ZEROS OF THE SUCCESSIVE DERIVATIVES OF HADAMARD GAP SERIES

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Abstract. A complex number $z$ is in the final set of an analytic function $f$, as defined by Pólya, if every neighborhood of $z$ contains zeros of infinitely many $f^{(n)}$. If $f$ is a Hadamard gap series, then the part of the final set in the open disk of convergence is the origin along with a union of concentric circles.

1. Introduction

A complex number $z$ is in the final set $\Lambda(f)$ of an analytic function $f$ if every neighborhood of $z$ contains zeros of infinitely many $f^{(n)}$. Final sets of various functions have been determined by Pólya [4, 5] (who introduced the notion) and others (see [2] for references). A power series

$$f(z) = \sum_{k=0}^{\infty} c_k z^k,$$

with $c_k \neq 0$ for all $k$, has Hadamard gaps if there exists $L > 1$ such that

$$N_{k+1}/N_k > L \quad \text{for all } k \geq 0.$$

Theorem 1. Let $f$ be a function whose Maclaurin series has Hadamard gaps and (finite or infinite) radius of convergence $R$. Then $\Lambda(f) \cap \{|z| < R\} = \{0\} \cup \{|z| : |z| \in E\}$, where $E$ is closed in the topology of $(0, R)$.

Theorem 1 is best possible in the following sense.

Theorem 2. Let $R$ be in $(0, \infty]$, and let $E$ be closed in the topology of $(0, R)$. Then there exists a Hadamard gap series $f$ with radius of convergence $R$ such that $\Lambda(f) \cap \{|z| < R\} = \{0\} \cup \{|z| : z \in E\}$.

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2. Proof of Theorem 1

The proof of Theorem 1 depends on two lemmas, which I will prove in §§3 and 4, respectively, concerning functions $h$ of the form

$$h(z) = \sum_{k=0}^{\infty} a_k z^k.$$
Fix such an \( h \) and denote by \( R \) the radius of convergence of the series. Set
\[
\mu(r) = \max\{|a_k|r^{n_k} : k \geq 0\}, \quad \nu(r) = \max\{k : |a_k|r^{n_k} = \mu(r)\}.
\]
(This notation is not standard; see [6, p. 3].) Finally, call a number \( r \) in \((0, R)\) \( h \)-dominant if
\[
\sum_{k=0}^{\nu(r)-1} |a_k|r^{n_k} + \sum_{k=\nu(r)+1}^{\infty} n_k^{\nu(r)}|a_k|r^{n_k} < \mu(r),
\]
where the first sum is taken to be zero if \( \nu(r) = 0 \).

The first lemma is an adaptation of [3, Theorem 6, p. 605]. Denote by \( Z(s, t, \theta_1, \theta_2) \) the number of zeros (counting multiplicity) of \( h \) in the set \( \{re^{i\theta} : s \leq r \leq t \text{ and } \theta_1 \leq \theta \leq \theta_2\} \).

**Lemma 1.** Let \( h \) have the form (2.1) (not necessarily with Hadamard gaps), and let \( R, \mu(r), \) and \( \nu(r) \) be as above. If \( s \) and \( t \) are \( h \)-dominant, if \( s < t \), and if \( 0 < \theta_2 - \theta_1 < 2\pi \), then
\[
\frac{1}{2\pi} \left| Z(s, t, \theta_1, \theta_2) - \left(n_{\nu(t)} - n_{\nu(s)}\right) \frac{\theta_2 - \theta_1}{2\pi}\right| < \nu(t) + 2.
\]

**Lemma 2.** Let \( h \) and \( R \) be as in Lemma 1, and suppose that there exists \( L > 1 \) such that \( n_{k+1}/n_k > L \) for all \( k \geq 0 \). Suppose also that
\[
n_0 \geq \max\{9, \exp[\sqrt{(\log 6) \log L}]\}.
\]
Define
\[
\tau = 54e^{-2}/(\log L)(1 - 1/L)(1 - L^{-1/3}).
\]
Then there is at least one \( h \)-dominant point in each interval \((C, D) \subset (0, R)\) such that
\[
\log(D/C) > \tau/n_0^{1/3}.
\]

