

THE COHOMOLOGY ALGEBRA OF A COMMUTATIVE GROUP SCHEME

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ABSTRACT. Let k be a commutative ring with unit of characteristic $p > 0$ and let $G = \text{Spec}(A)$ be an affine commutative group scheme over k . Let $H^\bullet(G)$ be the graded Hochschild algebraic group cohomology algebra and, for M a rational G -module, let $H^\bullet(G, M)$ denote the graded Hochschild cohomology $H^\bullet(G)$ -module. We show that $H^\bullet(G)$ is, in general, a graded Hopf algebra. When $G = \mathbf{G}_{a,k}$, let α_{p^ν} denote the subgroup of p^ν -nilpotents and let F_ν denote the ν th power of the Frobenius. We show that for any finite M that there is a ν such that

$$H^\bullet(\mathbf{G}_{a,k}, M) \cong H^\bullet(\alpha_{p^\nu}, M) \otimes_k F_\nu^*(H^\bullet(\mathbf{G}_{a,k}))$$

where F_ν^* is the endomorphism of $H^\bullet(\mathbf{G}_{a,k})$ induced by F_ν . As a consequence, we can show that $H^\bullet(\mathbf{G}_{a,k}, M)$ is a finitely generated module over $H^\bullet(\mathbf{G}_{a,k})$ when M is a finite dimensional vector space over k .

1. INTRODUCTION

Let k be a commutative ring with unit and let $G = \text{Spec}(A)$ be an affine group scheme over k . If M is a G -module (flat over k and an A -co-module), let $H^\bullet(G, M)$ denote the graded k -module $\coprod_{n \geq 0} H^n(G, M)$, where $H^i(G, M)$ denotes the i th Hochschild cohomology of G in M . This is Ext in the category of rational G -modules (see [Demazure-Gabriel] or [Jantzen]). Then $H^\bullet(G)$ denotes $H^\bullet(G, k)$ for k the trivial G -module. It is well known (loc. cit) that there is a cup product so that $H^\bullet(G)$ is a graded algebra and $H^\bullet(G, M)$ is a graded $H^\bullet(G)$ -module. Further $H^\bullet(G, M)$ is contravariant in G and covariant in M and, if $G \rightarrow G'$ is a morphism of group schemes, then the map $H^\bullet(G', M) \rightarrow H^\bullet(G, M)$ is compatible with the algebra morphism $H^\bullet(G) \rightarrow H^\bullet(G')$. These facts are recalled and explained in the first few sections of this paper.

The main results of this paper are these: In §6, we show, for a commutative group scheme G , that the cohomology algebra $H^\bullet(G)$ is canonically a graded Hopf algebra. The results of §§6 and 7 apply to $\mathbf{G}_{a,k}$, the additive group scheme over a field k . Let $\alpha_{p^\nu, k}$ be the subgroup of p^ν nilpotents over k . Then it is known that the restriction map $H^\bullet(\mathbf{G}_{a,k}, M) \rightarrow H^\bullet(\alpha_{p^\nu, k}, M)$ splits, though noncanonically, for sufficiently large ν . Let $F_\nu^* : H^\bullet(\mathbf{G}_{a,k}) \rightarrow H^\bullet(\mathbf{G}_{a,k})$ be the

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map corresponding to the ν th power of the Frobenius. In §6, we show that, for ν large enough, depending upon the finite dimensional $\mathbf{G}_{a,k}$ -module M , there is an isomorphism:

$$H^*(\mathbf{G}_{a,k}, M) \cong H^*(\alpha_{p^\nu}, M) \otimes_k F_\nu^* H^*(\mathbf{G}_{a,k}).$$

This implies the main result of §7 which settles a long outstanding question concerning $\mathbf{G}_{a,k}$ cohomology. Namely, when M is a finite dimensional representation of $\mathbf{G}_{a,k}$, then the cohomology module $H^*(\mathbf{G}_{a,k}, M)$ is a finitely generated right $H^*(\mathbf{G}_{a,k})$ -module. This question was first communicated to us by Brian Parshall in 1987. We have subsequently encountered it in conversation with several other mathematicians.

2. PRELIMINARIES

Let k be a commutative ring with unit and let $\text{Spec}(A) = G$ be an affine group scheme over k . For convenience we assume that A is faithfully flat over k . Then A is a commutative Hopf algebra over k . Let $\mu : A \rightarrow A \otimes_k A$ be the co-multiplication, let $e : A \rightarrow k$ be the augmentation, and let $s : A \rightarrow A$ be the antipode.

Recall that a *rational G -module*, say M , is a k -module with a co-associative map

$$\alpha : M \rightarrow M \otimes_k A$$

such that $(1_M \otimes e) \circ \alpha = 1_M$. Then the Hochschild cohomology with coefficients in M is $H^i(G, M) = \text{Ext}^i(k, M)$ where the Ext is taken in the category of rational G -modules and k is the trivial module. (That is: the co-action $k \rightarrow k \otimes_k A$ is given by $a \rightarrow a \otimes 1$ for $a \in k$.)

