BESOV SPACES ON CLOSED SUBSETS OF $\mathbb{R}^n$

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Abstract. Motivated by the need in boundary value problems for partial differential equations, classical trace theorems characterize the trace to a subset $F$ of $\mathbb{R}^n$ of Sobolev spaces and Besov spaces consisting of functions defined on $\mathbb{R}^n$, if $F$ is a linear subvariety $\mathbb{R}^d$ of $\mathbb{R}^n$ or a $d$-dimensional smooth submanifold of $\mathbb{R}^n$. This was generalized in [2] to the case when $F$ is a $d$-dimensional fractal set of a certain type. In this paper, traces are described when $F$ is an arbitrary closed set. The result may also be looked upon as a Whitney extension theorem in $L^p$.

0. Introduction

In this paper the trace of the Besov space $B^{p,q}_\alpha(\mathbb{R}^n)$ to an arbitrary closed subset $F$ of $\mathbb{R}^n$ is characterized for $\alpha p > n$ and $\alpha < 1$. For $p = q$ the result is as follows. Consider a fixed measure $\mu$ with support $F$ satisfying the doubling condition, more precisely the conditions $(D_n)$ and $(3)$ in the next section; such a measure exists due to a result in Volberg and Konyagin [4]. Then the trace of $B^{p,q}_\alpha(\mathbb{R}^n)$ to $F$ consists of all functions $f$ defined on $F$ with finite norm

$$\|f\|_{p,\mu} + \left( \int_{|x-y| < 1} \frac{|f(x) - f(y)|^p}{|x-y|^{\alpha p-n}(\mu(B(x, |x-y|))^2 d\mu(x) d\mu(y)} \right)^{1/p},$$

where $B(x, r)$ denotes the open ball with center $x$ and radius $r$. We also consider results when the range of the parameters $\alpha$ and $p$ is wider, in which case one has to impose restrictions on $F$. In particular the case $p = q = 2$ and $\alpha = 1$ is included, which means that we have a result for the Sobolev spaces $W^{2}_1(\mathbb{R}^n)$.

The results contain the classical results on the restriction of Besov spaces to hyperplanes and their generalizations to $d$-sets given in [2] (see Example 1). However, we do not allow the range of the parameters to be as wide as in these cases.

The possibility to prove theorems like the one in this paper is suggested by the results in Dynkin [1], where the norm (1) is used to prove results on the...
trace to the boundary of the unit circle of analytic functions whose derivatives are in the Hardy space \( H^p \), under some additional conditions on \( F \).

1. Definitions and results

We assume throughout the paper that \( F \) is a closed subset of \( \mathbb{R}^n \), \( n \geq 1 \), and that \( \mu \) is a measure with support \( F \). Following [4], we say that \( \mu \) satisfies the condition \((D_q)\), \( 0 < q \leq n \), if for some constant \( c \) independent of \( x, r \), and \( k \),

\[
(D_q) \quad \mu(B(x, kr)) \leq c k^q \mu(B(x, r)), \quad x \in F, \ r > 0, \ k \geq 1, \ kr \leq 1
\]

(in [4], however, the limitation \( kr \leq 1 \) is not imposed). Note that \((D_q)\) is a generalization of the familiar doubling condition: If \( \mu \) satisfies \((D_q)\), then

\[
(2) \quad \mu(B(x, 2r)) \leq c \mu(B(x, r)), \quad x \in F, \ 0 < r \leq 1/2.
\]

In general the measures considered will be assumed to satisfy the assumption that for some constants \( c_1, c_2 > 0 \)

\[
(3) \quad c_1 < \mu(B(x, 1)) < c_2, \quad x \in F.
\]

In [4] it is shown that every closed set \( F \) carries a measure satisfying \((D_n)\), and (Proposition 1 below) it may also be assumed that this measure satisfies \((3)\). We furthermore say that \( \mu \) satisfies the condition \((L_d)\), \( 0 < d \leq n \), if for some constant \( c > 0 \)

\[
(L_d) \quad \mu(B(x, kr)) \geq c k^d \mu(B(x, r)), \quad x \in F, \ r > 0, \ k \geq 1, \ kr \leq 1.
\]

Note that \((L_0)\) imposes no restriction on \( \mu \).