**Proof of Theorem 1.** \( 0 \in \Lambda(f) \) by (1.2).

Define \( h_j, a_k = a_k(j), \) and \( n_k = n_k(j) \) by
\[
h_j(z) = z^j f^{(j)}(z) = \sum_{k=0}^{\infty} a_k z^{n_k}.
\]

Then by (1.2),
\[
\begin{align*}
(a) & \quad n_{k+1}/n_k > L > 1, \quad (b) \quad n_0 \geq j.
\end{align*}
\]

Define a set \( E \subset (0, R) \) as follows: \( r^* \in E \) if there exist an infinite set \( T \) of positive integers and a sequence \( \{r_j\}_{j \in T} \) such that
\[
\begin{align*}
(a) & \quad \lim_{j \to \infty, j \in T} r_j = r^*, \quad (b) \quad \text{no } r_j \text{ is } h_j \text{-dominant}.
\end{align*}
\]

I will show that if \( r^* \in E \) then \( \{|z| = r^*\} \subset \Lambda(f) \), whereas if \( r^* \notin E \) then \( \{|z| = r^*\} \cap \Lambda(f) \) is empty.

**Case I.** \( r^* \in E \). Choose \( \{r_j\} \) as above and define \( \tau \) by (2.5). By (2.8b) and (2.9a), \( r_j \exp\{2\tau/n_0^{1/3}\} < R \) for all large \( j \) in \( T \). Pick such a \( j > \max\{9, \exp[\sqrt{(\log 6) \log L}]\} \). By Lemma 2, there are \( h_j \)-dominant points \( s = s(j) \) in \((r_j \exp\{-2\tau/n_0^{1/3}\}, r_j)\) and \( t = t(j) \) in \((r_j, r_j \exp\{2\tau/n_0^{1/3}\})\).
Then
\[ (2.10) \quad \nu(s, h_j) < \nu(t, h_j). \]

For suppose that \( \nu(s) = \nu(t) \equiv p \), and set
\[
\psi(r) = \frac{1}{|a_p|r^{n_p}} \left( \sum_{k=0}^{p-1} |a_k|^n_k + \sum_{k=p+1}^{\infty} n_k^p |a_k|r^{n_k} \right).
\]

Then \( \psi(s) < 1 \) and \( \psi(t) < 1 \) by (2.3). Hence \( \psi(r_j) < 1 \) since \( \psi \) is convex [7, p. 172]. Thus \( r_j \) is \( h_j \)-dominant, contrary to the definition of \( r_j \). This proves (2.10).

Put
\[
U_j(\theta_1, \theta_2) = \{ re^{i\theta} : r_j \exp(-2\pi/n_0^{1/3}) \leq r \leq r_j \exp(2\pi/n_0^{1/3}) \text{ and } \theta_1 \leq \theta \leq \theta_2 \}.
\]

I will show that, if \( j \) is sufficiently large, then
\[ (2.11) \quad h_j \text{ has at least one zero in } U_j(\theta_1, \theta_2) \text{ whenever } \theta_2 - \theta_1 > 6\pi L^{-j}/(1 - L^{-1}). \]

For \( xL^{-x} \downarrow \) for large \( x \), so that, when \( j \leq k \), (2.8) and (2.10) give \( k/n_k < kL^{-k}/j \leq L^{-j} \) and \( (n_\nu(t) - n_\nu(s))/n_\nu(t) > 1 - L^{-[\nu(t) - \nu(s)]} \geq 1 - L^{-1} \); thus, by Lemma 1, the number of zeros in \( U_j(\theta_1, \theta_2) \) is at least
\[
Z(s, t, \theta_1, \theta_2) \geq (n_\nu(t) - n_\nu(s)) \frac{\theta_2 - \theta_1}{2\pi} \frac{\theta_2 - \theta_1}{2\pi} > 0,
\]
which establishes (2.11).

