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BESOV SPACES ON CLOSED SUBSETS OF $\mathbb{R}^n$

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ABSTRACT. Motivated by the need in boundary value problems for partial differential equations, classical trace theorems characterize the trace to a subset F of $\mathbb{R}^n$ of Sobolev spaces and Besov spaces consisting of functions defined on $\mathbb{R}^n$, if F is a linear subvariety $\mathbb{R}^d$ of $\mathbb{R}^n$ or a d-dimensional smooth submanifold of $\mathbb{R}^n$. This was generalized in [2] to the case when F is a d-dimensional fractal set of a certain type. In this paper, traces are described when F is an arbitrary closed set. The result may also be looked upon as a Whitney extension theorem in L^p.

0. INTRODUCTION

In this paper the trace of the Besov space $B_\alpha^{p,q}(\mathbb{R}^n)$ to an arbitrary closed subset F of $\mathbb{R}^n$ is characterized for $\alpha p > n$ and $\alpha < 1$. For $p = q$ the result is as follows. Consider a fixed measure μ with support F satisfying the doubling condition, more precisely the conditions (D_n) and (3) in the next section; such a measure exists due to a result in Volberg and Konyagin [4]. Then the trace of $B_\alpha^{p,q}(\mathbb{R}^n)$ to F consists of all functions f defined on F with finite norm

$$(1) \qquad \|f\|_{p,\mu} + \left(\iint_{|x-y|<1} \frac{|f(x)-f(y)|^p}{|x-y|^{\alpha p-n}(\mu(B(x,|x-y|)))^2}\, d\mu(x)\, d\mu(y) \right)^{1/p}$$

where $B(x, r)$ denotes the open ball with center x and radius r. We also consider results when the range of the parameters α and p is wider, in which case one has to impose restrictions on F. In particular the case $p = q = 2$ and $\alpha = 1$ is included, which means that we have a result for the Sobolev spaces $W_1^2(\mathbb{R}^n)$.

The results contain the classical results on the restriction of Besov spaces to hypherplanes and their generalizations to d-sets given in [2] (see Example 1). However, we do not allow the range of the parameters to be as wide as in these cases.

The possibility to prove theorems like the one in this paper is suggested by the results in Dynkin [1], where the norm (1) is used to prove results on the

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trace to the boundary of the unit circle of analytic functions whose derivatives are in the Hardy space H^p, under some additional conditions on F.

1. Definitions and results

We assume throughout the paper that F is a closed subset of $\mathbb{R}^n$, $n \geq 1$, and that μ is a measure with support F. Following [4], we say that μ satisfies the condition (D_s), $0 < s \leq n$, if for some constant c independent of x, r, and k,

$$(D_s) \qquad \mu(B(x, kr)) \leq ck^s \mu(B(x, r)), \qquad x \in F, r > 0, k \geq 1, kr \leq 1$$

(in [4], however, the limitation $kr \leq 1$ is not imposed). Note that (D_s) is a generalization of the familiar doubling condition: If μ satisfies (D_s), then

$$(2) \qquad \mu(B(x, 2r)) \leq c\mu(B(x, r)), \qquad x \in F, 0 < r \leq 1/2.$$

In general the measures considered will be assumed to satisfy the assumption that for some constants $c_1, c_2 > 0$

$$(3) \qquad c_1 < \mu(B(x, 1)) < c_2, \qquad x \in F.$$

In [4] it is shown that every closed set F carries a measure satsfying (D_n), and (Proposition 1 below) it may also be assumed that this measure satisfies (3). We furthermore say that μ satisfies the condition (L_d), $0 \leq d \leq n$, if for some constant $c > 0$

$$(L_d) \qquad \mu(B(x, kr)) \geq ck^d \mu(B(x, r)), \qquad x \in F, r > 0, k \geq 1, kr \leq 1.$$

Note that (L_0) imposes no restriction on μ.

Let assumptions on α, p, q, s, d, and μ, be as in Theorem 1 below. The *Besov space* $B_\alpha^{p,q}(F)$ is defined as the class of functions defined on F with finite norm

$$\|f\|_{p,\mu} + \left(\sum_{\nu=0}^{\infty} \left(2^{\nu(\alpha - n/p)} \left(\iint_{|x-y|<2^{-\nu}} \frac{|f(x) - f(y)|^p}{m_\nu(x) m_\nu(y)} \, d\mu(x) \, d\mu(y) \right)^{1/p} \right)^q \right)^{1/q},$$

where $m_\nu(x)$ denotes $\mu(B(x, 2^{-\nu}))$. For $p = \infty$, the terms in the ν-summation should be interpreted as $(2^{\nu\alpha} \sup_x \sup_{|x-y|<2^{-\nu}} |f(x) - f(y)|)^q$. This norm depends on μ, but if follows from Theorem 1 that different μ give rise to equivalent norms in a certain sense, see §3.5. One obtains an equivalent norm if the domain of integration is replaced by $|x - y| < c2^{-\nu}$, where $c > 0$ is a constant (cf. [3, p. 434]). For $p = q$, the norm is equivalent to the one given in the introduction (Proposition 2). If $F = \mathbb{R}^n$ and μ is the n-dimensional Lebesgue measure, then the space $B_\alpha^{p,q}(F)$ is equivalent to the classical Besov space $\Lambda_\alpha^{p,q}(\mathbb{R}^n)$ ([2, p. 128]; the definition of $\Lambda_\alpha^{p,q}(\mathbb{R}^n)$ will be recalled in §3).

The main result of this paper is the following trace theorem. To state it, we must define what we mean with the restriction Rf to F of a function f in $\Lambda_\alpha^{p,q}(\mathbb{R}^n)$, since functions in this space are defined a.e. with respect to the Lebesgue measure m in $\mathbb{R}^n$, only. We put

$$Rf(x) = \lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x,r)} f(y) \, dy$$

at every point $x \in F$ where the limit exists. It is part of the conclusion of the theorem that the limit exists μ-a.e. on F.

Theorem 1. *Let* $0 \le d \le n$, $d \le s \le n$, $s > 0$, $1 \le p, q \le \infty$, $(n-d)/p < \alpha < 1 + (n-s)/p$, *and let* μ *satisfy the conditions* (D_s), (L_d), *and* (3). *Then* $B^{p,q}_\alpha(F)$ *is the trace to* F *of* $\Lambda^{p,q}_\alpha(\mathbb{R}^n)$ *in the following sense*:

(a) *The restriction operator* R *is a continuous linear operator from* $\Lambda^{p,q}_\alpha(\mathbb{R}^n)$ *to* $B^{p,q}_\alpha(F)$.

(b) *There is a continuous linear extension operator* $\mathscr{E}$ *from* $B^{p,q}_\alpha(F)$ *to* $\Lambda^{p,q}_\alpha(\mathbb{R}^n)$ *such that* $R(\mathscr{E})$ *is the identity operator.*

Part (a) is proved in §4 and part (b) in §3. We point out that as a special case we have the following result, in which the space $B^{p,q}_\alpha(F)$ is defined by means of a measure μ with support F satisfying (D_n) and (3).

Corollary 1. *Let* F *be a closed subset of* $\mathbb{R}^n$, $1 \le p, q \le \infty$, $\alpha < 1$, *and* $\alpha p > n$. *Then the trace of* $\Lambda^{p,q}_\alpha(\mathbb{R}^n)$ *to* F *is* $B^{p,q}_\alpha(F)$.

We conclude this section by giving some examples.

Example 1. If μ is a d-set, $0 < d \le n$, as defined in [2], then μ satisfies (D_d), (L_d), and (3), and the restriction on α in Theorem 1 becomes $0 < \beta < 1$ where $\beta = \alpha - (n-d)/p$. The space $B^{p,q}_\alpha(F)$ is in this case equivalent to the space $B^{p,q}_\beta(F)$ defined in [2], and Theorem 1 reduces to the trace theorem in [2, p. 142]. However, in [2] the trace on a d-set F of $\Lambda^{p,q}_\alpha(\mathbb{R}^n)$ is characterized for $\beta = \alpha - (n-d)/p > 0$.