Let assumptions on \( \alpha, \beta, \gamma, s, d, \) and \( \mu, \) be as in Theorem 1 below. The Besov space \( B^{\alpha, \gamma}_0(F) \) is defined as the class of functions defined on \( F \) with finite norm

\[
\|f\|_{\alpha, \beta, \gamma} = \left( \sum_{\nu=0}^{\infty} \left( 2^{\nu(\alpha-n)/p} \left( \frac{\int_{|x-y|<2^{-\nu}} (|f(x)-f(y)|^p}{m_\nu(x)m_\nu(y)} \mu(dx)\mu(dy) \right)^{1/p} \right)^{\gamma/p} \right)^{1/\gamma}
\]

where \( m_\nu(x) \) denotes \( \mu(B(x, 2^{-\nu})) \). For \( p = \infty \), the terms in the \( \nu \)-summation should be interpreted as \( 2^{\nu\alpha} \sup_x \sup_{|x-y|<2^{-\nu}} |f(x)-f(y)|^\alpha \). This norm depends on \( \mu \), but if follows from Theorem 1 that different \( \mu \) give rise to equivalent norms in a certain sense, see §3.5. One obtains an equivalent norm if the domain of integration is replaced by \( |x-y| < c2^{-\nu} \), where \( c > 0 \) is a constant (cf. [3, p. 434]). For \( p = q \), the norm is equivalent to the one given in the introduction (Proposition 2). If \( F = \mathbb{R}^n \) and \( \mu \) is the \( n \)-dimensional Lebesgue measure, then the space \( B^{\alpha, \gamma}_0(F) \) is equivalent to the classical Besov space \( \Lambda^{\alpha, q}_0(\mathbb{R}^n) \) ([2, p. 128]; the definition of \( \Lambda^{\alpha, q}_0(\mathbb{R}^n) \) will be recalled in §3).

The main result of this paper is the following trace theorem. To state it, we must define what we mean with the restriction \( Rf \) to \( F \) of a function \( f \) in \( \Lambda^{\alpha, q}(\mathbb{R}^n) \), since functions in this space are defined a.e. with respect to the Lebesgue measure \( m \) in \( \mathbb{R}^n \), only. We put

\[
Rf(x) = \lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) \, dy
\]

at every point \( x \in F \) where the limit exists. It is part of the conclusion of the theorem that the limit exists \( \mu \)-a.e. on \( F \).
Theorem 1. Let $0 < d \leq n$, $d \leq s \leq n$, $s > 0$, $1 \leq p$, $q \leq \infty$, $(n-d)/p < \alpha < 1+(n-s)/p$, and let $\mu$ satisfy the conditions $(D_d)$, $(L_d)$, and (3). Then $B_{\alpha}^{p,q}(F)$ is the trace to $F$ of $\mathcal{N}_{\alpha}^{p,q}(\mathbb{R}^n)$ in the following sense:

(a) The restriction operator $R$ is a continuous linear operator from $\mathcal{N}_{\alpha}^{p,q}(\mathbb{R}^n)$ to $B_{\alpha}^{p,q}(F)$.

(b) There is a continuous linear extension operator $E$ from $B_{\alpha}^{p,q}(F)$ to $\mathcal{N}_{\alpha}^{p,q}(\mathbb{R}^n)$ such that $R(E) = I$. The proof of part (a) is given in §4 and part (b) in §3. We point out that as a special case we have the following result, in which the space $B_{\beta}^{p,q}(F)$ is defined by means of a measure $\mu$ with support $F$ satisfying $(D_n)$ and (3).

Corollary 1. Let $F$ be a closed subset of $\mathbb{R}^n$, $1 \leq p$, $q \leq \infty$, $\alpha < 1$, and $\alpha p > n$. Then the trace of $\mathcal{N}_{\alpha}^{p,q}(\mathbb{R}^n)$ to $F$ is $B_{\alpha}^{p,q}(F)$.

We conclude this section by giving some examples.

Example 1. If $\mu$ is a $d$-set, $0 < d \leq n$, as defined in [2], then $\mu$ satisfies $(D_d)$, $(L_d)$, and (3), and the restriction on $\alpha$ in Theorem 1 becomes $0 < \beta < 1$ where $\beta = \alpha - (n-d)/p$. The space $B_{\beta}^{p,q}(F)$ is in this case equivalent to the space $B_{\beta}^{p,q}(F)$ defined in [2], and Theorem 1 reduces to the trace theorem in [2, 142]. However, in [2] the trace on a $d$-set $F$ of $B_{\theta}^{p,q}(W)$ is characterized for $\beta = \alpha - (n-d)/p > 0$.