Now by (2.9a) and (2.8b), \( r_j \exp(-2\pi/n_0^{1/3}) \to r^* \) and \( r_j \exp(2\pi/n_0^{1/3}) \to r^* \) as \( j \to \infty \) in \( T \). Thus (2.11) implies that every point of \( \{ |z| = r^* \} \) is a limit point of zeros of \( \{ h_j \}_{j \in T} \), so that, by (2.7), \( \{ |z| = r^* \} \subseteq \Lambda(f) \).

Case II. \( r^* \notin E \). For all large \( j \) and small \( \varepsilon \), \( r \) is \( h_j \)-dominant for \( r \) in \( I \equiv (r^* - \varepsilon, r^* + \varepsilon) \). So by (2.3),
\[
|h_j(z)| \geq \mu(r, h_j) - \sum_{k=0}^{\nu(r, h_j)-1} |a_k|r^{n_k} - \sum_{k=\nu(r, h_j)+1}^{\infty} |a_k|r^{n_k} > 0,
\]
whenever \( |z| = r \in I \). This completes the proof of Theorem 1.

3. Proof of Lemma 1

We need two more lemmas. The first is a variation on [6, Problem 66, p. 45]; the second is an adaptation of [3, Lemma 7]. Let \( D \) denote differentiation.

**Lemma 3.** Let \( J \equiv (a, b) \subset \mathbb{R}^+ \), let \( g : J \to \mathbb{C} \) be differentiable, and let \( \alpha \in \mathbb{Z}^+ \). If \( \text{Im}\{g\} \) changes sign at least twice in \( J \), then \( \text{Im}\{(rD - \alpha)g\} \) changes sign there at least once.

**Proof.** For real \( r \),
\[
\text{Im}\{(rD - \alpha)g(r)\} = \text{Im}\left\{ r^{\alpha+1} \frac{d}{dr}[r^{-\alpha}g(r)] \right\} = r^{\alpha+1} \frac{d}{dr}[r^{-\alpha} \text{Im}\{g(r)\}],
\]
and the lemma follows from Rolle's Theorem.
For a function $H$ analytic on a contour $C$, denote by $\Delta(H, C)$ the variation over $C$ of any continuous branch of $\arg H$.

**Lemma 4.** Let $h$ have the form (2.1) with radius of convergence $R$, let $[s, t] \subset (0, R)$, and suppose that $t$ is $h$-dominant. Set $I = I(\theta) = \{re^{i\theta} : s \leq r \leq t\}$. If $h \neq 0$ on $I$, then $|\Delta(h, I)| \leq \pi [\nu(t) + 1]$.

**Proof.** We may assume that $\theta = 0$. Set $q = \nu(t)$. Choose $\phi$ so that $e^{i\phi}a_q$ is positive imaginary, and put

$$H(r) = e^{i\phi}(rD - n_0)(rD - n_1)\cdots(rD - n_{q-1})h(r) \equiv \sum_{k=n_q}^{\infty} b_k r^{n_k},$$

where

$$b_k = (n_k - n_0)(n_k - n_1)\cdots(n_k - n_{q-1})e^{i\phi}a_k.$$  

I claim that $\text{Im}\{H\}$ does not change sign in $(s, t)$. If the claim is correct, then $q$ applications of Lemma 3 show that $\text{Im}\{h\}$ changes sign at most $q$ times in $(s, t)$, so that the curve $h(I)$ crosses the real axis at most $q$ times. Therefore $|\Delta(h, I)| \leq \pi (q + 1)$, and the lemma follows.