Example 2. Consider $F \subset \mathbb{R}^2 = \{(x_1, x_2)\}$ defined by $F = F_1 \cup F_2$, where $F_1 = \{(x_1+1)^2 + x_2^2 \le 1\}$ and $F_2 = \{0 \le x_1 \le 2, x_2 = 0\}$. Let m_n denote the n-dimensional Lebesgue measure, for $n = 1$ distributed over the x_1-axes, and put $d\lambda = x_1 dm_1$. Then $\mu = m_2|F_1 + \lambda|F_2$ satisfies (D_2), (L_1), and (3), and thus Theorem 1 yields, with this μ, a trace theorem in the range $1/p < \alpha < 1$.

Example 3. Let $F \subset \mathbb{R}^2$ be the set $F = \{0 \le x_1 \le 1, 0 \le x_2 \le x_1^\gamma\}$ where $\gamma > 1$, and put $d\nu = x_1^{1-\gamma} dm_2$ and $\mu = \nu|F$. Then it can be shown that μ satisfies (D_2), (L_1), and (3), so we get a trace theorem for $1/p < \alpha < 1$ using this measure μ.

Example 4. Let $F \subset \mathbb{R}^1$, $F = \{0\} \cup \{a_k, k \ge 0\}$ where $a_k = 2^{-k}$, and let μ be defined by $\mu(\{0\}) = 0$ and $\mu(\{a_k\}) = a^{-k}$, where a is a constant, $1 < a \le 2$. Then μ satisfies (D_s) with $s = \ln a / \ln 2$, and we get a trace theorem for $1/p < \alpha < 1 + (1-s)/p$. We now simplify by taking $q = p$. It can be shown that the $B^{p,p}_\alpha(F)$-norm can be simplified to an equivalent norm

$$\left(\sum_{i=0}^{\infty} |f(a_i)|^p a^{-i}\right)^{1/p} + \left(\sum_{i=0}^{\infty} 2^{i(\alpha p - 1)} |f(a_i) - f(a_{i+1})|^p\right)^{1/p};$$

also, it can be shown directly that these norms are equivalent for $a > 1$. It follows that we have a trace theorem for $1/p < \alpha < 1 + 1/p$, with the norm in the trace space given by the above expression with, e.g., $a = 2$.

2. Remarks on the preceding definitions

In the conditions (D_s) and (L_d), it is assumed that $kr \le 1$. However, for $r_0 > 1$, the class of measures satisfying (L_d) is unchanged if $kr \le 1$ is

replaced by $kr \leq r_0$, and so is the class satisfying the conditions (D_s) and (3) simultaneously. To show the latter statement, one uses the observation that if $\mu(B(x, 1)) \leq c_2$, $x \in F$, then

$$\mu(B(x, r)) \leq cr^n, \qquad r \geq 1, x \in \mathbb{R}^n. \tag{4}$$

It is also easy to realize that if μ satisfies (L_d), then the condition defining (L_d) is satisfied not only for $x \in F$ but for $x \in \mathbb{R}^n$ and that if (D_s), (L_d), or (3) hold for open balls, they hold for closed.

If μ satisfies (D_s), then the set F has Hausdorff dimension $\leq s$, and if μ satisfies (L_d), then it has dimension $\geq d$ at every point in the sense that every set $F \cap B(x, r)$, $x \in F$, has dimension $\geq d$. These statements can be proved in the same way as similar results for d-sets in [2, p. 32]. The reason for this is that, putting $A(d) = r_1^{-d}\mu(B(x, r_1))$, we have for $x \in F$ and $r \leq r_1$ (cf. (18)) the inequalities

$$\mu(B(x, r)) \leq cA(d)r^d \quad \text{and} \quad \mu(B(x, r)) \geq cA(s)r^s \tag{5}$$

if μ satisfies (L_d) or (D_s), respectively; these conditions are similar to those used in the definition of a d-set. Note also that it follows that μ has no point mass if μ satisfies (L_d), $d > 0$. One can think of the significance of these observations in relation to the assumptions in Theorem 1 as follows. The condition (D_s) guarantees that F is so small that $B_\alpha^{p,q}(F)$ should be defined by means of first differences, and (L_d) that F is so big that the trace on F of a function in $B_\alpha^{p,q}(F)$ exists μ-a.e. In [4] it is shown that every closed set is the support of a measure satisfying (D_n). We shall verify that things may be arranged so that (3) is fulfilled, too.

Proposition 1. *For every closed set $F \subset \mathbb{R}^n$, there is a measure μ with support F satisfying (D_n) and* (3).

Proof. Cover $\mathbb{R}^n$ with the closed balls of radii $r_0 = 6\sqrt{n}$ which have centers in points with integer coordinates. Denote the balls intersecting F by B_i, $i = 1, 2, \ldots$, and let ν_i be a probability measure on $B_i \cap F$ satisfying the condition (D_n), without the limitation $kr \leq 1$; ν_i exists due to [4]. Let φ be given by $\varphi(x) = 1$ if $|x| \leq r_0/2$, $\varphi(x) = ((r_0 - |x|)2/r_0)^n$ if $r_0/2 \leq |x| \leq r_0$, and $\varphi(x) = 0$ elsewhere, and let φ_i be the translation of φ such that the support of φ_i is B_i. Put $d\mu_i = \varphi_i d\nu_i$ and $\mu = \sum \mu_i$. Then μ has the desired properties, as we shall see.

Denote the center of B_i by x_i and the ball $B_i(x_i, r_0/2)$ by $B_i/2$. For any $x \in F$ there is an x_i with $|x - x_i| \leq \sqrt{n}/2$, so $B(x, 1)$ is contained in $B_i/2$ and thus $\mu(B(x, 1)) \geq \mu_i(B(x, 1)) = \nu_i(B(x, 1))$. Since ν_i satisfies (D_n), with the constant c in the condition (D_n) depending only on n according to the construction in [4], we have $1 = \nu_i(B(x, 2r_0)) \leq c\nu_i(B(x, 1))$, and the lower bound in (3) follows. The upper bound follows from the fact that only a bounded number of B_i's intersect $B(x, 1)$.

To show that (D_n) holds, consider a ball $B(x, r)$, $x \in F$, and assume $kr \leq 1$. We shall show that for any i we have $\mu_i(B(x, kr)) \leq ck^n\mu(B(x, r))$ which gives (D_n). We consider two cases.

Case 1. The distance d from $B(x, kr)$ to the complement of B_i is $\geq r$. Assume first that $B(x, kr)$ does not intersect $B_i/2$, and note that if a is the distance from a point $x \in B_i \backslash (B_i/2)$ to the boundary of B_i, then $\varphi_i(x) =$

$(2a/r_0)^n$. Then from $\mu_i = \varphi_i \nu_i$ it follows that

$$\begin{aligned}\mu_i(B(x, kr)) &\le (2(d+2kr)/r_0)^n \nu_i(B(x, kr)) \\ &\le (2(d+2kr)/r_0)^n ck^n \nu_i(B(x, r)) \\ &\le (2(d+2kr)/r_0)^n ck^n (2(d+(k-1)r)/r_0)^{-n} \mu_i(B(x, r)) \\ &\le c3^n k^n \mu_i(B(x, r)).\end{aligned}$$

If $B(x, kr)$ intersects $B_i/2$ we instead have, using $kr \le 1$,

$$\mu_i(B(x, kr)) \le \nu_i(B(x, kr)) \le ck^n \nu_i(B(x, r)) \le ck^n (2/r_0)^{-n} \mu_i(B(x, r)).$$

Case 2. $d < r$. Then

$$\mu_i(B(x, kr)) \le (2(r+2kr)/r_0)^n \nu_i(B(x, kr)) \le (6kr/r_0)^n.$$

But $B(x, r)$ is contained in some $B_j/2$, so $\mu(B(x, r)) \ge \nu_j(B(x, r))$, and we also have

$$1 = \nu_j(B(x, 2r_0)) = \nu_j(B(x, (2r_0/r)r)) \le cr^{-n} \nu_j(B(x, r)).$$

Together, these estimates give $\mu_i(B(x, kr)) \le ck^n r^n \le ck^n \mu(B(x, r))$.

