THE LIMITING BEHAVIOR OF THE KOBAYASHI-ROYDEN PSEUDOMETRIC

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Abstract. We study the limit of the sequence of Kobayashi metrics of Riemann surfaces (when these Riemann surfaces form an analytic fibration in such a way that the total space of fibration becomes a complex surface), as the fibers approach the center fiber which is not in general smooth. We prove that if the total space is a Stein surface and the smooth part of the center fiber contains a component biholomorphic to a quotient of the disk by a Fuchsian group of first kind, then the Kobayashi metrics of the near-by fibers converge to the Kobayashi metric of this component as fibers tend to the center fiber.

Introduction

Let \( \Phi: M \to \Delta \) be a holomorphic mapping from a complex surface \( M \) on the disc \( \Delta = \{ z \in \mathbb{C} \mid |z| < 1 \} \). Suppose that for each \( c \neq 0 \), \( \Gamma_c = \Phi^{-1}(c) \) is a smooth noncompact Riemann surface and \( \Gamma_0^* \) is a smooth part of \( \Gamma_0 = \Phi^{-1}(0) \). We shall investigate relations between the Kobayashi-Royden pseudometric \( k_{\Gamma_c} \) on \( \Gamma_0^* \) and the limit of the Kobayashi-Royden pseudometric on nearby fibers. More precisely, we shall study the problem when the equality

\[
\lim_{c \to 0} k_{\Gamma_c} = k_{\Gamma_0^*}
\]

holds. In general, it is not so. In [PS, §2.2] there is an example of such mapping \( \Phi: M \to \Delta \), where \( M \) is a holomorphically convex region in \( \mathbb{C}^2 \), every \( \Gamma_c \) is a disc, but \( \lim_{c \to 0} k_{\Gamma_c} \neq k_{\Gamma_0^*} \). Zaidenberg found certain sufficient conditions, which imply (1) [Z]. But his result does not give the answer to the question whether (1) holds, when \( \Phi \) is a polynomial of two complex variables and \( M = \Phi^{-1}(\Delta) \). He supposed that the answer was positive. Let \( G \) be a Fuchsian group of the first kind. The Main Theorem of this paper says that, if \( M \) is a Stein surface and \( \Gamma_0^* \) contains a component \( R \), which is biholomorphically equivalent to \( \Delta/G \), then \( \lim_{c \to 0} k_{\Gamma_c} = k_R \). In particular, the Zaidenberg’s conjecture is true. The last fact was announced in [Ka], where it was used to classify isotrivial polynomials on \( \mathbb{C}^2 \).

The paper is organized as follows. We present some terminology and formulate our main results in the first section. The second section contains a technical lemma about Fuchsian groups and its corollaries needed for the proof of the Main Theorem. This lemma asserts that two noncommutative nonelliptic elements of a Fuchsian group cannot move any point \( z \in \Delta \) by a distance less than...
a certain $\varepsilon > 0$ at the same time. Next we handle the case of hyperbolic fibers $\{\Gamma_{b_j}\}$ with $b_j \to 0$. We consider universal holomorphic covering $f_j: \Delta \to \Gamma_{b_j}$ and find out when $\{f_j\}$ converge to an unramified mapping $f: D \to \Gamma_0$ on a certain maximal region $D \subset \Delta$. We also prove that $f(D)$ is a component of $\Gamma_0$ and, if $D = \Delta$, then the Kobayashi-Royden pseudometric on $f(D)$ coincides with the limit of the Kobayashi-Royden pseudometric of nearby fibers. The result of the forth section says that $D$ is simply connected in the case when $M$ is a Stein surface. The last section contains the proof of the Main Theorem.

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1. Formulation of the main theorem

First we fix terminology, notations and definitions that we shall use throughout the paper. Every manifold we are going to consider will be complex. If $Y$ is a manifold, then $TY$ is its holomorphic tangent bundle and $T_yY$ is a tangent space at a point $y \in Y$. Put $\Delta_r = \{z \in C \mid |z| < r\}$, $\Delta = \Delta_1$, and $\Delta^* = \Delta - 0$. By a curve $\eta$ in $Y$ we mean a continuous mapping $\eta: [0, 1] \to Y$. A loop $\gamma$ in $Y$ is a curve with $\gamma(0) = \gamma(1)$. In other words, $\gamma$ is a continuous mapping from $\partial \Delta$ to $Y$ (as is frequently done, we use the symbol $\partial$ to denote boundaries). If $x \in \gamma(\partial \Delta)$, then we write $x \in \gamma$. Recall that a differential pseudometric on a complex manifold $Y$ is a nonnegative homogeneous function on the tangent bundle $TY$, i.e., it is a function $p: TY \to \mathbb{R}$ such that $p(y, v) \geq 0$, $p(y, \lambda v) = |\lambda| p(y, v)$ for all $y \in Y$, $v \in T_yY$, and $\lambda \in \mathbb{C}$. When $p$ is continuous, we call the pseudometric continuous. If $Y$ is connected and for each piecewise smooth curve $\eta$ in $Y$ there exists the integral $P(\eta) = \int_0^1 p(\eta(t), d\eta(t)) dt$, one can define the integral pseudometric $P(x, y) = \inf \{P(\eta) \mid \eta(0) = x, \eta(1) = y\}$. Of course, the integral pseudometric exists, when a proper differential pseudometric is continuous. The Kobayashi-Royden differential pseudometric is given by the formula

$$k_Y(y, v) = \inf_{r} \{1/r \mid \phi \in \text{Hol}(\Delta_r, Y), \phi(0) = y, d\phi(0) = v\}.$$

By Royden’s theorem [R] it generates the integral pseudometric $K_Y$ which coincides with the Kobayashi pseudometric on $Y$ [Ko].

Throughout the paper $\Phi: M \to \Delta$ is a holomorphic mapping from a smooth complex surface $M$ on $\Delta$ such that for $c \neq 0$ $\Gamma_c = \Phi^{-1}(c)$ is a smooth Riemann surface. We shall say that $\Phi: M \to \Delta$ is a family of Riemann surfaces.