It is well known (see [Demazure-Gabriel] or [Jantzen]) that $H^*(G, M)$ is the cohomology of the complex $(C^*(G, M), \partial)$, where

$$(1) \quad C^q(G, M) = M \otimes_k \underbrace{A \otimes A \otimes \cdots \otimes A}_q = M \otimes A^{\otimes q},$$

$$(2) \quad \begin{aligned} \partial^q(m \otimes a_1 \otimes \cdots \otimes a_q) &= \alpha(m) \otimes a_1 \otimes \cdots \otimes a_q \\ &+ \sum_{j=1}^q (-1)^j m \otimes a_1 \otimes \cdots \otimes \mu(a_j) \otimes \cdots \otimes a_q \\ &+ (-1)^{q+1} m \otimes a_1 \otimes \cdots \otimes a_q \otimes 1 \quad (q \geq 0). \end{aligned}$$

When $M = k$, write $C^*(G)$ and $H^*(G)$ rather than $C^*(G, k)$ and $H^*(G, k)$. Then it is again well known [Jantzen, p. 62] that

$$C^*(G) = k \amalg A \amalg A^{\otimes 2} \amalg \cdots \amalg A^{\otimes q} \amalg \cdots = T(A)$$

is a ring. It is the tensor algebra of A regarded as a k -module.

If $u \in C^q(G) = A^{\otimes q}$, where $u = a_1 \otimes \cdots \otimes a_q$, then

$$(3) \quad \begin{aligned} \partial^q(u) &= 1 \otimes a_1 \otimes \cdots \otimes a_q \\ &+ \sum_{j=1}^q (-1)^j a_1 \otimes \cdots \otimes \mu(a_j) \otimes \cdots \otimes a_q \\ &+ (-1)^{q+1} a_1 \otimes \cdots \otimes a_q \otimes 1. \end{aligned}$$

If $u_p \in C^p(G)$ and $u_q \in C^q(G)$, then

$$\partial^{p+q}(u_p u_q) = \partial^p(u_p) u_q + (-1)^p u_p \partial^q(u_q).$$

It follows that the module of co-cycles, $Z^\bullet(G) = \coprod Z^q(G)$, is a subring of $C^\bullet(G)$ and that the module of co-boundaries, $B^\bullet(G) = \coprod B^q(G)$, is an ideal of $Z^\bullet(G)$.

Finally, it follows that $C^\bullet(G, M)$ is a right $C^\bullet(G)$ -module. This is clear because $C^\bullet(G, M) = M \otimes_k C^\bullet(G)$. An elementary computation shows, for $u \in C^p(G, M)$ and $v \in C^q(G)$, that

$$(4) \quad \partial^{p+q}(uv) = \partial^p(u)v + (-1)^p u \partial^q(v).$$

It follows immediately that $Z^\bullet(G, M)$ is a $Z^\bullet(G)$ -submodule of $C^\bullet(G, M)$, that

$$Z^\bullet(G, M)B^\bullet(G) \subseteq B^\bullet(G, M),$$

and so that

$$H^\bullet(G, M) = \coprod_r H^r(G, M)$$

is a graded $H^\bullet(G)$ -module.

In general, if $M = \coprod_r M^r$ is a graded module over the graded ring S , then $M[n]$ denotes the graded module with homogeneous components given by $M[n]^r = M^{r+n}$. If

$$0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$$

is a short exact sequence of rational G -modules, then the usual exact cohomology sequence becomes the exact sequence of graded $H^\bullet(G)$ -modules

$$\cdots \rightarrow H^\bullet(V') \rightarrow H^\bullet(V) \rightarrow H^\bullet(V'') \rightarrow H^\bullet(V')[1] \rightarrow \cdots.$$

Finally, we require some further terminology. We shall call the group scheme $G = \text{Spec}(A)$ homologically exact if $C^q(G)$, $B^q(G)$, and $H^q(G)$ are all flat k -modules for all $q \geq 0$. This is true if, for example, G is the base extension by k of a group scheme defined over a field.

3. CHANGE OF GROUPS

Suppose that G and G' are two affine group schemes over k and that $f : G \rightarrow G'$ is a homomorphism. Let $G = \text{Spec}(A)$, $G' = \text{Spec}(A')$, and write μ, s, e, μ', s', e' for the structural data of G and G' respectively. Then f corresponds to a Hopf algebra map which we will write $f^\circ : A' \rightarrow A$. Then f° commutes with co-multiplication, augmentation, and antipodes. Since the boundary ∂^q is an alternating sum, each term of which is either an application

of μ to one component or an inclusion $u \mapsto 1 \otimes u$, the diagram

$$\begin{CD} C^q(G') @>\partial^q>> C^{q+1}(G') \\ @Vf^{\circ\otimes q}VV @VVf^{\circ\otimes(q+1)}V \\ C^q(G) @>\partial^q>> C^{q+1}(G) \end{CD}$$

commutes.

The sum $\coprod_{r \geq 0} f^{\circ\otimes r}$ is surely a homomorphism of tensor algebras. So f induces a functorial homomorphism of algebras, which will be denoted by $H^\bullet(f)$, from $H^\bullet(G')$ to $H^\bullet(G)$.

Suppose that M' is a rational G' -module with structure map $\alpha' : M' \rightarrow M' \otimes A'$. Let $\alpha : M' \rightarrow M' \otimes A$ be the map given by $\alpha = (1_{M'} \otimes f^\circ) \circ \alpha'$. Then this is a rational G -module structure on M' .

Definition 3.1. Let $f : G \rightarrow G'$ be a homomorphism of affine algebraic group schemes over k . If M' is a rational G' -module, then write $f^!(M') = M$ for the rational G -module with underlying module M' and co-action $\alpha = (1_{M'} \otimes f^\circ) \circ \alpha'$.