Finally we give, for $p = q$, a norm in $B^{p,q}_\alpha(F)$ which in the case when $F = \mathbb{R}^n$ and μ is Lebesgue measure reduces to a classical norm in $\Lambda^{p,p}_\alpha(\mathbb{R}^n)$. Compare also with §0.

Proposition 2. *Let α, p, s, d, and μ be as in Theorem* 1. *Then the $B^{p,p}_\alpha(F)$-norm is equivalent to expression* (1).

Proof. The $B^{p,q}_\alpha(F)$ norm equals, if $p = q$, $\|f\|_{p,\mu}$ plus

$$\left(\sum_{\nu=0}^{\infty} 2^{\nu(\alpha p - n)} \sum_{\tau=\nu}^{\infty} \iint_{2^{-\tau-1} \le |x-y| < 2^{-\tau}} \frac{|f(x)-f(y)|^p}{m_\nu(x) m_\nu(y)} \, d\mu(x)\, d\mu(y)\right)^{1/p}. \tag{6}$$

Since μ satisfies (L_d) we have $m_\nu(x) = \mu(B(x, 2^{\tau-\nu}2^{-\tau})) \ge c2^{(\tau-\nu)d} m_\tau(x)$. Using this in (6) and reversing the order of the summations one obtains that (6) is less than

$$\left(c \sum_{\tau=0}^{\infty} 2^{-2\tau d} \sum_{\nu=0}^{\tau} 2^{\nu(\alpha p - n + 2d)} \iint_{2^{-\tau-1} \le |x-y| < 2^{-\tau}} \frac{|f(x)-f(y)|^p}{m_\tau(x) m_\tau(y)} \, d\mu(x)\, d\mu(y)\right)^{1/p},$$

where the sum with respect to ν may be replaced by $2^{\tau(\alpha p - n + 2d)}$. Since, by (2), $m_\tau(y) \ge cm_{\tau-1}(y) \ge c\mu(B(x, |x-y|))$ if $|x-y| < 2^{-\tau}$, it follows that the $B^{p,q}_\alpha(F)$-norm is less than (1).

The converse follows immediately if we write the double integral in the latter norm as a sum of integrals over $2^{-\nu-1} \le |x-y| < 2^{-\nu}$, and use that then we have $\mu(B(x, |x-y|)) \ge cm_\nu(x), cm_\nu(y)$.

3. The extension theorem

3.1. The proof of part (b) in Theorem 1 is similar to the proof of the extension theorem in [2], and we shall now and then refer to [2] for details. To define the extension operator, we need the Whitney decomposition of the complement of F, which has the following properties (cf. [2, p. 23]). It consists of a collection of closed cubes Q_i, $i = 1, 2, \ldots$, with mutually disjoint interiors and sides

parallel to the axes such that $\complement F = \bigcup Q_i$. Denote the center of Q_i by x_i, its diameter by l_i, and its sidelength, which we assume is of the form 2^{-M}, M integer, by s_i. Then

$$l_i \le d(Q_i, F) \le 4l_i, \tag{7}$$

where $d(Q_i, F)$ is the distance from Q_i to F, and if Q_i and Q_ν touch, then

$$1/4l_i \le l_\nu \le 4l_i. \tag{8}$$

Let $0 < \varepsilon < 1/4$, and put $Q_i^* = (1+\varepsilon)Q_i$. Then each point in $\complement F$ is contained in at most $N_0 = N_0(n)$ cubes Q_i^*, and, furthermore, Q_i^* intersects a cube Q_ν only if Q_i touches Q_ν. To this decomposition we associate a partition of unity, consisting of nonnegative functions φ_i such that $\varphi_i(x) = 0$ if $x \notin Q_i^*$, $\sum \varphi_i(x) = 1$, $x \in \complement F$, and

$$|D^j \varphi_i(x)| \le A_j (l_i)^{-|j|}. \tag{9}$$

We are now prepared to define the extension operator $\mathscr{E}$, and assume first that μ satisfies (D_s) for some $s < n$; the remaining case will be taken care of in §3.5. Putting $c_i = \mu(B(x_i, 6l_i))^{-1}$, we define $\mathscr{E}$ on $B_\alpha^{p,q}(F)$ by

$$\mathscr{E} f(x) = \sum_{i \in I} \varphi_i(x) c_i \int_{|t-x_i| \le 6l_i} f(t)\, d\mu(t), \qquad x \in \complement F,$$

where I means those i such that $s_i \le 1$. Note that since $s < n$, this defines $\mathscr{E} f$ a.e. on $\mathbb{R}^n$, by the discussion in the preceding section. We prove in §3.3 that $\mathscr{E} f \in \Lambda_\alpha^{p,q}(\mathbb{R}^n)$ and in §3.4 that $\mathscr{E} f | F = f$, but give first some preliminary estimates related to the above construction.

3.2. Assume that Q_i touches Q_χ. By (7), there is a point $p_i \in F$ with $|p_i - x_i| \le 5l_i$, and since, if $|t - x_\chi| \le 6l_\chi$, we have by (8) that

$$|t - x_i| \le |t - x_\chi| + |x_\chi - x_i| \le 6l_\chi + l_i + l_\chi \le 30 l_i$$

it follows from the doubling condition that $\mu(B(x_i, 6l_i)) \ge \mu(B(p_i, l_i)) \ge c\mu(B(p_i, 30l_i)) \ge c\mu(B(x_\chi, 6l_\chi))$, so

$$c_i \le c c_\chi. \tag{10}$$

As we saw, we have

$$|t - x_\chi| \le 30 l_\chi \quad \text{if } |t - x_i| \le 6l_i \tag{11}$$

and by (9) and (8) we have

$$|D^j \varphi_i| \le c l_\chi^{-|j|}. \tag{12}$$

Note that (10), (11), and (12) hold if $x \in Q_\chi$ and $\varphi_i(x) \ne 0$, since then Q_i and Q_χ touch. Using (8) it also follows that $\mathscr{E} f(x) = 0$ if $x \in Q_\chi$, $s_\chi > 4$ and $\sum_{i \in I} \varphi_i(x) = \sum \varphi_i(x)$ if $x \in Q_\chi$, $s_\chi \le 1/4$.

The following two lemmas will be needed. The first is a variation of Lemma D in [2, p. 111], and we omit its proof since it is similar to the proof in [2], the only essential difference being that we use (10) as a replacement for [2, formula (6)].

The second lemma is the same as Lemma 2 in [2]. Put

$$J_p(x_\chi, x_\tau) = \left(c_\chi c_\tau \iint_{\substack{|t-x_\chi|\le 30l_\chi \\ |s-x_\tau|\le 30l_\tau}} |f(t)-f(s)|^p \, d\mu(t)\, d\mu(s) \right)^{1/p}.$$

Lemma 1. *Let $1 \le p \le \infty$ and let $x \in Q_\chi$ and $y \in Q_\tau$, $s_\chi, s_\tau \le 1/4$. Then the following hold:*

(a) $|\mathscr{E}f(x) - \mathscr{E}f(y)| \le cJ_p(x_\chi, x_\tau)$.