The fiber $\Gamma_0 = \Phi^{-1}(0)$ can contain singular points. Denote the smooth part of $\Gamma_0$ by $\Gamma_0^s$. Let $\beta = \{b_j\} \subset \Delta^*$ be a sequence that tends to zero, let $R$ be a component of $\Gamma_0^s$. We say that $\lim_{j \to \infty} k_{\Gamma_{b_j}} = k_R$ (or $\overline{\lim}_{j \to \infty} k_{\Gamma_{b_j}} \leq k_R$), if for each sequence $\{w_j \in TR_{b_j}\}$ that converges to $w \in TR$ in the topology of $TM$ the equality $\lim_{j \to \infty} k_{\Gamma_{b_j}}(w_j) = k_R(w)$ (or inequality $\overline{\lim}_{j \to \infty} k_{\Gamma_{b_j}}(w_j) \leq k_R(w)$) holds. If $\lim_{j \to \infty} k_{\Gamma_{b_j}} = k_R$ for each sequence $\beta$ as above, then we say $\lim_{c \to 0} k_{\Gamma_c} = k_R$. In the same meaning $\overline{\lim}_{c \to \infty} k_{\Gamma_c} \leq k_R$. The following two results belong to Zaidenberg [Z].

**Proposition 1.1.** For each component $R$ of $\Gamma_0^s$ the inequality $\overline{\lim}_{c \to \infty} k_{\Gamma_c} \leq k_R$ holds.
Theorem 1.2. Let $\overline{M}$ be a smooth compact surface and $\overline{\Gamma} \subset \overline{M}$ be an analytic curve in $\overline{M}$. Suppose that $M \subset \overline{M} - \overline{\Gamma}$, $\overline{\Gamma}_0 = \bigcap_{r > 0} \Phi^{-1}(\Delta_r)$, and $\Gamma_0 = \overline{\Gamma}_0 - \overline{\Gamma}$. If every component of $\Gamma_0^*$ is hyperbolic, then $\lim_{c \to 0} k_{\Gamma_c} = k_R$.

Zaidenberg conjectured that, if $\Phi$ is a polynomial on $C^2$ and $M = \Phi^{-1}(\Delta)$, then the assumption that all the components of $\Gamma_0^*$ are hyperbolic can be omitted. We shall show that this hypothesis is correct. Recall that $G$ is a Fuchsian group of the first kind, if the closure of the orbit $\{g(0) | g \in G\}$ in $C$ contains $\partial \Delta$ [B]. In particular, in the polynomial case every hyperbolic component $R$ of $\Gamma_0^*$ has a representation $R \cong \Delta/G$, where $G$ is a Fuchsian group of the first kind.

Main Theorem. Let $\Phi: M \to \Delta$ be a family of Riemann surfaces. Suppose that $M$ is a Stein manifold and $\Gamma_0^*$ contains a component $R$ that is biholomorphically equivalent to $\Delta/G$, where $G$ is a Fuchsian group of the first kind. Then $\lim_{c \to 0} k_{\Gamma_c} = k_R$.

Note that, if $R$ is nonhyperbolic, such a fact follows from Proposition 1.1. Hence we have

Corollary. Let $\Phi: C^2 \to C$ be a polynomial. Then $\lim_{c \to 0} k_{\Gamma_c} = k_{\Gamma_0^*}$.

We shall restrict ourselves to the case of connected fibers for $c \neq 0$ (in general case the proof is the same, but instead of $\Gamma_c$ we have to use their components).

2. One property of Fuchsian groups

We shall denote the Kobayashi metric on $\Delta$ by $K_\Delta$.

Lemma 2.1. For every $r > 0$ there exists $\epsilon > 0$ such that for every Fuchsian group $G$, noncommutative elements $a', b' \in G$, and a point $z \in \Delta$ satisfying $0 < K_\Delta(z, a'(z)) < r$, either $K_\Delta(z, b'(z)) > \epsilon$ or $z$ is a fixed point of the mapping $b': \Delta \to \Delta$.

Proof. Assume, to reach a contradiction, that for a certain $r > 0$ and each $\epsilon > 0$ there exists a Fuchsian group $G_\epsilon$, noncommutative elements $a'_\epsilon, b'_\epsilon \in G_\epsilon$, and a point $z_\epsilon \in \Delta$ such that $0 < K_\Delta(z_\epsilon, a'_\epsilon(z_\epsilon)) < r$ and $0 < K_\Delta(z_\epsilon, b'_\epsilon(z_\epsilon)) < \epsilon$. We shall show that for a sufficiently small $\epsilon$ the group $G_\epsilon$ cannot be discontinuous. Without loss of generality, we set $z_\epsilon = 0$. Let id be the identity element of $G_\epsilon$.

Since $G_\epsilon$ is a discontinuous group, one can find elements $a_\epsilon$ and $b_\epsilon$ satisfying

\[(2.1) \quad K_\Delta(0, b_\epsilon(0)) = \min\{K_\Delta(0, g(0)) | g \in G_\epsilon, g(0) \neq 0\},\]
\[(2.2) \quad K_\Delta(0, a_\epsilon(0)) = \min\{K_\Delta(0, g(0)) | g \in G_\epsilon, g(0) \neq 0, \{b_\epsilon, g\} \neq \text{id}\}.\]

The mapping $a_\epsilon$ and $b_\epsilon$ can be represented in the form

\[a_\epsilon(z) = e^{i\theta_\epsilon}(z + \alpha_\epsilon)/(1 + \overline{\alpha_\epsilon}z), \quad |\theta_\epsilon| \in [0, \pi],\]
\[b_\epsilon(z) = e^{i\tau_\epsilon}(z + \beta_\epsilon)/(1 + \overline{\beta_\epsilon}z), \quad |\tau_\epsilon| \in [0, \pi].\]

We shall omit the index $\epsilon$ from now on, if it does not cause misunderstanding. Let us consider $b$ as a function of two variables $z$ and $\beta$. Expand $b$ in power series of $z$, $\beta$, and $\overline{\beta}$. Then $b(z) = e^{it}z + e^{it}\beta$ up to the nonlinear terms. Hence for every natural $m$ one can find a neighborhood of the origin
in $C^2 = \{(z, \beta)\}$ so that for all $n = 1, 2, \ldots, m$, 

$$b^n(z) = e^{int}z + \sum_{l=1}^{n} e^{ilt} \beta + O(|z|^2 + |\beta|^2)$$

in this neighborhood. Thus

$$b^n(0) = \beta \sum_{l=1}^{n} e^{ilt} + O(|\beta|^2) = \beta e^{int}(e^{int} - 1)/(e^{it} - 1) + O(|\beta|^2).$$

