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NONFIBERING SPHERICAL 3-ORBIFOLDS

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ABSTRACT. Among the finite subgroups of $SO(4)$, members of exactly 21 conjugacy classes act on S^3 preserving no fibration of S^3 by circles. We identify the corresponding spherical 3-orbifolds, i.e., for each such $\mathbf{G} < SO(4)$, we describe the embedded trivalent graph $\{x \in S^3 : \exists \mathbf{I} \neq \mathbf{g} \in \mathbf{G} \text{ s.t. } \mathbf{g}(x) = x\}/\mathbf{G}$ in the topological space $S^3/\mathbf{G}$ (which turns out to be homeomorphic to S^3 in all cases). Explicit fundamental domains (of Dirichlet type) are described for 9 of the groups, together with the identifications to be made on the boundary. The remaining 12 spherical orbifolds are obtained as mirror images or (branched) covers of these.

1. INTRODUCTION

W. P. Thurston [12] has conjectured that all irreducible compact 3-manifolds have canonical geometric decompositions, i.e., they can be split along incompressible tori (and Klein bottles) into manifolds whose interiors admit locally homogeneous Riemannian metrics. In fact, he conjectured that the same holds for (abad) irreducible compact 3-orbifolds. In particular, this would imply that an abad irreducible compact 3-orbifold whose universal orbifold cover was finite-sheeted would necessarily be diffeomorphic to a quotient of the three-sphere by a finite group of isometries (not acting freely, in general). The classical Poincaré conjecture would be a corollary, as would a long-standing conjecture that all smooth actions of finite groups on the three-sphere are topologically conjugate to isometric actions (in 1952, Bing [1] gave the first example of a wild topological involution of the three-sphere).

It is in this context that we will be investigating certain finite subgroups of $SO(4)$ acting on the three-sphere, or equivalently, certain orientable spherical 3-orbifolds. Roughly speaking, an n-orbifold is a Hausdorff topological space locally modelled on the orbit spaces of finite group actions on $\mathbf{R}^n$. In the case of a closed orientable 3-orbifold, this information can be summarized in the form of a triple consisting of the topological space, an embedded trivalent graph consisting of the points corresponding to fixed points of a group action (called the "singular set" of the orbifold), and a labelling function which assigns to each edge of the graph the order of the (cyclic) group which stabilizes (upstairs in $\mathbf{R}^n$) the points which represent that edge. For example, one can "make" a knot in the three-sphere into a closed 3-orbifold by considering the knot as the

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singular set and labelling it with an integer $n > 1$; this can be considered as a kind of Dehn surgery, with coefficient $0/n$. Readers interested in more details can consult [11, 9, or 2]. Though the final results can best be appreciated in light of the general theory, the methods also have appeal, and are phrased in language which is "orbifold-free".

In this paper, we identify the 21 spherical 3-orbifolds which do not fiber over 2-orbifolds, in the sense of specifying the triple (underlying space, singular set, labelling) in each case. In fact, since we identify the orbifold by finding a fundamental domain for the group action on the three-sphere, not only topological but also metric information is uncovered, e.g. lengths of edges in the singular set. Since the finite subgroups of $O(4)$ have been classified up to conjugacy (see Goursat [6] for $O(4)/\{\pm\mathbf{I}\}$, Threlfall and Seifert [10] for $SO(4)$, and DuVal [5, pp. 57–91] for the general case), we can name the group which corresponds to each orbifold. One application of this information would be to help determine the geometric decomposition of a given smooth 3-orbifold, in particular to determine whether or not it admits a spherical structure. A similar study of the hyperbolic 3-orbifolds corresponding to several Bianchi subgroups of $PSL(2, \mathbf{C})$ was done by A. Hatcher [7], based on the Ford domains for such actions obtained by R. Riley [8].

We begin with a review of this classification, followed by a discussion of how one can "read off" the way a rotation of the three-sphere appears in stereographic projection from its projection to $SO(3) \times SO(3)$. Then, the actions of 21 "exceptional" subgroups of $SO(4)$ on the three-sphere are examined in detail, and explicit fundamental domains constructed (together with boundary identifications) for 9 of them. Upon pasting these up, we can determine the corresponding 3-orbifolds. The other 12 "exceptional" orbifolds are then derived from these. All other finite subgroups of $SO(4)$ leave a Hopf fibration invariant and have orbit spaces which are orbifold fiber bundles over 2-orbifolds, with fiber a circle. Hence, these orbifolds can be analyzed by (roughly) the same methods applicable to S^1-bundles over surfaces, or more generally, Seifert-fibered manifolds. They are determined by the base orbifold and additional data, mostly local, concerning the fibration. The necessary theory can be found in [2] and in [4]. In particular, the fibered spherical 3-orbifolds which also have the three-sphere as underlying topological space fall into a finite number of infinite families; the singular sets are drawn in [3]. Very roughly, they look either like torus links (when the base 2-orbifold is topologically a two-sphere) or like necklaces of rational tangles, sometimes known as Montesinos links or star links (when the base 2-orbifold is topologically a disk). All other orientable spherical 3-orbifolds arise from these as orbit spaces of fixed-point-*free* symmetry groups.

2. Finite subgroups of $SO(4)$

We will be concerned with certain quotient spaces of the form $S^3/\mathbf{G}$, where $\mathbf{G}$ is a finite subgroup of $SO(4)$. First, we consider the algebraic side of the situation, and introduce some notation.

$$S^3 := \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : \Sigma(x_i)^2 = 1\}$$
$$\leftrightarrow \{x_1 + x_2 \cdot i + x_3 \cdot j + x_4 \cdot k \in \mathbf{H} : x_i \in \mathbf{R},\ \Sigma(x_i)^2 = 1\}.$$

$\kappa : S^3 \to SO(3)$ is a 2-1 cover (and Lie group homomorphism) induced by the action of S^3 on the quaternions $\mathbf{H} \leftrightarrow \mathbf{R}^4$ by conjugation, which leaves $\text{span}\{i, j, k\} = \{x_1 = 0\} \leftrightarrow \mathbf{R}^3$ invariant. $\kappa(p)$ is the map $v \mapsto pvp^{-1}$, $v \in \text{span}\{i, j, k\}$. We will use the basis $\{j, k, i\}$ when writing elements of $SO(3)$ as matrices, so e.g. $\kappa(\pm i) =$ 180-degree rotation about the z-axis.

$\sigma\colon S^3 \times S^3 \to SO(4)$ and $\rho\colon SO(4) \to SO(3) \times SO(3)$ are each 2-1 covers, and $\rho \circ \sigma = \kappa \times \kappa$. $\sigma(p, q)$ is the map $v \mapsto pvq^{-1}$, $v \in \mathbf{H}$. We will use the basis $\{1, i, j, k\}$ when writing elements of $SO(4)$ as matrices. $\rho(M)$ is the pair $(v \mapsto M(v)M(1)^{-1}, v \mapsto M(1)^{-1}M(v))$, $v \in \text{span}\{i, j, k\}$.

The figures drawn in this article will involve the use of stereographic projection from $S^3 - \{-1\}$ to $\mathbf{R}^3$, defined by

$$(x_1, x_2, x_3, x_4) \mapsto (x_3, x_4, x_2)/(1 + x_1).$$

Note that 1 is sent to $(0, 0, 0)$, $\pm i$ to $\pm(0, 0, 1)$, $\pm j$ to $\pm(1, 0, 0)$, and $\pm k$ to $\pm(0, 1, 0)$.

Given a finite subgroup G of $SO(3) \times SO(3)$, we can define the following associated groups, using the maps $\pi_i\colon SO(3) \times SO(3) \to SO(3)$ (projections onto the ith factor): $R := \pi_1(G)$, $L := \pi_2(G)$, $\textsc{r} := \pi_1(G \cap SO(3) \times I)$, $\textsc{l} := \pi_2(G \cap I \times SO(3))$. Note that $\textsc{r}$ is normal in R, and $\textsc{l}$ is normal in L. Furthermore, $R/\textsc{r} \cong G/(\textsc{r} \times \textsc{l}) \cong L/\textsc{l}$ (since if $(A, B) \in G$, then $(A, I) \in G$ iff $(I, B) \in G$). Let ϕ_G denote the composite isomorphism from $R/\textsc{r}$ to $L/\textsc{l}$.

R and L are so named (agreeing with [10], but not with [5]) because the elements of $\rho^{-1}(SO(3) \times I)$ act as *right*-handed screw motions of S^3 (using the orientation that $S^3 = \partial B^4$ inherits from $\mathbf{R}^4$, which corresponds to a right-handed orientation of $\mathbf{R}^3$ under the stereographic projection we are using). As a consequence of associativity of quaternion multiplication, they leave invariant the *left* Hopf fibration of S^3 which has as fibers the *left* cosets pS^1 of the subgroup $S^1 := \{\cos\theta + \sin\theta \cdot i\colon \theta \in \mathbf{R}\}$ of S^3. On the other hand, the elements of $\rho^{-1}(I \times SO(3))$ act as *left*-handed screw motions of S^3, and leave invariant the *right* Hopf fibration of S^3 which has as fibers the *right* cosets of S^1.

Conversely, given finite subgroups R and L of $SO(3)$, normal subgroups $\textsc{r}$ and $\textsc{l}$, and an isomorphism $\phi\colon R/\textsc{r} \to L/\textsc{l}$, we can construct a subgroup G of $SO(3) \times SO(3)$ such that $\pi_1(G) = R$, $\pi_2(G) = L$, $G \cap (SO(3) \times I) = \textsc{r} \times I$, $G \cap (I \times SO(3)) = I \times \textsc{l}$ and $\phi_G = \phi$. Namely, set $G := \{(A, B) \in R \times L\colon \phi[A] = [B]\}$, where $[A]$ is the coset of A in $R/\textsc{r}$ and $[B]$ is the coset of B in $L/\textsc{l}$. Note that if G and H are conjugate subgroups of $SO(3) \times SO(3)$, then their respective R's (and L's) are conjugate in $SO(3)$.

Furthermore, if ϕ and ψ differ only by an inner automorphism of $R/\textsc{r}$, then the groups $\{R, L, \textsc{r}, \textsc{l}, \phi\}$ and $\{R, L, \textsc{r}, \textsc{l}, \psi\}$ will be conjugate in $SO(3) \times SO(3)$. In fact, it suffices that ϕ and ψ differ by an automorphism of $R/\textsc{r}$ which can be induced by an inner automorphism of $SO(3)$ which leaves R and $\textsc{r}$ each invariant. For example, the automorphism of C_{mn}/C_n taking each element to its inverse can be induced by a 180-degree rotation about an axis perpendicular to that of a generator of C_{mn} (here C_k denotes the group generated by a rotation about the z-axis in R^3 by an angle of $2\pi/k$).

At this point, we can begin to classify the finite subgroups of $SO(4)$ up to conjugacy, via a classification of the finite subgroups of $SO(3) \times SO(3)$ up to conjugacy. We start by recalling the classification of finite subgroups of $SO(3)$. Any such group is cyclic, dihedral, or isomorphic to the orientation-preserving symmetries of a tetrahedron, octahedron, or icosahedron. More precisely, it is conjugate to exactly one of the following groups:

C_n, for some integer $n \geq 1$, defined as above.

D_n, for some integer $n > 1$, generated by C_n and a 180-degree rotation about the x-axis.

T, generated by D_2 and a 120-degree rotation having as axis the line through the origin and $(1, 1, 1)$.

O, generated by T and a 90-degree rotation about the x-axis.

J, generated by T and a 120-degree rotation having as axis the line through the origin and $(\sqrt{5}+1, 0, \sqrt{5}-1)$.

Next, we need to determine the nontrivial proper normal subgroups of these groups, and the corresponding quotient groups. They are:

C_n: C_d for any d which divides n; $C_n/C_d \cong C_{n/d}$;

D_n, n odd: C_d for any $d|n$; $D_n/C_d \cong D_{n/d}$;

D_n, n even: C_d for any $d|n$, $D_{n/2}$; $D_n/C_d \cong D_{n/d}$, $D_n/D_{n/2} \cong C_2$;

T: D_2; $T/D_2 \cong C_3$;

O: D_2, T; $O/D_2 \cong D_3$, $O/T \cong C_2$;

J: none.

The finite subgroups of $SO(3) \times SO(3)$ fall into three broad families: those for which R and L are each cyclic or dihedral, those for which one of these groups is cyclic or dihedral, and those for which neither R nor L is cyclic or dihedral. It is an easy matter to show that a finite subgroup of $SO(4)$ whose image under ρ belongs to the first family will leave invariant both Hopf fibrations of S^3 mentioned above; if the image belongs to the second family, the group will leave invariant one of the two Hopf fibrations. These fibrations will thus pass to the quotient space, yielding a fibration of the quotient space (thought of as an orbifold) over some 2-orbifold. In particular, if a subgroup of $SO(4)$ has a projection to $SO(3) \times SO(3)$ with L cyclic or dihedral, then the corresponding orbifold has a Seifert fibering with a *positive* (rational) Euler number; it is induced from the left Hopf fibration of S^3, which has Euler number $+1$. As mentioned in the introduction, these orbifolds are amenable to study *en masse*, although in practice there are so many cases that it is hard to give a formula that will translate a description of a subgroup of $SO(4)$ into a description of the corresponding orbifold (or vice versa).

However, consider the remaining family. There are 16 conjugacy classes, including three enantiomorphic (mirror-image) pairs.

$T \times T$,	$R = L = T$,	$\mathrm{R} = \mathrm{L} = T$,	XX
$T \times_{C_3} T$,	$R = L = T$,	$\mathrm{R} = \mathrm{L} = D_2$,	XXII
$T \times_T T$,	$R = L = T$,	$\mathrm{R} = \mathrm{L} = \{I\}$,	XXI
$O \times O$,	$R = L = O$,	$\mathrm{R} = \mathrm{L} = O$,	XXV
$O \times_{C_2} O$,	$R = L = O$,	$\mathrm{R} = \mathrm{L} = T$,	XXVIII
$O \times_{D_3} O$,	$R = L = O$,	$\mathrm{R} = \mathrm{L} = D_2$,	XXVII
$O \times_O O$,	$R = L = O$,	$\mathrm{R} = \mathrm{L} = \{I\}$,	XXVI
$J \times J$,	$R = L = J$,	$\mathrm{R} = \mathrm{L} = J$,	XXX
$J \times_J J$,	$R = L = J$,	$\mathrm{R} = \mathrm{L} = \{I\}$,	XXXI
$J \times_J^* J$,	$R = L = J$, $\phi \in \mathrm{Aut}(J) - \mathrm{Inn}(J)$	$\mathrm{R} = \mathrm{L} = J$,	XXXII

$$\begin{array}{llll}
T\times O, & R=\mathrm{R}=T, & L=\mathrm{L}=O, & \mathrm{XXIII} \\
O\times T, & R=\mathrm{R}=O, & L=\mathrm{L}=T, & \\
T\times J, & R=\mathrm{R}=T, & L=\mathrm{L}=J, & \mathrm{XXIV} \\
J\times T, & R=\mathrm{R}=J, & L=\mathrm{L}=T, & \\
O\times J, & R=\mathrm{R}=O, & L=\mathrm{L}=J, & \mathrm{XXIX} \\
J\times O, & R=\mathrm{R}=J, & L=\mathrm{L}=O. &
\end{array}$$

The right-hand column of Roman numerals refers to the numbering in [6, 5]. The notation in the left-hand column is meant to emphasize the structure of these groups as fibered products; $\{R, L, \mathrm{R}, \mathrm{L}, \phi\}$ would be denoted $R \times_H^\phi L$, where $R/\mathrm{R} \cong H \cong L/\mathrm{L}$. To confirm that the above list is complete, it is necessary to check that all automorphisms of C_2, C_3, D_3, T, and O are either inner automorphisms, or induced by conjugation in $SO(3)$, to check that the inner automorphisms of J are a subgroup of index 2 in the automorphism group, and to check that all possible choices for R, L, R, and L have been made. We assure the reader that this is a routine, and not unpleasant, exercise in abstract algebra. We move on to consider subgroups of $SO(4)$.

Given a subgroup G of $SO(3) \times SO(3)$, what are the subgroups of $SO(4)$ which the map ρ projects onto G? One is clearly $\mathbf{G} := \rho^{-1}(G)$ (the use of bold resp. italic fonts will distinguish subgroups of $SO(4)$ resp. $SO(3) \times SO(3)$ which correspond under ρ). There may also be "lifts of G," i.e., index-2 subgroups $\mathbf{G}^1, \mathbf{G}^2, \ldots$, of $\mathbf{G}$ which ρ maps isomorphically onto G. The "lifting problem" is resolved by the following proposition.

Proposition 1. *Let G be a finite subgroup of $SO(3) \times SO(3)$, with associated groups R, L, R, L. Then:*

(1) *If R or L contains an element of order 2 (i.e., if they are not both cyclic groups of odd order), then there are NO lifts.*

(2) *If R and L are both cyclic groups of odd order, then at least one lift exists.*

(3) *If G has at least one lift, then the total number of lifts is one more than the number of index-2 subgroups of G.*

Proof.

(1) It is an easy calculation that if A is an element of $SO(3)$ which has even order n, then both elements of $\rho^{-1}(A, I)$ and both elements of $\rho^{-1}(I, A)$ have order $2n$.

(2) This is proven on a case-by-case basis; see [10].

(3) This is immediate from the observation that any two distinct lifts must intersect in a group which has index 2 in each. Indeed, there is a homomorphism from G to $\mathbf{Z}/2\mathbf{Z}$ defined by the rule that g goes to 0 if the two lifts agree, and g goes to 1 if they do not agree. □

$T \times_T T$ has a lift $\mathbf{T} \times_{\mathbf{T}} \mathbf{T}^1$, constructed by lifting a pair of order-3 generators to (the unique) order-3 elements of $SO(4)$ which cover them. This lift is contained in the stabilizer of 1 in $SO(4)$, which is isomorphic to $SO(3)$. Hence its action on S^3 is a suspension of an action on S^2. Note that since $T \times_T T$ is a diagonal group, isomorphic to T, it has no subgroups of index 2, and hence no other lifts.

$O \times_O O$ has a lift $\mathbf{O} \times_{\mathbf{O}} \mathbf{O}^1$, as above, and also another lift $\mathbf{O} \times_{\mathbf{O}} \mathbf{O}^2$, whose intersection with $\mathbf{O} \times_{\mathbf{O}} \mathbf{O}^1$ equals $\mathbf{T} \times_{\mathbf{T}} \mathbf{T}^1$. These are the only lifts, since $O \times_O O$, which is isomorphic to O, has a single subgroup of index 2.

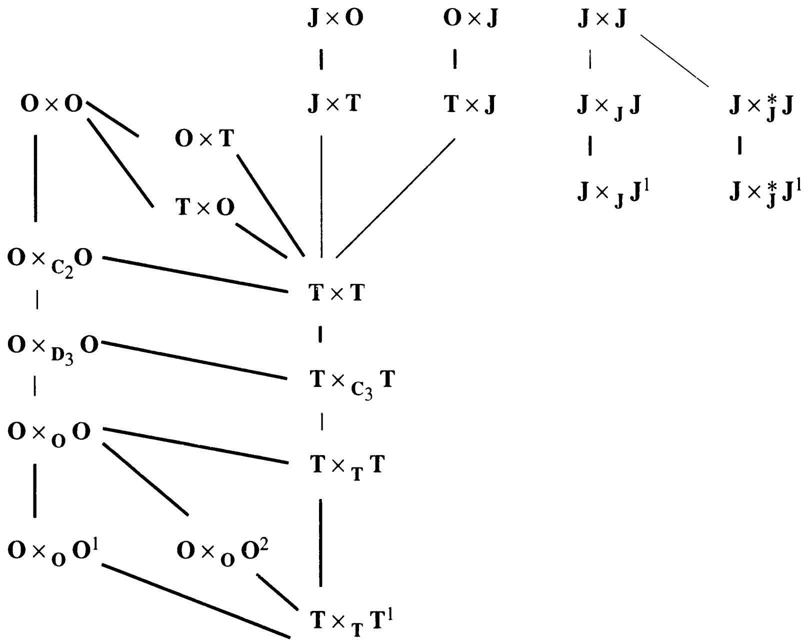


FIGURE 1. Subgroups

$J \times_J J$ has a lift $\mathbf{J} \times_\mathbf{J} \mathbf{J}^1$, and no other, since J has no subgroups of index 2. For similar reasons, $J \times_J^* J$ has exactly one lift, denoted $\mathbf{J} \times_\mathbf{J}^* \mathbf{J}^1$.

We can therefore completely describe the subgroups of $SO(4)$ with which we will be concerned.

Proposition 2. *If a finite subgroup* $\mathbf{G}$ *of* $SO(4)$ *is such that* $\pi_i(\rho(\mathbf{G}))$ *is neither cyclic nor dihedral for* $i = 1, 2$, *then* $\mathbf{G}$ *is conjugate in* $SO(4)$ *to exactly one of* 21 *subgroups* (*shown in Figure* 1; *heavy lines denote normal subgroups, light lines, other subgroups*).

There are 3 enantiomorphic pairs, hence 18 conjugacy classes in $O(4)$.

3. GEOMETRY OF THE ACTION

We would like to find fundamental domains in S^3 for the actions of these groups. Once the rules for gluing up the boundary of the domain are found, the quotient space can be determined. Actually, it will only be necessary to explicitly construct fundamental domains for 9 groups; the rest are normal subgroups of these (with index 2 or 3), or mirror-images of these, and their quotient spaces can be inferred. In general, we begin by forming a "prefundamental domain" centered at $1 \in S^3$, i.e., the set of all points in S^3 which are closer to 1 than to any point in the orbit of 1. This would be a Dirichlet fundamental domain, but for the nontriviality of the stabilizer of 1. Thus, the fundamental domain will be the intersection of a suitable cone (with vertex at 1) with the prefundamental domain. We begin by finding ways to describe the geometry of the action of elements of $SO(4)$ on S^3 via their projections to $SO(3) \times SO(3)$.

Proposition 3. *An element of $SO(4)$, acting on S^3, is either the identity, has fixed point set a great circle (in which case we will call it a "rotation"), or acts without fixed points.*

Since all 21 groups turn out to be generated by rotations, it is useful to have a convenient means of finding all the rotations in a particular group, and determining the position of the geodesics which they fix.

Proposition 4. (1) *If $\mathbf{g} \in SO(4)$ and $\rho(\mathbf{g}) = (A, B)$, where A and B are rotations by an angle $\pm\theta$, then either $\mathbf{g}$ or $-\mathbf{g}$ is a rotation about a great circle γ by an angle $\pm\theta$. If $\theta = \pi$, then $\mathbf{g}$ and $-\mathbf{g}$ are both 180-degree rotations, about great circles which lie in orthogonal planes in $\mathbf{R}^4$.*

(2) *If l and m denote the axes of the rotations A and B, then the closest approach of γ to 1 occurs (in stereographic projection) at a point on the line through the origin which is perpendicular to l and m, in fact in the direction of* minus (*right-hand end of l*) $\times$ (*right-hand end of m*).

(3) *If α denotes the angle between l and m (choosing the right-handed end of the axes if $\theta \neq \pi$), then the spherical distance from 1 to γ at perigee equals $\alpha/2$.*

Conversely, if $\mathbf{g} \in SO(4)$ is a rotation, then its action on S^3 is related to the actions of $\pi_i(\rho(\mathbf{G}))$, $i = 1, 2$ on $\mathbf{R}^3$ as described above.

Proof. To prove part (1), conjugate $\mathbf{g}$ until

$$\pi_1(\rho(\mathbf{g})) = \pi_2(\rho(\mathbf{g})) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \kappa(\cos(\theta/2) + \sin(\theta/2)i);$$

then

$$\begin{aligned} \mathbf{g} &= \pm\sigma(\cos(\theta/2) + \sin(\theta/2)i, \cos(\theta/2) + \sin(\theta/2)i) \\ &= \pm \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}, \end{aligned}$$

which is clearly (up to sign) a rotation by θ. The converse of (1) can be proven by essentially the same calculation.

To prove parts (2) and (3), and their converses, first consider the case $\theta = \pi$. By part (1), we can conjugate $\mathbf{g}$ so that $\pi_1(\rho(\mathbf{g}))$ is 180-degree rotation about the x-axis and $\pi_2(\rho(\mathbf{g}))$ is 180-degree rotation about a line in the xy-plane. Hence we can assume

$$\mathbf{g} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ -\sin\alpha & -\cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & \sin\alpha & -\cos\alpha \end{bmatrix} = \sigma(j, (\cos\alpha)j + (\sin\alpha)k),$$

$$\rho(\mathbf{g}) = \left(\begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} \cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & -\cos 2\alpha & 0 \\ 0 & 0 & -1 \end{bmatrix} \right).$$

By inspection, $\mathbf{g}$ is a rotation of order 2 about a geodesic passing through $\pm[\cos(\alpha/2)j + \sin(\alpha/2)k]$ and $\pm[\cos(\alpha/2) - \sin(\alpha/2)i]$. Using stereographic projection, we sketch the situation in Figure 2 (with $\alpha = 45$ degrees). The

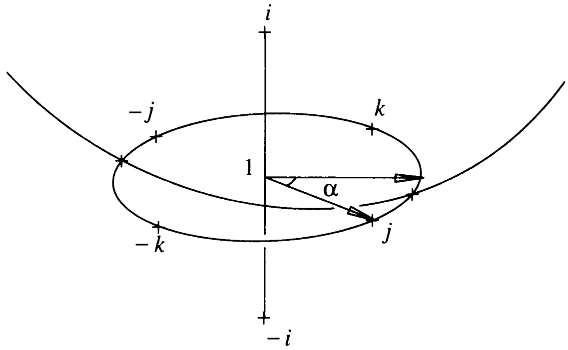


FIGURE 2. Fixed geodesic

assertions of parts (2) and (3) should now be obvious. To handle the general case (θ arbitrary), it suffices to repeat the above calculation with $\cos\theta+(\sin\theta)j$ and $\cos\theta+(\sin\theta)((\cos\alpha)j+(\sin\alpha)k)$ replacing j and $(\cos\alpha)j+(\sin\alpha)k$ as arguments of σ. □

Consequently, when determining the prefundamental domain for a subgroup $\mathbf{G}$ of $SO(4)$, we will be looking in $\rho(\mathbf{G})$ for pairs of rotations by equal angles, and with small angle between the axes. These correspond to rotations in $\mathbf{G}$ which move 1 only a little, and the great spheres which are the perpendicular bisectors of the segments from 1 to its images under these elements will thus contribute faces to the prefundamental domain. We then obtain the fundamental domain by taking account of the stabilizer of 1 in $\mathbf{G}$. Often, adjacent faces of the fundamental domain are paired by rotations about a common edge, or a face is self-identified by a 180-degree rotation about a diagonal. These rotations are found by using Proposition 4, after which the fundamental domain can be folded up and the orbifold $S^3/\mathbf{G}$ can be recognized. That is to say, we obtain the orbit space (always homeomorphic to S^3), together with the singular set (from the folded-up axes of rotations). The edges of the singular set are labelled with the order of the corresponding rotation (in the figures below, edges of order 2 are left unlabelled).

4. FUNDAMENTAL DOMAINS AND ORBIFOLDS

$\mathbf{T}\times_{\mathbf{T}}\mathbf{T}$, $\mathbf{O}\times_{\mathbf{O}}\mathbf{O}$, $\mathbf{J}\times_{\mathbf{J}}\mathbf{J}$. Each element of each of these groups either fixes 1, or sends 1 to -1. This follows immediately from Proposition 4, using the fact that the images of these groups in $SO(3)\times SO(3)$ are diagonal groups. Hence the prefundamental domain in each case is the region $\{(x_1, x_2, x_3, x_4)\in S^3 : x_1 \geq 0\}$ bounded by the great sphere perpendicular to 1. A fundamental domain is easy to find in each case, as are the identifications which must be made on the intersection of the great sphere with this fundamental domain. See Figure 3 for illustrations. Dash-dot-dotted edges of the fundamental domain are fixed by rotations which pair adjacent faces, dashed edges are fixed by rotations which do not pair adjacent faces, and solid edges are not fixed by any rotation.

$\mathbf{T}\times\mathbf{T}$. The closest images of 1 are $(1\pm i\pm j\pm k)/2$, which arise e.g. from group elements whose images in $SO(3)\times SO(3)$ are of the form (120-degree

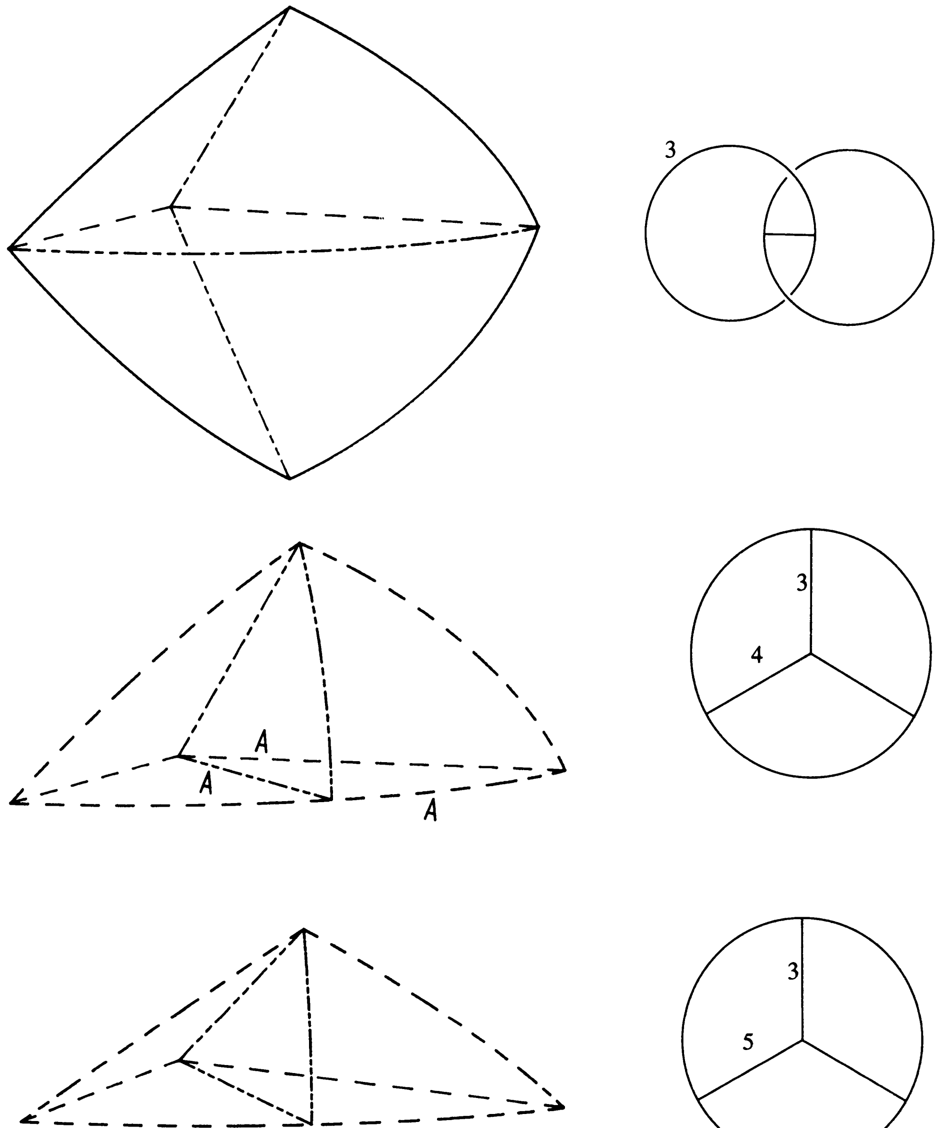


FIGURE 3. $\mathbf{T} \times_{\mathbf{T}} \mathbf{T}$, $\mathbf{O} \times_{\mathbf{O}} \mathbf{O}$, $\mathbf{J} \times_{\mathbf{J}} \mathbf{J}$: fundamental domains and orbifolds

rotation, I). The prefundamental region is a regular octahedron. It is easy to determine that the 6 vertices are $(1 \pm i)/\sqrt{2}$, $(1 \pm j)/\sqrt{2}$, and $(1 \pm k)/\sqrt{2}$.

Adjacent faces of this octahedron are identified by a 120-degree rotation about their common edge. The image in $SO(3) \times SO(3)$ of such a rotation is a pair of 120-degree rotations in T whose axes (or rather their right-hand ends) lie in adjacent octants of $\mathbf{R}^3$. The stabilizer of 1 is a tetrahedral group, which introduces additional identifications. See Figure 4 on page 132 for illustrations.

$\mathbf{O} \times \mathbf{O}$. The closest images of 1 are $(1 \pm i)/\sqrt{2}$, $(1 \pm j)/\sqrt{2}$, and $(1 \pm k)/\sqrt{2}$, which arise e.g. from group elements whose images in $SO(3) \times SO(3)$ are of the form (90-degree rotation, I). Next closest are the eight images arising from elements of the subgroup $\mathbf{T} \times \mathbf{T}$, which are named above. A little calculation shows that the prefundamental domain is combinatorially a truncated cube (adjacent inner great spheres intersect closer to 1 than adjacent outer great spheres). Furthermore, the vertices of this domain turn out to be $((\sqrt{2}+1) \pm (\sqrt{2}-1)i \pm j \pm k)/2\sqrt{2}$, plus the 16 additional points you can obtain by cyclically permuting the roles of i, j, k. In particular, the octagonal faces are regular, as are the triangular faces.

Each triangular face of the truncated cube is self-identified by 180-degree rotations about each of the three altitudes. These identifications correspond to pairs of 180-degree rotations in O whose axes make an angle of 60 degrees. Each octagonal face is self-identified by 180-degree rotations about diagonals connecting opposite vertices. These correspond to pairs of 180-degree rotations in O whose axes make an angle of 45 degrees. The stabilizer of 1 is an octahedral group, which introduces additional identifications. See Figure 5 (p. 133) for illustrations.

$\mathbf{J} \times \mathbf{J}$. The closest images of 1 are $((\sqrt{5}+1) \pm 2i \pm (\sqrt{5}-1)j)/4$, plus the 8 additional points obtained from cyclically permuting the roles of i, j, k. They arise e.g. from group elements whose projections to $SO(3) \times SO(3)$ are of the form (72-degree rotation, I). The prefundamental domain is a regular dodecahedron. Its vertices are $((3+\sqrt{5}) \pm (\sqrt{5}-1)i \pm (\sqrt{5}-1)j \pm (\sqrt{5}-1)k)/4\sqrt{2}$, $((3+\sqrt{5}) \pm (3-\sqrt{5})i \pm 2j)/4\sqrt{2}$, plus the 8 additional vertices obtained from cyclically permuting the roles of i, j, k in the latter expression.

Each pentagonal face is self-identified by 180-degree rotations about each of the five altitudes. These identifications correspond to pairs of 180-degree rotations in J whose axes make an angle of 36 degrees. Adjacent faces are identified by 120-degree rotations about their common edge. These correspond to pairs of 120-degree rotations in J whose axes make an angle of $\arccos(\sqrt{5}/3) \approx 41.8$ degrees. The stabilizer of 1 is an icosahedral group, which introduces additional identifications. See Figure 6 (p. 134) for illustrations.

$\mathbf{O} \times_{\mathbf{D}_3} \mathbf{O}$. This is the group of orientation-preserving symmetries of the hypercube with vertices at $(\pm 1 \pm i \pm j \pm k)/2$. The closest images of 1 are $\pm i$, $\pm j$, $\pm k$, which arise e.g. from group elements whose projections to $SO(3) \times SO(3)$ are of the form (α, β), where α and β are distinct nontrivial elements of D_2. The prefundamental domain is a cube. Its vertices are $(1 \pm i \pm j \pm k)/2$.

Each face is self-identified by 180-degree rotations about two diagonals connecting opposite vertices and two diagonals connecting opposite edge-midpoints. These identifications correspond to pairs of 180-degree rotations in O whose axes make an angle of 90 degrees and whose product lies in D_2, hence representing the same coset in $O/D_2 \cong D_3$. Adjacent faces are identified by 120-degree rotations about their common edge. These correspond to pairs of 120-degree rotations in O whose axes make an angle of $\arccos(-1/3) \approx 109.5$ degrees. Alternatively, they can be described as pairs of 120-degree rotations about the centers of adjacent faces of an octahedron, one clockwise and one counterclockwise. The stabilizer of 1 is an octahedral group. See Figure 7 (p. 135) for illustrations.

$\mathbf{J} \times_{\mathbf{J}}^{*} \mathbf{J}$. The fixed subgroup of any outer automorphism of J is relatively small, so it is convenient to choose the following group as the "nicest" representative of the conjugacy class, generated by:

$$\mathbf{h}_1 := \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}, \qquad \mathbf{h}_2 := \begin{bmatrix} -\frac{1}{4} & \sqrt{5}/4 & -\sqrt{5}/4 & \sqrt{5}/4 \\ -\sqrt{5}/4 & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ \sqrt{5}/4 & \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ \sqrt{5}/4 & -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix},$$

$$\mathbf{h}_3 := \begin{bmatrix} -\frac{1}{4} & -\sqrt{5}/4 & -\sqrt{5}/4 & -\sqrt{5}/4 \\ -\sqrt{5}/4 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \\ \sqrt{5}/4 & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} \\ \sqrt{5}/4 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \qquad \mathbf{h}_4 := -\mathbf{I}.$$

The set $\{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3\}$ generates the group of orientation-preserving symmetries of the regular 4-simplex with vertices $v_1 = 1$, $v_2 = -1/4 + \sqrt{5}/4(-i+j+k)$, $v_3 = -1/4+\sqrt{5}/4(i-j+k)$, $v_4 = -1/4+\sqrt{5}/4(i+j-k)$, $v_5 = -1/4+\sqrt{5}/4(-i-j-k)$. $\mathbf{h}_1$ performs the permutation (24)(35) on the vertices, $\mathbf{h}_2$ performs (123), and $\mathbf{h}_3$ performs (12345). It is easy to verify that $\{\pi_1(\rho(\mathbf{h}_i)) | i = 1, 2, 3\}$ generates J, and that $\{\pi_2(\rho(\mathbf{h}_i)) | i = 1, 2, 3\}$ generates a group $\overline{J}$ which is conjugate to J, e.g. by a 90-degree rotation about the x-axis. $J \cap \overline{J} = T$. $\overline{J}$ can also be described as the image of $J \subset \mathrm{GL}(\mathbf{Q}[\sqrt{5}])$ under the involution $a + b\sqrt{5} \leftrightarrow a - b\sqrt{5}$.

The set of nearest images of 1 is $\{-v_i \mid i = 2, 3, 4, 5\}$; the set of next closest images is $\{v_i \mid i = 2, 3, 4, 5\}$. The prefundamental domain is combinatorially a truncated tetrahedron. The vertices are $(\sqrt{5} \pm 3i \pm j \pm k)/4$ (just those combinations with an *odd* number of minus signs), plus the 8 other vertices obtained from these by cyclically permuting the coordinates of i, j, k, for a total of 12 vertices. The hexagonal faces are all regular, as are the triangular faces.

Each hexagonal face is self-identified by 180-degree rotations about three diagonals connecting opposite vertices and three diagonals connecting opposite edge-midpoints. These identifications correspond to pairs of 180-degree rotations, one in J and one in $\overline{J}$, whose axes make an angle of $\arccos(1/4) \approx 75.5$ degrees. Each triangular face is self-identified by 180-degree rotations about the three altitudes. These correspond to the *other* inverse image of the elements of $SO(3) \times SO(3)$ just mentioned. See Figure 8 (p. 136) for illustrations; note the similarities to the case of $\mathbf{O} \times \mathbf{O}$.

$\mathbf{J} \times \mathbf{O}$. This is the most complicated case. The prefundamental domain can be described as a "small" tetrahedron, each of whose 4 vertices are truncated by intersection with a "large" tetrahedron in dual position (producing 12 vertices), and then further truncated by intersection with a dodecahedron (producing 36 vertices). The small tetrahedron is formed by planes halfway between 1 and its image under group elements which project to pairs of 180-degree rotations, one

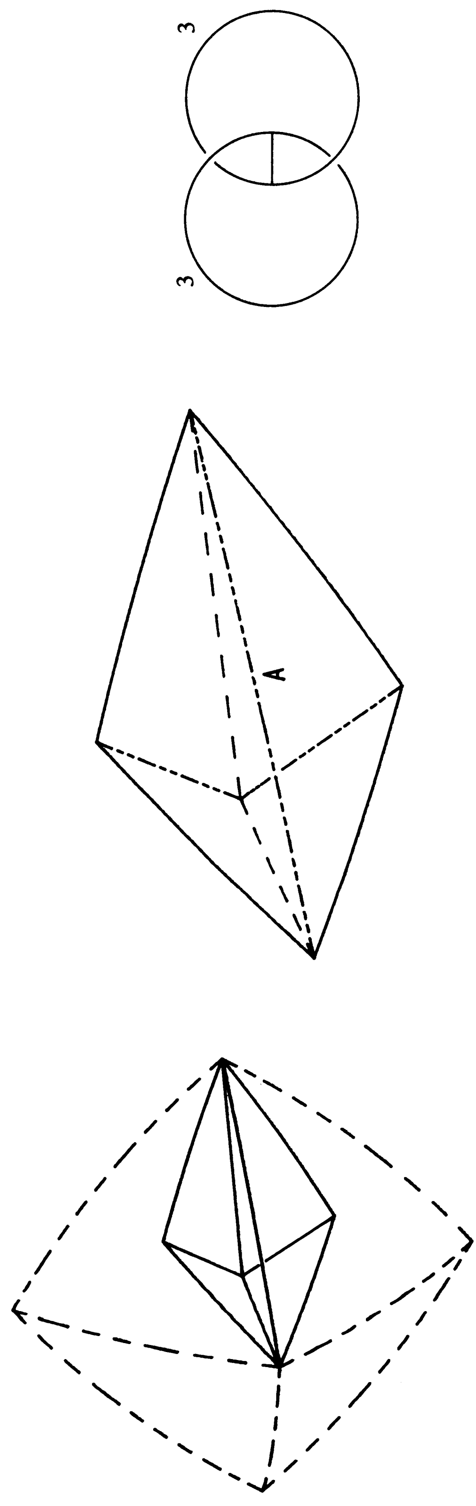


FIGURE 4

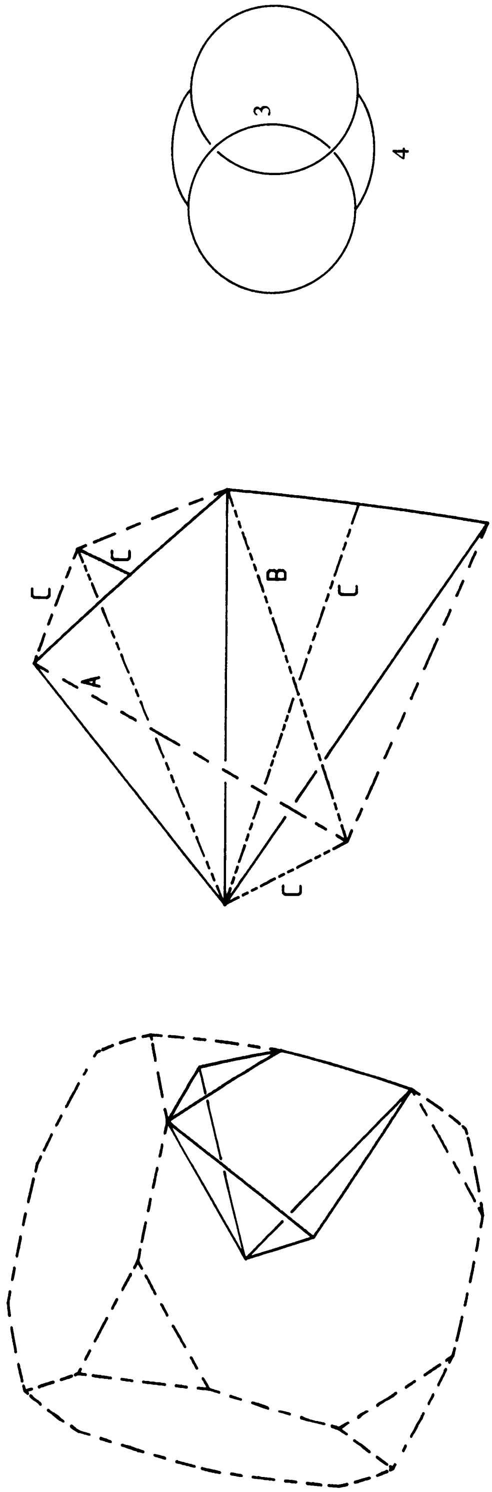


FIGURE 5

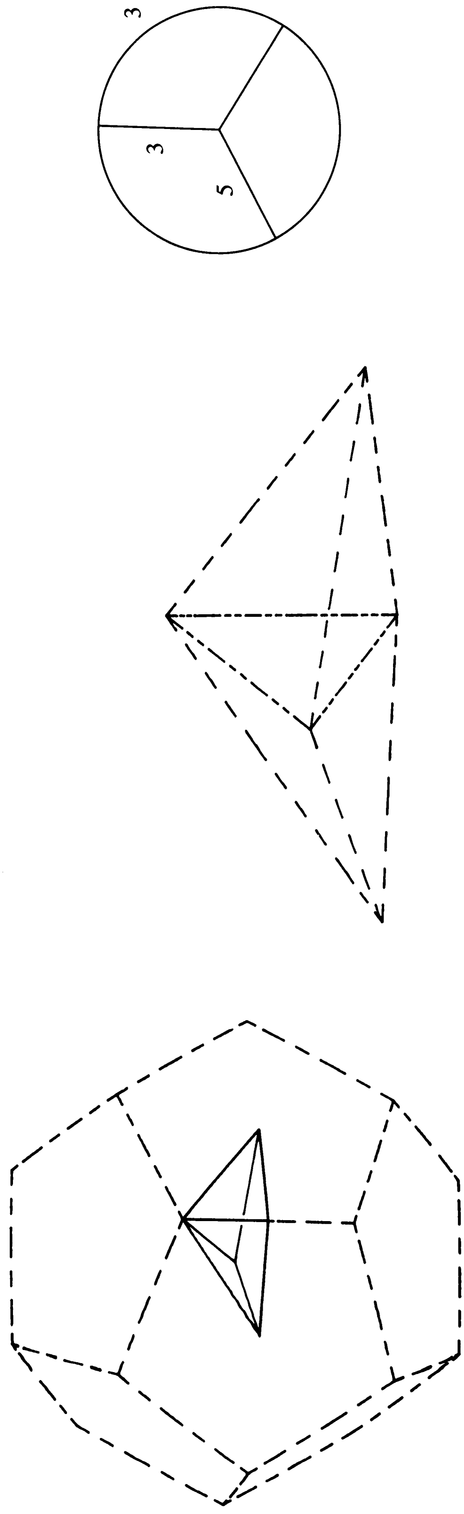


FIGURE 6

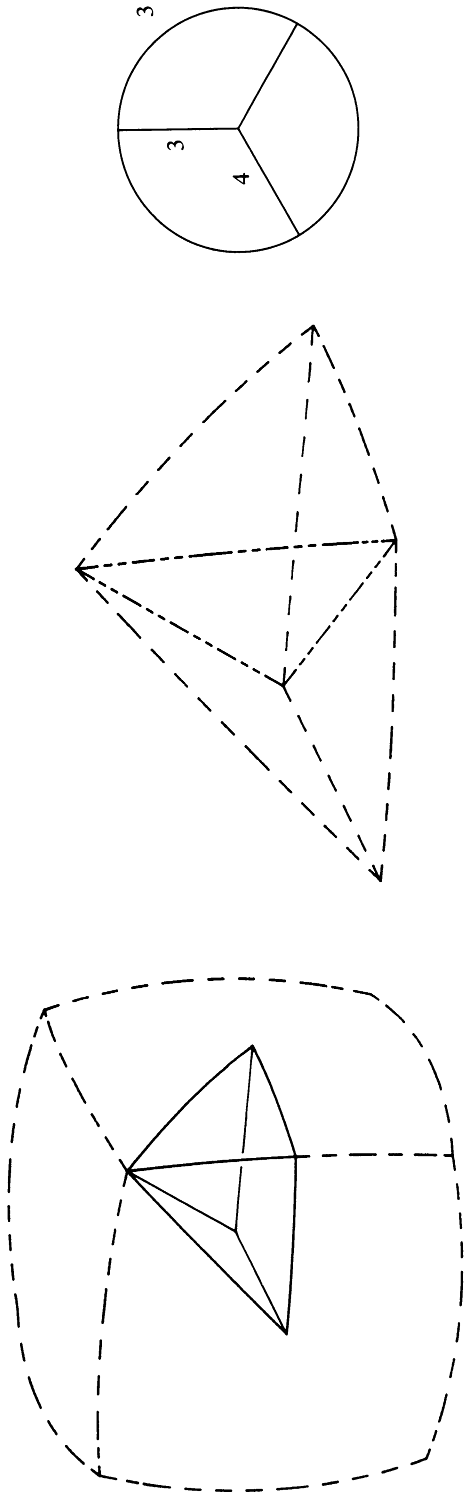


FIGURE 7

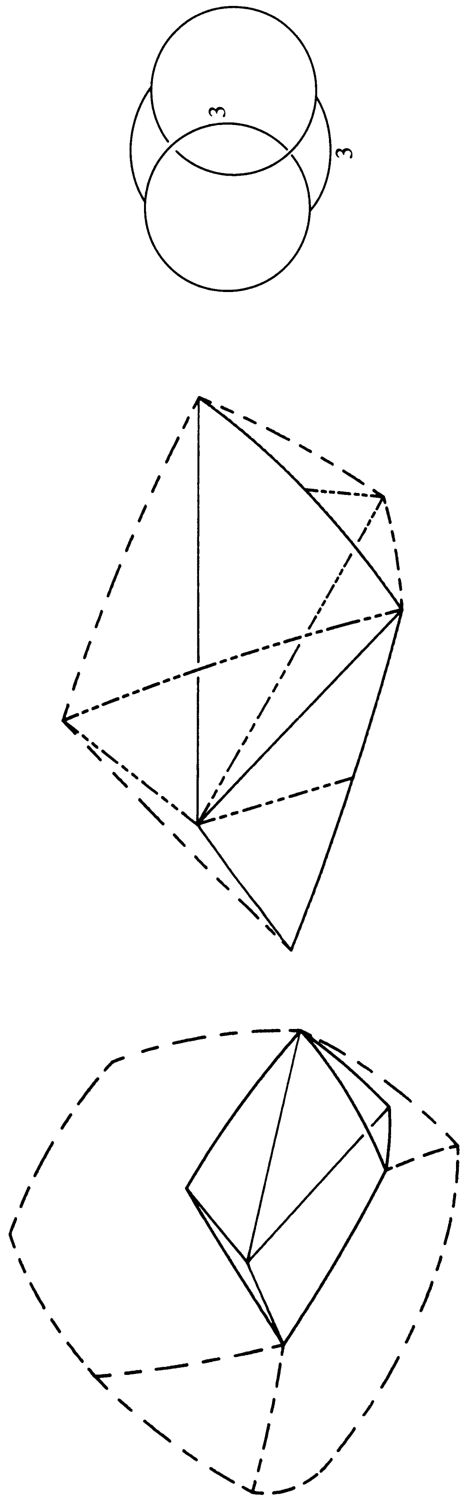


FIGURE 8

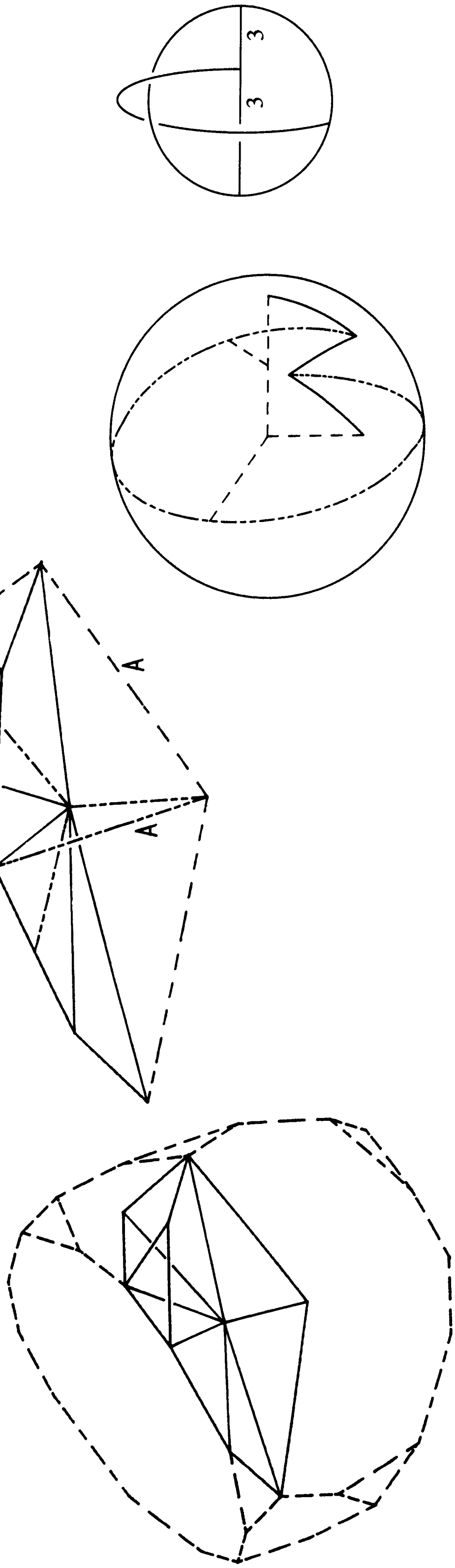


FIGURE 9

in J and one in O, whose axes make an angle of $\arccos((3+\sqrt{5})/4\sqrt{2}) \approx 22.2$ degrees. The large tetrahedron corresponds to pairs of 180-degree rotations in $J \times O$ which make an angle of $\arccos(\sqrt{5}/2\sqrt{2}) \approx 37.8$ degrees. The dodecahedron corresponds to pairs of 180-degree rotations in $J \times O$ which make an angle of $\arccos((1+\sqrt{5})/4) = 36$ degrees. The prefundamental domain thus has 12 (isosceles) triangular faces, which are remnants of the dodecahedron; each is self-identified by a single 180-degree rotation about its axis of symmetry. It also has four (equilateral, but not regular) hexagonal faces, which are remnants of the large tetrahedron; each is self-identified by 180-degree rotations about the three diagonals connecting opposite pairs of vertices. Finally, it has four twelve-sided faces, which are remnants of the small tetrahedron; each is self-identified by 180-degree rotations about the three diagonals connecting those pairs of opposite vertices for which the lengths of the incident edges are equal. The elements of $SO(3) \times SO(3)$ corresponding to these self-identifications were described above. See Figure 9 for illustrations; included is a half-way stage in the gluing process, in which the faces adjoining the tetrahedral vertex and one of the dihedral vertices have been identified, pushing these vertices and their incident (dashed) edges into the interior of the domain.

Having found the orbifolds corresponding to 9 of the 21 groups by brute force, we can determine the rest relatively easily on the basis of two elementary observations, which follow.

Proposition 5. *Conjugation by*

$$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

transforms a subgroup of $SO(4)$ with projection to $SO(3) \times SO(3)$ described by $\{R, L, \mathrm{R}, \mathrm{L}, \phi\}$ to one with projection described by $\{L, R, \mathrm{L}, \mathrm{R}, \phi^{-1}\}$. The quotients of S^3 by these two groups are orbifolds which are "mirror images" of each other.

Proposition 6. *If $\mathbf{K}$ is a normal subgroup of $\mathbf{G}$ (itself a finite subgroup of $SO(4)$), if $\mathbf{G}/\mathbf{K}$ is cyclic of prime order p, and if the edges of the singular set of $S^3/\mathbf{G}$ which correspond to elements of $\mathbf{G} - \mathbf{K}$ form a circle, then $S^3/\mathbf{K}$ equals the p-fold cyclic cover of $S^3/\mathbf{G}$ "branched" over the circle.*

Proof. The covering space theory of orbifolds assures us that $S^3/\mathbf{K}$ is a p-sheeted covering space of $S^3/\mathbf{G}$ (in the category of orbifolds). The unfolding about the circle in the singular set accounts for all sheets. □

$\mathbf{T} \times_{\mathbf{T}} \mathbf{T}^1$, $\mathbf{O} \times_{\mathbf{O}} \mathbf{O}^1$, $\mathbf{J} \times_{\mathbf{J}} \mathbf{J}^1$. The orbifolds corresponding to these groups $\mathbf{G}$ can be determined either by taking the appropriate 2-sheeted cover of $S^3/\rho^{-1}(\rho(\mathbf{G}))$ as illustrated in Figures 10–16 (the branch sets are dash-dot-dotted), or directly, using the fact that since $\mathbf{G}$ stabilizes 1, $S^3/\mathbf{G}$ is a suspension of a two-dimensional spherical orbifold. The elements of $\rho^{-1}(\rho(\mathbf{G})) - \mathbf{G}$ are those which interchange 1 and -1. The corresponding edges of the fundamental domain lie on the unit sphere (the boundary of the prefundamental domain).

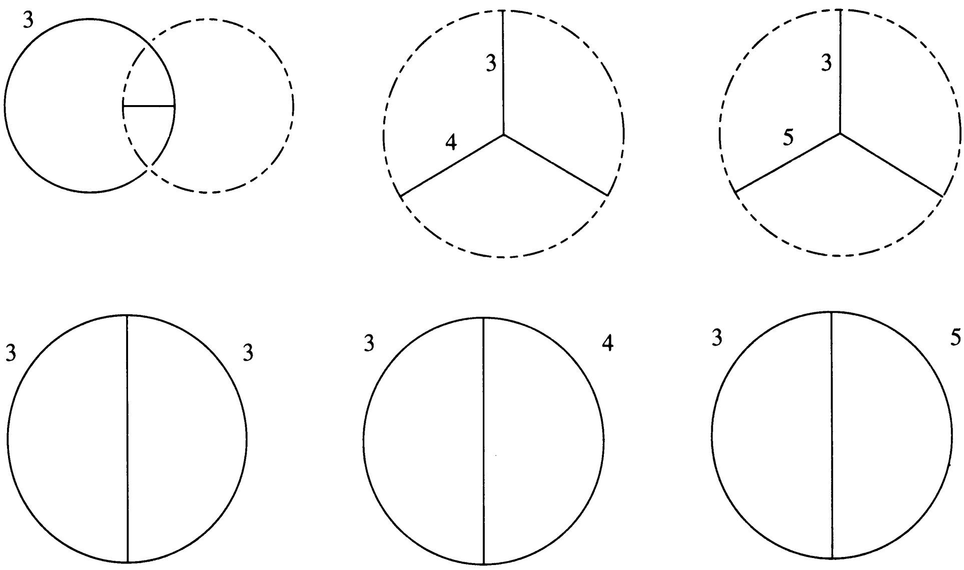


FIGURE 10. $\mathbf{T} \times_{\mathbf{T}} \mathbf{T}^1$, $\mathbf{O} \times_{\mathbf{O}} \mathbf{O}^1$, $\mathbf{J} \times_{\mathbf{J}} \mathbf{J}^1$: branch sets and orbifolds

$\mathbf{O} \times_{\mathbf{O}} \mathbf{O}^2$. The rotations in $\mathbf{O} \times_{\mathbf{O}} \mathbf{O} - \mathbf{O} \times_{\mathbf{O}} \mathbf{O}^2$ intersect the fundamental domain for $\mathbf{O} \times_{\mathbf{O}} \mathbf{O}$ along the segments labelled "A" in Figure 3.

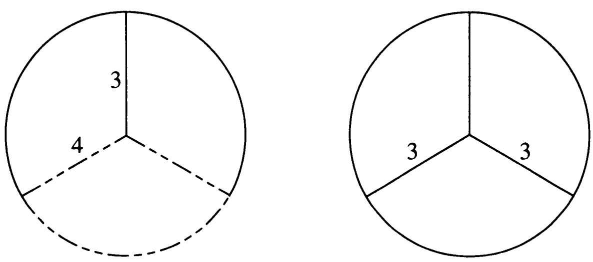


FIGURE 11. $\mathbf{O} \times_{\mathbf{O}} \mathbf{O}^2$: branch set and orbifold

$\mathbf{T} \times_{C_3} \mathbf{T}$. The rotations in $\mathbf{T} \times \mathbf{T} - \mathbf{T} \times_{C_3} \mathbf{T}$ intersect the fundamental domain for $\mathbf{T} \times \mathbf{T}$ along the segment labelled "A" in Figure 4.

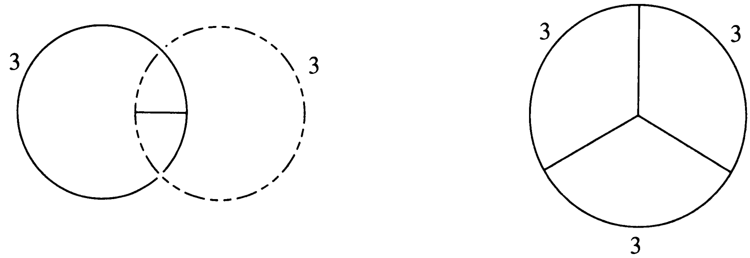


FIGURE 12. $\mathbf{T} \times_{C_3} \mathbf{T}$: branch set and orbifold

$\mathbf{O} \times_{C_2} \mathbf{O}$. This group is the (orientation-preserving) symmetry group of a tesselation of S^3 by regular octahedra with three meeting at each edge. The rotations in $\mathbf{O} \times \mathbf{O} - \mathbf{O} \times_{C_2} \mathbf{O}$ intersect the fundamental domain for $\mathbf{O} \times \mathbf{O}$ along the segments labelled " A " or " B " in Figure 5.