(b) $|D^j(\mathscr{E}f)(x)| \le cl_\chi^{-|j|} J_p(x_\chi, x_\tau)$, $|j| > 0$.

(c) $|\mathscr{E}f(x) - b| \le c(c_\chi \int_{|t-x_\chi|\le 30l_\chi} |f(t)-b|^p \, d\mu(t))^{1/p}$ *where b is a constant, and*

(d) *for any Q_χ and $|j| \ge 0$*

$$|D^j(\mathscr{E}f)(x)| \le cl_\chi^{-|j|} \left(c_\chi \int_{|t-x_\chi|\le 30l_\chi} |f(t)|^p \, d\mu(t) \right)^{1/p}.$$

Let Δ_ν denote the union of those cubes Q_χ which have sides of length $s_\chi = 2^{-\nu}$.

Lemma 2. *Let $a > 0$, let h be a nonnegative function defined on a closed set $F \subset \mathbb{R}^n$, and let μ be supported by F. Let g be given by*

$$g(x) = \int_{|t-x_x|\le al_x} h(t)\, d\mu(t), \qquad x \in \operatorname{int} Q_x,\ s_\chi = 2^{-\nu}.$$

Then, for $x_0 \in \mathbb{R}^n$ and $0 < r \le \infty$

$$\int_{\substack{x\in\Delta_\nu \\ |x-x_0|\le r}} g(x)\, dx \le c2^{-\nu n} \int_{|t-x_0|\le r+c_0 2^{-\nu}} h(t) d\mu(t); \tag{13}$$

in particular for $r = \infty$

$$\int_{x\in\Delta_\nu} g(x)\, dx \le c2^{-\nu n} \int h(t)\, d\mu(t). \tag{14}$$

Here the constants c and c_0 depend only on a and n.

Finally we note that it is shown in [2, p. 110], that

$$y \in \Delta'_\nu = \bigcup_{i=\nu-2}^{\nu+3} \Delta_i \quad \text{if } x \in \Delta_\nu \text{ and } |x-y| < 2^{-\nu}/2. \tag{15}$$

3.3. We are now ready to prove that $\mathscr{E}f \in \Lambda_\alpha^{p,q}(\mathbb{R}^n)$. Explicitly, we treat the case $p < \infty$ and $q < \infty$ only. Recall that the norm of $\mathscr{E}f$ in $\Lambda_\alpha^{p,q}(\mathbb{R}^n)$ can be given by

$$\|\mathscr{E}f\|_p + \left(\int_{|h|<H} \frac{\|\Delta_h^s(\mathscr{E}f)\|_p^q}{|h|^{n+\alpha q}}\, dh \right)^{1/q} \tag{16}$$

where s is the smallest integer greater than α and H is a positive number.

In order to estimate $\|\mathscr{E}f\|_p$, note first that, with p_χ as in the argument leading to (10), it follows from the doubling condition that if $|t - x_\chi| \le 30 l_\chi$, then

$$\mu(B(x_\chi, 6l_\chi)) \ge \mu(B(p_\chi, l_\chi)) \ge c\mu(B(p_\chi, 36l_\chi)) \ge c\mu(B(t, l_\chi)) \ge c\mu(B(t, s_\chi)),$$

so

$$c_\chi \le c/m_\nu(t) \quad \text{if } |t - x_\chi| \le 30 l_\chi,\ s_\chi = 2^{-\nu}. \tag{17}$$

If x belongs to some Q_χ with $s_\chi > 4$ then $\mathscr{E}f(x) = 0$, and otherwise it follows from (d) in Lemma 1 and (17) that

$$|\mathscr{E}f(x)|^p \le c \int_{|t-x_\chi| \le 30 l_\chi} \frac{|f(t)|^p}{m_\nu(t)}\, d\mu(t), \qquad x \in Q_\chi,\ s_\chi = 2^{-\nu}.$$

Using (14) in Lemma 2 with $h(t)$ equal to $h_\nu(t) = |f(t)|^p/m_\nu(t)$ and $g(x)$ equal to $g_\nu(x) = \int_{|t-x_\chi| \le 30 l_\chi} h_\nu(t)\, d\mu(t)$, $x \in \operatorname{int} Q_\chi$, $s_\chi = 2^{-\nu}$, one obtains

$$\begin{aligned}\|\mathscr{E}f\|_p^p &= \sum_{\nu=-2}^{\infty} \int_{\Delta_\nu} |\mathscr{E}f(x)|^p\, dx \le c \sum_{\nu=-2}^{\infty} \int_{\Delta_\nu} g_\nu(x)\, dx \\ &\le c \sum_{\nu=-2}^{\infty} 2^{-\nu n} \int \frac{|f(t)|^p}{m_\nu(t)}\, d\mu(t).\end{aligned}$$

Now, if $1 \ge r_1 > r_0 > 0$ and μ satisfies (D_s), then by writing $B(t, r_1) = B(t, (r_1/r_0) r_0)$ we see that

$$\mu(B(t, r_1)) \le c(r_1/r_0)^s \mu(B(t, r_0)), \qquad t \in F. \tag{18}$$

With $r_1 = 1$ and $r_0 = 2^{-\nu}$ we get $c_1 \le m_0(t) \le c 2^{\nu s} m_\nu(t)$, $t \in F$, where the first inequality is due to assumption (3). Using this estimate for m_ν above and performing the summation we get, since $s < n$, that $\|\mathscr{E}f\|_p \le c\|f\|_{p,\mu}$.

To treat the second term in (16), we for convenience choose $H = 2^{-6}/2s$ and put $h_m = 2^{-m}/2s$. Then the second term in (16) is less than

$$c\left(\sum_{m=6}^{\infty} 2^{m(n+\alpha q)} \int_{h_{m+1} \le |h| < h_m} \left(\int |\Delta_h^s(\mathscr{E}f)(x)|^p\, dx\right)^{q/p} dh\right)^{1/q}. \tag{19}$$

Since $\mathscr{E}f(x) = 0$ if $x \in \Delta_\nu$, $\nu \le -3$, it follows from (7) that $\Delta_h^s(\mathscr{E}f)(x) = 0$ if $x \in \Delta_\nu$, $\nu \le -5$ and $|h| < h_6$, and putting $G_m = \bigcup_{\nu=-4}^{m} \Delta_\nu$ and $F_{m+1} = \bigcup_{\nu=m+1}^{\infty} \Delta_\nu$, we see that (19) is less than $c(\sum A_m)^{1/q} + c(\sum B_m)^{1/q}$, where A_m and B_m are as the terms in (19), but with the x-integration taken over $x \in G_m$ and $x \in F_{m+1}$, respectively.

In [2, pp. 115–116] it is shown (cf. (15)) that

$$\int_{\Delta_\nu} |\Delta_h^s(\mathscr{E}f)(x)|^p\, dx \le c|h|^{sp} \sum_{|j|=s} \int_{\Delta'_\nu} |D^j(\mathscr{E}f)(x)|^p\, dx$$

if $|h| < h_\nu$, where $\Delta'_\nu = \bigcup_{i=\nu-2}^{\nu+3} \Delta_i$ (this does not depend on the form of $\mathscr{E}f$). Thus, if $|h| < h_m$,

$$\int_{G_m} |\Delta_h^s(\mathscr{E}f)(x)|^p\, dx \le c 2^{-msp} \sum_{|j|=s} \sum_{\nu=-2}^{m+3} \int_{\Delta_\nu} |D^j(\mathscr{E}f)(x)|^p\, dx$$

since $D^j(\mathscr{E}f)(x) = 0$ if $x \in \Delta_\nu$, $\nu \le -3$, and putting this into A_m, and performing the h-integration we get

$$\sum_{m=6}^{\infty} A_m \le c \sum_{m=6}^{\infty} 2^{m(\alpha-s)q} \left(\sum_{|j|=s} \sum_{\nu=-2}^{m+3} \int_{\Delta_\nu} |D^j(\mathscr{E}f)(x)|^p \, dx \right)^{q/p}$$

$$\le c \sum_{\nu=-2}^{\infty} 2^{\nu(\alpha-s)q} \left(\sum_{|j|=s} \int_{\Delta_\nu} |D^j(\mathscr{E}f)(x)|^p \, dx \right)^{q/p} = c \sum_{\nu=-2}^{\infty} A'_\nu,$$

where the latter inequality is a consequence of Hardy's inequality, [2, p. 121], since $\alpha < s$.