It is easy to check that for each $\tau_0 \neq 2\pi k$ there is a neighborhood $U$ of $\tau_0$ and integer $n \geq 2$ so that for every $\tau \in U$,

$$|(e^{int} - 1)/(e^{it} - 1)| < 1.$$

Now one can see that $\tau \to 0$ as $\varepsilon \to 0$. Indeed, the preceding assumption implies

$$0 < |\alpha| < \tilde{\tau}, \quad 0 < |\beta| < \tilde{\varepsilon},$$

where $\tilde{\varepsilon} = (e^\varepsilon + 1)$ and $\tilde{\tau} = (e^\tau - 1)/(e^{\tau} + 1)$. Thus $\lim_{\varepsilon \to 0} |\beta| = 0$. Assume $\lim_{\varepsilon \to 0} |\tau| = 1/m$. Then by (2.3) we can find $n \leq m$ with $|b^n(0)| < |b(0)|$. This contradicts (2.1). Thus $b_\varepsilon(z) \to z$ uniformly on compact subsets of $\Delta$ as $\varepsilon \to 0$. Let $\lim_{\varepsilon \to 0} |\alpha| = \alpha^0$. Since $|\alpha| < \tilde{\tau}$, $a_\varepsilon \circ b_\varepsilon \circ a_\varepsilon^{-1}(z) \to z$ as $\varepsilon \to 0$. In particular, for any sufficiently small $\varepsilon$ we have $|a_\varepsilon b_\varepsilon a_\varepsilon^{-1}(0)| < \alpha^0/2$. This implies either $\alpha^0 = 0$ or $b$ and $aba^{-1}$ are commutative. We shall prove that the last case does not hold. One can represent $a$ and $b$ as mappings of the upper half-plane. Then, if $a$ is a hyperbolic element, we may put $a(z) = \lambda z$ with $\lambda > 0$ and if $a$ is a parabolic element, we may put $a(z) = z + 1$ [A]. In both cases for any $b(z) = (pz + q)/(tz + s)$ with $p, q, t, s \in \mathbb{R}$ the direct computation shows that $[aba^{-1}, b] = 0$, i.e., $[a, b] = 0$. When $a$ is an elliptic element, one may consider $a$ as a mapping $a: \Delta \to \Delta$ given by the formula $a(z) = \lambda z$ with $\lambda^n = 1$ for a certain natural $n$. Again it is easy to show that $[aba^{-1}, b] = 0$, i.e., $[a, b] = 0$ for any Möbius transformation $b: \Delta \to \Delta$. But this a contradicts (2.2). Therefore $\lim_{\varepsilon \to 0} |\alpha| = 0$. Same arguments as above show that $\theta \to 0$ as $\varepsilon \to 0$. Hence for any sufficiently small $\varepsilon$ we have $|e^{it\varepsilon} - 1| + |e^{it\varepsilon} - 1| < 1/2$ and for an arbitrarily small $\alpha$ the following inequality holds

$$|b^{-1}a^{-1}ba(0)| \approx |e^{it\varepsilon} - 1| |\alpha| + |e^{it\beta} - 1| |\beta| < |\alpha|/2 < |a(0)|$$

but $b^{-1}a^{-1}ba$ and $b$ are not commutative, since $[a^{-1}, ba, b] \neq 0$. This is a contradiction. \(\square\)

**Corollary 2.2.** For every $r > 0$ there exists $\varepsilon > 0$ such that for every hyperbolic Riemann surface $R$, for every point $x \in R$, and for every couple of loops $\gamma$ and $\mu$ that generate noncommutative elements of the fundamental group $\pi_1(R, x)$, the inequalities $K_R(\gamma) < \varepsilon$ and $K_R(\mu) < r$ do not hold simultaneously.

The next three lemmas enable us to restate this corollary in a form which will be convenient for our following needs.

**Lemma 2.3.** Let $\gamma$ be a noncontractible loop on a Riemann surface $R$. Suppose that the corresponding element of the fundamental group $\pi_1(R)$ has a representation $[\gamma] = [\mu]^n$, where $[\mu] \in \pi_1(R)$ and the natural number $n \geq 2$. Then $\gamma$ has points of self-intersection.
Proof. Let $H$ be the upper half-plane, and let $f : H \to R$ be a universal holomorphic covering. Then we can define the Möbius transformation $b : H \to H$ corresponding to $[\mu]$. If $b$ is a hyperbolic transformation, one can choose $f$ so that $b(z) = \lambda z$ with $\lambda > 0$ [A]. Let $z_0$ be a point in the inverse image of a point $x_0 \in \gamma$. Obviously, each curve in $H$ that connects the points $z_0$ and $\lambda^n z_0$ contains points $z'$ and $z''$ such that $z' = \lambda z''$. But this means that $\gamma$ has the point of self-intersection $f(z')$. If $b$ is parabolic, we may suppose that $b(z) = z + 1$. Again each curve that connects the points $z_0$ and $z_0 + n$ contains points $z'$ and $z'' = z' + 1$. This implies the desired conclusion. \[ \square \]

Lemma 2.4. Let $\gamma$ and $\mu$ be disjoint noncontractible loops in a Riemann surface $R$. Suppose that neither $\gamma$ nor $\mu$ has points of self-intersection. Then $\gamma$ and $\mu$ are homotopically equivalent, iff there is a region $U \subset R$ such that $\partial U = \gamma \cup \mu$ and $U$ is topologically an annulus.

