THE GORENSTEINNESS OF THE SYMBOLIC BLOW-UPS FOR CERTAIN SPACE MONOMIAL CURVES

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Abstract. Let \( p = p(n_1, n_2, n_3) \) denote the prime ideal in the formal power series ring \( A = k[[X, Y, Z]] \) over a field \( k \) defining the space monomial curve \( X = T^{n_1}, Y = T^{n_2}, \) and \( Z = T^{n_3} \) with \( \gcd(n_1, n_2, n_3) = 1. \) Then the symbolic Rees algebras \( R_s(p) = \bigoplus_{n \geq 0} p^n \) are Gorenstein rings for the prime ideals \( p = p(n_1, n_2, n_3) \) with \( \min\{n_1, n_2, n_3\} = 4 \) and \( p = p(m, m + 1, m + 4) \) with \( m \neq 9, 13. \) The rings \( R_s(p) \) for \( p = p(9, 10, 13) \) and \( p = p(13, 14, 17) \) are Noetherian but non-Cohen-Macaulay, if \( \text{ch} k = 3. \)

1. Introduction

Let \( k \) be a field and let \( A = k[[X, Y, Z]] \) and \( S = k[[T]] \) be formal power series rings over \( k. \) Let \( p = p(n_1, n_2, n_3) \) denote, for positive integers \( n_1, n_2 \) and \( n_3 \) with \( \gcd(n_1, n_2, n_3) = 1, \) the kernel of the homomorphism \( f: A \to S \) of \( k \)-algebras defined by \( f(X) = T^{n_1}, f(Y) = T^{n_2}, \) and \( f(Z) = T^{n_3}. \) We put \( R_s(p) = \sum_{n \geq 0} p^n t^n \) (here \( t \) denotes an indeterminate over \( A \)) and call it the symbolic Rees algebra of \( p. \)

In the previous paper \([1]\) the authors studied the problem when \( R_s(p) \) is a Gorenstein ring and gave a criterion for the case in terms of the elements \( f \) and \( g \) of \( p \) in Huneke's condition \([6]\) for \( R_s(p) \) to be Noetherian. With the criterion the authors proved that \( R_s(p) \) are always Gorenstein for the prime ideals \( p = p(m, m + 1, m + 3) \) with \( m \geq 1 \) and \( p = p(n_1, n_2, n_3) \) with \( \min\{n_1, n_2, n_3\} = 3. \)

To be the next targets we would like to choose the prime ideals \( p = p(m, m + 1, m + 4) \) with \( m \geq 1 \) and \( p = p(n_1, n_2, n_3) \) with \( \min\{n_1, n_2, n_3\} = 4, \) and our conclusion for these ideals can be summarized into the following two theorems.

**Theorem (1.1).** \( R_s(p) \) is a Gorenstein ring for \( p = p(m, m + 1, m + 4), \) if \( m \neq 9, 13. \)

**Theorem (1.2).** \( R_s(p) \) is a Gorenstein ring for \( p = p(n_1, n_2, n_3), \) if \( \min\{n_1, n_2, n_3\} = 4. \)

In Theorem (1.2) the fact that \( R_s(p) \) is Noetherian is due to \([6].\) Our contribution is its Gorensteinness. For \( m = 9, 13 \) in Theorem (1.1) the rings \( R_s(p) \) are Noetherian but not Cohen-Macaulay, if \( \text{ch} k = 3 \) (cf. \([7]\) and \((3.4).)\)

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Theorem (1.1) (resp. Theorem (1.2)) shall be proved in §3 (resp. §4). Section 2 is devoted to some preliminary steps. In his remarkable paper [6] Huneke gave a criterion for $R_s(p)$ to be Noetherian, by which he guaranteed the Noetherian property of $R_s(p)$ for $p = p(n_1, n_2, n_3)$ with $\min\{n_1, n_2, n_3\} = 4$. To prove Theorem (1.2) we need his arguments as well as his results (that we will briefly summarize in §4). However the key is the criterion given by the authors [1] for $R_s(p)$ to be a Gorenstein ring, which we will recall in §2 for the sake of completeness.

Throughout this paper let $(A, m)$ be a regular local ring of $\dim A = 3$ and $p$ a prime ideal in $A$ with $\dim A/p = 1$. For each finitely generated $A$-module $M$ let $l_A(M)$ and $\mu_A(M)$ respectively denote the length of $M$ and the number of elements in a minimal system of generators for $M$.

2. Preliminaries

First of all let us recall Huneke’s criterion.

**Proposition (2.1)** [6]. If there exist $f \in p^{(k)}$ and $g \in p^{(l)}$ with positive integers $k, l$ such that $l_A(A/(f, g, x)A) = kl \cdot l_A(A/p + xA)$ for some $x \in m \setminus p$, then $R_s(p)$ is Noetherian. When the field $A/m$ is infinite, the converse is also true.

The criterion given by the authors for $R_s(p)$ to be a Gorenstein ring is based on (2.1) and is stated as follows.

**Theorem (2.2)** [1]. Let $f$ and $g$ be as in (2.1). Then the following two conditions are equivalent.

1. $R_s(p)$ is a Gorenstein ring.
2. $A/(f, g) + p(n)$ is a Cohen-Macaulay ring for any $1 \leq n \leq k + l - 2$.

When this is the case, the $A$-algebra $R_s(p)$ is generated by $\{p^{(n)}t^n\}_{1 \leq n \leq k + l - 2}$, $ft^k$ and $gt^l$, and the rings $A/(f) + p(n)$, $A/(g) + p(n)$ and $A/(f, g) + p(n)$ are Cohen-Macaulay for all $n \geq 1$.

Here let us note the following lemma that we will use to calculate the length of certain modules.