Write $T(f)$ for the co-product $\coprod_{q \geq 0} f^{\circ\otimes q}$ from $C^\bullet(G')$ to $C^\bullet(G)$. Since M and M' share the same underlying k -module, there is a natural map

$$T(f, M') : C^\bullet(G', M') \rightarrow C^\bullet(G, M).$$

Namely, $C^q(G', M') = M' \otimes C^q(G') = M' \otimes A'^{\otimes q}$ and $C^q(G, M) = M \otimes C^q(G) = M' \otimes A^{\otimes q}$. Then $T^q(f, M')$ is just $1_{M'} \otimes f^{\otimes q}$.

The map $T(f, M')$ is evidently a module map with respect to the homomorphism of rings $T(f)$. That is, if $\beta = T(f, M')$, $\phi = T(f)$, then $\beta(ua) = \beta(u)\phi(a)$. Since $T(f, M')$ commutes with the differential, it induces a homomorphism

$$H^\bullet(f, M') : H^\bullet(G', M') \rightarrow H^\bullet(G, M).$$

Because $T(f, M')$ is compatible with the ring morphism $T(f)$, it follows that $H^\bullet(f, M')$ is compatible with the $H^\bullet(\)$ -module structure on $H^\bullet(G, M)$ and $H^\bullet(G', M')$ respectively. This establishes the following proposition.

Proposition 3.1. Let $G = \text{Spec}(A)$ and $G' = \text{Spec}(A')$ be affine algebraic group schemes over k , let $f : G \rightarrow G'$ be a homomorphism of group schemes, let M' be a rational G' -module, and let $M = f^!M'$. Then

- (1) the natural map $H^\bullet(f) : H^\bullet(G') \rightarrow H^\bullet(G)$ is a homomorphism of k -algebras and
- (2) the map $H^\bullet(f, M') : H^\bullet(G', M') \rightarrow H^\bullet(G, M)$ is an $H^\bullet(G')$ -module map, where $H^\bullet(G', M')$ is an $H^\bullet(G')$ -module by means of the algebra map $H^\bullet(f)$.

4. A DIRECT LIMIT REPRESENTATION; A KÜNNETH FORMULA

In this section assume that k is a field and that the groups involved are of finite type over k . If $G = \text{Spec}(A)$ and $G' = \text{Spec}(A')$, then the algebras A and A' are finitely generated over k .

Lemma 4.1. *Let $G = \text{Spec}(A)$ be an affine algebraic group scheme of finite type over k . Then there is a finite dimensional k vector subspace E of A containing k and such that*

- (1) $\mu(E) \subset E \otimes_k E$,
- (2) $s(E) = E$,
- (3) $A = k[E]$; that is A is generated as an algebra by E .

Proof. Consider first $\mathbf{GL}(n, k)$ the coordinate ring of which is

$$k[x_{11}, \dots, x_{nn}, 1/\delta],$$

where the $\{x_{ij}\}$ are n^2 indeterminates and $\delta = \det(x_{ij})$, and which is denoted by \tilde{A}_n . Let $y_{ij} = s(x_{ij})$. Then y_{rs} is the minor in (x_{ij}) corresponding to x_{sr} multiplied by $(-1)^{r+s}/\delta$. Let \tilde{E}_n be the linear span of the $2n^2 + 3$ elements $1, \delta, 1/\delta, \{x_{ij}\}, \{y_{ij}\}, 1 \leq i, j \leq n$. Then

$$\begin{aligned} \mu(1) &= 1 \otimes 1, & \mu(\delta^{\pm 1}) &= \delta^{\pm 1} \otimes \delta^{\pm 1}, \\ \mu(x_{ij}) &= \sum_{l=1}^n x_{il} \otimes x_{lj}, & \mu(y_{ij}) &= \sum_{l=1}^n y_{il} \otimes y_{lj}. \end{aligned}$$

Clearly \tilde{E}_n satisfies the conditions given in the lemma.

Now suppose that $G = \text{Spec}(A)$ is any affine algebraic group scheme satisfying the conditions of the lemma. Choose a faithful representation $\rho : G \rightarrow \mathbf{GL}(n, k)$. This determines a surjective map $\rho^\circ : \tilde{A}_n \rightarrow A$ which is in fact a Hopf algebra morphism. Let $E = \rho^\circ(\tilde{E}_n)$. Then E satisfies the conclusion of the lemma. \square

Corollary 4.1. *Let $G = \text{Spec}(A)$ be an affine algebraic group scheme of finite type over k . Then there is a sequence $\{E^{(n)}; n \geq 1\}$ of finite dimensional sub-coalgebras of A each with antipode and augmentation so that*

- (1) $A = \bigcup_n E^{(n)}$ and
- (2) $E^{(i)} \cdot E^{(j)} \subset E^{(i+j)}$.

Proof. Choose $E^{(1)} = E$ as in the Lemma 4.1. Set

$$E^{(n)} = \overbrace{E^{(1)} \dots E^{(1)}}^n$$

the n -fold product (in A) of $E^{(1)}$ with itself. (That is: the span of degree n monomials in elements in $E^{(1)}$.) Since $k \subset E^{(1)}$, it follows that

$$E^{(1)} \subset E^{(2)} \subset \dots \subset E^{(n)} \subset \dots$$

and since E generates A over k , the $E^{(n)}$ exhaust A . The last condition is clear. \square

Definition 4.1. *Let $G = \text{Spec}(A)$ be an affine algebraic group scheme of finite type over k . An ascending exhaustive filtration of A by finite dimensional s -stable sub-coalgebras each containing k is called a coalgebraic filtration of A .*

Suppose now that M is a finite dimensional rational G -module. For any basis $\{m_1, \dots, m_j\}$ of M , let

$$\alpha(m_i) = \sum_j m_j \otimes a_{ji}, \quad a_{ji} \in A.$$

If $\{E^{(r)}\}_{r \geq 1}$ is a co-algebraic filtration of A , then all of the a_{ij} are in $E^{(r)}$ for some r . Starting from $E^{(r)}$ for this r rather than from $E^{(1)}$ if necessary, we see that for any finite rational G -module M , there is a co-algebraic filtration $\{E^{(\bullet)}\}$ so that $\alpha(M) \subset M \otimes_k E^{(i)}$ for each i . When this is so, then we will say that M is $E^{(\bullet)}$ -compatible.

Now the Hochschild complex differential employs only the co-action on the coefficient module, the co-multiplication, and the injection of $M \otimes A^{\otimes n}$ to $M \otimes A^{\otimes(n+1)}$ via $u \mapsto u \otimes 1$. Each of these maps carries $M \otimes (E^{(i)})^{\otimes n}$ to $M \otimes (E^{(i)})^{\otimes(n+1)}$ when M is $E^{(\bullet)}$ -compatible. Define subspaces of the Hochschild cochains for M by the equation

$$(5) \quad C^q(E^{(i)}; M) = M \otimes (E^{(i)})^{\otimes q}.$$

Then

$$\partial^q(C^q(E^{(i)}; M)) \subset C^{q+1}(E^{(i)}; M).$$

Hence $C^\bullet(E^{(i)}; M)$ is a subcomplex of the Hochschild complex $C^\bullet(G, M)$. Furthermore, if M and N are finite and both $E^{(\bullet)}$ -compatible, then the cup-product (which is just the tensor product) induces the map

$$C^q(E^{(i)}; M) \otimes C^r(E^{(j)}; N) \rightarrow C^{q+r}(E^{(i+j)}; M \otimes N)$$

compatible with all the structures involved.

Write $H^\bullet(E^{(i)}, M)$ for the cohomology groups of the complex $C^\bullet(E^{(i)}; M)$. Notice that

$$C^\bullet(G, M) = \varinjlim_i C^\bullet(E^{(i)}; M).$$

The inclusions $C^\bullet(E^{(i)}; M) \subset C^\bullet(E^{(i+j)}; M)$ induce maps of cohomology

$$(6) \quad H^\bullet(E^{(i)}, M) \rightarrow H^\bullet(E^{(i+j)}, M),$$

$$(7) \quad H^\bullet(E^{(i)}, M) \rightarrow H^\bullet(G, M)$$

so that there is induced a natural homomorphism

$$\varinjlim_i H^\bullet(E^{(i)}, M) \rightarrow H^\bullet(G, M).$$

This leads to the next lemma.

Lemma 4.2. *Let $G = \text{Spec}(A)$ be an affine group scheme of finite type over k , let $E^{(\bullet)}$ be a co-algebraic filtration of A , and let M be a finite $E^{(\bullet)}$ -compatible rational G -module. Then*

$$\varinjlim_i H^\bullet(E^{(i)}, M) \rightarrow H^\bullet(G, M)$$

is an isomorphism.

Proof. Clearly

$$Z^q(G, M) \cap C^q(E^{(i)}, M) = Z^q(E^{(i)}, M).$$

Suppose that $u \in Z^q(E^{(i)}, M)$ is a co-boundary in $C^q(G, M)$. Then there is $u' \in C^{q-1}(G, M)$ such that $u = \partial^{q-1}(u')$. There is some j with $j \geq i$ such that $u' \in C^{q-1}(E^{(j)}, M)$. Then u is a co-boundary in $C^q(E^{(j)}, M)$. The result follows. \square

Suppose that A is the coordinate ring of an affine algebraic group scheme of finite type over k and let $E^{(\bullet)}$ be a co-algebraic filtration of A . Let M be a finite $E^{(\bullet)}$ -compatible rational G -module. Let $\check{E}_i = \text{Hom}_k(E^{(i)}, k)$ be the linear dual of $E^{(i)}$. Then by standard duality theory \check{E}_i is an algebra over k with unit $e|_{E^{(i)}}$. Write e for this map. The definition of the multiplication on \check{E}_i assures that $f_1 \cdot f_2 = (f_1 \otimes f_2) \circ \mu$ for all $f_1, f_2 \in \check{E}_i$. In particular, if $a \in E^{(i)}$ with $\mu(a) = \sum_{j=1}^q a'_j \otimes a''_j$, then $(f_1 \cdot f_2)(a) = \sum_{j=1}^q f_1(a'_j) f_2(a''_j)$.

Now M admits a bimodule structure over \check{E}_i . If $f \in \check{E}_i$ and $m \in M$, then

$$f \cdot m = ((1_M \otimes f) \circ \alpha)(m) \quad \text{and} \quad m \cdot f = f(1)m.$$

(Recall that $k \subset E^{(i)}$.) Clearly there is a canonical identification

$$C^q(E^{(i)}; M) = \text{Hom}_k(\check{E}_i^{\otimes q}, M).$$

It is just a matter of definition that after rewriting co-chains as elements in $\text{Hom}_k(\check{E}_i^{\otimes q}, M)$, we may write the differential as

$$\begin{aligned} (\partial^q)(u)(f_1, f_2, \dots, f_{q+1}) &= f_1 u(f_2, \dots, f_{q+1}) \\ (8) \quad &+ \sum_{j=1}^{q-1} (-1)^j u(f_1, \dots, f_{j-1}, f_j \cdot f_{j+1}, \dots, f_{q+1}) \\ &+ (-1)^{q+1} f_{q+1} u(f_1, \dots, f_q). \end{aligned}$$

This proves the following lemma.

Lemma 4.3. *The complex $C^\bullet(E^{(i)}; M)$ is isomorphic as a complex to the Hochschild algebra cohomology complex $C^\bullet(\check{E}_i, M)$. The cohomology groups $H^\bullet(\check{E}_i, M)$ are the Hochschild algebra cohomology groups of \check{E}_i with coefficients in the bi-module M .*

This lemma brings us to the main result of this section. Let G_1 and G_2 be two affine algebraic group schemes of finite type over k . Then their product $G_1 \times G_2$ is an algebraic group scheme of finite type and the projections $p_i : G_1 \times G_2 \rightarrow G_i$ are morphisms. If M_1 is a rational G_1 -module and M_2 is a rational G_2 -module, then $M_1 \otimes_k M_2$ is a rational $G_1 \times G_2$ -module. We may also apply 3.1 to each projection p_i . Write M_i also for the pull-back $p_i^!(M_i)$. Then, in this sense, both M_1 and M_2 are $G_1 \times G_2$ -modules and the tensor product gives a natural pairing from the M_i into $M_1 \otimes M_2$ over $G_1 \times G_2$. Hence cup product induces natural pairings

$$H^r(G_1, M_1) \otimes H^s(G_2, M_2) \rightarrow H^{r+s}(G_1 \times G_2, M_1 \otimes M_2).$$

Hence there is a natural cup product map

$$(9) \quad \xi_n : \coprod_{r+s=n} H^r(G_1, M_1) \otimes H^s(G_2, M_2) \rightarrow H^n(G_1 \times G_2, M_1 \otimes M_2).$$

Theorem 4.1. *Let G_1 and G_2 be two affine algebraic group schemes over k and let M_1 and M_2 be finite dimensional rational modules over G_1 and G_2 respectively. Then the cup product map*

$$(10) \quad \xi_n : \coprod_{r+s=n} H^r(G_1, M_1) \otimes H^s(G_2, M_2) \rightarrow H^n(G_1 \times G_2, M_1 \otimes M_2)$$

is an isomorphism for each n .

Proof. Let $G_i = \text{Spec}(A_i)$ for $i = 1, 2$. Let $E_i^{(\bullet)}$ be a co-algebraic filtration of A_i with respect to which M_i is compatible, again for $i = 1, 2$. Observe that the maps $C^\bullet(E_i^{(j)}; M_i) \rightarrow C^\bullet(G_i, M_i)$ are functorial with respect to cup product as are the cohomology maps.

Let $E_{1,2}^{(j)} = E_1^{(j)} \otimes E_2^{(j)}$. Then $E_{1,2}^{(\bullet)}$ is a co-algebraic filtration in $A_1 \otimes A_2$. Hence there is a cup-product map corresponding to each term in the filtration that makes the following diagram commute:

$$\begin{array}{ccc} \coprod_{r+s=n} H^r(E_1^{(j)}, M_1) \otimes H^s(E_2^{(j)}, M_2) & \xrightarrow{\xi_n^{(j)}} & H^n(E_{1,2}^{(j)}, M_1 \otimes M_2) \\ \downarrow & & \downarrow \\ \coprod_{r+s=n} H^r(G_1, M_1) \otimes H^s(G_2, M_2) & \xrightarrow{\xi_n} & H^n(G_1 \times G_2, M_1 \otimes M_2). \end{array}$$

By Lemma 4.3, the cohomology groups $H^\bullet(E_i^{(j)}, M_i)$ and $H^\bullet(E_{1,2}^{(j)}, M_1 \otimes M_2)$ are Hochschild algebra cohomology groups for the finite dimensional k -algebras $\check{E}_i^{(j)}$ and $\check{E}_{1,2}^{(j)}$ with coefficients in finite bi-modules. Hence [Mac Lane, Theorem VIII, 1.2, p. 222] applies, so $\xi_n^{(j)}$ is an isomorphism for each n . Applying Lemma 4.2 and passing to the direct limit, noting that direct limit commutes with tensor product over a field, we see that ξ_n is an isomorphism for each n . \square

Corollary 4.2. *Let G_1, G_2 be two affine algebraic groups schemes of finite type over k . Then*

$$(11) \quad H^\bullet(G_1 \times G_2) = H^\bullet(G_1) \otimes H^\bullet(G_2)$$

where the isomorphism is an isomorphism of graded algebras and the tensor product is the product of graded algebras (i.e.

$$\alpha \otimes \beta \cdot \gamma \otimes \delta = (-1)^{\text{deg}(\beta) \text{deg}(\gamma)} (\alpha\gamma) \otimes (\beta\delta)$$

where the elements are homogeneous elements.)

Proof. The theorem shows that the two groups in (11) are isomorphic. That it is an isomorphism of graded objects follows because the isomorphism sends $\alpha \otimes \beta$ to $p_1^*(\alpha) \cup p_2^*(\beta)$. \square

5. COMMUTATIVE GROUP SCHEMES; THE HOPF STRUCTURE

Suppose that G is a commutative affine algebraic group. Then it is a group object in the category of k -group schemes. This is equivalent to the fact that

the morphisms

$$\mu : G \times G \rightarrow G, \quad s : G \rightarrow G, \quad \epsilon : \text{Spec}(k) \rightarrow G$$

that describe the group are group homomorphisms. Also, recall that for a graded Hopf algebra, the co-multiplication takes the Hopf algebra to its graded tensor product.

We are now prepared to state one of the main results of this paper.

Theorem 5.1. *Let G be a commutative affine algebraic group scheme of finite type over k . Then $H^\bullet(G)$ is a graded Hopf algebra with co-multiplication $H^\bullet(\mu)$, antipode $H^\bullet(s)$, and augmentation $H^\bullet(\epsilon)$. If $\phi : G \rightarrow G'$ is a homomorphism of commutative group schemes, then $H^\bullet(\phi) : H^\bullet(G') \rightarrow H^\bullet(G)$ is a morphism of Hopf algebras.*

Proof. Write $\mu^\bullet = H^\bullet(\mu)$, $s^\bullet = H^\bullet(s)$, and $\epsilon^\bullet = H^\bullet(\epsilon)$. By Corollary 4.2, there is a natural, graded isomorphism

$$\xi_\bullet : H^\bullet(G) \otimes_k H^\bullet(G) \rightarrow H^\bullet(G \times G).$$

The map is the one induced by sending $\alpha \otimes \beta$ to $p_1^*(\alpha) \cup p_2^*(\beta)$, where p_1 and p_2 are the two projections. Hence μ^\bullet may be viewed as a map

$$\mu^\bullet : H^\bullet(G) \rightarrow H^\bullet(G) \otimes_k H^\bullet(G).$$

Recall that the group axioms are:

$$(12) \quad \mu \circ (\mu \times 1_G) = \mu \circ (1_G \times \mu),$$

$$(13) \quad \mu \circ (\epsilon \times 1_G) = 1_G = \mu \circ (1_G \times \epsilon),$$

and

$$(14) \quad \mu \circ (1_G \times s) \circ \Delta = \epsilon \circ \pi = \mu \circ (s \times 1_G) \circ \Delta$$

where $\Delta : G \rightarrow G \times G$ is the diagonal and $\pi : G \rightarrow \text{Spec}(k)$ is the one point projection. All of these maps, without exception, are group homomorphisms (since G is commutative) and so we can apply the contravariant functor $H^\bullet(\cdot)$ to obtain

$$(15) \quad H^\bullet(\mu \times 1_G) \circ H^\bullet(\mu) = H^\bullet(1_G \times \mu) \circ H^\bullet(\mu),$$

$$(16) \quad H^\bullet(\epsilon \times 1_G) \circ H^\bullet(\mu) = H^\bullet(1_G) = H^\bullet(1_G \times \epsilon) \circ H^\bullet(\mu),$$

$$(17) \quad \begin{aligned} H^\bullet(\Delta) \circ H^\bullet(1_G \times s) \circ H^\bullet(\mu) \\ = H^\bullet(\pi) \circ H^\bullet(\epsilon) = H^\bullet(\Delta) \circ H^\bullet(s \times 1_G) \circ H^\bullet(\mu). \end{aligned}$$

Each of these equations incorporates certain minor inconsistencies whose resolution we leave to the reader. For example, the axiom $\mu \circ 1_G = 1_G$ should really be written $\mu \circ 1_G \times \epsilon \circ q = 1_G$ where $q : G \cong G \times e$ is the isomorphism of G with $G \times_k \text{Spec}(k)$.

Nonetheless, each of these equations can be interpreted as a verification of the corresponding axioms in the definition of a Hopf algebra. Rather than subject the reader to these technicalities, we will select a part of (17), which we believe

to be the most subtle, to interpret and trust the reader to verify the remaining equations.

We will show that

$$(18) \quad H^\bullet(\Delta) \circ H^\bullet(s \times 1_G) \circ H^\bullet(\mu) = H^\bullet(\pi) \circ H^\bullet(e)$$

which can be interpreted as the co-inverse axiom.

First, $H^\bullet(\mu) = \mu^\bullet$ is the co-multiplication and $H^\bullet(s \times 1_G) = H^\bullet(s) \otimes H^\bullet(1_G) = s^\bullet \otimes \text{id}$ under the Künneth identification. Furthermore $H^\bullet(\pi)$ is the map from $H^\bullet(e)$ to $H^\bullet(G)$. Since $H^\bullet(e)$ is just k in degree 0, it is clear that $H^\bullet(e)$ is the algebra inclusion $j : k \rightarrow H^\bullet(G)$.

Consider $H^\bullet(\Delta)$. Since Δ is a group morphism, this gives an algebra morphism $H^\bullet(G) \otimes H^\bullet(G) \rightarrow H^\bullet(G)$. Here the main problem involved is the question of sign and degree, which is addressed by the fact that the tensor product is the graded tensor product. Hence we can write $m = H^\bullet(\Delta)$, where m is cup product with a sign adjustment. Rewriting (17), now with m replacing $H^\bullet(\Delta)$, gives

$$m \circ s^\bullet \otimes \text{id} \circ \mu^\bullet = j \circ \epsilon^\bullet.$$

This is the left co-inverse axiom for the Hopf algebra structure with co-multiplication μ^\bullet , antipode s^\bullet , and augmentation ϵ^\bullet .

This is just one of the equations in the list of defining equations for a Hopf algebra. \square

6. ANOTHER KÜNNETH DECOMPOSITION

Consider the affine algebraic group scheme $G = G_{a,k}$. Thus the coordinate ring of $G_{a,k}$ is $A = k[x]$. For each $\nu \geq 0$, let $A_\nu = k \amalg kx \amalg kx^2 \amalg \dots \amalg kx^{\nu-1}$ and let $F_\nu^\circ A = k[x^{\nu}]$. Then $F_\nu^\circ A$ is a sub-Hopf algebra of $k[x]$ with all of its structures. If needed, we will write $F_\nu^\circ A = k[y]$, $y = x^{\nu}$.

Lemma 6.1. *The coordinate ring of $G_{a,k} = \text{Spec}(A) = \text{Spec}(k[x])$ is isomorphic to $A_\nu \otimes_k F_\nu^\circ A$. Furthermore, this is an isomorphism of augmented co-algebras with antipode.*

Proof. The map is multiplication

$$\begin{aligned} A_\nu \otimes_k F_\nu^\circ A &\rightarrow A, \\ x^r \otimes y^s &\rightarrow x^{r+sp^\nu}. \end{aligned}$$

No further proof is required. \square

Now suppose that V is a finite $G_{a,k}$ -module with co-module structure map $\alpha : V \rightarrow V \otimes_k A$. Since V is finite dimensional, there is some ν such that $\alpha(V) \subset V \otimes A_\nu$.

Now $F_\nu^\circ A$ is the coordinate ring of a finite dimensional affine algebraic group. Hence there is a co-algebraic filtration E^\bullet such that $F_\nu^\circ A = \bigcup_i E^{(i)}$. Finally observe that defining $F_\nu^\circ : A \rightarrow A$ by $F_\nu^\circ(x) = x^{\nu}$ gives the Frobenius morphism

$F_\nu : \mathbf{G}_{a,k} \rightarrow \mathbf{G}_{a,k}$ yielding a contravariant morphism of Hochschild complexes

$$(19) \quad F_\nu^{\circ*} : C^n(\mathbf{G}_{a,k}, k) \rightarrow C^n(\mathbf{G}_{a,k}, k)$$

and hence a contravariant morphism of cohomology rings

$$(20) \quad H^\bullet(F_\nu) : H^\bullet(\mathbf{G}_{a,k}) \rightarrow H^\bullet(\mathbf{G}_{a,k}).$$

The image of $F_\nu^{\circ*}$ is a complex that we will denote by $C^\bullet(F_\nu^\circ A)$. Its r th term is $(F_\nu^\circ A)^{\otimes r}$. Notice that $C^\bullet(F_\nu^\circ A)$ is closed under the cup product multiplication, which is just tensor product.

Likewise the complex $C^\bullet(A_\nu, V)$ is a subcomplex of $C^\bullet(\mathbf{G}_{a,k}, V)$ closed under tensor multiplication by elements of $C^\bullet(A_\nu, k)$. Furthermore, the inclusion of the subgroup of p^ν -nilpotents, $j_\nu : \alpha_{p^\nu} \rightarrow \mathbf{G}_{a,k}$, corresponds to a surjective mapping of Hopf algebras $j_\nu^\circ : k[x] \rightarrow k[x]/x^{p^\nu}k[x]$ which induces an isomorphism of augmented coalgebras with antipode by restriction to A_ν . In particular (see [Jantzen, p. 69]), j_ν° induces an isomorphism of $H^\bullet(A_\nu, V)$ with $H^\bullet(\alpha_p^\nu, V)$. Since this identifies $H^\bullet(\alpha_p^\nu, V)$ with a direct summand of $H^\bullet(\mathbf{G}_{a,k}, V)$, it proves that $H^\bullet(A_\nu, V)$ is an $H^\bullet(A_\nu)$ -sub-module of $H^\bullet(\mathbf{G}_{a,k}, V)$. Since there are maps $H^\bullet(A_\nu, V) \rightarrow H^\bullet(\mathbf{G}_{a,k}, V)$ and $H^\bullet(F_\nu^\circ A) \rightarrow H^\bullet(\mathbf{G}_{a,k})$, the cup product yields a map

$$(21) \quad \eta_\nu : H^\bullet(A_\nu, V) \otimes H^\bullet(F_\nu^\circ A) \rightarrow H^\bullet(\mathbf{G}_{a,k}, V).$$

Theorem 6.1. *Let V be a finite dimensional representation of $\mathbf{G}_{a,k}$ and let $A_\nu \subset k[\mathbf{G}_{a,k}]$ be the sub-co-algebra $A_\nu = k \amalg kc \amalg kx^2 \amalg \dots \amalg kx^{p^\nu-1}$. Let $F_\nu^\circ : H^\bullet(\mathbf{G}_{a,k}) \rightarrow H^\bullet(\mathbf{G}_{a,k})$ be the contravariant map associated to the ν th order Frobenius covering. Then the cup product induces an isomorphism*

$$(22) \quad \eta_\nu : H^\bullet(A_\nu, V) \otimes H^\bullet(F_\nu^\circ A) \rightarrow H^\bullet(\mathbf{G}_{a,k}, V).$$

This is evidently a map of $H^\bullet(F_\nu^\circ A)$ -modules.

Proof. We proceed just as in the proof of Theorem 4.1. Begin by observing that $F_\nu^\circ(H^\bullet(F_\nu^\circ A))$ is exactly the image of $H^\bullet(F_\nu^\circ A)$ in $H^\bullet(\mathbf{G}_{a,k})$. Now consider the co-algebraic filtration $E^{(\bullet)}$ of $F_\nu^\circ A$. Then we may write

$$C^\bullet(F_\nu^\circ A) = \varinjlim_n C^\bullet(E^{(n)}).$$

Consider the complexes $C^\bullet(E^{(r)})$ and $C^\bullet(A_\nu, V)$. Then \tilde{E}_r , the dual of $E^{(r)}$, and D_ν , the dual of A_ν , are finite k -algebras and Lemma 4.3 applies to show that $C^\bullet(A_\nu, V)$ and $C^\bullet(E^{(r)})$ are the Hochschild algebra co-chains of respectively D_ν in V and \tilde{E}_r in k . Hence, by [Mac Lane], *loc. cit.*, the cup product induces isomorphisms

$$(23) \quad \eta_{\nu,r} : H^\bullet(A_\nu, V) \otimes H^\bullet(E^{(r)}) \rightarrow H^\bullet(A_\nu \otimes E^{(r)}, V).$$

Taking direct limits, as in Lemma 4.2, yields an isomorphism

$$\eta_\nu : H^\bullet(A_\nu, V) \otimes H^\bullet(F_\nu^\circ A) \rightarrow H^\bullet(A_\nu \otimes F_\nu^\circ A, V \otimes k).$$

But by the remarks preceding the statement of the theorem, $A_\nu \otimes F_\nu^\circ A \cong A$ as Hopf algebras and it is similarly true that $V \otimes k \cong A$ under this identification. But then the result is established. \square

Corollary 6.1. *Let V be any finite dimensional $G_{a,k}$ -representation. Then for some ν with $\nu \gg 0$, there is an isomorphism of $F_\nu^*H^\bullet(G_{a,k})$ -modules*

$$(24) \quad H^\bullet(\alpha_{p^\nu}, V) \otimes F_\nu^*H^\bullet(G_{a,k}) \rightarrow H^\bullet(G_{a,k}, V).$$

Proof. For some ν , it follows that $\alpha(V) \subset V \otimes A_\nu$. Then apply the theorem and observe that $H^\bullet(A_\nu, V) \cong H^\bullet(\alpha_{p^\nu}, V)$. \square

Corollary 6.2. *Let V be a finite dimensional representation of $G_{a,k}$. Then there is a natural isomorphism of graded Hopf algebras*

$$(25) \quad H^\bullet(\alpha_{p^\nu}) \otimes F_\nu^*H^\bullet(G_{a,k}) \cong H^\bullet(G_{a,k})$$

compatible with the isomorphism of Corollary 6.1.

*That is: $H^\bullet(G_{a,k}, V)$ is the module $H^\bullet(\alpha_{p^\nu}, V)$ with coefficients extended by $F_\nu^*H^\bullet(G_{a,k})$.*

Proof. The only remark necessary to make is that $H^\bullet(A_\nu)$ is easily seen to be a sub-Hopf algebra of $H^\bullet(G_{a,k})$. the rest, which does not use this fact, is clear.

To establish that $H^\bullet(A_\nu)$ is canonically a (graded) Hopf algebra, note that the commutativity of $G_{a,k}$ means that $G_{a,k} \times G_{a,k} \rightarrow G_{a,k}$ is a group morphism. Hence $A \rightarrow A \otimes A$, the co-multiplication, is a co-algebra morphism. Hence $A_\nu \rightarrow A_\nu \otimes A_\nu$ is a co-algebra morphism. Hence $H^\bullet(A_\nu)$ maps onto a graded sub-Hopf algebra of $H^\bullet(G_{a,k}) = H^\bullet(A)$. \square

7. THE COHOMOLOGY IS FINITELY GENERATED.

In this final section we show that the cohomology $H^\bullet(G_{a,k}, V)$ of a finitely dimensional rational $G_{a,k}$ -module is finitely generated as a $H^\bullet(G_{a,k})$ -module. This follows easily from the results in the last section.

By a direct computation, for example [Jantzen, Proposition 4.27, p. 70], it is seen that $H^\bullet(\alpha_{p^\nu})$ is Noetherian. It is in fact a ring of commuting polynomial variables over a finite dimensional k -algebra.

Proposition 7.1. *Let V be a finite dimensional rational α_{p^ν} -module. Then $H^\bullet(\alpha_{p^\nu}, V)$ is finitely generated as a $H^\bullet(\alpha_{p^\nu})$ -module.*

Proof. The proof is by induction on $\dim_k V$. If $\dim_k V = 1$, then $V = k$ and the result is clear. Suppose that $\dim_k V > 1$. Then there is a sub- α_{p^ν} -module V' of V such that $\dim_k(V/V') = 1$, or that

$$0 \rightarrow V' \rightarrow V \rightarrow k \rightarrow 0$$

is exact.

Consider the resulting exact sequence (where we are writing $H^\bullet(V)$ for $H^\bullet(\alpha_{p^\nu}, V)$):

$$\dots \rightarrow H^\bullet(k)[-1] \rightarrow H^\bullet(V') \rightarrow H^\bullet(V) \rightarrow H^\bullet(k) \rightarrow H^\bullet(V')[1] \rightarrow \dots$$

By induction, $H^\bullet(V')$ is finitely generated. As $H^\bullet(k)$ is Noetherian, it follows that

$$\text{Coker}(H^\bullet(k)[-1] \rightarrow H^\bullet(V'))$$

and

$$\text{Ker}(H^\bullet(k) \rightarrow H^\bullet(V')[1])$$

are finitely generated. Hence $H^\bullet(V)$ is also finitely generated. \square

Theorem 7.1. *Let V be a finite dimensional, rational $\mathbf{G}_{a,k}$ -module. Then $\mathbf{H}^*(\mathbf{G}_{a,k}, V)$ is finitely generated as a $\mathbf{H}^*(\mathbf{G}_{a,k})$ -module.*

Proof. By Corollaries 6.1 and 6.2 of the previous section, there is a ν with $\nu \gg 0$ such that

$$\mathbf{H}^*(\mathbf{G}_{a,k}, V) \cong \mathbf{H}^*(\alpha_{p^\nu}, V) \otimes \mathbf{F}_\nu^*(\mathbf{H}^*(\mathbf{G}_{a,k}, k)).$$

Since $\mathbf{H}^*(\alpha_{p^\nu}, V)$ is of finite type, the result follows. \square

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