Example 2. Consider $F \subset \mathbb{R}^2 = \{(x_1, x_2)\}$ defined by $F = F_1 \cup F_2$, where $F_1 = \{(x_1+1)^2 + x_2^2 \leq 1\}$ and $F_2 = \{0 \leq x_1 \leq 2, x_2 = 0\}$. Let $m_n$ denote the $n$-dimensional Lebesgue measure, for $n=1$ distributed over the $x_1$-axes, and put $d\lambda = x_1 dm_1$. Then $\mu = m_2[F_1 + \lambda]F_2$ satisfies $(D_2)$, $(L_1)$, and (3), and thus Theorem 1 yields, with this $\mu$, a trace theorem in the range $1/p < \alpha < 1$.

Example 3. Let $F \subset \mathbb{R}^2$ be the set $F = \{0 \leq x_1 \leq 1, 0 \leq x_2 \leq x_1^2\}$ where $\gamma > 1$, and put $d\nu = x_1^{-\gamma} dm_2$ and $\mu = \nu|F$. Then it can be shown that $\mu$ satisfies $(D_2)$, $(L_1)$, and (3), so we get a trace theorem for $1/p < \alpha < 1$ using this measure $\mu$.

Example 4. Let $F \subset \mathbb{R}^1$, $F = \{0\} \cup \{a_k, k \geq 0\}$ where $a_k = 2^{-k}$, and let $\mu$ be defined by $\mu(\{0\}) = 0$ and $\mu(\{a_k\}) = a^{-k}$, where $a$ is a constant, $1 < a < 2$. Then $\mu$ satisfies $(D_1)$ with $s = \ln a/\ln 2$, and we get a trace theorem for $1/p < \alpha < 1 - (1-s)/p$. We now simplify by taking $q = p$. It can be shown that the $B_{\alpha}^{p,p}(F)$-norm can be simplified to an equivalent norm

\[
\left( \sum_{i=0}^{\infty} |f(a_i)|^p a^{-i} \right)^{1/p} + \left( \sum_{i=0}^{\infty} 2^{i(\alpha p - 1)} |f(a_i) - f(a_{i+1})|^p \right)^{1/p} ;
\]

also, it can be shown directly that these norms are equivalent for $a > 1$. It follows that we have a trace theorem for $1/p < \alpha < 1 - 1/p$, with the norm in the trace space given by the above expression with, e.g., $a = 2$.

2. Remarks on the preceding definitions

In the conditions $(D_d)$ and $(L_d)$, it is assumed that $kr \leq 1$. However, for $r_0 > 1$, the class of measures satisfying $(L_d)$ is unchanged if $kr \leq 1$ is
replaced by \( kr \leq r_0 \), and so is the class satisfying the conditions \((D_s)\) and \((3)\) simultaneously. To show the latter statement, one uses the observation that if \( \mu(B(x, 1)) \leq c_2, x \in F \), then
\[
(4) \quad \mu(B(x, r)) \leq cr^n, \quad r \geq 1, x \in \mathbb{R}^n.
\]
It is also easy to realize that if \( \mu \) satisfies \((L_d)\), then the condition defining \((L_d)\) is satisfied not only for \( x \in F \) but for \( x \in \mathbb{R}^n \) and that if \((D_s), (L_d), \) or \((3)\) hold for open balls, they hold for closed.

If \( \mu \) satisfies \((D_s)\), then the set \( F \) has Hausdorff dimension \( \leq s \), and if \( \mu \) satisfies \((L_d)\), then it has dimension \( \geq d \) at every point in the sense that every set \( F \cap B(x, r), x \in F \), has dimension \( \geq d \). These statements can be proved in the same way as similar results for \( d \)-sets in [2, p. 32]. The reason for this is that, putting \( A(d) = r^d \mu(B(x, r)) \), we have for \( x \in F \) and \( r \leq r_1 \) (cf. \(18)\) the inequalities
\[
(5) \quad \mu(B(x, r)) \leq cA(d)r^d \quad \text{and} \quad \mu(B(x, r)) \geq cA(s)r^s
\]
if \( \mu \) satisfies \((L_d)\) or \((D_s)\), respectively; these conditions are similar to those used in the definition of a \( d \)-set. Note also that it follows that \( \mu \) has no point mass if \( \mu \) satisfies \((L_d)\), \( d > 0 \). One can think of the significance of these observations in relation to the assumptions in Theorem 1 as follows. The condition \((D_s)\) guarantees that \( F \) is so small that \( B_{\mu,p}^q(F) \) should be defined by means of first differences, and \((L_d)\) that \( F \) is so big that the trace on \( F \) of a function in \( B_{\alpha,q}^p(F) \) exists \( \mu \)-a.e. In [4] it is shown that every closed set is the support of a measure satisfying \((D_n)\). We shall verify that things may be arranged so that \((3)\) is fulfilled, too.

