0 + iI(w), \]

with \( c_0 = \frac{1}{2}(c - 1) \) and \( I(w) \) is a smooth real-valued function on \( A_\delta \). Hence we obtain
\[ g(w) = c_0e^{-i\ln|w|^2} + iI(w)e^{-i\ln|w|^2} \in H(A_\delta). \]

Locally one can multiply equation (3.5) by \( e^{2i\ln w} \) to get a new well-defined holomorphic function, and get
\[ g(w)e^{2i\ln w} = c_0e^{-2\text{Arg} w} + iI(w)e^{-2\text{Arg} w}. \]
The real part of \( g(w)e^{2i\ln w} \) is a harmonic function. So let \( w = u + iv \), by direct computation we get
\[
\Delta_w (c_0 e^{-2\text{Arg} w}) = c_0 \Delta_w (e^{-2\tan^{-1} v/u}) = \frac{4c_0}{u^2 + v^2} e^{-2\tan^{-1} v/u} \equiv 0.
\]

It follows that \( c_0 = 0 \), and hence \( c = 1 \). This reduces (3.5) to
\[
(3.7) \quad -ig(w) = I(w)e^{-i\ln |w|^2} \in H(A_\delta).
\]

Then repeat the same argument, we see that
\[
-ig(w)e^{2i\ln w} = I(w)e^{-2\text{Arg} w} = c_1,
\]
where \( c_1 \) is a global constant. Hence
\[
I(w) = c_1 e^{2\text{Arg} w}
\]
is a well-defined function on \( A_\delta \). It forces \( c_1 = 0 \). This shows \( g(w) \equiv 0 \), and the proof of our claim is now completed.

It follows then from (3.4) that we have \( b(w) \equiv 0 \) on \( A_\delta \), and equation (3.3) can be simplified to
\[
(3.8) \quad f_1(z, w) = z \quad \text{on } \Omega_r.
\]
Our main theorem now follows from (3.2) and (3.8). So we are done.

References