To prove the claim, pick $r$ in $(s, t)$ and set

$$\psi(r) = \frac{1}{|b_q|r^{n_q}} \left( \sum_{k=n_{q+1}}^{\infty} |b_k|r^{n_k} \right).$$

We have $|b_k| \leq n^q_k|a_k|$ and $|b_q| \geq |a_q|$ by (3.2). Thus, since $t$ is $h$-dominant and $q = \nu(t)$, (2.3) gives

$$\psi(t) \leq \frac{1}{|a_q|r^{n_q}} \left( \sum_{k=n_{q+1}}^{\infty} n^q_k|a_k|t^{n_k} \right) = \frac{1}{\mu(t)} \left( \sum_{k=n_{\nu(t)+1}}^{\infty} n^{\nu(t)}_k|a_k|t^{n_k} \right) < 1.$$ 

Now $\psi$ increases, so, by (3.1),

$$|H(r) - b_q r^{n_q}| \leq \sum_{k=q+1}^{\infty} |b_k|r^{n_k} = \psi(r)|b_q|r^{n_q} < |b_q|r^{n_q}.$$ 

But our choice of $\phi$ makes $b_q r^{n_q}$ positive imaginary, so that $H(r)$ is in the upper half-plane. This establishes the claim and Lemma 4.

**Proof of Lemma 1.** Let $\Gamma = I_1 \cup C_1 \cup I_2 \cup C_s$, where $I_1 = \{re^{i\theta_1} : s \leq r \leq t\}$, $I_2 = \{re^{i\theta_2} : s \leq r \leq t\}$, $C_s = \{se^{i\theta_2} : \theta_1 \leq \theta \leq \theta_2\}$, and $C_t = \{te^{i\theta_1} : \theta_1 \leq \theta \leq \theta_2\}$. Also put $P(z) = a_{\nu(s)}z^{\nu(s)}$ and $Q(z) = a_{\nu(t)}z^{\nu(t)}$.

First assume that $h \neq 0$ on $\Gamma$. Then

$$\Delta(h, \Gamma) - \Delta(P, C_s) - \Delta(Q, C_t) = \Delta(h/P, C_s) + \Delta(h/Q, C_t) + \Delta(h, I_1) + \Delta(h, I_2).$$

Also, (2.3) gives $|h(z)/P(z) - 1| < 1$, and hence $\text{Re}\{h(z)/P(z)\} > 0$, for $z \in C_s$. So $|\Delta(h/P, C_s)| \leq \pi$. Similarly, $|\Delta(h/Q, C_t)| \leq \pi$. Thus, by (3.3) and Lemma 4,

$$|\Delta(h, \Gamma) - (n_{\nu(t)} - n_{\nu(s)})(\theta_2 - \theta_1)| \leq 2\pi + 2\pi [\nu(t) + 1].$$
and Lemma 1 follows from the argument principle. If $h$ has zeros on $\Gamma$, apply (3.4) to a nearby contour $\Gamma'$ on which $h \neq 0$ and let $\Gamma' \to \Gamma$.

4. Proof of Lemma 2

**Lemma 5.** Let $h$, $R$, and $L$ be as in Lemma 2, and let (2.4) hold. Pick $m \geq 0$ and $[A, B] \subset (0, R)$, and suppose that

\[ |a_k|s^{nk} \leq |a_m|s^{nm} \quad \text{for all } k \geq 0 \text{ and } s \in [A, B] \]

and

\[ \log \frac{B}{A} > \frac{6}{(\log L)(1 - 1/L)} \frac{(\log n_m)^2}{n_m} \]

Then there exists $r \in (A, B)$ such that (2.3) holds with $\nu(r) = m$.

**Proof of Lemma 2.** For each $m \geq 0$, set $I_m = \{r \geq 0 : |a_m|s^{nm} = \mu(r, h)\}$. Denote by $(A, B)$ the interior of $I_m \cap (C, D)$. If $(A, B)$ has no $h$-dominant points, then (4.2) must fail. Therefore, since $\bigcup I_m = \mathbb{R}^+$,

\[ \log \frac{C}{D} = \int_{C}^{D} \frac{dx}{x} = \sum_{m=0}^{\infty} \int_{I_m \cap (C, D)} \frac{dx}{x} \]

\[ \leq \frac{6}{(\log L)(1 - 1/L)} \sum_{m=0}^{\infty} \frac{(\log n_m)^2}{n_m} \]

Now $(\log x)x^{1/3} \leq 3/e$ for $x > 0$; also, $n_m^{1/3} > (Lm n_0)^{1/3}$ by (2.8a). Thus $(\log n_m)^2/n_m \leq 9e^{-2}n_m^{2/3}/n_m < 9e^{-2}(Lm n_0)^{-1/3}$. So by (4.3) and (2.5),

\[ \log \frac{C}{D} \leq \frac{1}{n_0^{1/3}} \frac{6}{(\log L)(1 - 1/L)} \frac{9}{e^2} \sum_{m=0}^{\infty} L^{-m/3} = \frac{\tau}{n_0^{1/3}}. \]

But this contradicts (2.6), and the proof is complete.