**Proposition 1.** For every closed set \( F \subset \mathbb{R}^n \), there is a measure \( \mu \) with support \( F \) satisfying \((D_n)\) and \((3)\).

**Proof.** Cover \( \mathbb{R}^n \) with the closed balls of radii \( r_0 = 6\sqrt{n} \) which have centers in points with integer coordinates. Denote the balls intersecting \( F \) by \( B_i, i = 1, 2, \ldots \), and let \( \nu_i \) be a probability measure on \( B_i \cap F \) satisfying the condition \((D_n)\), without the limitation \( kr \leq 1 \); \( \nu_i \) exists due to [4]. Let \( \phi \) be given by \( \phi(x) = 1 \) if \( |x| \leq r_0/2, \phi(x) = ((r_0 - |x|)2/r_0)^n \) if \( r_0/2 \leq |x| \leq r_0 \), and \( \phi(x) = 0 \) elsewhere, and let \( \phi_i \) be the translation of \( \phi \) such that the support of \( \phi_i \) is \( B_i \). Put \( d\mu_i = \phi_id\nu_i \) and \( \mu = \sum \mu_i \). Then \( \mu \) has the desired properties, as we shall see.

Denote the center of \( B_i \) by \( x_i \) and the ball \( B_i(x_i, r_0/2) \) by \( B_i/2 \). For any \( x \in F \) there is an \( x_i \) with \( |x - x_i| \leq \sqrt{n}/2 \), so \( B(x, 1) \) is contained in \( B_i/2 \) and thus \( \mu(B(x, 1)) \geq \mu_i(B(x, 1)) = \nu_i(B(x, 1)) \). Since \( \nu_i \) satisfies \((D_n)\), with the constant \( c \) in the condition \((D_n)\) depending only on \( n \) according to the construction in [4], we have \( 1 = \nu_i(B(x, 2r_0)) \leq c\nu_i(B(x, 1)) \), and the lower bound in \((3)\) follows. The upper bound follows from the fact that only a bounded number of \( B_i \)'s intersect \( B(x, 1) \).

To show that \((D_n)\) holds, consider a ball \( B(x, r), x \in F \), and assume \( kr \leq 1 \). We shall show that for any \( i \) we have \( \mu_i(B(x, kr)) \leq c^k\mu(B(x, r)) \) which gives \((D_n)\). We consider two cases.

**Case 1.** The distance \( d \) from \( B(x, kr) \) to the complement of \( B_i \) is \( \geq r \). Assume first that \( B(x, kr) \) does not intersect \( B_i/2 \), and note that if \( a \) is the distance from a point \( x \in B_i\setminus(B_i/2) \) to the boundary of \( B_i \), then \( \phi_i(x) = \)
(2a/r_0)^n$. Then from $\mu_i = \phi_i \nu_i$ it follows that

$$\mu_i(B(x, kr)) \leq (2(d + 2kr)/r_0)^n \nu_i(B(x, kr))$$

$$\leq (2(d + 2kr)/r_0)^n c_k^n \nu_i(B(x, r))$$

$$\leq (2(d + 2kr)/r_0)^n c_k^n (2(d + (k - 1)r)/r_0)^{-n} \mu_i(B(x, r))$$

$$\leq c_3^n k^n \mu_i(B(x, r)).$$

If $B(x, kr)$ intersects $B_j/2$ we instead have, using $kr \leq 1$,

$$\mu_i(B(x, kr)) \leq \nu_i(B(x, kr)) \leq c_k^n \nu_i(B(x, r)) \leq c_k^n (2/r_0)^{-n} \mu_i(B(x, r)).$$

**Case 2.** $d < r$. Then

$$\mu_i(B(x, kr)) \leq (2(r + 2kr)/r_0)^n \nu_i(B(x, kr)) \leq (6kr/r_0)^n.$$

But $B(x, r)$ is contained in some $B_j/2$, so $\mu(B(x, r)) \geq \nu_j(B(x, r))$, and we also have

$$1 = \nu_j(B(x, 2r_0)) = \nu_j(B(x, (2r_0/r)r)) \leq cr^{-n} \nu_j(B(x, r)).$$

Together, these estimates give $\mu_i(B(x, kr)) \leq c_k^n r^n \leq c_k^n \mu(B(x, r))$.

Finally we give, for $p = q$, a norm in $B_{p}^{p,q}(F)$ which in the case when

$$F = \mathbb{R}^n \text{ and } \mu \text{ is Lebesgue measure reduces to a classical norm in } \Lambda_{\alpha,p}(\mathbb{R}^n).$$

Compare also with §0.

**Proposition 2.** Let $\alpha, p, s, d, \mu$ be as in Theorem 1. