AN ATRIODIC SIMPLE-4-OD-LIKE CONTINUUM WHICH IS NOT SIMPLE-TRIOD-LIKE

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ABSTRACT. The paper contains an example of a continuum $K$ such that $K$ is the inverse limit of simple 4-ods, $K$ cannot be represented as the inverse limit of simple triods and each proper subcontinuum of $K$ is an arc.

1. INTRODUCTION

All topological spaces considered in this paper are metric. A continuum is a connected and compact space. A simple $n$-od is the union of $n$ arcs meeting at a common endpoint and which are mutually disjoint otherwise. A simple 3-od is called a simple triod. If $X$ and $Y$ are continua, we say that $Y$ is $X$-like provided that $Y$ is the inverse limit of a sequence of copies of $X$. A continuum is atriodic if does not contain three subcontinua $A$, $B$ and $C$ such that none of them is contained in the union of the remaining two and $\emptyset \neq A \cap B \cap C = A \cap B = A \cap C = B \cap C$.

In 1972, W. T. Ingram gave his brilliant example of an atriodic continuum which is simple-triod-like and not arc-like [4]. Note that an arc-like continuum is simple-2-od-like. The Ingram continuum is not only atriodic, but each of its proper subcontinuums is an arc. S. Young asked whether there exists a simple-4-od-like continuum which is not simple-triod-like and whose every proper subcontinuum is an arc [7, Problem 115]. A similar question was asked by H. Cook, W. T. Ingram and A. Lelek. They asked whether there exists an atriodic simple-4-od-like continuum which is not simple-triod-like [1, Problem 5]. Of course, if every proper subcontinuum is an arc, then the continuum is atriodic [3], so a positive answer to Young's question implies a positive answer to the question by Cook, Ingram and Lelek. Even after a perfunctory glance at the problems, it becomes apparent that they should have a positive answer. It is very easy to get an example of a simple-4-od like continuum such that every proper subcontinuum is an arc. Most of such continua appear not to be simple-triod-like and it is very likely that they really are not. So the only difficulty is a proof. Ingram proved that his continuum [4] is not arc-like (chainable) by showing that it has a positive span, and it was proved earlier by Lelek that chainable continua have the span zero [6]. The same method was subsequently

Received by the editors December 27, 1990.
1991 Mathematics Subject Classification. Primary 54F15.

Key words and phrases. Continua, simple-triod-like, simple-4-od-like, simplicial maps, graphs, factorization through a triod.
used by Ingram in [5] and by Davis and Ingram in [2]. A topological invariant different than the span is needed to distinguish between those continua which are simple-triod-like and those that are not. Another way of approaching the problem is to use a continuum with simplicial (piecewise linear) bonding maps and prove that they cannot be factored through a simple triod. In this paper we choose this way to prove that there is a simple-4-od-like but not simple-triod-like continuum $K$ such that every proper subcontinuum of $K$ is an arc. Recently, the author [8] introduced an operation $d$ assigning to a simplicial map between graphs a simplicial map between another pair of graphs and using it characterized simplicial maps which can be factored through an arc. This characterization yielded an alternate proof [8, Examples 5.12 and 5.14] of non-chainability for the Ingram and Davis-Ingram continua. In this paper we adapt the same idea to show that some maps cannot be factored through a simple triod.

2. SIMPLICIAL MAPS

In this section we introduce the notion of simplicial maps and prove some auxiliary propositions.

By a graph we understand one dimensional, finite simplicial complex. If $G$ is a graph then $\mathcal{V}(G)$ will denote the set of vertices and $\mathcal{E}(G)$ will denote the set of edges. By the order of a vertex $v$ we understand the number of edges containing $v$. Two points belonging to an edge are called adjacent. A simplicial map of a graph $G_1$ into a graph $G_0$ is a function from $\mathcal{V}(G_1)$ into $\mathcal{V}(G_0)$ taking every two adjacent vertices either onto a pair of adjacent vertices or onto a single vertex. A simplicial map is light if the image of each edge is nondegenerate.

In this paper the same notation is kept for a graph and for its geometric realization. We will assume that every graph is a subset of three dimensional Euclidean space and every edge is a straight linear closed segment between its vertices. In this convention a simplicial map is understood as an actual continuous mapping (linearly extended to the edges). But it is important to note that a graph, either abstract or geometric, has a fixed collection of vertices and any change in this collection changes the graph.

A graph with a geometric realization homeomorphic to an arc is simply called an arc. Observe that two arcs are isomorphic if and only if they have the same number of vertices. A connected graph without a simple closed curve is called a tree. A tree $T$ is a triod if it is the union of three arcs intersecting at a common endpoint. If $u$ and $v$ are two adjacent vertices of a graph, by $\langle u, v \rangle$ we will denote the edge between $u$ and $v$. Additionally, if $u$ and $v$ are two vertices of a tree, by $\langle u, v \rangle$ we will denote the arc between $u$ and $v$.

2.1 Proposition. Let $\{G_j, \varphi_j^i\}$ be an inverse system of graphs with simplicial and surjective bonding maps $\varphi_j^i: G_i \to G_j$. Let $K$ denote the inverse limit (in the topological sense) of $\{G_j, \varphi_j^i\}$. Suppose that $K$ is simple-triod-like, i.e. $K$ is the inverse limit of simple 3-ods with continuous and not necessarily simplicial bonding maps. Then for each positive integer $j$ there is a positive integer $i$ such that $\varphi_j^i$ can be factored through a (simplicial) triod.
Proof. Let $p_n$ denote the projection of $K$ onto $G_n$. For each $v \in \mathcal{V}(G_j)$, let $U(v)$ denote a small ball around $v$ in $G_j$ such that $U(v_1) \cap U(v_2) = \emptyset$ for each $v_1, v_2 \in \mathcal{V}(G_j)$ and $v_1 \neq v_2$. Let $\mathcal{U}$ be the open covering of $G_j$ consisting of the sets $U(v)$ and all open edges of $G_j$. (By an open edge we understand an edge without its endpoints.) The collection $\mathcal{H} = \{ p_j^{-1}(U) | U \in \mathcal{U} \}$ is an open covering of $K$. Since $K$ is simple-triod-like, there is an open covering $\mathcal{I}$ of $K$ such that $\mathcal{I}$ subdivides $\mathcal{H}$ and the nerve of $\mathcal{I}$ is a triod. By a chain of elements of $\mathcal{I}$ we understand a sequence of sets $t_1, t_2, \ldots, t_k$ such that $t_n \cap t_m \neq \emptyset$ if and only if $|n - m| \leq 1$. Since the nerve of $\mathcal{I}$ is a triod, for any two elements $a$ and $b$ of $\mathcal{I}$ there is exactly one chain $\text{ch}(a, b)$ with the first element $a$ and the last $b$. ($\text{ch}(a, a)$ denote the chain reduced to one element $a$.)

If $A$ is a subset of $\mathcal{I}$, then by conv($A$) we will denote the union of all chains $\text{ch}(a, b)$, where $a, b \in A$. Let $t$ be the only element of $\mathcal{I}$ intersecting three other elements of $\mathcal{I}$, and let $a_1, a_2$ and $a_3$ be the three elements of $\mathcal{I}$ such that $\text{ch}(t, a_1) \cup \text{ch}(t, a_2) \cup \text{ch}(t, a_3) = \mathcal{I}$.

For each $v \in \mathcal{V}(G_j)$, let $\mathcal{I}(v)$ be the set of the elements of $\mathcal{I}$ contained in $p_j^{-1}(U(v))$. Denote by $\hat{\mathcal{I}}$ the union of $\mathcal{I}(v)$, where $v \in \mathcal{V}(G_j)$. We will define an equivalence relation $\equiv$ on $\hat{\mathcal{I}}$ in the following way: $t_1 \equiv t_2$ if there is $v \in \mathcal{V}(G_j)$ such that $t_1, t_2 \in \mathcal{I}(v)$ and $\text{ch}(t_1, t_2) \subset (\mathcal{I} \setminus \hat{\mathcal{I}}) \cup \mathcal{I}(v)$. Let $\Theta$ denote the set $\hat{\mathcal{I}}/\equiv$. If $\tau \in \Theta$, then by $v(\tau)$ we will denote the vertex of $\mathcal{V}(G_j)$ such that $\tau \subset \mathcal{I}(v(\tau))$. Observe that if $\tau_1$ and $\tau_2$ are two distinct elements of $\Theta$, then the sets conv($\tau_1$) and conv($\tau_2$) are disjoint. Note also that conv($\tau_1 \cup \tau_2$) $\subset$ $(\mathcal{I} \setminus \hat{\mathcal{I}}) \cup \tau_1 \cup \tau_2$ if and only if there are elements $t_1 \in \tau_1$ and $t_2 \in \tau_2$ such that $\text{ch}(t_1, t_2) \subset (\mathcal{I} \setminus \hat{\mathcal{I}}) \cup \tau_1 \cup \tau_2$.