**Proof of Lemma 5.** Set

\[ \sigma = \exp\{(\log n_m)^2/(n_m \log L)\} \]

Then

\[ \sigma > 1, \]

\[ n_k^m \leq \sigma^n \quad \text{for all } k \geq m, \]

\[ 2\sigma^n \frac{\sigma(A/B)^{1/2}}{\nu}^{(1 - 1/L)n_m} < \frac{1}{3}, \quad \text{and} \quad \sigma(A/B)^{1/2} < 1. \]

**Proof of (4.6).** By (2.8a) and (2.4), $k < (\log n_k)/(\log L)$. Also, $(\log x)^2/x$ decreases for $x > e^2$. Hence, by (4.4),

\[ m \log n_k \leq k \log n_k \leq \frac{(\log n_k)^2}{\log L} = \frac{(\log n_k)^2}{n_k} \frac{n_k}{\log L} \]

\[ \leq \frac{(\log n_m)^2}{n_m} \frac{n_k}{\log L} = n_k \log \sigma. \]
Proof of (4.7). By (4.2) and (4.4), \((A/B)^{(1-1/L)n_m/2} < \sigma^{-3n_m}\). By (2.4), \(\sigma^{n_m} \geq \sigma^{n_0} \geq 6\). Therefore

\[
\{\sigma(A/B)^{1/2}\}^{(1-1/L)n_m} < \sigma^{(1-1/L)n_m-3n_m} < \sigma^{-n_m}\sigma^{-n_m} \leq \sigma^{-n_m/6}.
\]

This yields (4.7a), and (b) follows from (a) and (4.5).

We are now ready to prove (2.3) with \(\nu(r) = m\) and

\[
r = (AB)^{1/2}.
\]

When \(k \geq m + 1\), (2.8a) implies that

\[
n_k - n_m = \sum_{\gamma=m+1}^{k} (n_{\gamma} - n_{\gamma-1}) \geq \sum_{\gamma=m+1}^{k} (L-1)n_{\gamma-1} \geq (L-1)n_m(k-m).
\]

By (4.6), (4.1) with \(s = B\), and (4.8), and by (4.9), (which we may apply because of (4.7b)),

\[
n_k^m|a_k|r^{n_k} \leq |a_m|B^{n_m}\sigma(A/B)^{1/2}\chi_{n_m} \\
\leq |a_m|r^{n_m}\sigma^{n_m}\chi_{n_m}(A/B)^{1/2}\chi_{k-m}.
\]

Next, if \(0 \leq k < m - 1\), then (2.8a) gives

\[
n_k \leq n_m - n_{m-1} \leq n_m - (1 - 1/L)n_m.
\]

Thus, by (4.1) with \(s = A\) and (4.8),

\[
|a_k|r^{n_k} \leq |a_m|A^{n_m}\sigma(A/B)^{1/2}\chi_{n_m} \leq |a_m|r^{n_m}(A/B)^{1/2}(1-1/L)n_m.
\]

We have \(m \leq n_m \leq \sigma^{n_m}\) from (2.4) and (4.6). Also, \(L-1 > 1-1/L\) by (2.8a). Thus (4.11), (4.10), (4.5), and (4.7a) give

\[
\sum_{k=0}^{m-1} |a_k|r^{n_k} + \sum_{k=m+1}^{\infty} n_k^m|a_k|r^{n_k} \\
< |a_m|r^{n_m}\left\{m \left[\frac{A}{B}\right]^{1/2}\chi_{n_m} + \sigma^{n_m} \chi_{n_m} \left[\frac{\sigma(A/B)^{1/2}(1-1/L)n_m}{1 - \sigma(A/B)^{1/2}(1-1/L)n_m}\right]\right\} \\
\leq |a_m|r^{n_m}\left\{\frac{2\sigma^{n_m}}{\sigma(A/B)^{1/2}(1-1/L)n_m} \frac{1/3}{1 - 1/6} < |a_m|r^{n_m}.
\]

This yields (2.3) and completes the proof of Lemma 5.