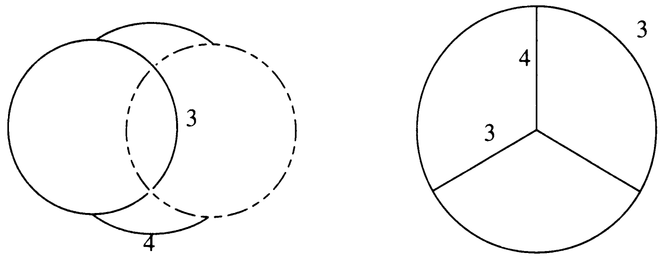


FIGURE 13. $\mathbf{O} \times_{C_2} \mathbf{O}$: branch set and orbifold

$\mathbf{O}\times\mathbf{T}$, $\mathbf{T}\times\mathbf{O}$. The rotations in $\mathbf{O}\times\mathbf{O}-\mathbf{O}\times\mathbf{T}$ intersect the fundamental domain for $\mathbf{O} \times \mathbf{O}$ along the segments labelled " A " or " C " in Figure 5. $S^3/(\mathbf{T} \times \mathbf{O})$ is a mirror image of $S^3/(\mathbf{O} \times \mathbf{T})$.

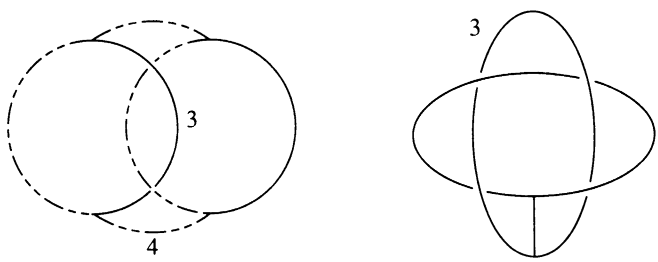


FIGURE 14. $\mathbf{O} \times \mathbf{T}$: branch set and orbifold

$\mathbf{J} \times \mathbf{T}$, $\mathbf{O} \times \mathbf{J}$, $\mathbf{T} \times \mathbf{J}$. The rotations in $\mathbf{J} \times \mathbf{O} - \mathbf{J} \times \mathbf{T}$ intersect the fundamental domain for $\mathbf{J} \times \mathbf{O}$ along the segments labelled " A " in Figure 9. $S^3/(\mathbf{T} \times \mathbf{J})$ and $S^3/(\mathbf{O} \times \mathbf{J})$ are mirror images of $S^3/(\mathbf{J} \times \mathbf{T})$ and $S^3/(\mathbf{J} \times \mathbf{O})$, respectively.

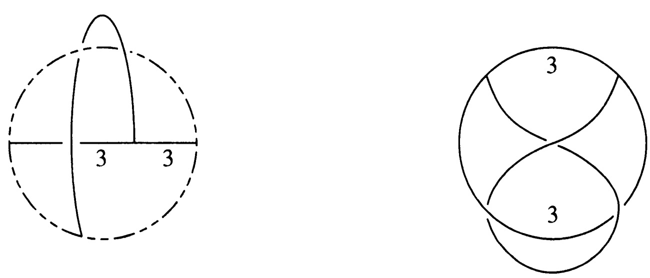


FIGURE 15. $\mathbf{J} \times \mathbf{T}$: branch set and orbifold

$\mathbf{J} \times_{\mathbf{J}}^{*} \mathbf{J}^1$. The quotient of S^3 by this group can be determined either by taking an appropriate 2-sheeted cover of $S^3/(\mathbf{J} \times_{\mathbf{J}}^{*} \mathbf{J})$, as illustrated below, or directly, using the fact that it is, as observed above, the (orientation-preserving) symmetry group of a regular 4-simplex.

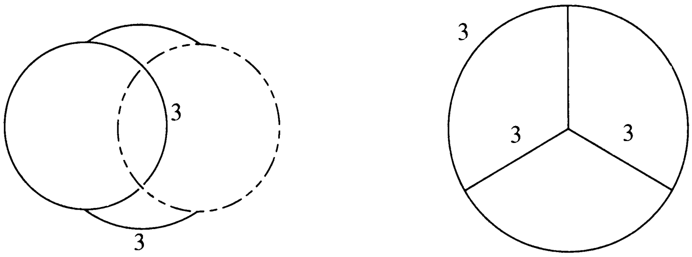


FIGURE 16. $\mathbf{J}\times^*_{\mathbf{J}}\mathbf{J}^1$: branch set and orbifold

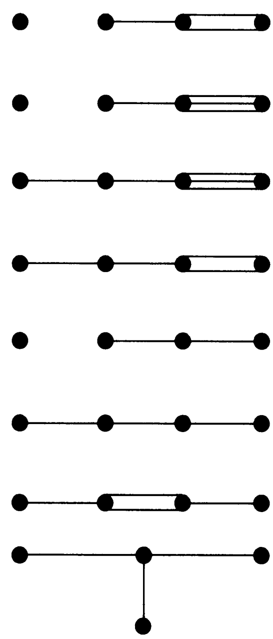
5. FINAL REMARKS

All of the 21 spherical orbifolds which we have investigated have S^3 as their underlying topological space. We do not know of any substantially faster way to prove this fact than the way shown above. Since the fundamental group of the underlying space of an orbifold is isomorphic to the quotient of the fundamental group of the orbifold by the normal subgroup generated by elements which act on the universal cover with fixed points, it follows that these 21 subgroups are generated by rotations. Again, we know of no general reason that would establish this property.

A posteriori, it can be observed that not only do these orbifolds not admit a "geometric" fibering over a 2-orbifold (since no fibration of S^3 by geodesics is invariant under the actions of the corresponding subgroups of $SO(4)$), but furthermore no "exotic" fibering is possible, if only for the reason that all the orbifolds have vertices in their singular sets corresponding to fixed points of the corresponding action on S^3 with tetrahedral, octahedral, or icosahedral isotropy, which can never occur in a 3-orbifold fibering over a 2-orbifold. The existence of such vertices in fact follows from the observation that all the "third-family" subgroups of $SO(3)\times SO(3)$, listed in §2, contain $T\times_T T$ as a subgroup (for the case of $J\times^*_J J$, recall the isomorphism between J and $\overline{J}$ fixing T).

Three of the 21 are suspensions of 2-dimensional spherical orbifolds, having singular sets which are combinatorically equivalent to a lower-case Greek letter theta. Eight of them are orientable double covers of nonorientable spherical orbifolds which arise as orbit spaces of groups generated by reflections in great spheres; these have singular sets which are combinatorially equivalent to a complete graph on 4 vertices (i.e., to the 1-skeleton of a 3-simplex). The corresponding Dynkin diagrams are shown as Figure 17.

Nonorientable spherical orbifolds all arise as quotients of orientable orbifolds by orientation-reversing involutions. By inspection, it is easy to find 25 such involutions of the orbifolds listed above: $\mathbf{T}\times_{\mathbf{T}}\mathbf{T}^1$ covers 4 nonorientable spherical orbifolds; $\mathbf{T}\times_{\mathbf{T}}\mathbf{T}$, $\mathbf{T}\times_{\mathbf{C}_3}\mathbf{T}$, $\mathbf{O}\times_{\mathbf{O}}\mathbf{O}^1$, $\mathbf{O}\times_{\mathbf{O}}\mathbf{O}^2$, $\mathbf{O}\times_{\mathbf{C}_2}\mathbf{O}$, $\mathbf{J}\times_{\mathbf{J}}\mathbf{J}^1$, and $\mathbf{J}\times^*_{\mathbf{J}}\mathbf{J}^1$ each cover 2 nonorientable spherical orbifolds; $\mathbf{T}\times\mathbf{T}$, $\mathbf{O}\times_{\mathbf{O}}\mathbf{O}$, $\mathbf{O}\times_{\mathbf{D}_3}\mathbf{O}$, $\mathbf{O}\times\mathbf{O}$, $\mathbf{J}\times_{\mathbf{J}}\mathbf{J}$, $\mathbf{J}\times^*_{\mathbf{J}}\mathbf{J}$, $\mathbf{J}\times\mathbf{J}$ each cover 1 nonorientable spherical orbifold. The list in [5, 3.22] seems to suggest that there are only 2, rather than 4, orbifolds whose orientable double covers correspond to lifts of $\mathbf{O}\times_{\mathbf{O}}\mathbf{O}$, but otherwise agrees with the above figures. The underlying spaces of 22 of these orbifolds are balls, and the remaining ones are suspensions of the projective plane.



$O \times_O O$

$J \times_J J$

$J \times J$

$O \times_{D_3} O$

$O \times_O O^2$

$T \times_{C_3} T$

$O \times_{C_2} O$

$J \times_J^* J^1$

FIGURE 17

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