**Lemma (2.3)** [1]. Let $R$ be a two-dimensional Cohen-Macaulay local ring and let $x, y$ be a system of parameters of $R$. For given sequences $p_0 = 0 < p_1 \leq p_2 \leq \cdots \leq p_n$ and $q_0 \geq q_1 \geq \cdots \geq q_{n-1} > q_n = 0$ of integers, let

$$I = (x^{p_i}, y^{q_i})|0 \leq i \leq n)R.$$  

Then

$$l_R(R/I) = l_R(R/(x, y)) \cdot \sum_{i=1}^{n} q_{i-1} (p_i - p_{i-1}).$$

**Proof.** We may assume that $n \geq 2$ and that our assertion is true for $n - 1$. Then considering the sequences $p_i' = p_i$ $(0 \leq i \leq n - 1)$, $q_i' = q_i$ $(0 \leq i \leq n - 2)$ and $q_{n-1}' = 0$, we get by the hypothesis on $n$ that

$$l_R(R/I') = l_R(R/(x, y)) \cdot \sum_{i=1}^{n-1} q_{i-1} (p_i - p_{i-1}),$$

The formulation of this lemma is due to the referee. The authors are grateful to the referee for his suggestion.
where \( I' = (x^{p_i}y^{q_i}) | 0 \leq i \leq n-1) R \). Since \( I' = I + (x^{p_n-1}) \) and \( I: x^{p_n-1} = (x^{p_n-p_{n+1}}, y^{q_n-1}) \), we have

\[
l_R(R/I) = l_R(R/I') + l_R(I + (x^{p_n-1})/I) = l_R(R/(x, y)) \cdot \sum_{i=1}^{n-1} q_{i-1}(p_i - p_{i-1}) + l_R(R/(x^{p_n-p_{n+1}}, y^{q_n-1})) = l_R(R/(x, y)) \cdot \sum_{i=1}^{n} q_{i-1}(p_i - p_{i-1})
\]
as required.

Now let us assume that our ideal \( p \) is generated by the maximal minors of the matrix

\[
M = \begin{bmatrix}
X^\alpha & Y^\beta & Z^\gamma \\
Y^\beta & Z^\gamma & X^\alpha
\end{bmatrix},
\]

where \( X, Y, Z \) is a regular system of parameters for \( A \) and \( \alpha, \beta, \gamma, \alpha', \beta', \gamma' \) are positive integers. Then after suitable permutations of the rows and columns of \( M \), we may assume that the matrix \( M \) is one of the following type.

(I) \( \alpha \leq \alpha', \beta \leq \beta' \) and \( \gamma \leq \gamma' \),

(II) \( \alpha' < \alpha, \beta < \beta' \) and \( \gamma < \gamma' \).

As was proved by Herzog and Ulrich [3], \( p \) is self-linked (resp. not self-linked) if and only if \( M \) has type (I) (resp. type (II)). And in any case it is already known that \( \mu_A(p^{(2)}/p^2) = 1 \) and \( p^{(n)} \neq p^n \) for all \( n \geq 2 \) (cf. [5]). However, later we will need so frequently the assertions for the prime ideals \( p \) whose matrices \( M \) have type (I) that we would like to give a brief proof for the case. (See [7] for the case of type (II).)

So assume that \( \alpha \leq \alpha', \beta \leq \beta' \) and \( \gamma \leq \gamma' \). Let \( a = Z^{\gamma+\gamma'} - X^\alpha Y^\beta, \ b = X^{\alpha+\alpha'} - Y^\beta Z^{\gamma'}, \ c = Y^\beta + Z^\gamma - X^\alpha \). Hence \( p = (a, b, c) \) and any pair of \( a, b \) and \( c \) forms a regular system of parameters for \( A_p \). We begin with the following

**Lemma (2.4).** \( \alpha < \alpha', \beta < \beta' \) or \( \gamma < \gamma' \).

**Proof.** Suppose that \( \alpha = \alpha', \beta = \beta' \) and \( \gamma = \gamma' \). Then since \( a - b = (X^\alpha + Y^\beta + Z^\gamma)(Z^\gamma - X^\alpha) \), we have \( X^\alpha + Y^\beta + Z^\gamma \in p \) or \( Z^\gamma - X^\alpha \in p \), while \( p \subseteq (X^\alpha, Y^\beta, Z^\gamma)^2 \). Hence \( Z^\gamma \in (X^\alpha, Y^\beta, Z^\gamma)^2 \), which is absurd.

**Proposition (2.5).** There exists \( d_2 \in p^{(2)} \) such that

\[
X^\alpha d_2 = acZ^{\gamma-\gamma} - b^2 Y^{\beta-\beta},
Y^\beta d_2 = ab - c^2 X^{\alpha-\alpha} Z^{\gamma-\gamma} \quad \text{and} \quad Z^\gamma d_2 = -a^2 + bcX^{\alpha-\alpha} Y^{\beta-\beta}.
\]

If \( \alpha < \alpha' \), then \( d_2 \equiv -Z^{\gamma+2\gamma} \mod(X) \).

**Proof.** Because \( X^\alpha a + Y^\beta b + Z^{\gamma} c = Y^\beta a + Z^{\gamma} b + X^{\alpha} c = 0 \), we see

\[
(X^\alpha a + Y^\beta b) \cdot b = -Z^\gamma bc = (Y^\beta a + X^\alpha c) \cdot cZ^{\gamma-\gamma}
\]

so that \( X^\alpha(ab - c^2 X^{\alpha-\alpha} Z^{\gamma-\gamma}) = Y^\beta(acZ^{\gamma-\gamma} - b^2 Y^{\beta-\beta}) \), whence \( X^\alpha d_2 = acZ^{\gamma-\gamma} - b^2 Y^{\beta-\beta} \) and \( Y^\beta d_2 = ab - c^2 X^{\alpha-\alpha} Z^{\gamma-\gamma} \) for some \( d_2 \in p^{(2)} \). Notice
that
\[(Z \gamma \ d_2) b = (Z \gamma b) d_2 = (-Y^\beta a - X^{a'} c) d_2
\]
\[= (Y^\beta d_2)(-a) + (X^{a'} d_2)(-cX^{a'-a})
\]
\[= -a^2 + bcX^{a'-a}Y^{\beta'-\beta} b\]
and we get \(Z \gamma \ d_2 = -a^2 + bcX^{a'-a}Y^{\beta'-\beta}\), too. If \(a < a'\), we have \(Y^\beta d_2 \equiv ab \equiv -Y^\beta Z^{\gamma+2\gamma'} \mod(X)\) so that \(d_2 \equiv -Z^{\gamma+2\gamma'} \mod(X)\).