Proof. Let $x_1 \in \gamma$ and $y_1 \in \mu$. Choose a curve $\nu_1 : [0, 1] \to R$ so that $\nu_1(0) = x_1$, $\nu_1(1) = y_1$, $\nu_1$ has no points of self-intersection and $\nu_1$ intersects $\gamma \cup \mu$ at the points $x_1$ and $y_1$ only. Choose an analogous curve $\nu_2$ so that $\nu_2$ connects points $x_2 \in \gamma$ and $y_2 \in \mu$, and $\nu_2$ is sufficiently close to, but disjoint from $\nu_1$. Then $\gamma - (x_1 \cup x_2)$ consists of two components $\gamma_1$ and $\gamma_2$, and $\gamma_1$ is small enough. In the same way $\mu - (y_1 \cup y_2) = \mu_1 \cup \mu_2$, and $\mu_1$ is small. Then there exists an open disc $D \subset R$ with $\partial D = \nu_1 \cup \nu_2 \cup \gamma_1 \cup \mu_1$. One can construct the loop $\eta = \nu_1 \cup \mu_2 \cup \nu_2 \cup \gamma_2$. Since $\gamma$ and $\mu$ are homotopically equivalent, $\eta$ must be contractible. By our construction $\eta$ has no points of self-intersection. This implies the existence of the disc $U \subset R$ with $\partial U = \eta$. If $U \supset D$, then $U - D$ contains the two components $U_1$ and $U_2$. Each of them is a disc, $\partial U_1 = \gamma$ and $\partial U_2 = \mu$. This contradicts the assumption that $\gamma$ and $\mu$ are noncontractible. Hence $U \cap D = \emptyset$. Obviously, $\overline{D} \cup U$ is topologically a closed annulus and $\partial(\overline{U} \cup \overline{D}) = \gamma \cup \mu$. This completes the proof of the lemma.

Lemma 2.5. Let $\gamma$ and $\mu$ be noncontractible loops on a Riemann surface $R$, and neither $\gamma$ nor $\mu$ has points of self-intersection. Suppose that $R - (\gamma \cup \mu)$ does not contain components that are topologically an annulus. Then for each $\varepsilon > 0$ there exists $r > 0$ such that, if $K_R(\gamma) < \varepsilon$ and $K_R(\mu) < \varepsilon$, then the distance between $\gamma$ and $\mu$ in the Kobayashi metric is greater than $r$.

Proof. Let $\nu : [0, 1] \to R$ be a curve that connects $\gamma$ and $\mu$ so that $K_R(\nu)$ coincides with the distance between $\gamma$ and $\mu$. Let $\nu(0) = x_0 \in \gamma$. By Corollary 2.2 it is enough to verify that $\gamma$ and $\gamma' = \nu^{-1} \circ \mu \circ \nu$ generate noncommutative elements $[\gamma]$ and $[\gamma']$ in $\pi_1(R, x_0)$. Since the group $\pi_1(R, x_0)$ is free, $[\gamma]$ and $[\gamma']$ are commutative, iff they belong to a cyclic subgroup. This implies that $[\gamma] = [\nu]^n$ and $[\gamma'] = [\nu]^l$ for a certain $[\nu] \in \pi_1(R, x_0)$. By Lemma 2.3 $k = l = 1$. Hence $[\gamma] = [\gamma']$. Therefore $\gamma$ and $\mu$ must be homotopically equivalent. But this contradicts Lemma 2.4. \[ \square \]

3. Limiting behavior of hyperbolic metric

From now on by $R$ we denote a connected hyperbolic component of $\Gamma_0$.

Lemma 3.1. Let $\alpha$ be a sequence of points in $\Delta^*$ that tends to zero. Suppose that for each $c \in \alpha$ the fiber $\Gamma_c$ is a hyperbolic Riemann surface. Then for a certain infinite subsequence $\beta = \{b_j\} \subset \alpha$ there exists a differential pseudometric $\alpha_\beta$
on $R$ such that $\alpha_\beta = \lim_{j \to \infty} k_{\Gamma_{b_j}}$. Moreover, $\alpha_\beta$ is a continuous pseudometric and the equality $\alpha_\beta(v) = 0$ for a vector $v \in TR$ implies $\alpha_\beta \equiv 0$.

**Proof.** Let $\phi: \Delta \to R$ be a holomorphic embedding and $\phi(\Delta) = U$. It is easy to construct holomorphic embeddings $\phi_j: \Delta \to U_j \subset \Gamma_{b_j}$ so that $\phi_j(z) \to \phi(z)$ as $j \to \infty$ (e.g., see [Z]). Let $\nu_z$ denote the point $(z, d/dz) \in T\Delta$. We set $s_j = \phi_j(\nu_z)$ and $s_z = \phi(\nu_z)$ (where $\phi_*$ and $\phi_*$ are the induced mappings of the tangent bundles). Then $s_j \to s_z$ in topology of $TM$. Let $f_j: \Delta \to \Gamma_{b_j}$ be a universal holomorphic covering. Choose a connected component $V_j$ of $f_j^{-1}(U_j)$ and a holomorphic mapping $g_j: \Delta \to \Delta$ so that the restriction of $g_j \circ \phi_j^{-1} \circ f_j$ to $V_j$ is the identity mapping. One may suppose that $0 \in V_j$ and $g_j(0) = 0$. Let $s_j \in TV_j$ belong to the inverse image of the vector $s_z$ under the mapping $f_j$. Then $g_j(\nu_z) = s_j$. On the other hand $g_j(\nu_z) = g_j'(z)\nu_{g_j(z)}$. Hence, taking into consideration the equalities $k_\Delta(s_j) = k_\Delta(s_z)$ and $k_\Delta(\nu_z) = 1/(1 - |\nu_z|^2)$, we have $k_{\Gamma_{b_j}}(s_j) = |g_j'(z)|/(1 - |g_j(z)|^2)$. Passing to a subsequence, if necessary, we suppose that $g_j(z) \to g(z)$ uniformly on compact subsets of $\Delta$. By Hurwitz’s theorem either $g'(z) \neq 0$ for every $z \in \Delta$ or $g'(z) \equiv 0$ (in the last case $g(z) \equiv 0$, since $g(0) = 0$). Therefore $\lim_{j \to \infty} k_{\Gamma_{b_j}}(s_j) = |g'(z)|/(1 - |g(z)|^2)$. Let $s_j = (x_j(z), t_j(z))$, where $x_j(z) \in U_j$ and $t_j(z) \in T_{x_j(z)}U_j$ (the notation, $s_j = (x(z), t(z))$ has the same meaning). A sequence $\{v_j\} v_j = (x(z_j), \lambda_j t_j(z_j)); \lambda \in C\}$ converges to $v = (x(z), \lambda t(z))$ in the topology of $TM$, iff $z_j \to z$ and $\lambda_j \to \lambda$. Hence $\lim_{j \to \infty} k_{\Gamma_{b_j}}(v_j) = |\lambda g'(z)|/(1 - |g(z)|^2)$ and a proper limiting pseudometric exists on $U$. Let $\{U_j\}$ be a cover on $R$ and each $U_j$ be an open disc. We can repeat the above construction of the limiting pseudometric for each $U_j$ instead of $U$. Application of the diagonal process completes the proof of the lemma. \[ \square \]

**Definition.** Let $\beta = \{b_j\} \subset \Delta^*$ be a sequence that converges to zero, and let every fiber $\Gamma_{b_j}$ hyperbolic. We shall say that $\beta$ is an admissible sequence if there is a continuous differential pseudometric $\alpha_\beta$ on $R$ such that $\alpha_\beta = \lim_{j \to \infty} k_{\Gamma_{b_j}}$, and the quality $\alpha_\beta(v) = 0$ for a vector $v \in TR$ implies $\alpha_\beta \equiv 0$. We will denote the corresponding integral pseudometric by $A_\beta$, and throughout the rest of the paper we will fix these notations $\beta$, $\alpha_\beta$, and $A_\beta$ for the above objects.

**Lemma 3.2.** Suppose that the $\alpha_\beta$ is a metric. Let $F = \{f_j\}$, where $f_j: \Delta \to \Gamma_{b_j}$ is a holomorphic universal covering with $f_j(0) \to x_0 \in R$ as $j \to \infty$. Then there is a nonempty open subset $D \subset \Delta$ that contains $0$ and a subsequence $F_1 \subset F$ that converges to a mapping $f: D \to R$. Moreover

(i) $f: D \to R$ is an unramified covering;

(ii) $F$ transforms the metric $k_{A_\beta}|_D$ into the metric $\alpha_\beta$.

**Proof.** First assume that $f$ exists and prove (ii). Let $z \in D$, $x = f(z)$, and $x_j = f_j(z)$. Then $x_j \to x$ as $j \to \infty$. Choose a sequence $\{v_j\} v_j \in T_{x_j}\Gamma_{b_j}$ that converges to a nonzero vector $v \in T_xR$ in the topology of $TM$. Let $\tilde{v}_j \in T\Delta$ belong to the inverse image of $v_j$ under the mapping $f_j$. Since $\beta$ is admissible, $k_{\Delta}(\tilde{v}_j) = k_{\Delta}(v_j) = \alpha_\beta(v)$. Thus for every $j$, $k_{\Delta}(\tilde{v}_j)$ is less than a certain common constant. Hence we may suppose that there is the limiting vector $\tilde{v}$ for the sequence $\{\tilde{v}_j\}$. Clearly, $k_{\Delta}(\tilde{v}) = \alpha_\beta(v)$ and $f_*\tilde{v} = v$.\[ \square \]
This implies (ii). Property (ii) means that, if \( f \) exists, then it must be locally homeomorphic. Let \( P \) be a sufficiently small neighborhood of \( X_0 \) such that \( P \) is biholomorphically equivalent to a ball, and all \( U_j = P \cap \Gamma_{\beta_j} \) and \( U = \Gamma_0 \cap P \) are discs. For every manifold \( N \) we will denote by \( B(y, r, N) \subset N \) the ball of radius \( r \) in the metric \( K_{N} \) with the center at \( y \). Let \( B(x, r, A_{\beta}, R) \subset U \) be the analogous ball in the metric \( A_{\beta} \) with the center at \( x \). Since \( A_{\beta} \) is a metric, there exists \( r > 0 \) such that \( B(x_0, r, A_{\beta}, R) \subset U \). Hence \( B(f_j(0), r, \Gamma_{\beta_j}) \subset U_j \), when \( j \) is sufficiently large. The restriction of \( f_j \) to \( H_0 = B(0, r, \Delta) \) is a homeomorphism between \( H_0 \) and \( B(f_j(0), r, \Gamma_{\beta_j}) \). The family \( \{f_j|_{H_0} : H_0 \rightarrow P\} \) is normal. Pick out a converging subsequence \( F_1 \subset F \) in this family. Let \( f : H_0 \rightarrow B(x, r, A_{\beta}, R) \) be the limiting mapping. We have proved that \( D \supset H_0 \), i.e., \( D \) is not empty. Since \( f \) is locally homeomorphic and each \( f_j|_{H_0} \) is a homeomorphism, one can easily check that the limiting mapping \( f|_{H_0} \) is also homeomorphism. Set \( H = B(x_0, r/3, A_{\beta}, R) \). Suppose there is a point \( z \in D - H_0 \) with \( y = f(z) \in H \). Let \( H_1 = B(y, 2r/3, A_{\beta}, R) \) and \( \tilde{H}_1 = B(z, 2r/3, \Delta) \). Clearly \( H_1 \subset U \). Repeating the above arguments we can choose a subsequence \( F_2 \subset F_1 \) so that the restriction \( F_2 \) to \( \tilde{H}_1 \) converges to a homeomorphism \( g : \tilde{H}_1 \rightarrow H_1 \). For every subsequence \( F_3 \subset F_1 - F_2 \) that converges to a mapping \( h : \tilde{H}_1 \rightarrow H_1 \) we have \( g|_{\tilde{H}_1 \cap H} = h|_{\tilde{H}_1 \cap H} = f|_{\tilde{H}_1 \cap H} \). By the uniqueness theorem \( h = g \). Thus one can take \( F_1 \) itself as \( F_2 \) and \( D \supset \tilde{H}_1 \). Since \( H_1 \supset H \), \( \tilde{H}_1 \) contains a disc \( H \) such that \( f|_{\tilde{H}_1} : H \rightarrow H \) is a homeomorphism and \( \tilde{H}_1 \cap H = \emptyset \) (indeed, \( z \notin \tilde{H}_1 \) and the restriction of \( f \) to \( \tilde{H}_1 \) is also a homeomorphism). We can consider \( H \) as a neighborhood of \( x_0 \). Of course, analogous arguments enable us to find such a neighborhood for every point \( x \in F(D) \). Hence \( f : D \rightarrow F(D) \) is an unramified covering. In particular, \( f(D) \) is an open set.

To check the equality \( R = f(D) \) it is enough to prove that the set \( f(D) \) is closed in \( R \). Let \( x \) belong to the closure of \( f(D) \) in \( R \). Let \( P' \) be a sufficiently small neighborhood of \( x \). Suppose that \( P' \) is biholomorphically equivalent to a ball, and \( U' = P' \cap \Gamma_0 \) and \( \{U'_j = P' \cap \Gamma_{\beta_j}\} \) are discs. Choose \( r > 0 \) with \( B(x, r, A_{\beta}, R) \subset U' \). Then we can find a point \( y \in f(D) \cap B(x, r/3, A_{\beta}, R) \). As we have seen, in this case \( f(D) \supset B(y, 2r/3, A_{\beta}, R) \). Hence, \( x \in f(D) \), which is the desired conclusion.