Now we use (b) in Lemma 1 with $\chi = \tau$ and get for $x \in Q_\chi \subset \Delta_\nu$, $\nu \ge 2$, and $|j| = s$ that

$$|D^j(\mathscr{E}f)(x)|^p \le c2^{\nu sp} \int_{|t-x_\chi| \le 30 l_\chi} \int_{|t-s|<c2^{-\nu}} |f(t) - f(s)|^p \, d\mu_1,$$

where we temporarily put $d\mu_1 = d\mu(s)\, d\mu(t)/(m_\nu(t) m_\nu(s))$. To obtain this we put c_χ under the integral sign in J_p, applied (17), and used that $|s - x_\chi| \le 30 l_\chi$ and $|t - x_\chi| \le 30 l_\chi$ imply $|s - t| \le 60 l_\chi < c2^{-\nu}$. Using (14) in Lemma 2 with the integral with respect to s above as $h(t)$, we obtain

$$\sum_{\nu=2}^{\infty} A'_\nu \le c \sum_{\nu=2}^{\infty} 2^{\nu(\alpha-s)q} \left(2^{\nu sp} 2^{-\nu n} \iint_{|t-s|<c2^{-\nu}} |f(t) - f(s)|^p \, d\mu_1 \right)^{q/p},$$

which is less than a constant times the $B_\alpha^{p,q}(F)$-norm of f raised to the power q by the remarks given after the definition of $B_\alpha^{p,q}(F)$. The estimation of A'_ν, $\nu < 2$, is simpler; by means of (d) in Lemma 1 and (14) in Lemma 2 one easily obtains that $A'_\nu \le c\|f\|_{p,\mu}^q$, $-2 \le \nu \le 1$.

In order to treat $\sum B_m$, we start from a simple estimate given in [2, p. 117], which gives that if $|h| < h_m$, then

$$\int_{F_{m+1}} |\Delta_h^s(\mathscr{E}f)(x)|^p \, dx \le c \int_{F_{m-2}} |\mathscr{E}f(x) - \mathscr{E}f(x+h)|^p \, dx.$$

It is easy to show (we perform it at the end of this paragraph), that the right-hand side is, if $|h| < h_m$, less than

$$c|h|^{-n} \int_{|t|<2|h|} \int_{x,\, x+t \in F_{m-4}} |\mathscr{E}f(x) - \mathscr{E}f(x+t)|^p \, dx\, dt. \tag{20}$$

Thus

$$\sum_{m=6}^{\infty} B_m \le c \sum_{m=6}^{\infty} 2^{m\alpha q} \left(2^{mn} \iint_{\substack{x,\, y \in F_{m-4} \\ |x-y|<2h_m}} |\mathscr{E}f(x) - \mathscr{E}f(y)|^p \, dx\, dy \right)^{q/p}.$$

Now we proceed as in [2, p. 118], with small modifications, and get for ν, $N \ge m-4$, $m \ge 6$,

$$\iint_{\substack{x \in \Delta_\nu,\, y \in \Delta_N \\ |x-y|<2h_m}} |\mathscr{E}f(x) - \mathscr{E}f(y)|^p \, dx\, dy$$

$$\le c2^{-\nu n} 2^{-Nn} \iint_{|t-s|<c2^{-m}} \frac{|f(t) - f(s)|^p}{m_\nu(t) m_N(s)} \, d\mu(t)\, d\mu(s).$$

The calculation involves starting with the estimate (a) in Lemma 1 for $|\mathscr{E}f(x) - \mathscr{E}f(y)|^p$, then replacing c_χ, as we may by (17), by $1/m_\nu(t)$ if $s_\chi = 2^{-\nu}$ and integrating over $x \in \Delta_\nu$, $|x-y| < 2h_m$ by means of (13) in Lemma 2, replacing then c_τ by $1/m_N(s)$ and integrating over $y \in \Delta_N$ applying (14). From (18) we get that $m_\nu(t) \geq c2^{(m-\nu)s} m_m(t)$, $\nu \geq m$, an inequality which is obviously valid for $m-4 \leq \nu < m$, also, and using this and the analogous estimate for $m_N(s)$ one obtains

$$\sum_{m=6}^{\infty} B_m \leq c \sum_{m=6}^{\infty} 2^{m\alpha q} \left(\sum_{\nu,\, N=m-4}^{\infty} 2^{mn} 2^{-2ms} 2^{-\nu(n-s)} 2^{-N(n-s)} I_\nu \right)^{q/p},$$

where I_m is the double integral with respect to t and s above with m_ν and m_N replaced by m_m. Since we assume $s < n$, we can perform the inner summations and get the desired estimate, which completes the proof of the result $\mathscr{E}f \in \Lambda_\alpha^{p,q}(\mathbb{R}^n)$.

We conclude this section by showing how (20) follows. For $M \subset \mathbb{R}^n$ we have for a certain constant c_1

$$\begin{aligned} c_1|h|^n \int_{x\in M} |f(x+h) - f(x)|^p\, dx &= \int_{|t|<|h|} dt \int_{x\in M} |f(x+h) - f(x)|^p\, dx \\ &\leq c \int_{|t|<|h|} \int_{x\in M} |f(x+t) - f(x)|^p\, dx \\ &\quad + c \int_{|t|<|h|} \int_{x\in M} |f(x+h) - f(x+t)|^p\, dx, \end{aligned}$$

where the latter term equals $c \int_{|t|<|h|} \int_{x\in M_h} |f(x) - f(x+t-h)|^p\, dx$, where M_h is a certain translation of M. Thus the right-hand side of the inequality above is less than $c \int_{|t|<2|h|} \int_{x\in M_h \cup M} |f(x+t) - f(x)|^p\, dx$. In our case we have $M = F_{m-2}$ and $|h| < h_m$, and (20) follows with the aid of (7).

3.4. It remains to prove that $\mathscr{E}f|F = f$. From (c) in Lemma 1 it follows (cf. [2, p. 119]) with the aid of (17) and (13) in Lemma 2 that

$$\int_{\substack{x\in\Delta_\nu \\ |x-t_0|\leq r}} |\mathscr{E}f(x) - f(t_0)|^p\, dx \leq c2^{-\nu n} \int_{|t-t_0|\leq r+c2^{-\nu}} \frac{|f(t)-f(t_0)|^p}{m_\nu(t)}\, d\mu(t).$$