**Corollary (2.6) [5].**

1. \(p^{(2)} = (d_2) + p^2\).
2. \(\mu_A(p^{(2)}) \leq 5\).
3. \(p^{(n)} \neq p^n\) if \(n \geq 2\).

**Proof.** By (2.4) we may assume that \(a < a'\). Then as \(d_2 \equiv -Z^{\gamma+2\gamma'} \mod(X)\) by (2.5) and as \((X) + p = (X) + (Z^{\gamma+\gamma'}, Y^\beta Z^{\gamma'}, Y^{\beta'+\beta'})\), we have

\[(*) \ (X, d_2) + p^2 = (X) + (Z^{\gamma+2\gamma'}, Y^{\beta+\beta'} Z^{\gamma'+\gamma'}, Y^{\beta'+\beta'} Z^{\gamma'}, Y^{2(\beta'+\beta')})\]
whence \(l_A(A/(X, d_2) + p^2) = 3(\beta' \gamma + \beta' \gamma' + \beta' \gamma')\) by (2.3). Let \(e_{XA}(A/p^{(2)})\) denote the multiplicity of \(A/p^{(2)}\) relative to the parameter \(X\). Then

\[l_A(A/(X) + p^{(2)}) = e_{XA}(A/p^{(2)})\]

since \(A/p^{(2)}\) is a Cohen-Macaulay ring, while we get by the associative formula [8, p. 126] of multiplicity that
\[e_{XA}(A/p^{(2)}) = l_A(A/p^2 A_p) \cdot e_{XA}(A/p) = 3 \cdot l_A(A/(X) + p)\]
\[= 3 \cdot l_A(A/(X) + (Z^{\gamma+\gamma'}, Y^\beta Z^{\gamma'}, Y^{\beta'+\beta'}))\]
\[= 3(\beta' \gamma + \beta' \gamma' + \beta' \gamma')\]
(cf. (2.3)). Hence \(l_A(A/(X, d_2) + p^2) = l_A(A/(X) + p^{(2)})\), which yields \((X) + p^{(2)} = (X, d_2) + p^2\) so that \(p^{(2)} = (d_2) + p^2 + Xp^{(2)}\). Thus Nakayama's lemma proves the assertion (1). Notice that \(\mu_A(p^{(2)}) = \mu_A((X) + p^{(2)}/(X)) \leq 5\) by the above equality (*) and we have the assertion (2). As \((X) + p^2 \subseteq (X, Y, Z^{2(\gamma+\gamma')})\) and as \(d_2 \equiv -Z^{\gamma+2\gamma'} \mod(X)\), we have \(d_2 \notin (X) + p^2\) so that \(d_2 \notin p^2\); hence \(p^{(2)} \neq p^2\). Let \(n \geq 3\) be an integer and assume that \(p^{(n)} = p^n\). Hence \(d_2^{(n)} \cdot (at)^{n-2} \in p^n t^{n-1}\). We put \(R = \bigoplus_{i \geq 0} p^i t^i\) and \(G = R/pR = \bigoplus_{i \geq 0} p^i /p^{i+1}\). Then because \(at\) is \(G\)-regular (cf., e.g., [4, 2.1]), we have \(d_2 t \in pR\), that is \(d_2 \in p^2\) which cannot happen as we have checked above. Thus \(p^{(n)} \neq p^n\) for all \(n \geq 2\).

3. **Proof of Theorem (1.1)**

We begin with the following

**Theorem (3.1).** Suppose that \(p\) is generated by the maximal minors of the matrix
\[
\begin{bmatrix}
X & Y^3 & Z^{n+1} \\
Y & Z^3 & X^n
\end{bmatrix},
\]
where \(X, Y, Z\) is a regular system of parameters for \(A\) and \(n\) is a positive integer. Then \(R_s(p)\) is a Gorenstein ring.
Proof. If \( n = 1 \), then after renaming \( X, Y \) and \( Z \), we may assume that \( p \) is generated by the maximal minors of the matrix

\[
M = \begin{bmatrix} X & Y & Z^3 \\ Y & Z^2 & X^3 \end{bmatrix}.
\]

Let us maintain the same notation as in §2. Then the matrix \( M \) is of type (I) and so we have by (2.5) that \( d_2 \equiv -Z^8 \mod(X) \). Hence \((c, d_2, X) = (X, Y^2, Z^8)\) and

\[
l_A(A/(c, d_2, X)) = 16 = 1 \cdot 2 \cdot l_A(A/(X)+p),
\]

because \( l_A(A/(X)+p) = l_A(A/(X)+(Z^5, YZ^3, Y^2)) = 8 \) (cf. (2.3)). Thus \( R_5(p) \) is a Gorenstein ring by (2.2).

Suppose that \( n \geq 2 \) and recall that \( Xd_2 = acZ^{n-2} - b^2Y^2 \) and \( Yd_2 = ab - c^2X^{n-1}Z^{n-2} \) (cf. (2.5)). Then as

\[
(Xd_2 + b^2Y^2)b = Z^{n-2}abc = (Yd_2 + c^2X^{n-1}Z^{n-2})cZ^{n-2},
\]

we have \( X(bd_2 - c^3X^{n-2}Z^{2n-4}) = Y(cd_2Z^{n-2} - b^3Y) \) so that

1. \( Xd_3 = cd_2Z^{n-2} - b^3Y \) and
2. \( Yd_3 = bd_2 - c^3X^{n-2}Z^{2n-4} \),

for some \( d_3 \in p(3) \). When \( n = 2 \), we have \( d_2 \equiv -Z^9 \mod(X) \) (cf. (2.5)). Hence as \( Yd_3 \equiv (Z^{12} - Y^{11})Y \mod(X) \) by the equation (2), we get \( d_3 \equiv Z^{12} - Y^{11} \mod(X) \). Therefore \((b, d_3, X) = (X, YZ^3, Z^{12} - Y^{11})\) so that

\[
l_A(A/(b, d_3, X)) = l_A(A/(X, Y, Z^{12} - Y^{11})) + l_A(A/(X, Z^3, Z^{12} - Y^{11}))
\]

\[
= 45 = 1 \cdot 3 \cdot l_A(A/(X)+p),
\]

since \( l_A(A/(X)+p) = l_A(A/(X)+(Z^6, YZ^3, Y^4)) = 15 \). Thus \( R_5(p) \) is Noetherian by (2.1). Because \( p(2) = (d_2) + p^2 \) (cf. (2.6)(1)), we have \((X, b) + p(2) = (X, YZ^3, Y^8)\) whence

\[
l_A(A/(X, b) + p(2)) = 30 = e_X(A/(b) + p(2)),
\]

that is \( A/(b) + p(2) \) is Cohen-Macaulay and so \( R_5(p) \) is a Gorenstein ring by (2.2).