Let $T$ be the graph defined in the following way: $\mathcal{V}(T) = \Theta$ and two vertices $\tau_1$ and $\tau_2$ of $T$ are adjacent if $\text{conv}(\tau_1 \cup \tau_2) \subset (\mathcal{I} \setminus \hat{\mathcal{I}}) \cup \tau_1 \cup \tau_2$. Let $\beta : T \rightarrow G_j$ be defined by the formula $\beta(\tau) = v(\tau)$ for $\tau \in \mathcal{V}(T)$. Clearly, $\beta$ is a simplicial map.

We will prove that $T$ is a triod (possibly degenerate). Note that if $\hat{\mathcal{I}}$ is contained in the union of two of the chains $\text{ch}(t, a_1)$, $\text{ch}(t, a_2)$ and $\text{ch}(t, a_3)$, then $T$ is an arc (or a single vertex). So we can assume that for each $k = 1, 2, 3$, there is $b_k \in \Theta$ such that $b_k$ intersects the chain $\text{ch}(t, a_k)$ and no other element of $\Theta$ intersects conv($b_k \cup \{t\}$). Let $V_k$ denote the set of elements of $\Theta$ contained in conv($b_k \cup \{a_k\}$), and let $A_k$ be the subgraph of $T$ spanned by $V_k$. Observe that $A_k$ is an arc (possibly degenerate) and $b_k$ is an end point of $A_k$. Suppose that $T$ is not a triod. Then the vertices $b_1$, $b_2$ and $b_3$ are distinct and each of them is adjacent to the remaining two. Since for each two of the sets conv($b_1$), conv($b_2$) and conv($b_3$) there is a chain in $\mathcal{I}$ between them which does not intersect the third, we may assume that $t \notin \hat{\mathcal{I}}$ and $b_k \subset \text{ch}(t, a_k)$ for $k = 1, 2, 3$. Let $s_k$ be the first element of $\text{ch}(t, a_k)$ belonging to $\hat{\mathcal{I}}$. Clearly, $s_k \in b_k$. Since $t \notin \hat{\mathcal{I}}$, there is an open edge $e$ of $G_j$ such that $p_j(t) \subset e$. Let $v'$ and $v''$ be the vertices of $e$. Observe that $p_j(z) \subset e$ for each $z \in \text{ch}(t, s_k) \setminus \{s_k\}$. It follows that $v(s_k)$ is either $v'$ or $v''$ and consequently two of $b_1$, $b_2$ and $b_3$ coincide. This contradiction proves that $T$ is a triod.

Let $\epsilon$ be a Lebesgue number for the covering $\mathcal{I}$ (i.e. $\epsilon$ is a positive number
such that for each subset $Y$ of $K$, if the diameter of $Y$ is less than $\varepsilon$, then $Y$ is contained in some element of $\mathcal{T}$. There is a positive integer $i$ such that the diameter $p^{-1}_i(z)$ is less than $\varepsilon$ for each $z \in G_i$. Since the bonding maps of the inverse system defining $K$ are surjective, we have that $p_i(p^{-1}_i(z)) = z$. For each $w \in \mathcal{V}(G_i)$, let $a(w)$ be an element of $\mathcal{T}$ containing $p^{-1}_i(w)$. Note that $a(w) \in \mathcal{T}(\phi^i_j(w))$. Let $\alpha(w)$ be the vertex of $T$ representing $a(w)$. Clearly, $\beta \circ \alpha = \phi^i_j$. To complete the proof it is enough to show that $\alpha$ is a simplicial map.

Let $w$ and $w'$ be two adjacent vertices of $G_i$. We will prove that $\alpha(w)$ and $\alpha(w')$ are adjacent vertices of $T$. Let $I_1, I_2, \ldots, I_n$ be a chain covering of $\langle w, w' \rangle$ such that $p^{-1}_i(I_k) \subset a(w)$, $p^{-1}_i(I_k) \subset a(w')$ and the diameter $p^{-1}_i(I_k)$ is less than $\varepsilon$ for each $k = 1, \ldots, n$. Let $B_k$ be an element of $\mathcal{T}$ containing $p^{-1}_i(I_k)$ with $B_1 = a(w)$ and $B_n = a(w')$.

Consider two cases $\phi^i_j(w) = \phi^i_j(w')$ and $\phi^i_j(w) \neq \phi^i_j(w')$. If $\phi^i_j(w) = \phi^i_j(w')$, then $\phi^i_j(I_k) = \phi^i_j(w)$ and thus $B_k \in \mathcal{T}(\phi^i_j(w))$ for each $k = 1, \ldots, n$. It follows that in this case $\alpha(w) = \alpha(w')$. So we may assume that $\phi^i_j(w) \neq \phi^i_j(w')$. Let $e$ be the edge between $\phi^i_j(w)$ and $\phi^i_j(w')$. Since $p_j(B_k) \cap e \neq \emptyset$, we have the result that $p_j(B_k) \subset e \cup U(\phi^i_j(w)) \cup U(\phi^i_j(w'))$, and thus $B_k \in (\mathcal{F} \setminus \mathcal{F}(\phi^i_j(w))) \cup \mathcal{T}(\phi^i_j(w'))$. Let $m$ be the greatest integer such that $B_m \in \mathcal{T}(\phi^i_j(w))$. Since $\phi^i_j(I_1 \cup \cdots \cup I_m) \subset U(\phi^i_j(w))$ and $U(\phi^i_j(w)) \cap U(\phi^i_j(w')) = \emptyset$, we have the result that $B_k \in (\mathcal{F} \setminus \mathcal{F}(\phi^i_j(w)))$ for each $k = 1, \ldots, m$. It follows that $B_1 \cong B_m$. Let $m'$ be the least integer such that $B_{m'} \in \mathcal{T}(\phi^i_j(w'))$. Clearly, $m' > m$. By the same argument as the one above we infer that $B_{m'} \cong B_n$. Since the collection $B_m, B_{m+1}, \ldots, B_{m'}$ contains the chain $\text{ch}(B_m, B_{m'})$, we have the result that $\alpha(w)$ and $\alpha(w')$ are adjacent vertices of $T$. 

We need to recall the following definitions from [8, 5.1 and 5.3].

2.2 Definition. We will say that a graph $G'$ subdivides a graph $G$ if $G'$ is a graph obtained from $G$ by adding vertices on some of its edges. More precisely, $G'$ is a graph such that $\mathcal{V}(G) \subset \mathcal{V}(G')$ and for every edge $e \in \mathcal{E}(G)$ there is an arc $(e, G')$ contained in $G'$ such that

(i) $(e, G')$ has the same endpoints as $e$,

(ii) $(d, G') \cap (e, G') = d \cap e$ for $d, e \in \mathcal{E}(G)$ and $d \neq e$, and

(iii) every vertex from $\mathcal{V}(G')$ belongs to some $(e, G')$ and every edge from $\mathcal{E}(G')$ is an edge of some $(e, G')$.

If $v$ is a vertex of $G$ and $e$ is an edge of $G$ containing $v$, then by $(v, e, G')$ we denote the edge of $(e, G')$ containing $v$.