\textbf{Corollary 3.3.} \textit{If the assumptions of Lemma 3.2 hold and \( D = \Delta \), then \( k_R = \lim_{j \to \infty} k_{\Gamma_{\beta_j}} \).}

\textit{Proof.} We shall use the notation of the proof of Lemma 3.2. If \( D = \Delta \), then \( f : D \rightarrow R \) is a universal holomorphic covering and

\[ \lim_{j \to \infty} k_{\Gamma_{\beta_j}}(v_j) = \lim_{j \to \infty} k_{\Delta}(\tilde{v}_j) = k_{\Delta}(\tilde{v}) = k_D(\tilde{v}) = k_R(v). \]

\textbf{4. Stein case}

From now on \( M \) is a Stein surface, and we will use the same notations \( R, \beta = \{\beta_j\}, a_{\beta}, f_j : \Delta \rightarrow \Gamma_{\beta_j}, F = \{f_j\} \) and \( f : D \rightarrow R \) as in the preceding section. Let a Riemann surface \( A \) be topologically an annulus. Denote the minimum of lengths of noncontractible loops in \( A \) by \( l(A) \).
Proposition 4.1. To each number $t > 0$ corresponds a positive number $r < 1$ so that the assumptions:

(i) $L$ is a compact in $\Delta$;
(ii) $0 \in L$;
(iii) $\Delta - L$ is topologically an annulus;
(iv) $l(\Delta - L) < t$

imply that $L \subset \Delta$.

Proof. Assume that the contrary. Then for a certain $t$ and every $r < 1$ there is compact $L_r$ that contains a point $z_r$ with $|z_r| > r$ and satisfies (i)-(iv). Clearly $l(L - \Delta)$ is greater than $2K_\Delta(0, z_r)$. But $K_\Delta(0, z_r) \to \infty$ as $r \to 1$, and we have a contradiction with (iv). $\square$

According to [S] the Stein subvariety $R$ has a tubular neighborhood $V \subset M$ that is biholomorphically equivalent to a neighborhood of the zero section in the normal bundle to $R$ in $M$. Thus we have a holomorphic retraction $\tau : V \to R$. Let $Q$ be a region in $R$ with the compact closure. Then for a sufficiently small $\varepsilon$ and every $c \in \Delta$, the restriction $\tau$ to $\tau^{-1}(Q) \cap \Gamma_\varepsilon$ is a holomorphic unramified covering, whose multiplicity over $Q$ is equal to the multiplicity to zero of the function $\Phi$ on $R$.

Lemma 4.2. Let $\gamma$ be a loop in $R$ without points of self-intersection. Let $\{\phi_j \phi_j : \Delta \to \Gamma_{b_j}\}$ be continuous embeddings that are holomorphic on $\Delta$. Suppose that $\gamma_j = \phi_j (\partial \Delta)$ belong to $\tau^{-1}(\gamma)$. Then $\gamma$ is contractible.

Proof. We shall consider the Stein manifold $M$ as a closed analytic submanifold in $C^n$ (e.g., see [GR]). Then each $\phi_j$ has the following coordinate representation $\phi_j(z) = (\phi_{j1}, \ldots, \phi_{jn})$. Denote the restriction $\phi_j$ to $\Delta$ by $\phi_j$, and let $\phi' = (\phi'_{j1}, \ldots, \phi'_{jn})$ be the derivation of $\phi_j$. As usual we shall use the symbol $\|\phi_j'(z)\|$ to denote the Euclidean length of the vector $\phi_j'(z)$. Suppose that the functions $\|\phi_j'(z)\|$ converges to zero uniformly on compact subsets of $\Delta$. Then there exists a sequence of points $\{z_j\} \subset \Delta$ with $|z_j| \to 1$ that satisfies $\|\phi_j'(z_j)\| \geq t/(1 - |z_j|^2)$ for a certain positive $t$. Indeed, otherwise it is easy to show that the maximal Euclidean distance between the points of $\gamma_j$, tends to zero as $j \to \infty$. But $\gamma_j$ is close to $\tau(\gamma_j)$. This implies that $\gamma$ must be a constant mapping, and we have a contradiction. Put $\psi_j = \phi_j \circ \mu_j$, where $\mu_j(z) = (z + z_j)/(1 + z_j z)$. Let $\psi_j = \psi_j|\Delta$. The loop $\gamma$ belongs to a ball $B$ in $C^n$. Hence for an arbitrary large $j$ we have $\psi_j(\partial \Delta) \subset B$. By the Maximum Principle $\psi_j(\Delta) \subset B$. Therefore the family $\{\psi_j\}$ is normal. Passing to a subsequence, if necessary, we can suppose that $\{\psi_j\}$ converge to a mapping $\psi : \Delta \to \overline{R}$. Obviously, $\|\psi'(0)\| \geq t$, and, therefore, $\psi$ is not constant. According to [Z, Lemma 2.2] $\psi(\Delta) \subset R$. Using a Möbius transformation again, if necessary, one may suppose that $\psi(0) \notin \gamma$. Choose an arbitrary small neighborhood $N$ of $\gamma$ in $R$ so that $N$ is topologically an annulus and $\psi(0) \notin N$. Then $N - \gamma$ consists of two components $N_1$ and $N_2$, which are also annuli. Let $\mu_k$ be the component of the boundary of $N_k$ other than $\gamma$. Obviously, $\psi_j(\Delta)$ must contain a component of either $\tau^{-1}(N_1) \cap \Gamma_{b_j}$ or $\tau^{-1}(N_2) \cap T_{b_j}$. Denote this component by $L_j$. Passing to a subsequence, we may suppose that $\tau(L_j) = N_1$ and $\tau|L_j$ is a $s$-sheeted unramified covering, where $s$ does not exceed the multiplicity of zero of the function $\Phi$ on $R$. Hence the Riemann surfaces $\{L_j\}$ are pairwise biholomorphically equivalent, and $l(L_j) = l(\psi_j^{-1}(L_j))$ does not
depend on \( j \). Since \( 0 \notin \varphi_j^{-1}(L_j) \), we see by Proposition 4.1 that there is a positive \( r < 1 \) such that \( \Delta - \varphi_j^{-1}(L_j) \subset \Delta_r \). Hence \( \mu_1 \subset \varphi(\Delta_r) \). This implies that \( \mu_1 \) is contractible, and, therefore, \( \gamma \) is also contractible. \( \square \)

**Lemma 4.3.** The pseudometric \( a_\beta \) generated by an admissible sequence \( \beta \) is a metric on \( R \) in the case when \( R \) is different from \( \Delta, \Delta^* \), or an annulus.