Now assume that \( n \geq 3 \). Then since

\[
(Xd_3 + b^3Y)\equiv bcd_2Z^{n-2} = (Yd_3 + c^3X^{n-2}Z^{2n-4})cZ^{n-2}
\]

by the equations (1) and (2), we have \( X(bd_3 - c^4X^{n-3}Z^{3n-6}) = Y(cd_3Z^{n-2} - b^4) \) so that

3. \( Yd_4 = bd_3 - c^4X^{n-3}Z^{3n-6} \)

for some \( d_4 \in p(4) \). Notice that \( d_3 \equiv Z^{3n+6} \mod(X) \) by the equation (2) and we get \( d_4 \equiv -Z^{4n+7} - X^{n-3}Y^{15}Z^{3n-6} \mod(X) \) by the equation (3). Hence \((c, d_4, X) = (X, Y^4, Z^{4n+7})\) so that

\[
l_A(A/(c, d_4, X)) = 4 \cdot (4n + 7) = 1 \cdot 4 \cdot l_A(A/(X)+p).
\]

Thus \( R_6(p) \) is Noetherian by (2.1). To check that \( R_6(p) \) is Gorenstein, it is enough by (2.2) to see that \( A/(c) + p(2) \) and \( A/(c) + p(3) \) are Cohen-Macaulay. As \((X, c) + p(2) = (X) + (Z^{2n+5}, Y^2Z^{2n+2}, Y^4)\) (cf. (2.6)(1)), we have

\[
l_A(A/(X, c) + p(2)) = 2 \cdot (4n + 7) = e_X(A/(c) + p(2))
\]
whence \( A/(c) + p(2) \) is Cohen-Macaulay. Because \( d_3 \equiv Z^{3n+6} \mod(X) \), we have

\[
\begin{align*}
(X, d_3) + pp(2) &= (X) + (Z^{3n+6}, Y^3Z^{3n+3}, Y^4Z^{2n+5}, Y^6Z^{2n+2}, \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad Y^8Z^{n+4}, Y^9Z^{n+1}, Y^{12})
\end{align*}
\]

by (2.6)(1). Therefore

\[
l_A(A/(X, d_3) + pp(2)) = 6 \cdot (4n + 7) = l_A(A/(X) + p(3))
\]

so that \( (X) + p(3) = (X, d_3) + pp(2) \). Hence

\[
(X, c) + p(3) = (X) + (Z^{3n+6}, Y^3Z^{3n+3}, Y^4)
\]

and so we get

\[
l_A(A/(X, c) + p(3)) = 3 \cdot (4n + 7) = e_A(A/(c) + p(3)).
\]

Thus \( A/(c) + p(3) \) is Cohen-Macaulay.

To prove Theorem (1.1) we need one more result.

**Proposition (3.2).** Suppose that \( p \) is generated by the maximal minors of the matrix

\[
\begin{bmatrix}
X^2 & Y^2 & Z^3 \\
Y & Z^2 & X^2
\end{bmatrix}
\]

where \( X, Y, Z \) is a regular system of parameters for \( A \). Then \( R_3(p) \) is a Gorenstein ring.

**Proof.** The matrix has type (I) and so by (2.5), \( Yd_2 = ab - c^2Z \) and \( Z^2d_2 = -a^2 + bcY \). Therefore as

\[
(Yd_2 + c^2Z)a = a^2b = (bcY - Z^2d_2)b,
\]

we get \( Y(ad_2 - b^2c) = Z(-ac^2 - bd_2Z) \) so that \( Yd_3 = -ac^2 - bd_2Z \) and \( Zd_3 = ad_2 - b^2c \) for some \( d_3 \in p(3) \). Notice that

\[
d_2 \equiv -Z^8 \mod(Y), \quad d_2 \equiv -X^6Y \mod(Z), \\
d_3 \equiv -Z^{12} + X^{10}Z \mod(Y) \quad \text{and} \quad d_3 \equiv X^2Y^7 \mod(Z).
\]

Then we have \( c^2d_2 + bd_3 \equiv 0 \mod(Z) \), whence \( Zd_4 = c^2d_2 + bd_3 \) for some \( d_4 \in p(4) \). Because \( d_4 \equiv X^{14} - 2X^4Z^{11} \mod(Y) \), we see

\[
l_A(A/(d_2, d_4, Y)) = l_A(A/(X^{14}, Y, Z^8)) = 112 = 2 \cdot 4 \cdot l_A(A/(Y) + p).
\]

Thus \( R_3(p) \) is Noetherian by (2.1). To check that \( R_3(p) \) is Gorenstein, let \( I = (d_2, d_3) + p^3 (\subseteq (d_2) + p(3)) \). Then

\[
(Y) + I = (Y) + (Z^8, X^6Z^6, X^8Z^4, X^{10}Z, X^{12})
\]

so that \( l_A(A/(Y) + I) = 70 \) by (2.3), while

\[
e_{YA}(A/d_2 + p(3)) = l_{A_p}(A_p/d_2A_p + p^3A_p) \cdot e_{YA}(A/p) \\
= 5 \cdot 14 = 70
\]

by the associative formula of multiplicity (cf. [1, (3.1)(3)], too). Hence by the inequalities

\[
l_A(A/(Y) + I) \geq l_A(A/(Y, d_2) + p(3)) \geq e_{YA}(A/(d_2) + p(3)),
\]

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we get that $A/(d_2) + p^{(3)}$ is Cohen-Macaulay. Let $J = (d_2, d_4) + d_3p + p^4 (\subseteq (d_2) + p^{(4)})$. Then

$$(Y) + J = (Y) + (Z^8, X^{10}Z^6, X^{12}Z^3, X^{14})$$

so that $l_A(A/(Y) + J) = 98 = e_{Y_A}(A/(d_2) + p^{(4)})$, whence by the inequalities

$$l_A(A/(Y) + J) \geq l_A(A/(Y, d_2) + p^{(4)}) \geq e_{Y_A}(A/(d_2) + p^{(4)}),$$

we find that $A/(d_2) + p^{(4)}$ is Cohen-Macaulay. Thus $R_s(p)$ is a Gorenstein ring by (2.2).