Let $\phi : G_1 \rightarrow G_0$ be a simplicial map between graphs. Let $G'_0$ be a graph subdividing $G_0$ and let $\phi'$ be a simplicial map of a graph $G'_1$ subdividing $G_1$ onto $G'_0$. We will say that $\phi'$ is a subdivision of $\phi$ matching $G'_0$ provided that $\phi'(v) = \phi(v)$ for each vertex $v \in \mathcal{V}(G_1)$, and for each edge $e \in \mathcal{E}(G_1)$ we have that

- if $\phi(e)$ is degenerate then $(e, G'_1) = e$, and
- if $\phi(e)$ is an edge of $G_0$ then $\phi'$ is an isomorphism of $(e, G'_1)$ onto $(\phi(e), G_0)$.
2.3 Proposition. Suppose $\psi: G_1 \to G_0$ is a simplicial map between connected graphs. Let $G'_0$ be a graph subdividing $G_0$ and let $\psi': G'_1 \to G'_0$ be a subdivision of $\psi$ matching $G'_0$. Then $\psi$ can be factored through a triod if and only if $\psi'$ can be factored through a triod.

Proof. Observe that by [8, Proposition 5.4], if $\psi$ can be factored through a triod, then $\psi'$ also can be factored through a triod. Suppose that there is a triod $T'$ and there are simplicial maps $\alpha': G'_1 \to T'$ and $\beta': T' \to G'_0$ such that $\beta' \circ \alpha' = \psi'$. In view of [8, Proposition 5.13] we can assume that $\alpha'$ is surjective. Let $t$ denote the only order 3 vertex of $T'$. Let $t_0$, $t_1$ and $t_2$ be the endpoints of $T'$. Let $V = \{v \in T'(G_1) | \beta'(v) \in \mathcal{V}(G_0)\}$. Observe that $\alpha'(\mathcal{V}(G_1)) = V$ and $\alpha'(\mathcal{V}(G'_1) \setminus \mathcal{V}(G_1)) = \mathcal{Y}(T') \setminus V$. Let $V_i = V \cap \{t, t_i\}$ and let $w_i$ be the vertex of $V_i$ which is the closest to $t$. Let $A_i$ denote the graph with $V_i$ as its set of vertices such that any two vertices of $V_i$ are adjacent if there are no other points of $V$ between them. Observe that $A_i$ is an arc. We will prove the following claim.

Claim. We can assume that $\beta'(w_0) = \beta'(w_2)$.

If $t \in V$, then $t = w_0 = w_1 = w_2$ and the claim is true. So we can assume that $t \notin V$. Since $\alpha'$ is surjective and $G_1$ is connected there are two pairs $a$, $b$, $b'$ of vertices of $G_1$ such that $a$ and $a'$ are adjacent in $G_1$, $b$ and $b'$ are adjacent in $G_1$, and the set $\{\alpha'(a), \alpha'(a'), \alpha'(b), \alpha'(b')\}$ consists of all three vertices $w_0$, $w_1$ and $w_2$.

Without loss of generality we can assume that $\alpha'(a) = w_0$, $\alpha'(a') = w_1 = \alpha'(b)$ and $\alpha'(b') = w_2$. Let $e_0$ be the edge of $G_0$ joining $\beta'(w_0) = \beta'(\alpha'(a)) = \psi(a)$ and $\beta'(w_1) = \beta'(\alpha'(a')) = \psi(a')$. Let $e_1$ be the edge of $G_0$ joining $\beta'(w_1) = \beta'(\alpha'(b)) = \psi(b)$ and $\beta'(w_2) = \beta'(\alpha'(b')) = \psi(b')$. Since $(e_0, G_0)$ and $(e_1, G_0)$ have two common vertices $\beta'(w_1)$ and $\beta'(t)$, $e_0$ and $e_1$ coincide. Thus $\beta'(w_0) = \beta'(w_2)$.

We will define $T$ considering two cases $\beta'(w_0) = \beta'(w_2) = \beta'(w_1)$ and $\beta'(w_0) = \beta'(w_2) \neq \beta'(w_1)$. In the first case $T$ is the union of $A_0$, $A_1$ and $A_2$ with $w_0$, $w_1$ and $w_2$ identified to one vertex. If $\beta'(w_0) = \beta'(w_2) \neq \beta'(w_1)$, $T$ is the union of $A_0$, $A_1$ and $A_2$ with $w_0$ and $w_2$ identified to one vertex, and with an edge between the result of the identification and $w_1$.

Let $\beta: T \to G_0$ be such that $\beta(v) = \beta'(v)$ for each $v \in V$. Note that $\beta$ is a simplicial map. Let $\alpha: G_1 \to T$ be such that $\alpha(v) = \alpha'(v)$ for each $v \in \mathcal{V}(G_1)$. One can verify that $\alpha$ is a simplicial map and $\beta \circ \alpha = \psi$. ⊓⊔

3. Construction of the example

In this section we will construct a simplicial map $\varphi$ between two subdivisions of a simple 4-od. Then we consider the inverse system with subdivisions of $\varphi$ as bonding maps. We define $K$ to be the inverse limit of this system. We show in this section that each proper subcontinuum of $K$ is an arc. We will show later than $K$ is not simple-triod-like.

Let $X$ be a tree with its vertices named as in Figure 1.

Let $X'$ be a subdivision of $X$ with twelve new vertices $u_0, u_1, \ldots, u_{11}$ added as shown in Figure 2. Let $\varphi: X' \to X$ be the simplicial map defined by Table 1 and the following equality $\varphi(u_0) = \varphi(u_2) = \varphi(u_4) = \varphi(u_5) = \varphi(u_7) = \varphi(u_9) = \varphi(u_{10}) = v_0$. 

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3.1 Proposition. \( \varphi^4((u_{11}, u_6)) = X \) and for each vertex \( x \) of \( X' \), \( \varphi^4(x) \) is either \( v_6 \) or \( v_7 \). (\( \varphi^n \) is the \( n \)th iteration of \( \varphi \).) \( \square \)

We need to recall another definition from [8].

3.2 Definition. Let \( n \) be a positive integer and let \( N \) denote either the set \( \{0, 1, \ldots, n\} \) or the set of all nonnegative integers. Denote by \( N_1 \) the set \( N\{0\} \). Let \( G_0, G_1, G_2, \ldots \) be a sequence of graphs with \( N \) as the set of indices. Let \( \Sigma \) be a sequence of simplicial maps \( \psi_1, \psi_2, \ldots \) such that for each \( j \in N_1 \), \( \psi_j \) maps a graph \( G'_j \) subdividing \( G_j \) into \( G_{j-1} \). Using inductively Proposition 5.4 from [8], we can define a sequence of simplicial maps \( \sigma_1, \sigma_2, \ldots \) such that \( \sigma_1 = \psi_1 \) and for each \( j \in N_1 \{1\} \), \( \sigma_j \) subdivides \( \psi_j \).
3.3 **Proposition.** Every proper nondegenerate subcontinuum of $K$ is an arc.

**Proof.** Let $P$ be a proper subcontinuum of $K$. Suppose that $P$ is not an arc. Since $\varphi$ is simplicial on $X'$, there is a vertex $x$ of $X'$ such that $x \in p_j(P)$ for infinitely many $j$. In view of Proposition 3.1, we have that either $v_6 \in P_j(P)$ for all $j$ or $v_7 \in P_j(P)$ for all $j$. Since $\varphi$ restricted to $\langle u_{11}, v_7 \rangle$ is an embedding, $u_{11} \in p_j(P)$ for infinitely many $j$. Since $\langle u_{11}, v_6 \rangle \subset \langle u_{11}, v_7 \rangle$ and $\varphi^4(\langle u_{11}, v_6 \rangle) = X$, $p_j(P) = X$ for each $j$. Thus $P$ is not proper. $\square$

3.4 **Proposition.** Let $n$ be a positive integer. Let $e$ be an edge of $X$. Let $\bar{e}$ denote the interior of $e$. Suppose $C \subset \Phi_n$ is a component of $(\Phi_1^n)^{-1}(\bar{e})$. Then the closure of $C$ is mapped by $\Phi_1^n$ isomorphically onto $e$.

**Proof.** Since $\Phi_0^{n-1}$ is a simplicial map onto $X$, any component of $(\Phi_0^{n-1})^{-1}(\bar{e})$ is the interior of an edge of $\Phi_{n-1}$. Since $\Phi_1^n$ is a simplicial subdivision of $\Phi_0^{n-1}$ matching $X'$, $C$ is a subdivision of the interior of an edge of $\Phi_{n-1}$. Since $\Phi_0^{n-1}$ maps isomorphically edges of $\Phi_{n-1}$ onto edges of $X$, $C$ is mapped by $\Phi_1^n$ isomorphically onto $e$. $\square$

3.5 **Proposition.** Let $n$ be a positive integer. Suppose that $a$ is a vertex of $\Phi_n$ such that $\Phi_0^n(a) = v_0$. Then $a$ is a vertex of order 2 in $\Phi_n$. Moreover, if $b$ and $c$ are the two vertices of $\Phi_n$ adjacent to $a$, then $\Phi_0^n$ is an embedding of $\langle b, c \rangle$ into $X$.