5. PROOF OF THEOREM 2

The following construction is similar to that in [1].

**Proof of Theorem 2.** Pick positive sequences \(\{r_P\}_{P=0}^{\infty}, \{R_P\}_{P=0}^{\infty}, \) and \(\{\varepsilon_P\}_{P=0}^{\infty}\) so that

\[
\begin{align*}
&\text{(a) the set of limit points of } \{r_P\} \text{ in } (0, R) \text{ is } E, \\
&\text{(b) } r_pe^{\varepsilon_p} < R_P < R, \\
&\text{(c) } \varepsilon_P \to 0, \\
&\text{(d) } R_P \to R.
\end{align*}
\]

Choose a function \(\psi : \mathbb{Z}^+ \to \mathbb{R}^+\) such that

\[
[\psi(x)]^{1/x} \downarrow 1/R \text{ as } x \to \infty.
\]

Define \(f\) by (1.1), where \(\{c_k\}\) and \(\{N_k\}\) are defined inductively as follows.

Set

\[
\begin{align*}
&\text{(a) } N_0 = 3, \quad \text{(b) } c_{-1} = 1.
\end{align*}
\]
Having chosen $c_{-1}, \ldots, c_{2P-1} > 0$ and $N_0, \ldots, N_{2P}$, pick $c_{2P}, c_{2P+1}, N_{2P+1}, \text{ and } N_{2P+2}$ as follows. Using (5.2) and (5.1b), pick $N_{2P+1}$ large enough so that
\begin{equation}
N_{2P+1} / N_{2P} > 2, \quad (b) \quad N_{2P+1}^{-N_{2P}} > (e^{-\varepsilon P/2})^{N_{2P+1} - N_{2P}},
\end{equation}
and
\begin{equation}
\begin{aligned}
N_{2P+1} \log \left( \frac{y_{2P+1}}{1+y_{2P+1}} \right) &> (e^{-\varepsilon})^{N_{2P+1} - N_{2P}}, \\
\left( \frac{y_{2P+1}}{1+y_{2P+1}} \right)^{N_{2P+1} - N_{2P}} &< c_{2P+1}^{1/N_{2P}}.
\end{aligned}
\end{equation}

Set
\begin{equation}
\begin{aligned}
(a) & \quad c_{2P+1} = \psi(N_{2P+1}), \\
(b) & \quad c_{2P} = c_{2P+1} r_{N_{2P+1} - N_{2P}}.
\end{aligned}
\end{equation}

Finally, pick $N_{2P+2}$ large enough so that
\begin{equation}
N_{2P+2} / N_{2P+1} > 2, \quad (b) \quad 2N_{2P+1}^{2} \frac{\log N_{2P+2}}{N_{2P+3}} < 3.
\end{equation}

The function $f$ just constructed satisfies
\begin{equation}
l_2P \psi(N_{2P+1}) \geq l_2P \psi(N_{2P+1}) \quad \text{for all } P \geq 0 \text{ and } k \geq 2P + 1.
\end{equation}

For odd $k$, (5.9) follows from (5.7a) and (5.2). If $k = 2Q$ is even, (5.9) follows from (5.7) and (5.6) (both with $P$ replaced by $Q$):
\begin{equation}
c_{2Q}^{1/N_{2Q}} = (c_{2Q+1}^{1/N_{2Q+1}} r_{Q})^{N_{2Q+1} - N_{2Q}} < c_{2Q-1}^{1/N_{2Q-1}} \leq c_{2P}^{1/N_{2P}}.
\end{equation}

$f$ has radius of convergence $R$ by (5.7a), (5.2), and (5.9). By (1.1) and (5.8a),
\begin{equation}
0 \in \Lambda(f).
\end{equation}

We need the following lemma, proved at the end of the paper, to show that $\Lambda(f) \cap \{0 < |z| < R\} = \{z: |z| \in E\}$.