**Remark (3.3).** The prime ideal $p = p(11, 14, 10)$ corresponds to the ideal considered in (3.2).

**Proof of Theorem (1.1).** We write $m = 4n + r$ with $0 \leq r < 4$. If $r = 0$, then $p = (X^{n+1} - Z^n, Y^4 - X^3Z)$ which is a complete intersection in $A = k[[X, Y, Z]]$. Hence $p^n = p^n$ for any $n \geq 1$ and we have an isomorphism $R_s(p) \cong A[T_1, T_2]/(f)$ of $A$-algebras, where $A[T_1, T_2]$ is a polynomial ring and $0 \neq f \in A[T_1, T_2]$. Thus $R_s(p)$ is certainly Gorenstein.

(1) $(r = 1)$. If $n = 0$, then $Y - X^2 \in p$ and $p$ is a complete intersection in $A$. If $n = 1$, then $p = (Y^3 - Z^2, X^3 - YZ)$, which is a complete intersection in $A$. Thus we may assume $n \geq 2$. Then $p$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^3 & Y^3 & Z^n \\ Y & Z^2 & X^{n-1} \end{bmatrix}$$

(cf. [2]), whence the assertion follows from [1, (4.1)] if $n \geq 4$. The cases $n = 2, 3$ are the exceptional ones, that is $m = 9, 13$.

(2) $(r = 2)$. We may assume $n \geq 1$, because $Z - Y^2 \in p$ if $n = 0$. Hence $p$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^3 & Y^2 & Z^n \\ Y^2 & Z & X^n \end{bmatrix}$$

so that the assertion follows from [1, (4.1)] if $n \geq 3$. When $n = 1$, notice that $p$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} Y^2 & Z & X^3 \\ Z & X & Y^2 \end{bmatrix}$$

and we have $R_s(p)$ to be a Gorenstein ring again by [1, (4.1)]. If $n = 2$, $p$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} Y^2 & Z^2 & X^3 \\ Z & X^2 & Y^2 \end{bmatrix}$$

so that $R_s(p)$ is Gorenstein by (3.2).

(3) $(r = 3)$. We may assume $n \geq 1$, as $Z - XY \in p$ if $n = 0$. Hence $p$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} Z & Y^3 & X^{n+1} \\ Y & X^3 & Z^n \end{bmatrix}$$

so that the assertion follows from (3.1). This completes the proof of Theorem (1.1).

The symbolic Rees algebras $R_s(p)$ for $p = p(9, 10, 13)$ is Noetherian but not Cohen-Macaulay, if $\text{ch} k = 3$ (cf. [7]). The same is true for $p = p(13, 14, 17)$ too, if $\text{ch} k = 3$. We shall prove it in the following.
Example (3.4). Let $p = p(13, 14, 17)$ and let $M$ denote the unique graded maximal ideal of $R_5(p)$. Then $R_5(p)$ is a Noetherian ring with $\dim R_5(p)_M = 4$ and $\depth R_5(p)_M = 3$, if $\text{ch} k = 3$.

Proof. The ideal $p$ is generated by the maximal minors of the matrix

$$M = \begin{bmatrix} X^3 & Y^3 & Z^3 \\ Y & Z & X^2 \end{bmatrix}$$

of type (II). Let $a = Z^4 - X^2Y^3$, $b = X^5 - YZ^3$ and $c = Y^4 - X^3Z$ (hence $p = (a, b, c)$). Then as $X^3a + Y^3b + Z^3c = Ya + Zb + X^2c = 0$, we have $Y^3a^3 + Z^3b^3 + X^6c^3 = 0$. Therefore because

$$(Z^3b^3 + X^6c^3)b = -Y^3a^3b = (X^3a + Z^3c)a^3$$

we see $X^3(a^4 - bc^3X^3) = Z^3(b^4 - a^3c)$ so that $Z^3d_4 = a^4 - bc^3X^3$ for some $d_4 \in p^{(4)}$. Notice that $c \equiv Y^4$ and $d_1 \equiv Z^{13} \mod(X)$ and we find

$$l_4(A/(c, d_4, X)) = 52 = 1 \cdot 4 \cdot l_4(A/(X + p),$$

whence $R_5(p)$ is Noetherian by (2.1) but non-Cohen-Macaulay by (2.2) and [7, (2.4)]. Because $\depth R_5(p)_M \geq 3$ by [1, (2.1) and (3.7)(3)] and $\dim R_5(p)_M = 4$, we get $\depth R_5(p)_M = 3$ as required.

4. Proof of Theorem (1.2)

Let $p = p(n_1, n_2, n_3)$ with $n_1 = 4$ and assume that $p$ is not a complete intersection in $A = A[[X, Y, Z]]$. Hence by [2] the ideal $p$ is generated by maximal minors of a matrix of the following form

$$\begin{bmatrix} X^\alpha & Y^\beta & Z^\gamma \\ Y^\beta & Z^\gamma & X^\alpha' \end{bmatrix}$$

with positive integers $\alpha, \beta, \gamma, \alpha', \beta'$ and $\gamma'$. Then as $(X) + p = (X) + (Z^{\gamma+\gamma'}, Y^\beta Z^{\gamma'}, Y^\beta Z^{\gamma'})$, we have $l_4(A/(X) + p) = \beta \gamma + \beta \gamma' + \beta' \gamma'$ (cf. (2.3)), while $e_{X^\alpha}(A/p) = 4 (= n_1)$. Hence $\beta = \gamma = 1$ and $\gamma + \beta' = 3$, as $\beta \gamma + \beta \gamma' + \beta' \gamma' = 4$. We may assume $\gamma = 1$ and $\beta = 2$ in that solving the equations

$$4(\alpha + \alpha') = n_2 + n_3, \quad 3n_2 = 4\alpha + n_3, \quad 2n_3 = 4\alpha + 2n_2,$$

we get $n_2 = 2\alpha + \alpha'$ and $n_3 = 2\alpha + 3\alpha'$; hence $\alpha'$ is odd, as $\gcd(4, n_2, n_3) = 1$. Thus Theorem (1.2) follows from the next more general