**Proof.** Since $\Phi_1^n(a)$ is a vertex of $X'$ but not of $X$, the proposition follows from Proposition 3.4.
3.6 Proposition. Suppose that $Y$ is a triod which is the union of three arcs $A_1$, $A_2$ and $A_3$ meeting at a common endpoint $y$. Let $y_1, y_1', y_2, \ldots, y_k$ denote the sequence of consecutive vertices of $A_i$. Let $\beta : Y \to X$ be a simplicial map such that $\beta(y) = v_0$ and the points $\beta(y_1)$, $\beta(y_1')$ and $\beta(y_3')$ are three different vertices from the set $\{v_1, v_2, v_3, v_5\}$. Suppose there is a positive integer $n$ and there is a simplicial map $\alpha : \Phi_n \to Y$ such that $\beta \circ \alpha = \Phi_n$. Then there is a triod $Y'$ with its vertex of order three denoted by $y'$ and there are simplicial maps $\beta' : Y' \to X$ and $\alpha' : \Phi_n \to Y'$ such that $\beta' \circ \alpha' = \Phi_n$ and $\beta'(y') \neq v_0$.

Proof. Let $W_{1,2}, W_{1,3}$ and $W_{2,3}$ denote the sets $\alpha^{-1}(A_1) \cap \alpha^{-1}(A_2), \alpha^{-1}(A_1) \cap \alpha^{-1}(A_3)$ and $\alpha^{-1}(A_2) \cap \alpha^{-1}(A_3)$, respectively. By Proposition 3.5, the sets $W_{1,2}, W_{1,3}$ and $W_{2,3}$ are mutually exclusive and $\alpha^{-1}(y)$ is their union.

3.6.1 Claim. If one of the sets $W_{1,2}, W_{1,3}$ and $W_{2,3}$ is empty then the proposition is true.

Since $A_1$, $A_2$ and $A_3$ play the same role in the statement of the proposition, we may assume that $W_{1,2} = \emptyset$. Define $\gamma'(Y')$ to be $\gamma(Y)$ with $y$ replaced by two points $w_1$ and $w_2$. Let $\beta'(Y')$ consists of $\langle y_1, w_1 \rangle$, $\langle y_1, w_2 \rangle$, $\langle w_1, y_1 \rangle$, $\langle w_2, y_1 \rangle$, and all edges of $Y$ not containing $y$. Define $\beta'(w_1) = \beta'(w_2) = v_0$ and $\beta'(w) = \alpha(v)$ for $w \in Y'$ and $\gamma'(Y') \cap \gamma(Y')$. Define $\alpha'(v)$ for $v \in \gamma'\Phi_n) \alpha^{-1}(y)$, $\alpha'(v) = w_1$ for $v \in W_{1,3}$ and $\alpha'(v) = w_2$ for $v \in W_{2,3}$. One can verify that the hypothesis of the proposition is satisfied with $y' = y_3$.

3.6.2 Claim. Suppose that $\beta(y_1) = v_1$ and $\beta(y_3') = v_2$. Suppose also that either $\beta(y_1') = v_3$ or $\beta(y_1') = v_5$ and $\beta(y_3') = v_6$. If $W_{1,2} \neq \emptyset$, then $\alpha(\Phi_n) \subset A_1 \cup A_2$.

By Proposition 3.5, there are three vertices $w_1, w$ and $w_2$ of $\Phi_n$ such that $w_1$ is adjacent to $w$, $w$ is adjacent to $w_2$, $\alpha(w_1) = y_1$, $\alpha(w) = y$ and $\alpha(w_2) = y_1'$. Since $v_2 - u_4 - u_3$ is the only pair of intersecting edges of $X'$ mapped by $\varphi$ onto $v_1 - v_0 - v_2$, we have the result that $\Phi_i(w_1) = v_2$, $\Phi_i(w) = u_4$ and $\Phi_i(w_2) = u_3$. Let $x$ be an arbitrary vertex of $\Phi_n$. Let $s_0 = w$, $s_1$, $\ldots$, $s_k = x$ be vertices of $\langle w, x \rangle$ such that for $i = 0, 1, \ldots, k-1$, $\alpha(s_i) = y$ and $\alpha(a) \neq y$ for any vertex $a$ from the interior of $\langle s_i, s_{i+1} \rangle$.

In order to prove the claim, it is enough to show that if $\Phi_i(s_i) = u_4$, then $\alpha(\langle s_i, s_{i+1} \rangle) \subset A_1 \cup A_2$ and if $i + 1 < k$, then also $\Phi_i(s_{i+1}) = u_4$. Observe that $\alpha(\langle s_i, s_{i+1} \rangle)$ is contained in one of the arcs $A_1$, $A_2$ and $A_3$. Let $a_0 = s_{i_1}, a_1, a_2, \ldots, a_m = s_{i_{k+1}}$ be vertices of $\langle s_i, s_{i+1} \rangle$ listed in the natural order. Clearly, either $\Phi_i(a_1) = u_3$ or $\Phi_i(a_1) = v_2$. We will consider each of these cases separately.

Suppose $\Phi_i(a_1) = u_3$. Since $\beta \circ \alpha = \Phi_n = \varphi \circ \Phi_i$, $\alpha(a_1) = y_1'$ and consequently $\alpha(\langle s_i, s_{i+1} \rangle) \subset A_2$. By Proposition 3.4, there is a vertex $b$ of $\Phi_n$ such that $\Phi_b$ maps consecutive vertices of $\langle a_1, b \rangle$ onto the sequence $v_2, v_0, v_5, v_0, v_3$. To match this pattern, $\alpha$ must map $\langle a_1, b \rangle$ into $A_2$ and we must have that $\beta(y_2') = v_0$, $\beta(y_3') = v_5$, $\beta(y_3') = v_0$ and $\beta(y_5') = v_3$. Now, assume that $\alpha(s_{i+1}) = y$. Clearly, $m > 5$ and $\alpha(a_2) = y_2'$, $\alpha(a_3) = y_3'$, $\alpha(a_4) = y_2'$ and $\alpha(a_5) = y_3'$. Let $j < m$ be the greatest integer such that $\alpha(a_j) = y_4'$. Note that $m \geq j + 4$. Since $\beta \circ \alpha = \Phi_n$, it follows from Proposition 3.5 that $\alpha(a_{j-1}) = y_5'$ and $\alpha(a_{j+1}) = y_3'$. Observe that $\Phi_b(a_{j-1}) = v_3$, $\Phi_b(a_1) = v_0$ and $\Phi_b(a_{j+1}) = v_5$. Since $v_0 - u_0 - u_1$ is the only pair of intersecting edges of $X'$ mapped by $\varphi$ onto $v_3 - v_0 - v_5$, $\Phi_b(a_{j+1}) = u_1$. By Propo-
sition 3.4, there is a vertex \( c \) of \( \Phi_n \) such that \( a_j \in \langle a_{j-1}, c \rangle \) and \( \langle a_{j-1}, c \rangle \) is mapped by \( \Phi_n^* \) isomorphically onto \( \langle v_0, v_2 \rangle \). It follows from Proposition 3.4 that the vertices \( a_{j+1}, a_{j+2}, a_{j+3} \) and \( a_{j+4} \) belong to \( \langle a_{j-1}, c \rangle \). Since \( \langle a_{j-1}, c \rangle \) has seven vertices, \( a_{j+4} \) is adjacent to \( c \). Since \( \alpha(a_{j-1}) = y_{j-1}^2 \), \( \alpha(a_j) = y_j^2 \), \( \alpha(a_{j+1}) = y_{j+1}^2 \) and \( \Phi_0^* \) maps consecutive vertices of \( \langle a_{j-1}, c \rangle \) onto the sequence \( v_3, v_0, v_5, v_0, v_2, v_0, v_1 \), we have the result that \( \alpha(c) = y_1 \). Thus \( \alpha(a_{j+4}) = y_j, s_{j+1} = a_{j+4} \) and consequently \( \Phi_n^*(s_{j+1}) = u_4 \). This completes the proof of the claim in the case where \( \Phi_n^*(a_1) = u_3 \).

Suppose \( \Phi_n^*(a_1) = v_2 \). The proof in this case is essentially the same as in the previous case. Since \( \beta \circ \alpha = \Phi_0 \circ \Phi_n^* \), \( \alpha(a_1) = y_1 \) and consequently \( \alpha((s_1, s_{j+1})) \subset A_1 \). Now, suppose that \( \alpha(s_{j+1}) = y \). Clearly, \( m \geq 2 \) and \( \Phi_1^*(a_2) = u_4 \). Observe that either \( \alpha(a_2) = y \) or \( \alpha(a_2) = y_{j+1}^2 \). In the first case the claim is satisfied, so we can assume that \( \alpha(a_2) = y^2 \). By Proposition 3.4, there is a vertex \( b \) of \( \Phi_n \) such that \( \Phi_n^* \) maps consecutive vertices of \( \langle a_1, b \rangle \) onto the sequence \( v_1, v_0, v_2, v_0, v_5, v_0, v_3 \). To match this pattern, \( \alpha \) must map \( \langle a_1, b \rangle \) into \( A_1 \) and must have that \( \beta(y_2^1) = v_0, \beta(y_2^3) = v_2, \beta(y_3^2) = v_0, \beta(y_3^4) = v_2, \beta(y_4^1) = v_0, \beta(y_4^3) = v_1, \beta(y_5^1) = v_5, \beta(y_5^3) = v_0, \beta(y_5^5) = v_2, \beta(y_6^1) = v_0, \beta(y_6^3) = v_1 \) and \( \beta(y_7^1) = v_3, \beta(y_7^3) = v_0, \beta(y_7^5) = v_2 \). Clearly, \( m > 7 \) and \( \alpha(a_3) = y_1^3 \), \( \alpha(a_4) = y_1^4 \), \( \alpha(a_5) = y_1^5 \), \( \alpha(a_6) = y_1^6 \) and \( \alpha(a_7) = y_1^7 \). Let \( j < m \) be the greatest integer such that \( \alpha(a_j) = y_1^j \). Note that \( m \geq j + 6 \). Since \( \beta \circ \alpha = \Phi_n \), it follows from Proposition 3.5 that \( \alpha(a_{j-1}) = y_{j-1}^1 \) and \( \alpha(a_{j+1}) = y_{j+1}^1 \). Observe that \( \Phi_0^*(a_{j-1}) = v_3, \Phi_0^*(a_j) = v_0 \) and \( \Phi_0^*(a_{j+1}) = v_5 \). Since \( v_0 - u_0 - u_1 \) is the only pair of intersecting edges of \( X' \) mapped by \( \phi \) onto \( v_3 - v_0 - v_5, \Phi_1^*(a_{j+1}) = u_4 \). By Proposition 3.4, \( \langle a_{j-1}, a_{j+5} \rangle \) is mapped by \( \Phi_n^* \) isomorphically onto \( \langle v_0, v_2 \rangle \). In particular \( \Phi_n^*(a_{j+5}) = v_2 \) and consequently \( \Phi_1^*(a_{j+6}) = u_4 \). Since \( \alpha(a_{j-1}) = y_1^1 \), \( \alpha(a_j) = y_1^j \), \( \alpha(a_{j+1}) = y_{j+1}^2 \) and \( \Phi_n^* \) maps consecutive vertices of \( \langle a_{j-1}, a_{j+5} \rangle \) onto the sequence \( v_3, v_0, v_5, v_0, v_2, v_0, v_1 \), we have the result that \( \alpha(a_{j+5}) = y_{j+1}^1 \). If \( \alpha(a_{j+6}) = y \), then \( s_{j+1} = a_{j+6} \) and the claim is true. Suppose that \( \alpha(a_{j+6}) \neq y \) and thus \( \alpha(a_{j+6}) = y_{j+2}^2 \). By Proposition 3.4, there is a vertex \( c \) of \( \Phi_n \) such that \( a_{j+6} \in \langle a_{j+5}, c \rangle \) and \( \langle a_{j+5}, c \rangle \) is mapped by \( \Phi_n^* \) isomorphically onto \( \langle v_0, v_2 \rangle \). Since \( \alpha(a_{j+5}) = y^2_1, \alpha(a_{j+6}) = y_1^3 \) and \( \Phi_n^* \) maps consecutive vertices of \( \langle a_{j+5}, c \rangle \) onto the sequence \( v_1, v_0, v_2, v_0, v_5, v_0, v_3 \), we have the result that \( \alpha(a_{j+5}) = y_{j+1}^1 \). It follows that \( m > j + 11 \) and \( \alpha(a_{j+10}) = y_{j+1}^6 \). This contradicts the choice of \( j \). So the claim is true.

We will consider the following four cases: Case (i). \( \beta(y_1^1) = v_1, \beta(y_1^3) = v_2 \) and \( \beta(y_1^5) = v_3 \), Case (ii). \( \beta(y_1^1) = v_1, \beta(y_1^3) = v_2 \) and \( \beta(y_1^5) = v_5 \), Case (iii). \( \beta(y_1^1) = v_1, \beta(y_1^3) = v_3 \) and \( \beta(y_1^5) = v_5 \), and Case (iv). \( \beta(y_1^1) = v_2, \beta(y_1^3) = v_3 \) and \( \beta(y_1^5) = v_5 \).

Case (i). \( \beta(y_1^1) = v_1, \beta(y_1^3) = v_2 \) and \( \beta(y_1^5) = v_3 \).

By 3.6.2, if \( W_{1,2} \neq \emptyset \), then \( \alpha(\Phi_n) \) is an arc and the proposition is trivially satisfied. So, we can assume that \( W_{1,2} = \emptyset \) and infer the proposition from 3.6.1.

Case (ii). \( \beta(y_1^1) = v_1, \beta(y_1^3) = v_2 \) and \( \beta(y_1^5) = v_5 \).

If \( W_{1,3} \neq \emptyset \), then the proposition is true by 3.5.1. So we can assume that \( W_{1,3} = \emptyset \). By Proposition 3.5, there are three vertices \( w_1, w \) and \( w_3 \) of \( \Phi_n \) such that \( w_1 \) is adjacent to \( w \), \( w \) is adjacent to \( w_3 \), \( \alpha(w_1) = y_1^1 \), \( \alpha(w) = y \) and \( \alpha(w_3) = y_{j+1}^1 \). Since \( v_5 - u_{10} - u_1 \) is the only pair of intersecting edges of \( X' \) mapped by \( \phi \) onto \( v_1 - v_0 - v_5 \), we have the result that \( \Phi_n^*(w_1) = v_5, \Phi_n^*(w) = u_{10} \) and \( \Phi_n^*(w_3) = u_{11} \). By Proposition 3.4, there is a vertex \( c \) of \( \Phi_n \)
such that $c$ is adjacent to $w_3$ and $\Phi_0^r(c) = v_6$. Since $\Phi_0^r(c) = v_6$, $\alpha(c) = y_2^3$ and $\beta(y_2^3) = v_6$. Now, we can use 3.6.2. If $W_{1,2} \neq \emptyset$, then $\alpha(\Phi_n)$ is an arc and the proposition is trivially satisfied. So, we can assume that $W_{1,2} = \emptyset$ and infer the proposition from 3.6.1.

Case (iii). $\beta(y_1^1) = v_1$, $\beta(y_2^2) = v_3$ and $\beta(y_3^3) = v_5$.

If $W_{1,3} = \emptyset$, then the proposition is true by 3.6.1. So we can assume that $W_{1,3} \neq \emptyset$. By Proposition 3.5, there are three vertices $w_1$, $w$ and $w_3$ of $\Phi_n$ such that $w_1$ is adjacent to $w$, $w$ is adjacent to $w_3$, $\alpha(w_1) = y_1^1$, $\alpha(w) = y$ and $\alpha(w_3) = y_3^3$. Since $v_5 - u_1 - u_11$ is the only pair of intersecting edges of $X'$ mapped by $\varphi$ onto $v_1 - v_0 - v_5$, we have the result that $\Phi_0^r(w_1) = v_5$, $\Phi_0^r(w) = u_10$ and $\Phi_0^r(w_3) = u_11$. By Proposition 3.4, there is a vertex $c$ of $\Phi_n$ such that $c$ is adjacent to $w_3$ and $\Phi_0^r(c) = v_6$. Since $\Phi_0^r(c) = v_6$, $\alpha(c) = y_2^3$ and $\beta(y_2^3) = v_6$.

If $W_{2,3} = \emptyset$, then the proposition is true by 3.6.1. So we can assume that $W_{2,3} \neq \emptyset$. By Proposition 3.5, there are three vertices $u_2$, $u$ and $u_3$ of $\Phi_n$ such that $u_2$ is adjacent to $u$, $u$ is adjacent to $u_3$, $\alpha(u_2) = y_2^2$, $\alpha(u) = y$ and $\alpha(u_3) = y_3^3$. Since $v_0 - u_0 - u_1$ is the only pair of intersecting edges of $X'$ mapped by $\varphi$ onto $v_3 - v_0 - v_5$, we have the result that $\Phi_0^r(u_2) = v_0$, $\Phi_0^r(u) = u_0$ and $\Phi_0^r(u_3) = u_1$. By Proposition 3.4, there are vertices $a$ and $b$ of $\Phi_n$ such that $a$ is adjacent to $u_3$, $b$ is adjacent to $a$, $\Phi_0^r(a) = u_2$ and $\Phi_0^r(b) = u_3$. Since $\Phi_0^r(a) = v_0$ and $\beta(y_3^3) = v_6$, we have the result that $\alpha(a) = y$. Thus $\alpha(b)$ must be one of the points $y_1^1$, $y_2^2$ and $y_3^3$. This is a contradiction, because $\beta(\alpha(b)) = \Phi_0^r(b) = v_2$, $\beta(y_1^1) = v_1$, $\beta(y_2^2) = v_3$ and $\beta(y_3^3) = v_5$.

Case (iv). $\beta(y_1^1) = v_2$, $\beta(y_2^2) = v_3$ and $\beta(y_3^3) = v_5$.

If $W_{2,3} = \emptyset$, then the proposition is true by 3.6.1. So we can assume that $W_{2,3} \neq \emptyset$. By Proposition 3.5, there are three vertices $w_2$, $w$ and $w_3$ of $\Phi_n$ such that $w_2$ is adjacent to $w$, $w$ is adjacent to $w_3$, $\alpha(w_2) = y_2^2$, $\alpha(w) = y$ and $\alpha(w_3) = y_3^3$. Since $v_0 - u_0 - u_1$ is the only pair of intersecting edges of $X'$ mapped by $\varphi$ onto $v_3 - v_0 - v_5$, we have the result that $\Phi_0^r(w_2) = v_0$, $\Phi_0^r(w) = u_0$ and $\Phi_0^r(w_3) = u_1$. By Proposition 3.4, there is a vertex $b$ of $\Phi_n$ such that $w_3 \in (w_2, b)$ and $\Phi_0^r$ maps $(w_2, b)$ isomorphically onto $(v_0, v_2)$. Note that $\Phi_0^r$ maps the consecutive vertices of $(w_3, b)$ onto the sequence $v_5, v_0, v_2, v_0, v_1$. To match this pattern we must have that

$$\text{(3.6.3) either } \beta(y_3^3) = v_0, \text{ or } \beta(y_3^3) = v_0 \text{ and } \beta(y_1^1) = v_1.$$  

If $W_{1,2} = \emptyset$, then the proposition is true by 3.6.1. So we can assume that $W_{1,2} \neq \emptyset$. By Proposition 3.5, there are three vertices $z_1$, $z$ and $z_2$ of $\Phi_n$ such that $z_1$ is adjacent to $z$, $z$ is adjacent to $z_2$, $\alpha(z_1) = y_1^1$, $\alpha(z) = y$ and $\alpha(z_2) = y_2^3$. Since $v_0 - u_5 - u_6$ is the only pair of intersecting edges of $X'$ mapped by $\varphi$ onto $v_3 - v_0 - v_2$, we have the result that $\Phi_0^r(z_2) = v_0$, $\Phi_0^r(z) = u_5$ and $\Phi_0^r(z_1) = u_6$. By Proposition 3.4, there is a vertex $c$ of $\Phi_n$ such that $z_1 \in (z_2, c)$ and $\Phi_0^r$ maps $(z_2, c)$ isomorphically onto $(v_0, v_3)$. Note that $\Phi_0^r$ maps the consecutive vertices of $(z_1, c)$ onto the sequence $v_2, v_0, v_5, v_6$. To match this pattern we must have that

$$\text{(3.6.4) either } \beta(y_2^3) = v_0 \text{ and } \beta(y_1^1) = v_5, \text{ or } \beta(y_2^3) = v_6.$$  

We will prove that

$$\text{(3.6.5) either } \beta(y_2^3) = v_0 \text{ and } \beta(y_1^1) = v_1, \text{ or } \beta(y_2^3) = v_6.$$  

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Again, by 3.6.1, we can assume that $W_{1,3} \neq \emptyset$. By Proposition 3.5, there are three vertices $x_1$, $x$, and $x_3$ of $\Phi_n$ such that $x_1$ is adjacent to $x$, $x$ is adjacent to $x_3$, $\alpha(x_1) = y_1^1$, $\alpha(x) = y$ and $\alpha(x_3) = y_3^1$. Since $u_6 - u_7 - u_8$ and $u_3 - u_2 - u_1$ are the only two pairs of intersecting edges of $X'$ mapped by $\varphi$ onto $v_2 - v_0 - v_5$, we have the result that either $\Phi'_{\varphi}(x_1) = u_6$, $\Phi'_{\varphi}(x) = u_7$ and $\Phi'_{\varphi}(x_3) = u_8$, or $\Phi'_{\varphi}(x_1) = u_3$, $\Phi'_{\varphi}(x) = u_2$ and $\Phi'_{\varphi}(x_3) = u_1$. Suppose that $\Phi'_{\varphi}(x_3) = u_8$. Then by Proposition 3.4, there is a vertex $s$ of $\Phi_n$ such that $s$ is adjacent to $x_3$ and $\Phi'_{\varphi}(s) = v_3$. This forces $\beta(y^1_2) = v_6$ and (3.6.5) is true. So, we can assume that $\Phi'_{\varphi}(x_1) = u_3$, $\Phi'_{\varphi}(x) = u_2$ and $\Phi'_{\varphi}(x_3) = u_1$. By Proposition 3.4, there is a vertex $q$ of $\Phi_n$ such that $x_1 \in \langle x_3, q \rangle$ and $\Phi'_{\varphi}$ maps $\langle x_1, q \rangle$ isomorphically onto $\langle u_3, v_2 \rangle$. Note that $\Phi'^n$ maps the consecutive vertices of $\langle x_1, q \rangle$ onto the sequence $v_2, v_0, v_1$. To match this pattern we must have that $\beta(y^1_2) = v_0$ and $\beta(y^2_1) = v_1$. So (3.6.5) is true.

Combining (3.6.3), (3.6.4) and (3.6.5) we get the result that

$$\beta(y^1_2) = v_0, \quad \beta(y^2_1) = v_1 \quad \text{and} \quad \beta(y^1_2) = v_6. \quad (3.6.6)$$

Figure 4 shows the map $\beta$ on the "central" part of $Y$. As before, the dotted line graph represents the domain, $Y$, while the solid black represents the range, $X$, and each vertex of the domain is mapped onto the nearest vertex of the range.

Define $\mathcal{V}(Y')$ to be $\mathcal{V}(Y)$ with $y$ replaced by four points $g_0, g_2, g_5$ and $g_7$, with $y^1_1$ replaced by two points $g_3$ and $g_6$, and with $y^1_3$ replaced by two points $g_1$ and $g_8$. Let $\mathcal{E}(Y')$ consists of $\langle y^1_2, g_3 \rangle$, $\langle y^1_3, g_8 \rangle$, $\langle y^2_1, g_0 \rangle$, $\langle y^2_3, g_5 \rangle$, $\langle g_0, g_1 \rangle$, $\langle g_1, g_2 \rangle$, $\langle g_2, g_3 \rangle$, $\langle g_3, g_5 \rangle$, $\langle g_5, g_6 \rangle$, $\langle g_6, g_7 \rangle$, $\langle g_7, g_8 \rangle$ and all edges of $Y$ not containing any of the vertices $y, y^1_1$ and $y^1_3$. Observe that $Y'$ is a triod and $y^1_3$ is its vertex of order 3. (See Figure 5.) Define $\beta'(g_0) = \beta'(g_2) = \beta'(g_3) = \beta'(g_5) = \beta'(g_7) = v_0$, $\beta'(g_3) = \beta'(g_6) = v_2$, $\beta'(g_1) = \beta'(g_8) = v_5$ and $\beta'(w) = \beta(w)$ for $w \in \mathcal{V}(Y') \cap \mathcal{V}(Y)$. Figure 5 shows the map $\beta'$ on the "central" part of $Y'$. As usual, the dotted line graph represents the domain, $Y'$, while the solid black represents the range, $X$, and each vertex of the domain is mapped onto the nearest vertex of the range. Compare this figure with Figures 3 and 4.

Let $G$ denote the set $\{v \in \mathcal{V}(\Phi_n) | \alpha(v) \text{ is either } y, \text{ or } y^1_1, \text{ or } y^1_3\}$. Let
Figure 5

$G_i$ denote the set $\{v \in G | \Phi^i(v) = u_i\}$ where $i = 0, 1, 2, 3, 5, 6, 7$ or $8$. Using 3.5.6 and Proposition 3.4 one can prove that $G = G_0 \cup G_1 \cup G_2 \cup G_3 \cup G_5 \cup G_6 \cup G_7 \cup G_8$. Define $\alpha'(v) = \alpha(v)$ for $v \in \mathcal{V}(\Phi_n) \setminus G$ and $\alpha'(v) = g_i$ for $v \in G_i$, $i = 0, 1, 2, 3, 5, 6, 7, 8$. Clearly, $\alpha'$ is a simplicial map and $\beta' \circ \alpha' = \Phi^n_0$. 

4. The operation $d$

In this section we will recall combinatorial methods introduced in [8] and apply them to the map $\phi$.

4.1 Definition. For a graph $G_0$, let $D(G_0)$ denote the graph such that

(i) the set of vertices of $D(G_0)$ consists of edges of $G_0$ and

(ii) two vertices of $D(G_0)$ are adjacent if and only if they intersect (as edges of $G_0$).

Let $\psi : G_1 \to G_0$ be a simplicial map between graphs. For every (closed) edge $e \in \mathcal{E}(G_0)$, let $\mathcal{H}(e)$ denote the set of components of $\psi^{-1}(e)$ which are mapped by $\psi$ onto $e$. Denote by $\mathcal{H}(\psi)$ the union of all $\mathcal{H}(e)$. Let $D(\psi, G_1)$ be the graph such that

(i) the vertices of $D(\psi, G_1)$ are elements of $\mathcal{H}(\psi)$, and

(ii) two vertices of $D(\psi, G_1)$ are adjacent if and only if they intersect (as subgraphs of $G_1$).

Let $d[\psi] : D(\psi, G_1) \to D(G_0)$ be the map defined by the formula $d[\psi](v) = \psi(v)$ for every vertex $v$ of $D(\psi, G_1)$.

Every vertex $v \in \mathcal{V}(D(\psi, G_1))$ is also a subgraph of $G_1$. To avoid confusion we will denote this subgraph by $v^*$.

Let $\sigma$ be simplicial maps of a graph $G_2$ into $G_1$. Let $d[\psi, \sigma] : D(\psi \circ \sigma, G_2) \to D(\psi, G_1)$ be the map such that for every vertex $v$ of $D(\psi \circ \sigma, G_2)$, $d[\psi, \sigma](v)$ is the vertex of $D(\psi, G_1)$ containing $\sigma(v^*)$.

4.2 Proposition. Suppose that $Y$ is a triod which is the union of three arcs $A_1$, $A_2$ and $A_3$ meeting at a common endpoint $y$. Let $y, y_1, y_2, \ldots, y_{k(i)}$ denote the sequence of consecutive vertices of $A_i$. Suppose $\psi$ is a simplicial map of $Y$ into a graph $G$. Let $p$ be the least integer such that $\psi(y_{p+1}) \neq \psi(y)$, and let $q$
be the least integer such that $\psi(y^2_2) \neq \psi(y)$. If $\psi(y^1_p) = \psi(y^2_p)$, then $D(\psi, Y)$ is a triod (possibly degenerate).

**Proof.** Clearly, $\psi((y, y^2_p)) = \psi((y, y^2_p))$ is an edge of $G$. Denote this edge by $e$. Let $t$ be the vertex of $D(\psi, Y)$ representing the component of $\psi^{-1}(e)$ containing $(y, y^2_p)$ and observe that for any vertex $z \neq t$ of $D(\psi, Y)$, $z^*$ is contained in one of the arcs $A_1$, $A_2$ and $A_3$. Let $Z_t = \{z \in D(\psi, Y) | z \neq t$ and $z^* \subset A_i\}$. Let $z$ be an arbitrary point of $Z_t$ and let $j$ be an index such that $(y^i_j, y^i_{j+1}) \subset z^*$ and $\psi((y^i_j, y^i_{j+1}))$ is a nondegenerate edge of $G$. Observe that, if $s$ is an element of $D(\psi, Y)$ different than $z$, then either $s^* \subset (y^i_{j+1}, y^i_{k(i)})$ or $s^* \subset Y \setminus (y^i_{j+1}, y^i_{k(i)})$. It follows that $Z_t$ can be arranged into a sequence $z^1_t, z^2_t, \ldots, z^m_t$ such that $(z^1_t)^* \cap t^* \neq \emptyset$ and $(z^j_t)^* \cap (z^i_n)^* \neq \emptyset$ if and only if $|j - n| \leq 1$. Observe that $(t, z^1_{m(1)}), (t, z^2_{m(2)})$ and $(t, z^3_{m(3)})$ are three arcs intersecting at $t$. Clearly, $D(\psi, Y)$ is the union of these arcs. □

The following proposition follows immediately from 4.2.

4.3 **Proposition.** Suppose that $Y$ is a triod with its point of order 3 denoted by $y$. Suppose $\beta$ is a simplicial map of $Y$ into $X$. If $\beta(y) \neq v_0$, then $D(\beta, Y)$ is a triod (possibly degenerate). □

Figure 6 shows $D(X)$ with its vertices labeled by the corresponding to them edges of $X$. Figure 7 indicates $d[\phi] : D(\phi, X') \to D(X)$. The dotted line graph represents the domain, $D(\phi, X')$, while the solid black is the range, $D(X)$, and each vertex of the domain is mapped onto the nearest vertex of the range. The vertices of $D(\phi, X')$ are labeled $t_0, t_1, \ldots, t_{12}$ as shown in Figure 7. Table 2 shows the subgraphs of $X'$ corresponding to the vertices of $D(\phi, X')$.

Let $S$ be a function assigning to every vertex of $X$ a set of edges of $X$ defined in the following way: $S(v_0) = \{(v_0, v_2), (v_0, v_3), (v_0, v_5)\}$, $S(v_1) = \{(v_0, v_1)\}$, $S(v_2) = \{(v_0, v_2)\}$, $S(v_3) = \{(v_0, v_3)\}$, $S(v_4) = \{(v_3, v_4)\}$, $S(v_5) = \{(v_0, v_5)\}$, $S(v_6) = \{(v_5, v_6)\}$ and $S(v_7) = \{(v_6, v_7)\}$. Note that $v_i$ belongs to each edge from $S(v_i)$ for $i = 0, \ldots, 6$. So, $S$ is an edge selection on $X$ according to [8, Definition 5.5]. Observe that

(i) $\phi((v_i, X')) \in S(\phi(v_i))$ for each $v_i \in \mathcal{Y}(X)$ and each $e \in S(v_i)$, where $(v_i, e, X')$ denote the edge of $(e, X')$ containing $v_i$.

Observe also that
(ii) if \( e \) and \( e' \) are two different edges of \( X' \) intersecting at a common vertex \( q \) then at least one of the edges \( \phi(e) \) and \( \phi(e') \) belongs to \( S(\phi(q)) \).