**Proof.** Let \( \gamma_1, \ldots, \gamma_k \) be disjoint noncontractible loops in \( R \) without points of self-intersection such that they are not pairwise homotopically equivalent, for each \( i \) the set \( R - \gamma_i \) is not connected, and every \( \gamma_i \) is a component of the boundary of a compact \( L \subset R \). Let \( L_j \) be a component of \( \tau^{-1}(L) \cap \Gamma_{b_j} \).

One may suppose that \( \tau|_{L_j} : L_j \to L \) is a \( s \)-sheeted unramified covering for all \( j \). Let \( \{\gamma_{ij}^l | l = 1, \ldots, l_{ij} \leq s\} \) be the components of \( \tau^{-1}(\gamma_i) \cap L_j \). If \( R \) has a positive genus, we can suppose that \( L \) contains a loop \( \mu \) without points of self-intersection so that \( L - \mu \) is connected and \( \mu \cap \bigcup_{j=1}^k \gamma_j = \emptyset \). In this case we denote one of the components of \( \gamma_i^{-1}(\mu) \cap \Gamma_{b_j} \) by \( \mu_j \). Assume, to reach a contradiction, that \( a_\beta = 0 \). Then \( K_{b_j}(\gamma_{ij}^l), K_{b_j}(\mu_j) \to 0 \) as \( j \to \infty \) and the distance between each pair of these loops in the Kobayashi metric on \( \Gamma_{b_j} \) also tends to zero. By Lemma 2.5 all of these loops must be homotopically equivalent. Since \( \tau|_{L_j} : L_j \to L \) is an unramified covering, \( L_j - \mu_j \) is connected. Hence by Lemma 2.4 \( \mu_j \) cannot be homotopically equivalent to any component of the boundary of \( L_j \), or in other words, to any \( \gamma_{ij}^l \). Therefore it remains to consider the case when \( R \) is biholomorphically equivalent to a region in \( C \). Then under the assumptions of the lemma one may suppose that \( k \geq 3 \). Thus we have, at least, three loops \( \gamma_1, \gamma_2, \) and \( \gamma_3 \). By Lemma 2.5 there is a region \( U_j \subset \Gamma_{b_j} \) such that \( \partial U_j = \gamma_{1j}^1 \cup \gamma_{2j}^1 \) and \( U_j \) is topologically an annulus. Note that \( U_j \) does not belong to \( L_j \) (otherwise, using Lemmas 2.3 and 2.4 it is easy to show that \( \gamma_1 \) and \( \gamma_2 \) are homotopically equivalent). Moreover, since the component of \( \Gamma_{b_j} - L_j \) whose boundary contains \( \gamma_{3j}^1 \) is different from a disc according to Lemma 4.2, \( U_j \) does not contain \( L_j \). Hence \( U_j \) is a component of \( \Gamma_{b_j} - L_j \). Taking \( \gamma_{3j}^2 \) instead of \( \gamma_{2j}^2 \) we can construct a component \( V_j \) of \( \Gamma_{b_j} - L_j \) so that \( \partial V_j = \gamma_{1j}^1 \cup \gamma_{3j}^1 \) and \( V_j \) is topologically an annulus. Since \( \partial V_j \cap \partial U_j = \gamma_{1j}^1, V_j = U_j \). Then \( \partial U_j = \partial V_j \), and this leads to a contradiction. Therefore \( a_\beta \) is not trivial. By Lemma 3.1 \( a_\beta(v) \neq 0 \) for each \( v \in TR \). This completes the proof of the lemma. \( \square \)

**Lemma 4.4.** Let \( M \) be a Stein surface and let \( D \) be the same as in Lemma 3.2. Then \( D \) is simply connected.

**Proof.** Assume that \( D \) is not simply connected. Then there is a couple of discs \( d \) and \( d' \) such that \( d \subset \Delta, d' \subset d \), \( d \) does not belong to \( D \), and \( d - d' \subset D \). We again consider \( M \) as a submanifold in \( C^n \). The set \( f(d - d') \) belongs to a certain ball in \( C^n \). Same arguments as in Lemma 4.2 show that the family \( \{f_j|_d\} \) is normal. Let \( \tilde{f} : d \to R \) be a limiting mapping. This mapping is unique, since it coincides with \( f \) on \( d - d' \). In particular, it is nonconstant. The set \( f(d) \) does not contain singular points of \( \Gamma_0 \), because otherwise \( f_j(d) \) must intersect \( \Gamma_0 \) for an arbitrary large \( j \) \([Z]\). Hence \( \tilde{f}(d) \subset R \), i.e., \( d \subset D \). But this contradicts our assumption. \( \square \)
Corollary 4.5. Lemma 4.2 holds without the condition that $\gamma$ has no point of self-intersection.

5. Proof of the main theorem

We keep the same notation $R$, $\beta$, $a_\beta$, $F = \{f_j\}$, $f: D \to R$ as in the preceding section. By Lemmas 3.2, 4.3, and 4.4 we suppose that the family $F$ converges to the mapping $f: D \to R$ on a nonempty simply connected region $D \subset \Delta$ with $0 \in D$, and $f$ is an unramified covering, which transforms the metric $k_\Delta|D$ into the metric $a_\beta$. Let $G_j$ be the Fuchsian group such that $f_j(z) = f_j(z')$ iff $z' = g(z)$ for a certain $g \in G_j$. We say that a Möbius transformation $h$ is limiting for $\{G_j\}$, if there is a sequence $\{g_j\} : g_j \in G_j$ that converges to $h$ uniformly on compact subsets of $\Delta$. Let $G$ be the group of holomorphic one-to-one mappings $D$ to $D$ such that $f(z) = f(z')$, if $z' = g(z)$ for a certain $g \in G$.