Theorem (4.1). Let $p$ be a prime ideal in a 3-dimensional regular local ring $A$ and assume that $p$ is generated by the maximal minors of a matrix of the form

$$\begin{bmatrix} X^q & Y^2 & Z \\ Y & Z & X^p \end{bmatrix}$$

where $X, Y, Z$ is a regular system of parameters for $A$ and $p, q$ are positive integers with $p$ odd. Then $R_s(p)$ is a Gorenstein ring.

We divide the proof of Theorem (4.1) into a few parts. First we put $a = Z^2 - X^p Y^2$, $b = X^{p+q} - YZ$ and $c = Y^3 - X^q Z$. Hence $p = (a, b, c)$ and any pair of $a, b$ and $c$ forms a regular system of parameters for $A_p$. Choose $0 \leq k \in Z$ so that $kp < q \leq (k + 1)p$. Then we get by [6, Proof of 3.14] the following
Lemma 4.2. There exist elements $e_n \in p^{(n)}$ (1 ≤ $n$ ≤ $k + 2$) and $f \in p^{(2k+3)}$ such that

\[
e_n \equiv Y^{2n+1} \mod(X) \quad (1 \leq n \leq k+1), \\
e_{k+2} \equiv (-1)^k Z^{2k+3} \mod(X) \quad \text{if } q < (k+1)p, \\
e_{k+2} \equiv Y^{2k+5} + (-1)^k Z^{2k+3} \mod(X) \quad \text{if } q = (k+1)p, \\
f \equiv -Z^{4k+4} \mod(X) \quad \text{if } q - kp < (k+1)p - q, \\
f \equiv Y^{4k+8} \mod(X) \quad \text{if } q - kp > (k+1)p - q > 0.
\]

The Noetherian property of $R_s(p)$ now directly follows from (2.1) and (4.2), because

\[l_A(A/(b, e_{k+2}, X)) = l_A(A/(X, YZ, Y^{2k+5} + (-1)^k Z^{2k+3})) = 1 \cdot (k+2) \cdot 4 \quad \text{if } q = (k+1)p,
\]

\[l_A(A/(e_{k+1}, f, X)) = l_A(A/(X, Y^{2k+3}, Z^{4k+4})) = (k+1) \cdot (2k+3) \cdot 4 \quad \text{if } q - kp < (k+1)p - q
\]

and

\[l_A(A/(e_{k+2}, f, X)) = l_A(A/(X, Y^{4k+8}, Z^{2k+3})) = (k+2) \cdot (2k+3) \cdot 4 \quad \text{if } q - kp > (k+1)p - q > 0
\]

(notice that $q - kp \neq (k+1)p - q$, as $p$ is odd).

To see that $R_s(p)$ is a Gorenstein ring we need further informations about the ideals $p^{(n)}$. We begin with the following

Proposition (4.6). $p^{(n)} = p^n + \sum_{j=1}^n e_j p^{n-j}$ for $1 \leq n \leq k + 1$.

Proof. Let $I = p^n + \sum_{j=1}^n e_j p^{n-j}$ and 

\[J = (X) + (Z^{2n}, YZ^{2n-1}, \ldots, Y^{n-1}Z^{n+1}, Y^n Z^n) + (Y^{n+2} Z^{n-1}, Y^{n+3} Z^{n-2}, \ldots, Y^{2n} Z, Y^{2n+1}).
\]

Then $(X) + I \supseteq J$, because

\[a^{n-j} b^j \equiv Y^j Z^{2n-j} \mod(X) \quad \text{for } 0 \leq j \leq n,
\]

\[b^{n-1-j} e_{j+1} \equiv Y^{n+2+j} Z^{n-1-j} \mod(X) \quad \text{for } 0 \leq j \leq n-1.
\]

As $l_A(A/J) = 4^{n+1}/2 = e_{X_A}(A/p^{(n)})$ (cf. (2.3)), by the canonical inequalities

\[l_A(A/J) \geq l_A(A/(X) + I) \geq l_A(A/(X) + p^{(n)}) \geq e_{X_A}(A/p^{(n)})
\]

we get $J = (X) + I = (X) + p^{(n)}$. Hence $p^{(n)} = I + X p^{(n)}$ so that $p^{(n)} = I$ by Nakayama’s lemma.

Corollary (4.7). $R_s(p)$ is a Gorenstein ring, if $q = (k+1)p$.

Proof. By (4.6) and its proof we see $(X, b) + p^{(n)} = (X) + (Z^{2n}, YZ, Y^{2n+1})$ so that

\[l_A(A/(X, b) + p^{(n)}) = 4n = e_{X_A}(A/(b) + p^{(n)})
\]
for $1 \leq n \leq k + 1$. Hence $A/(b) + p^{(n)}$ is a Cohen-Macaulay ring, which proves by (2.2) and (4.3) the assertion.

**Proposition (4.8).** Suppose $q < (k + 1)p$. Then $p^{(n)} = p^n + \sum_{j=1}^{k+2} e_j p^{(n-j)}$ for $k + 2 \leq n \leq 2k + 2$.