**Lemma 6.** Let $f$ be as above, and define $\phi$ by
\begin{equation}
\phi_j(z) = \sum_{N_k \geq j} c_{N_k} N_{N_k} (N_k - 1) \cdots (N_k - j + 1) z^{N_k - j} \phi_{jN_k}(z).
\end{equation}

If $P \geq 0$, and if
\begin{equation}
\begin{aligned}
(a) & \quad j \leq N_{2P+1} \quad \text{and} \quad (b) \quad |z| \leq R_P,
\end{aligned}
\end{equation}
then
\begin{equation}
\sum_{k=2P+2}^{\infty} |\phi_{jN_k}(z)| \leq |\phi_j, 2P+1(z)| / 2.
\end{equation}

**Proof that** $\Lambda(f) \cap \{0 < |z| < R\} \subset \{z: |z| \in E\}$. By (5.1), it is enough to show that $f^{(j)}(z) \neq 0$ whenever either
\begin{equation}
j \in (N_{2P}, N_{2P+1}] \quad \text{and} \quad 0 < |z| \leq R_P
\end{equation}
or
\begin{equation}
j \in (N_{2P-1}, N_{2P}] \quad \text{and} \quad z \in \{0 < |z| \leq r_P e^{-\varepsilon P} \} \cup \{r_P e^{\varepsilon} \leq |z| \leq R_P\}.
\end{equation}

But if (5.13) holds, then (5.10) and (5.12) give
\begin{equation}
|f^{(j)}(z)| \geq |\phi_j, 2P+1(z)| - \sum_{k=2P+2}^{\infty} |\phi_{jN_k}(z)| \geq |\phi_j, 2P+1(z)| / 2 > 0.
\end{equation}
If \( (5.14) \) holds, define

\[
G(j, P) = \frac{N_{2P}(N_{2P} - 1) \cdots (N_{2P} - j + 1)}{N_{2P+1}(N_{2P+1} - 1) \cdots (N_{2P+1} - j + 1)}.
\]

Then \( G(j, P) > 1/N_{2P+1}^j > N_{2P+1}^{-N_{2P}} \), so that, by \( (5.4b) \),

\[
(5.16) \quad (e^{-\varepsilon_P/2})^{N_{2P+1} - N_{2P}} < G(j, P) < 1.
\]

Also, by \( (5.10), (5.15) \), and \( (5.7b) \),

\[
(5.17) \quad \left| \frac{\phi_j,2P(z)}{\phi_j,2P+1(z)} \right| = \frac{C_{2P}}{C_{2P+1}} G(j, P) |z|^{N_{2P} - N_{2P+1}} = G(j, P) \left( \frac{r_P}{|z|} \right)^{N_{2P+1} - N_{2P}}.
\]

If \( r_P e^{\varepsilon_P} \leq |z| \leq r_P \), then \( |\phi_j,2P(z)/\phi_j,2P+1(z)| < (e^{-\varepsilon_P})^{N_{2P+1} - N_{2P}} \leq 3^{-6} \) by \( (5.17), (5.16), (5.4b) \), and \( (5.3) \). Hence, by \( (5.10) \) and \( (5.12) \),

\[
|f^{(j)}(z)| \geq \phi_j,2P+1(z) - \sum_{k=2P+2}^{\infty} |\phi_j,k(z)| - |\phi_j,2P(z)|
\]

\[
\geq |\phi_j,2P+1(z)| - |\phi_j,2P+1(z)|/2 - |\phi_j,2P+1(z)|/3^6 > 0.
\]