The above two conditions mean exactly that

4.4 **Proposition.** \( \phi \) preserves \( (S, S) \) (in the sense of [8, Definition 5.7]). □

Observe that

(iii) \( (v_i, e, X') \subset t_i^* \) for each \( i = 0, 1, \ldots, 7 \) and each \( e \in S(v_i) \) and

(iv) \( t_8^* = \langle u_0, u_1 \rangle \cup \langle u_1, u_2 \rangle \subset \langle v_0, v_2 \rangle, \quad t_9^* = \langle u_2, u_3 \rangle \cup \langle u_3, u_4 \rangle \subset \langle v_0, v_2 \rangle, \quad t_{10}^* = \langle u_5, u_6 \rangle \cup \langle u_6, u_7 \rangle \subset \langle v_0, v_3 \rangle, \quad t_{11}^* = \langle u_7, u_8 \rangle \subset \langle v_0, v_3 \rangle \) and \( t_{12}^* = \langle u_{10}, u_{11} \rangle \subset \langle v_0, v_5 \rangle \).

Let \( X'' \) be a subdivision of \( X \) with five new vertices: \( v_8 \) added between \( v_0 \) and \( v_2 \), \( v_9 \) added between \( v_8 \) and \( v_2 \), \( v_{10} \) added between \( v_0 \) and \( v_3 \), and \( v_{12} \) added between \( v_{10} \) and \( v_3 \), and \( v_{12} \) added between \( v_5 \) and \( v_6 \). Let \( \lambda : X'' \to D(\phi, X') \) be defined by the formula \( \lambda(v_i) = t_i \) for \( i = 0, 1, \ldots, 12 \). Observe that \( \lambda \) is an isomorphism. Conditions (iii) and (iv) mean exactly that

<table>
<thead>
<tr>
<th>( t_i )</th>
<th>( d<a href="t_i">\phi</a> )</th>
<th>( t_i^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_0 )</td>
<td>( \langle v_0, v_3 \rangle )</td>
<td>( \langle v_0, u_0 \rangle \cup \langle v_0, u_5 \rangle \cup \langle v_0, u_9 \rangle )</td>
</tr>
<tr>
<td>( t_1 )</td>
<td>( \langle v_3, v_4 \rangle )</td>
<td>( \langle v_0, v_1 \rangle )</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>( \langle v_0, v_1 \rangle )</td>
<td>( \langle u_4, v_2 \rangle )</td>
</tr>
<tr>
<td>( t_3 )</td>
<td>( \langle v_5, v_6 \rangle )</td>
<td>( \langle u_8, v_3 \rangle )</td>
</tr>
<tr>
<td>( t_4 )</td>
<td>( \langle v_6, v_7 \rangle )</td>
<td>( \langle v_3, v_4 \rangle )</td>
</tr>
<tr>
<td>( t_5 )</td>
<td>( \langle v_0, v_1 \rangle )</td>
<td>( \langle u_9, v_5 \rangle \cup \langle v_5, u_{10} \rangle )</td>
</tr>
<tr>
<td>( t_6 )</td>
<td>( \langle v_5, v_6 \rangle )</td>
<td>( \langle u_{11}, v_6 \rangle )</td>
</tr>
<tr>
<td>( t_7 )</td>
<td>( \langle v_6, v_7 \rangle )</td>
<td>( \langle v_6, v_7 \rangle )</td>
</tr>
<tr>
<td>( t_8 )</td>
<td>( \langle v_0, v_5 \rangle )</td>
<td>( \langle u_0, u_1 \rangle \cup \langle u_1, u_2 \rangle )</td>
</tr>
<tr>
<td>( t_9 )</td>
<td>( \langle v_0, v_2 \rangle )</td>
<td>( \langle u_2, u_3 \rangle \cup \langle u_3, u_4 \rangle )</td>
</tr>
<tr>
<td>( t_{10} )</td>
<td>( \langle v_0, v_2 \rangle )</td>
<td>( \langle u_5, u_6 \rangle \cup \langle u_6, u_7 \rangle )</td>
</tr>
<tr>
<td>( t_{11} )</td>
<td>( \langle v_0, v_5 \rangle )</td>
<td>( \langle u_7, u_8 \rangle )</td>
</tr>
<tr>
<td>( t_{12} )</td>
<td>( \langle v_0, v_5 \rangle )</td>
<td>( \langle u_{10}, u_{11} \rangle )</td>
</tr>
</tbody>
</table>
4.5 Proposition. \( \varphi \) is consistent on \( S \) and \( \lambda \) is a consistency isomorphism (see [8, Definition 5.7]). \( \square \)

The following proposition can be readily verified by Figure 7.

4.6 Proposition. \( D[\varphi] \) is light and for each \( e \in \mathcal{E}(D(X)) \), each component of \( (D[\varphi])^{-1}(e) \) is either a vertex or an edge of \( D(\varphi, X') \). \( \square \)

5. \( K \) IS NOT SIMPLE-4-O-D-LIKE

5.1 Proposition. Suppose \( \Phi^n_\lambda \) can be factored through a triod. Then the map \( d[\varphi, \Phi^n_\lambda] : D(\Phi^n_\lambda, \Phi_n) \to D(\varphi, X') \) can also be factored through a triod.

Proof. Let \( Y \) be a triod with its point of order 3 denoted by \( y \). Let \( \alpha : \Phi_n \to Y \) and \( \beta : Y \to X \) be simplicial maps such that \( \alpha \circ \beta = \Phi^n_\lambda \). By Proposition 3.5, we can assume that \( \beta(y) \neq v_0 \). It follows from Proposition 4.3, \( D(\beta, Y) \) is a triod.

Observe that \( d[\beta, \alpha] : D(\Phi^n_\lambda, \Phi_n) \to D(\beta, Y), \ D[\beta] : D(\beta, Y) \to D(X), \ d[\varphi, \Phi^n_\lambda] : D(\Phi^n_\lambda, \Phi_n) \to D(\varphi, X') \) and \( d[\varphi] : D(\varphi, X') \to D(X) \) are light simplicial maps. By Proposition 4.6, it follows from [8, Theorem 4.3] that there is \( \beta' : D(\beta, Y) \to D(\varphi, X') \) such that \( \beta' \circ d[\beta, \alpha] = d[\varphi, \Phi^n_\lambda] \). \( \square \)

5.2 Proposition. The map \( \Phi^n_0 \) cannot be factored through a triod.

Proof. Clearly, the proposition is true if \( n = 0 \). Now, suppose that the proposition is true for \( n - 1 \). Let \( \Gamma \) denote the sequence \( D[\varphi_1] \circ \lambda, \ \varphi_2, \varphi_3, \ldots, \varphi_n \), where \( \varphi_i = \varphi \), and let \( \{\Gamma_j, \Gamma_j^n\}_{j=0}^n \) we denote the system generated by \( \Gamma \).

It follows from Propositions 4.4, 4.5 and [8, Theorem 5.11] that the system \( \{D(\Phi^n_\lambda, \Phi_j), d[\Phi^n_0, \Phi_j]\}_{j=0}^n \) is isomorphic to \( \{\Gamma_j, \Gamma_j^n\}_{j=0}^n \).

Suppose \( \Phi^n_0 \) can be factored through a triod. Then, by Proposition 5.1, \( d[\Phi^n_0, \Phi^n_\lambda] \) and consequently \( \Gamma^n_0 \) can be factored through a triod. Since the system \( \{\Gamma_j, \Gamma_j^n\}_{j=1}^n \) is generated by subdivisions of \( \varphi_2, \ldots, \varphi_n \), according to the inductive assumption and Proposition 2.3, \( \Gamma^n_0 \) cannot be factored through a triod. This contradiction proves the proposition. \( \square \)

5.3 Theorem. \( K \) is a simple-4-o-d-like but not simple-triod-like continuum and each proper nondegenerate subcontinuum of \( K \) is an arc.

Proof. \( K \) is simple-4-o-d-like as the inverse limit of the system \( \{\Phi_j, \Phi_j^n\} \) of subdivisions of \( X \). By Proposition 3.3, each proper nondegenerate subcontinuum of \( K \) is an arc. Suppose that \( K \) is triod-like. Then by Proposition 2.1, \( \Phi^n_0 \) can be factored through a triod for some \( n \), contrary to Proposition 5.2. \( \square \)

REFERENCES


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