**Proof.** Let $I = p^n + \sum_{j=1}^{k+2} e_j p^{(n-j)}$ and

$$J = (X) + (Z^{2n-1}, YZ^{2n-2}, \ldots, Y^{n-k} Z^{n+k+1})$$

$$+ (Y^{n-k} Z^{n+k}, Y^{n-k+1} Z^{n+k-1}, \ldots, Y^n Z^n)$$

$$+ (Y^{n+2} Z^{n-1}, Y^{n+3} Z^{n-2}, \ldots, Y^{n+k+2} Z^{n-k-1})$$

$$+ (Y^{n+k+4} Z^{n-k-2}, Y^{n+k+5} Z^{n-k-3}, \ldots, Y^{2n+2}).$$

Then $(X) + I \supseteq J$, because

$$a^{n-k-2-j} b^j e_{k+2} \equiv (1)^{k+j} Y^j Z^{2n-1-j} \mod(X)$$
for $0 \leq j \leq n - k - 2$,

$$a^{k-j} b^{n-k+j} \equiv (1)^{n-j} Y^{n-j+k} Z^{n+k-j} \mod(X)$$
for $0 \leq j \leq k$,

$$b^{n-j-1} e_{j+1} \equiv (1)^{n-j-1} Y^{n+j+2} Z^{n-j-1} \mod(X)$$
for $0 \leq j \leq k$,

$$b^{n-k-2-j} e_{k+1} e_{j+1} \equiv (1)^{n-k-j} Y^{n+k+4+j} Z^{n-k-2-j} \mod(X)$$
for $0 \leq j \leq n - k - 2$.

Therefore as $l_A(A/J) = e_X e_A/(p^{(n)})$, we get similarly as in the proof of (4.6) that $J = (X) + I = (X) + p^{(n)}$. Hence $p^{(n)} = I$.

**Proposition (4.9).** Suppose that $q - kp < (k + 1)p - q$. Then

$$p^{(n)} = p^n + f p^{(n-2k-3)} + \sum_{j=1}^{k+2} e_j p^{(n-j)}$$

for $2k + 3 \leq n \leq 3k + 3$.

**Proof.** We put $I = p^n + f p^{(n-2k-3)} + \sum_{j=1}^{k+2} e_j p^{(n-j)}$ and

$$J = (X) + (Z^{2n-2}, YZ^{2n-3}, \ldots, Y^{n-2k-3} Z^{n+2k+1})$$

$$+ (Y^{n-2k-1} Z^{n+2k}, Y^{n-2k} Z^{n+2k-1}, \ldots, Y^{n-k-2} Z^{n+k+1})$$

$$+ (Y^{n-k} Z^{n+k}, Y^{n-k+1} Z^{n+k-1}, \ldots, Y^n Z^n)$$

$$+ (Y^{n+2} Z^{n-1}, Y^{n+3} Z^{n-2}, \ldots, Y^{n+k+2} Z^{n-k-1})$$

$$+ (Y^{n+k+4} Z^{n-k-2}, Y^{n+k+5} Z^{n-k-3}, \ldots, Y^{n+2k+4} Z^{n-2k-2})$$

$$+ (Y^{n+2k+6} Z^{n-2k-3}, Y^{n+2k+7} Z^{n-2k-4}, \ldots, Y^{2n+3}).$$

Then $(X) + I \supseteq J$, because

$$a^{n-2k-3-j} b^j f \equiv (1)^{j+1} Y^j Z^{2n-2-j} \mod(X)$$
for $0 \leq j \leq n - 2k - 3$,

$$a^{k-j-1} b^{n-2k-1+j} f \equiv (1)^{n+j} Y^{n-2k+1+j} Z^{n+2k-j} \mod(X)$$
for $0 \leq j \leq k - 1$,

$$a^{k+j} b^{n-k-j} \equiv (1)^{n-k-j} Y^{n-k-j} Z^{n+k+j} \mod(X)$$
for $0 \leq j \leq k$,

$$b^{n-1-j} e_{j+1} \equiv (1)^{n-1-j} Y^{n+2+j} Z^{n-1-j} \mod(X)$$
for $0 \leq j \leq k$,

$$b^{n-k-2-j} e_{k+1} e_{j+1} \equiv (1)^{n-k-j} Y^{n+k+4+j} Z^{n-k-2-j} \mod(X)$$
for $0 \leq j \leq k$ and

$$b^{n-2k-3-j} (e_{k+1})^2 e_{j+1} \equiv (1)^{n-1-j} Y^{n+2k+6+j} Z^{n-2k-3-j} \mod(X)$$
for $0 \leq j \leq n - 2k - 3$.

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Hence we have \( J = (X) + I = (X) + p^{(n)} \) for \( 2k + 3 \leq n \leq 3k + 3 \) by the same reason as in the proof of (4.6). Thus \( p^{(n)} = I \).

**Corollary (4.10).** \( R_*(p) \) is a Gorenstein ring, if \( q - kp < (k + 1)p - q \).

**Proof.** It suffices to see that \( A/(e_{k+1}, f) + p^{(n)} \) is a Cohen-Macaulay ring for each \( k + 2 \leq n \leq 3k + 2 \) (cf. (2.2) and (4.4)); that is enough to check \( l_A(A/(X, e_{k+1}, f) + p^{(n)}) \leq e \chi_A(A/(e_{k+1}, f) + p^{(n)}) \). However, because

\[
   e \chi_A(A/(e_{k+1}, f) + p^{(n)}) = 4 \cdot l_p(A/(e_{k+1}, f))A_p + p^nA_p
\]

by the associative formula of multiplicity (cf. [8]) and because \( e_{k+1}, f \) forms a super regular sequence in \( A_p \) (cf. [1, (3.1)(3)]) we can easily compute the exact value of \( e \chi_A(A/(e_{k+1}, f) + p^{(n)}) \) in terms of \( n \) and \( k \), that is

\[
   e \chi_A(A/(e_{k+1}, f) + p^{(n)}) = 2(2n - k)(k + 1) \quad (k + 2 \leq n \leq 2k + 2) \;
   = 2(6kn - 5k^2 - 11k - n^2 + 7n - 6) \quad (2k + 3 \leq n \leq 3k + 2),
\]

while we now explicitly have the ideal \( (X, e_{k+1}, f) + p^{(n)} \) by (4.6), (4.8) and (4.9) (cf. their proofs, too). Therefore the required inequality \( l_A(A/(X, e_{k+1}, f) + p^{(n)}) \leq e \chi_A(A/(e_{k+1}, f) + p^{(n)}) \) can be directly checked, which we would like to leave to the readers.