Similarly, if \( 0 < |z| < r_P e^{-\varepsilon_P} \), then

\[
|\phi_j,2P(z)/\phi_j,2P+1(z)| > (e^{\varepsilon_P/2})^{N_{2P+1} - N_{2P}} \geq 3^3
\]

and

\[
|f^{(j)}(z)| \geq |\phi_j,2P(z)| - \sum_{k=2P+2}^{\infty} |\phi_j,k(z)| - |\phi_j,2P(z)|
\]

\[
\geq |\phi_j,2P(z)| - \left( \frac{3}{2} \frac{1}{3^3} \right) |\phi_j,2P+1(z)| > 0.
\]

Proof that \( \{ z : |z| \in E \} \subset \Lambda(f) \). Fix \( P \) and set

\[
(5.18) \quad r = [G(2P, P)]^{1/(N_{2P+1} - N_{2P})} r_P.
\]

Then, by \( (5.16) \),

\[
(5.19) \quad r \in (r_P e^{-\varepsilon_P}, r_P).
\]

Set \( h_j(z) = z^j f^{(j)}(z) \). For \( |z| = r \), we have

\[
|\phi_{2P,2P}(z)| = |\phi_{2P,2P+1}(z)| > \sum_{k=2P+2}^{\infty} |\phi_{2P,k}(z)|
\]

by \( (5.18), (5.17) \), and \( (5.12) \). Thus, by \( (5.10) \) and \( (2.2) \),

\[
\mu(r, h_2) = |z^{2P} \phi_{2P,2P}(z)| = |z^{2P} \phi_{2P,2P+1}(z)|.
\]

Therefore \( r \) violates the definition \( (2.3) \) of \( h_{2P} \)-dominance. It now follows from \( (5.19) \) and \( (5.1a) \) that \( E \) is in the set of limit points of the points which are not \( h_{2P} \)-dominant. Thus \( \{ z : |z| \in E \} \subset \Lambda(f) \) by the paragraph containing \( (2.9) \). This completes the proof of Theorem 2.

Proof of Lemma 6. Pick \( k \geq 2P + 2 \). By \( (5.9) \) and \( (5.7a) \),

\[
(5.20) \quad \log \frac{c_k}{C_{2P+1}} = \log c_k N_k - \frac{\log C_{2P+1}}{N_{2P+1}} N_{2P+1} \leq \frac{\log \psi(N_{2P+1})}{N_{2P+1}}.
\]
Also, \((\log x)/x\) decreases for \(x > N_{2P+1}\) by (5.3a). Thus, by (5.8),
\[
\frac{\log N_k}{N_k - N_{2P+1}} = \frac{N_{2P+1}}{1 - N_{2P+1}/N_k} \frac{\log N_k}{N_k} < \frac{N_{2P+1}}{2} \frac{\log N_{2P+2}}{N_{2P+2}} \leq \frac{\log 3}{N_{2P+1}}.
\]
(5.21)

By (5.10), (5.11), (5.20), (5.21), and (5.5),
\[
\log \left| \frac{\phi_{jk}(z)}{\phi_{j,2P+1}(z)} \right| \leq \log \frac{c_k}{c_{2P+1}} + N_{2P+1} \log N_k + (N_k - N_{2P+1}) \log R_P
\]
\[
\leq (N_k - N_{2P+1}) \left[ \log(\{\psi(N_{2P+1})\}^{1/N_{2P+1}}R_P) + \log 3 \right] = \left( N_k - N_{2P+1} \right) \left( -\frac{\log 3}{N_{2P+1}} \right).
\]
(5.22)

But \((N_k - N_{2P+1})/N_{2P+1} \geq k - 2P - 1\) by (5.4a), (5.8a), and (4.9) (with \(L = 2\), \(N_k\) in place of \(n_k\), and \(m = 2P + 1\)). Thus (5.22) gives
\[
\sum_{k=2P+2}^{\infty} \left| \frac{\phi_{jk}(z)}{\phi_{j,2P+1}(z)} \right| \leq \sum_{k=2P+2}^{\infty} e^{-(\log 3)(k-2P-1)} = \frac{1/3}{1 - 1/3} = 1/2.
\]

References