Lemma 5.1. The set $H$ of limiting Möbius transformations is a subgroup of $G$ of finite index.

Proof. By construction, $H$ is a group and for each pair $z$, $z' \in D$ the equality $h(z) = z'$ for an element $h \in H$ implies $f(z) = f(z')$. Hence $H \subset G$.

As in the preceding section $\tau : V \to R$ is a holomorphic retraction of a Stein neighborhood $V$ of $R$. Consider all the loops $\{y : \partial V \to R\}$ such that $y(1) = f(0)$ and for an arbitrary large $j$ there is a loop $\gamma_j$ in $\Gamma_{h_j}$ with $\gamma_j(1) = f_j(0)$ and $\gamma = \tau \circ \gamma_j$. These loops generate a subgroup $H_1$ of finite index in $\pi_1(R, f(0))$. This index does not exceed the multiplicity of zero of the function $\Phi$ on $R$. Since $\pi_1 \cong G$, one can consider $H_1$ as a subgroup in $G$ as well. Let $\gamma$ be a loop in $R$ with $[\gamma] \in H_1$ and $\{\gamma_j \in \Gamma_{h_j}\}$ be the corresponding loops, which converge to $\gamma$ uniformly. Consider the mappings $\nu_j : R \to \Delta$ and $\nu : R \to D$ such that $f_j \circ \nu_j(t) = \gamma_j(e^{2\pi i t})$, $f \circ \nu(t) = \gamma(e^{2\pi i t})$, and $\nu(0) = \nu_j(0) = 0$. Since $\gamma_j \to \gamma$ and $f_j \to f$, one can see that $\nu_j \to \nu$ uniformly. By $\tilde{\gamma}_j$ and $\tilde{\gamma}$ we will denote the elements of the Fuchsian groups $G_j$ and $G$ that correspond $[\gamma_j]$ and $[\gamma]$ respectively. Clearly, $\tilde{\gamma}_j^k(t) = \nu_j(t + k)$ and $\tilde{\gamma}^k(t) = \nu(t + k)$ for each integer $k$. This means $\tilde{\gamma}_j \to \tilde{\gamma}$ as $j \to \infty$. Hence $H_1 \subset H$ and $H$ is a subgroup of $G$ of finite index. □

Let $\tilde{R} \to R$ be an unramified covering that corresponds to the subgroup $H \subset \pi_1(R)$. Then, since $D$ is simply connected, the mapping $\tilde{f} : D \to \tilde{D} / H \cong \tilde{R}$ is a universal holomorphic covering. Recall that by the hypotheses of Main Theorem $G$ is isomorphic to a Fuchsian group of the first kind $G'$, acting on $\Delta$. More precisely, there is a biholomorphic mapping $\phi : \Delta \to D$ such that $\phi$ generates isomorphism between $G$ and $G'$. Therefore $H$ is isomorphic to a subgroup $H'$ of finite order in $G'$. Hence $H'$ is a Fuchsian group of the first kind as well. According to [G, §3, Lemma 3] it is easy to check now that, since the closure of the orbits $\{h(0)\} h \in H'$ coincides with $\partial \Delta$, the closure of orbits $\{h(0)\} h \in H$ must coincide with $\partial D$. Assume that $z$ is a point of $\partial D \cap \Delta$. Choose an arbitrary small neighborhood $U$ of $z$ and element $\tilde{\nu}$, $\tilde{\eta} \in H$ so that $\tilde{\nu}(0)$ and $\tilde{\eta} \circ \tilde{\nu}(0) \in U \cap D$. Let $\tilde{\mu}$, $\tilde{\eta} \in H$ be noncommutative elements. Then $\tilde{\eta} \circ \tilde{\nu}^{-1}$, $\tilde{\gamma} \circ \tilde{\nu}^{-1}$, $\tilde{\mu} \circ \tilde{\nu}^{-1}$ cannot belong to a cyclic subgroup of $H$. Hence one of the pairs $\tilde{\eta}$ and $\tilde{\nu}^{-1}$, $\tilde{\eta}$ and $\tilde{\gamma} \circ \tilde{\nu}^{-1}$ or $\tilde{\eta}$ and $\tilde{\mu} \circ \tilde{\nu}^{-1}$ are not commutative. Consider the corresponding noncommutative pair of elements in $G_j$ for a sufficiently large
Put \( z' = \tilde{\nu}(0) \). Application of Lemma 2.1 to the above pair and the point \( z' \) leads to a contradiction. Thus \( D = \Delta \) and by Corollary 3.3 \( k_R = \lim_{j \to \infty} k_{\Gamma_{b_j}} \).

This implies immediately that for every sequence \( \{b_j\} \subset \Delta^* \) with hyperbolic fibers \( \{\Gamma_{b_j}\} \) and \( b_j \to 0 \) \( k_R = \lim_{j \to \infty} k_{\Gamma_{b_j}} \). The last thing we need to confirm is that if there exists a sequence \( \{b_j\} \to 0 \) with nonhyperbolic fibers \( \{\Gamma_{b_j}\} \) then \( R \) cannot be hyperbolic. Assume that such a sequence exists. Then \( \Gamma_{b_j} \) is biholomorphically equivalent to \( C \) or \( C^* \). Hence \( R \) has no handle, for if it had, then all of the fibers \( \Gamma_{b_j} \) would have handles as well for sufficiently large \( j \). Since a Fuchsian group of the first kind corresponds to the Riemann surface \( R \), \( R \) is different from \( \Delta, \Delta^* \) or an annulus. Thus \( \pi_1(R) \) has, at least, two generators \( \gamma_1 \) and \( \gamma_2 \). One may suppose that the loops \( \gamma_1 \) and \( \gamma_2 \) have no points of self-intersection. Note that the proof of Lemma 4.2 does not use the assumption that \( \{\Gamma_{b_j}\} \) are hyperbolic, i.e., it remains true without this assumption. Thus, since \( \Gamma_{b_j} \) is biholomorphically equivalent to \( C \) or \( C^* \) either \( \gamma_1^k \) or \( \gamma_2^k \) must be approximated by contractible loops in \( \{\Gamma_{b_j}\} \) for a certain integer \( k \). This contradicts Lemma 4.2. Hence there is no sequence \( \{b_j\} \to 0 \) with nonhyperbolic fibers \( \{\Gamma_{b_j}\} \). The main theorem is proved.

**References**


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