If $|x-t_0| \leq r$ and $x \in \Delta_\nu$, then (see [2]) $\sqrt{n}2^{-\nu} \leq r$, so if τ is the integer such that $2^{-\tau} \leq r/\sqrt{n} < 2^{-(\tau-1)}$, we get integration over $|x-t_0| \leq r$ in the left-hand side if we perform summation from τ to ∞, and using again that, by (18), $m_\nu(t) \geq c2^{(\tau-\nu)s} m_\tau(t)$ and that $s < n$, we get

$$\int_{|x-t_0|\leq r} |\mathscr{E}f(x) - f(t_0)|^p\, dx \leq c2^{-\tau n} \mu(B(t_0, 2^{-\tau})) g_\tau(t_0), \tag{21}$$

where

$$g_\tau(t_0) = \int_{|t-t_0|<c2^{-\tau}} \frac{|f(t)-f(t_0)|^p}{m_\tau(t) m_\tau(t_0)}\, d\mu(t)$$

and where we also extended the fraction by $m_\tau(t_0)$. By the definition of $B_\alpha^{p,q}(F)$ and the remarks given after it, we have

$$\sum_{\tau=0}^{\infty} \left(2^{\tau(\alpha-n/p)} \left(\int g_\tau(t_0)\, d\mu(t_0) \right)^{1/p} \right)^q < \infty,$$

so $\int g_\tau(t_0)\,d\mu(t_0) \le c2^{-\tau(\alpha p-n)}$. Now, if we divide both sides in (21) by $m(B(t_0, r))$ and use (15), we obtain on the right-hand side

$$2^{-\tau d} g_\tau(t_0) = 2^{-\tau\varepsilon} 2^{-\tau(d-\varepsilon)} g_\tau(t_0).$$

But by the estimate above we have

$$\int \sum_{\tau=0}^{\infty} 2^{-\tau(d-\varepsilon)} g_\tau(t_0)\,d\mu(t_0) \le \sum_{\tau=0}^{\infty} 2^{-\tau(\alpha p-n+d-\varepsilon)},$$

so if $\alpha p-(n-d)-\varepsilon > 0$, then $2^{-\tau(d-\varepsilon)} g_\tau(t_0)$ is uniformly bounded in τ for μ-almost all t_0. From our assumption $\alpha > (n-d)/p$ it follows that $\varepsilon > 0$ may be chosen so that $\alpha p-(n-d)-\varepsilon > 0$, and thus we get for μ-almost all t_0

$$\frac{1}{m(B(t_0, r))} \int_{|x-t_0|\le r} |\mathscr{E}f(x) - f(t_0)|^p\,dx \to 0, \qquad r \to 0,$$

which gives the desired result (cf. [2]).

3.5. **The case** $s = n$. In this case the assumption on α is $(n-d)/p < \alpha < 1$. We shall reduce this case to the case $s < n$ by considering $\mathbb{R}^n$ as a subset of $\mathbb{R}^{n+1}$. It is clear from the definition, that the space $B_\alpha^{p,q}(F)$, $F \subset \mathbb{R}^n$, is the same as $B_{\alpha+1/p}^{p,q}(F)$ when F is considered as a subset of $\mathbb{R}^{n+1}$. Let $f \in B_\alpha^{p,q}(F)$. Since $(n+1-d)/p < \alpha+1/p < 1+(n+1-s)/p$ and $s = n < n+1$, the already proved part of the extension theorem gives us a function $Ef \in \Lambda_{\alpha+1/p}^{p,q}(\mathbb{R}^{n+1})$ such that $Ef|F = f$ μ-a.e. The classical restriction theorem for Besov spaces tells us that $Ef|\mathbb{R}^n \in \Lambda_\alpha^{p,q}(\mathbb{R}^n)$, and we take $\mathscr{E}f = Ef|\mathbb{R}^n$ as our extension of f. Possibly, $\mathscr{E}f|F$ could be different from $Ef|F$, since the restriction operators are defined in different ways, but at least $\mathscr{E}f|F \in B_\alpha^{p,q}(F)$ by the restriction part of Theorem 1, which will be proved in the next section. However, using that both $g \to g|F$ and $g \to (g|\mathbb{R}^n)|F$ are continuous operators from $\Lambda_{\alpha+1/p}^{p,q}(\mathbb{R}^{n+1})$ to $B_\alpha^{p,q}(F)$, and that $C_0^\infty(\mathbb{R}^{n+1})$ is dense in $\Lambda_{\alpha+1/p}^{p,q}(\mathbb{R}^{n+1})$, it is easy to see that the two ways of taking restrictions coincide (cf. [2, p. 211]). Thus, $\mathscr{E}f|F = f$ a.e., and since all operators involved in the above argument are continuous, the extension theorem in the case $s = n$ follows.

To see that different measures μ on F give rise to equivalent spaces $B_\alpha^{p,q}(F)$, we argue in a similar way. Let f belong to the space $B_\alpha^{p,q}(F)$ defined by means of μ_1. By the extension theorem, there is a function $\mathscr{E}f \in \Lambda_\alpha^{p,q}(\mathbb{R}^n)$ such that $\mathscr{E}f|F = f$ except on a set E_1 with $\mu(E_1) = 0$. By the restriction theorem, $\mathscr{E}f|F$ exists except on E_2, $\mu_2(E_2) = 0$, and belongs to $\overline{B}_\alpha^{p,q}(F)$, the space defined by means of μ_2. Thus, it is possible to redefine f (as $\mathscr{E}f|F$) on a set with μ_1-measure zero $(E_1 \backslash E_2)$, so that f is defined μ_2-a.e. and belongs to $B_\alpha^{p,q}(F)$.

Finally we mention that the equivalence of $\Lambda_\alpha^{p,q}(\mathbb{R}^n)$ and $B_\alpha^{p,q}(\mathbb{R}^n)$ also follows by means of an argument similar to the one used above to prove the extension theorem when $s = n$.

4. The restriction theorem

Following [2], we shall prove the restriction part of Theorem 1 by first proving two lemmas (Lemmas 6 and 7) on Bessel potentials, and then use the real

interpolation method to obtain the result. We start by giving some preliminary estimates in a sequence of lemmas. Recall that by the comments in the beginning of §2, the conditions (D_s), (L_d), and (3), are valid in a slightly more general setting; we will use this without further comments.

Lemma 3. *Let* $r \le 1$, $\gamma \ge 0$, *and* $t \in \mathbb{R}^n$, *and let* μ *satisfy* (2) *and* (3).
(i) *If* $0 \le d \le n$ *and* μ *satisfies* (L_d), *then*

$$\int_{|t-y|<r} \frac{|t-y|^{-\gamma}}{\mu(B(y,r))}\, d\mu(y) \le cr^{-\gamma} \quad \textit{for } \gamma < d.$$

The inequality also holds for $\gamma = d = 0$.
(ii) *If* $0 < s \le n$ *and* μ *satisfies* (D_s) *and* (3), *then*

$$\int_{r\le|t-y|<1} \frac{|t-y|^{-\gamma}}{\mu(B(y,r))}\, d\mu(y) \le cr^{-\gamma} \quad \textit{for } \gamma > s.$$

Proof. (i) We assume that $r = 2^{-\nu_0}$ for some integer $\nu_0 \ge 0$. For $|t-y| < r$ we have by (2) that $\mu(B(y,r)) \ge c\mu(B(y,2r)) \ge c\mu(B(t,2r))$ which gives the inequality for $\gamma = d = 0$, and that the integral in statement (i) is less than $c\sum_{\nu=\nu_0}^{\infty} 2^{\nu\gamma}\mu(B(t,2^{-\nu}))/\mu(B(t,2^{-\nu_0}))$ for $d > 0$. Here we used dyadic rings and the fact that μ has no point mass if $d > 0$. From (L_d) we have

$$\mu(B(t,2^{-\nu_0})) \ge c2^{(\nu-\nu_0)d}\mu(B(t,2^{-\nu})),$$

so the sum is less than $c\sum_{\nu=\nu_0}^{\infty} 2^{\nu_0 d}2^{\nu(\gamma-d)} = c2^{\nu_0\gamma} = cr^{-\gamma}$ if $\gamma < d$.

(ii) We assume that $r = 2^{-\nu_0}$ for some $\nu_0 \ge 1$. With $k = 3|y-t|/r$ we get from (D_s) that $\mu(B(t,2|y-t|)) \le \mu(B(y,kr)) \le ck^s\mu(B(y,r))$. Putting this estimate into the integral in (ii) and using again dyadic rings, one obtains that it is less than

$$cr^{-s}\sum_{\nu=0}^{\nu_0-1} 2^{\nu\gamma}2^{-\nu s}\frac{\mu(B(t,2^{-\nu}))}{\mu(B(t,2\cdot 2^{-\nu-1}))} \le cr^{-s}2^{\nu_0(\gamma-s)} = cr^{\gamma} \quad \text{if } \gamma > s.$$

For the definition of the Bessel kernel G_α, see, e.g., [2, p. 6]. We also need some estimates on it which may be found in [2, p. 104], for example we have

$$|D^j G_\alpha(x)| \le c|x|^{\alpha-|j|-n}, \qquad x \ne 0,\ \alpha < n+|j|. \tag{22}$$

In the remaining part of this section, we will use the shorthand notation

$$d\nu_r = d\mu(x)\,d\mu(y)/\mu(B(x,r))\mu(B(y,r)).$$

Lemma 4. *Let* $0 < \alpha < n+1$, $\alpha \ne n$, $0 < d \le s \le n$, $0 < r \le 1$, $\theta > 0$, $t \in \mathbb{R}^n$, *and let* μ *satisfy* (D_s), (L_d), *and* (3). *Then*

$$J_1 = \iint_{\substack{|x-y|<r\\ |y-t|<2r}} |G_\alpha(x-t) - G_\alpha(y-t)|^\theta\, d\nu_r \le cr^{(\alpha-n)\theta} \quad \textit{if } \frac{d}{\theta} - (n-\alpha) > 0.$$

The same estimate holds for $s/\theta - (n-\alpha) < 1$ *if in the domain of integration* $|y-t| < 2r$ *is replaced by* $|y-t| \ge 2r$ *(we denote the corresponding integral by* J_2*)*.

Proof. Starting with J_1, we consider the cases $\alpha < n$ and $n < \alpha < n+1$ separately. If $n < \alpha < n+1$, the Bessel kernel is Lipschitz continuous with

exponent $\alpha - n$, so

$$J_1 \le cr^{(\alpha-n)\theta} \iint_{\substack{|x-y|<r \\ |y-t|<2r}} \frac{1}{\mu(B(x,r))\mu(B(y,r))}\, d\mu(x)\, d\mu(y).$$

If $|x-y| < r$ we have $B(y,r) \subset B(x,2r)$ and (2) gives $\mu(B(y,r)) \le c\mu(B(x,r))$, so replacing $\mu(B(x,r))$ by $c\mu(B(y,r))$ in the denominator and performing the x-integration we get

$$J_1 \le cr^{(\alpha-n)\theta} \int_{|y-t|<2r} \frac{1}{\mu(B(y,r))}\, d\mu(y) \le cr^{(\alpha-n)\theta},$$

where we used (i) in Lemma 3 with $\gamma = 0$. (To apply Lemma 1 here and below, we have to make some small adjustments: Replace $\mu(B(y,r))$ by $\mu(B(y,2r))$, which is allowed because of (2), and use that the limitation $r < 1$ in Lemma 3 may be replaced by $r \le r_0$.) If $\alpha < n$ we use the triangle inequality to split J_1 into two parts with $|G_\alpha(x-t)|^\theta$ and $|G_\alpha(y-t)|^\theta$ in the integrands. To treat the second part we replace as above $\mu(B(x,r))$ by $c\mu(B(y,r))$ and perform the x-integration which gives $c\int_{|y-t|<2r} |G_\alpha(y-t)|^\theta/\mu(B(y,r))\, d\mu(y)$ which is less than $r^{(\alpha-n)\theta}$ by (22) and Lemma 3 if $(n-\alpha)\theta < d$. The first part may be treated in the same way if one first replaces the domain of integration by $\{(x,y);\, |x-t| < 3r,\ |x-y| < r\}$.

To treat J_2, we first note that it is less than $J_2' + J_2''$, where in J_2' and J_2'' the domains of integration are as in J_2, but with $2r \le |y-t| < 2$ and $|y-t| \ge 2$, respectively. Using the mean value theorem on $G_\alpha(x-t) - G_\alpha(y-t)$ and estimating in the same way as in [2, p. 106], using (22), we get

$$J_2' \le cr^\theta \iint |y-t|^{(\alpha-n-1)\theta}\, d\nu_r,$$

the integral taken over $|x-y| < r$, $2r \le |y-t| < 2$. Replacing as above $\mu(B(x,r))$ by $c\mu(B(y,r))$, performing the x-integration and using part (ii) in Lemma 3 we get $J_2' \le cr^{(\alpha-n)\theta}$ if $(n+1-\alpha)\theta > s$. Finally a similar calculation (cf. [2]) gives, using the exponential decrease of G_α, (4), and (5), that $J_2'' \le cr^{\theta-s}$, so $J_2'' \le cr^{(\alpha-n)\theta}$ if $s/\theta - (n-\alpha) < 1$.

Lemma 5. *Let* $0 < \alpha < n+1$, $\alpha \ne n$, $\theta > 0$, $\gamma = n/\theta - (n-\alpha)$, $h \in \mathbb{R}^n$, $|h| \le 1$. *Then*

$$\int_{|t|<2|h|} |G_\alpha(t+h) - G_\alpha(t)|^\theta\, dt \le c|h|^{\gamma\theta} \quad \textit{if } \gamma > 0.$$

The same estimate holds for $\gamma < 1$ *if in the domain of integration* $|t| < 2|h|$ *is replaced by* $|t| \ge 2|h|$.

The proof of this lemma, which is essentially the same as Lemma B in [2, p. 106], is similar to but simpler than the proof of Lemma 4. From now on, u will denote the Bessel potential defined by $u(x) = \int G_\alpha(x-t) f(t)\, dt$, $f \in L^p(\mathbb{R}^n)$; cf. also [2, p. 107].

Lemma 6. *Let* $1 \le p \le \infty$, $0 \le d \le n$, $d \le s \le n$, $0 < r \le 1$, $s > 0$, $\alpha > 0$, $\alpha \ne n$, *and* $(n-d)/p < \alpha < 1 + (n-s)/p$. *Let furthermore* μ *satisfy* (D_s),

(L_d), and (3), and let $u = G_\alpha * f$, $f \in L^p(\mathbb{R}^n)$. Then

$$\left(\iint_{|x-y|<r} |u(x) - u(y)|^p \, d\nu_r\right)^{1/p} \le cr^{\alpha - n/p} \|f\|_p ,$$

where c is independent of r and f.

Proof. The left-hand side of the inequality to be proved is less than or equal to $A + B$, where

$$A = \left(\iint_{|x-y|<r} \left(\int_{|y-t|<2r} |G_\alpha(x-t) - G_\alpha(y-t)| \, |f(t)| \, dt\right)^p d\nu_r\right)^{1/p},$$

and B is given by the same expression but with $|y - t| \ge 2r$. We first estimate A. Let $0 \le a < 1$. From Hölder's inequality it follows that the integral with respect to t in A raised to the power p is less than or equal to

$$\int_{|y-t|<2r} |G_\alpha(x-t) - G_\alpha(y-t)|^{ap} |f(t)|^p \, dt \cdot I_A(x, y),$$

where, with $1/p' = 1 - 1/p$,

$$I_A(x, y) = \left(\int_{|y-t|<2r} |G_\alpha(x-t) - G_\alpha(y-t)|^{(1-a)p'} \, dt\right)^{p/p'}.$$

If $|x - y| < r$ and

$$\gamma = n/(1-a)p' - (n - \alpha) > 0 \tag{23}$$

then, by Lemma 5, $I_A(x, y) \le cr^{\gamma(1-a)p}$, and after a change of order in the integration we get

$$A \le cr^{\gamma(1-a)} \left(\int |f(t)|^p J_A(t) \, dt\right)^{1/p}, \tag{24}$$

where

$$J_A(t) = \iint_{\substack{|x-y|<r \\ |y-t|<2r}} |G_\alpha(x-t) - G_\alpha(y-t)|^{ap} \, d\nu_r.$$

To deal with J_A, we separate two cases. If $d = 0$ and so $\alpha p > n$ we take $a = 0$. Then clearly (23) holds, and estimates performed in the beginning of the proof of Lemma 4 show that $J_A \le c$. Consequently, $A \le cr^\gamma = cr^{\alpha - n/p}$. If $d > 0$ we obtain from Lemma 4 that $J_A \le cr^{(\alpha-n)ap}$ if

$$d - (n - \alpha)ap > 0, \tag{25}$$

and putting this estimate into (24) we get $A \le cr^{\alpha - n/p} \|f\|_p$. It remains to see that a with $0 < a < 1$ may be chosen so that (23) and (25) hold. But these inequalities can be simplified to, if $\alpha < n$ (the case $\alpha > n$ is obvious), $a > (n - \alpha p)/(n - \alpha)p$ and $a < d/(n - \alpha)p$, respectively, so a can be chosen since we assume $n - \alpha p < d$ and, in this calculation, $p > 1$ and $d > 0$.

Now we turn to the term B, and take an a with $0 < a < 1$, which need not be the same as above. We make the same calculations as above, and get analogous factors $I_B(x, y)$ and $J_B(t)$ with $|y - t| \ge 2r$ in the domains of

integration. By the second parts of Lemmas 5 and 4 we have with γ as in (23) that $I_B(x, y) \le cr^{\gamma(1-a)p}$ if $\gamma < 1$, and that (now we have no case $a = 0$) $J_B(x, y) \le cr^{(\alpha-n)ap}$ if $s-(n-\alpha)ap < ap$. Thus we get $B \le cr^{\alpha-n/p}$ if a may be chosen. The limitations may be simplified to $a < (n+p-\alpha p)/(n+1-\alpha)p$ and $a > s/(n+1-\alpha)p$, respectively, and thus a may be chosen since we assume $s < n+p-\alpha p$ and $p > 1$.

Lemma 7. *Let $1 \le p \le \infty$, $0 \le d \le n$, $\alpha > 0$, $\alpha \ne n$, and $(n-d)/p < \alpha$, let μ satisfy (L_d) and (3), and let $u = G_\alpha * f$, $f \in L^p(\mathbb{R}^n)$. Then $\|u\|_{p,\mu} \le c\|f\|_p$, where c is independent of f.*

Proof. We have (cf. the preceding proof) that $\|u\|_{p,\mu} \le A + B$, where

$$A = \left(\int \left(\int_{|x-t|<1} |G_\alpha(x-t)f(t)|\,dt\right)^p d\mu(x)\right)^{1/p}$$

and B is the same but with $|x-t| \ge 1$. To treat A, let $0 \le a < 1$. By means of Hölder's inequality we get

$$A \le I_A^{1/p'} \left(\int |f(t)|^p J_A(t)\,dt\right)^{1/p},$$

where $I_A = \int_{|t|<1} |G_\alpha(t)|^{(1-a)p'}\,dt$ and $J_A(t) = \int_{|x-t|<1} |G_\alpha(x-t)|^{ap}\,d\mu(x)$. Thus we get the desired estimate for A if I_A and J_A converge, the latter with a bound independent of t. If $\alpha > n$ they converge since the integrands are continuous. If $\alpha < n$, we get from (22) that I_A converges if

$$(n-\alpha)(1-a)p' < n. \tag{26}$$

To treat J_A, we consider first the case $d = 0$, so that $\alpha p > n$, and take $a = 0$. Then (26) is fulfilled so I_A converges, and we have $J_A(t) \le c$ as a consequence of (3). If $d > 0$ then (22) and the estimate $\mu(B(t, r)) \le cr^d$, $t \in \mathbb{R}^n$, easily give (cf. the proof of (i) in Lemma 3) that $J_\alpha(t) \le c$ if $(n-\alpha)ap < d$. As we saw in the proof of Lemma 6, a may be chosen so that this inequality and (26) hold simultaneously, so the desired estimate follows. To deal with B we take $0 < a < 1$ and use Hölder's inequality as above. Then it is easily seen that the correspondence factors I_B and $J_B(t)$ are convergent, due to the exponential decrease of G_α and (4).

Now we prove the restriction part of Theorem 1. The proof consists of an application of the real interpolation method to the estimates in Lemmas 6 and 7. We need the space $\mathscr{L}_\alpha^p(\mathbb{R}^n)$ of Bessel potentials. This consists of all functions u of the form $u = G_\alpha * f$, $f \in L^p(\mathbb{R}^n)$, and the norm of u in $\mathscr{L}_\alpha^p(\mathbb{R}^n)$ is defined as $\|f\|_p$.

Let α_0 and α_1 satisfy $0 < \alpha_0 < \alpha < \alpha_1$, $\alpha_0, \alpha_1 \ne n$, $(n-d)/p < \alpha_i < 1+(n-s)/p$, $i = 0, 1$, and define θ by $\alpha = (1-\theta)\alpha_0+\theta\alpha_1$. Using interpolation together with Lemma 7 in the same way as in [2], we get that the restriction operator is bounded from $\Lambda_\alpha^{p,q}(\mathbb{R}^n)$ to $L^p(\mu)$. In order to take care of the other part of the $B_\alpha^{p,q}(F)$-norm of f we proceed as follows. Define T_ν by

$$(T_\nu f)(x, y) = (f(x) - f(y))/(m_\nu(x)m_\nu(y))^{1/p}$$

if $|x-y| < 2^{-\nu}$ and $(T_\nu f)(x, y) = 0$ otherwise, and denote by $l_q^s(A)$ the space of all sequences $(a_\nu)_0^\infty$, $a_\nu \in A$, with norm $(\sum_{\nu=0}^\infty (2^{\nu s}\|a_\nu\|_A)^q)^{1/q} < \infty$, where

$A = L^p(d\mu(x)\,d\mu(y))$. We must prove that $Tf = (T_\nu f)_0^\infty$ is bounded from $\Lambda_\alpha^{p,q}(\mathbb{R}^n)$ to $l_q^{\alpha-n/p}(A)$. Lemma 6 shows that T is bounded from $\mathscr{L}_{\alpha_i}^p(\mathbb{R}^n)$ to $l_\infty^{\alpha_i-n/p}$, $i = 0, 1$, and thus by interpolation T is bounded from the intermediate space $(\mathscr{L}_{\alpha_0}^p(\mathbb{R}^n), \mathscr{L}_{\alpha_1}^p(\mathbb{R}^n))_{\theta_q} = \Lambda_\alpha^{p,q}(\mathbb{R}^n)$ to the intermediate space $(l_\infty^{\alpha_0-n/p}(A), l_\infty^{\alpha_1-n/p}(A))_{\theta_q} = l_q^{\alpha-n/p}(A)$.

References

1. E. M. Dynkin, *Free interpolation by functions with derivatives in* H^1, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) **126** (1983), 77–87; English transl. in J. Soviet Math. **27** (1984), 2475–2481.
2. A. Jonsson and H. Wallin, *Function spaces on subsets of* $\mathbb{R}^n$, Math. Rep., vol. 2, part 1, Harwood, Chur, Switzerland, 1984.
3. ——, *A trace theorem for generalized Besov spaces with three indexes*, Fourier Analysis and Approximation Theory (Budapest, 1976), Colloq. Math. Soc. János Bolyai, vol. 19, North-Holland, Amsterdam, 1979, pp. 429–449.
4. A. L. Volberg and S. V. Konyagin, *There is a homogeneous measure on any compact subset in* $\mathbb{R}^n$, Dokl. Akad. Nauk SSSR **278** (1984), 783–785; English transl. in Soviet Math. Dokl. **30** (1984),453–456.

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