**Proposition (4.11).** Suppose that \( q - kp > (k + 1)p - q > 0 \). Then we have

\[
   (1) \quad p^{(2k+3)} = p^{2k+3} + (f) + \sum_{j=1}^{k+2} e_j p^{(2k+3-j)}.
\]

\[
   (2) \quad p^{(n)} = p^n + f p^{(n-2k-3)} + \sum_{j=1}^{k+2} e_j p^{(n-j)} \quad for \quad 2k + 4 \leq n \leq 3k + 4.
\]

**Proof.** (1) Let \( I = p^{2k+3} + (f) + \sum_{j=1}^{k+2} e_j p^{(2k+3-j)} \) and

\[
   J = (X) + (Y^{4k+5}, YZ^{4k+4}, \ldots, Y^{k+1}Z^{3k+4})
\]

\[
   + (Y^{k+3}Z^{3k+3}, Y^{k+4}Z^{3k+2}, \ldots, Y^{2k+3}Z^{2k+3})
\]

\[
   + (Y^{2k+5}Z^{2k+2}, Y^{2k+6}Z^{2k+1}, \ldots, Y^{3k+5}Z^{k+2})
\]

\[
   + (Y^{3k+7}Z^{k+1}, Y^{3k+8}Z^k, \ldots, Y^{4k+8}).
\]

Then \( (X) + I \supseteq J \), because

\[
   a^{k+1-j}b^j e_{k+2} \equiv (-1)^{j+k} Y^j Z^{4k+5-j} \mod (X) \quad for \quad 0 \leq j \leq k + 1,
\]

\[
   a^{k-j}b^{k+3+j} \equiv (-1)^{k+1+j} Y^{k+3+j} Z^{3k+3-j} \mod (X) \quad for \quad 0 \leq j \leq k,
\]

\[
   b^{2k+2-j}e_{j+1} \equiv (-1)^{j} Y^{2k+5+j} Z^{2k+2-j} \mod (X) \quad for \quad 0 \leq j \leq k,
\]

\[
   b^{k+1-j}e_{k+1} \equiv (-1)^{k+1-j} Y^{3k+7+j} Z^{k+1-j} \mod (X) \quad for \quad 0 \leq j \leq k
\]

and

\[
   f \equiv Y^{4k+8} \mod (X).
\]

As \( l_A(A/J) = e \chi_A(A/p^{(2k+3)}) \), we get \( J = (X) + I = (X) + p^{(2k+3)} \) whence \( p^{(2k+3)} = I \).
Let $I = \mathfrak{p}^n + f\mathfrak{p}^{(n-2k-3)} + \sum_{j=1}^{k+2} e_j\mathfrak{p}^{(n-j)}$ and

$$J = (X) + (Z^{2n-2}, YZ^{2n-3}, \ldots, Y^n Z^{-2k-4} Z^{n+2k+2})$$

$$+ (Y^n Z^{-2k-2} Z^{n+2k+1}, Y^n Z^{-2k-1} Z^{n+2k}, \ldots, Y^n Z^{-2k} Z^{n+k+1})$$

$$+ (Y^n Z^{n+k}, Y^n Z^{n+k-1} Z^{n+k-1}, \ldots, Y^n Z^n)$$

$$+ (Y^{n+2} Z^{n-1}, Y^{n+3} Z^{n-2}, \ldots, Y^{n+k+2} Z^{n-k-1})$$

$$+ (Y^{n+k+4} Z^{n-k-2}, Y^{n+k+3} Z^{n-k-3}, \ldots, Y^{n+2k+5} Z^{-2k-3})$$

$$+ (Y^{n+2k+7} Z^{-2k-4}, Y^{n+2k+8} Z^{-2k-5}, \ldots, Y^{2n+3}).$$

Then $(X) + I \supseteq J$, as

$$a^{n-2k-4-j} b^j (e_{k+2})^2 \equiv (-1)^j Y^j Z^{2n-2-j} \mod(X) \quad \text{for } 0 \leq j \leq n - 2k - 4,$$

$$a^{k-j} b^{n-k-2+j} e_{k+2} \equiv (-1)^n Y^n Z^{n+2k+1-j} \mod(X) \quad \text{for } 0 \leq j \leq k,$$

$$a^{k-j} b^{n-k+j} \equiv (-1)^n Y^n Z^{n+k-2+j} \mod(X) \quad \text{for } 0 \leq j \leq k,$$

$$b^{n-1-j} e_{j+1} \equiv (-1)^n Y^n Z^{n+2j} \mod(X) \quad \text{for } 0 \leq j \leq k,$$

$$b^{n-k-2-j} e_{k+1} e_{j+1} \equiv (-1)^n Y^n Z^{n+2k+1+j} \mod(X) \quad \text{for } 0 \leq j \leq k,$$

$$b^{n-2k-3-j} f \equiv (-1)^n Y^n Z^{n+2k+7} \mod(X) \quad \text{and}$$

$$b^{n-2k-4-j} f e_{j+1} \equiv (-1)^n Y^n Z^{n+2k+7+j} \mod(X)$$

for $0 \leq j \leq n - 2k - 4$.

Because $l_A(A/J) = e_{XA}(A/(\mathfrak{p}^{(n)}))$, we have $J = (X) + I = (X) + \mathfrak{p}^{(n)}$, whence $\mathfrak{p}^{(n)} = I$.

**Corollary (4.12).** $R_s(\mathfrak{p})$ is a Gorenstein ring, if $q - kp > (k + 1)p - q > 0$.

**Proof.** By (2.2) and (4.5) we have only to check that $l_A(A/(X, e_{k+2}, f) + \mathfrak{p}^{(n)}) \leq e_{XA}(A/(e_{k+2}, f) + \mathfrak{p}^{(n)})$ for $k + 3 \leq n \leq 3k + 3$. Because we explicitly know the ideals $(X, e_{k+2}, f)$ by (4.6), (4.8) and (4.11) and because

$$e_{XA}(A/(e_{k+2}, f) + \mathfrak{p}^{(n)})$$

$$= 4k n - 2k^2 + 8n - 6k - 4 \quad (k + 3 \leq n \leq 2k + 3),$$

$$= 12k n - 2n^2 + 18n - 10k^2 - 26k - 16 \quad (2k + 4 \leq n \leq 3k + 3)$$

we are able to directly check the required inequality. This completes the proof of Theorem (4.1) as well as that of (4.12).

**References**


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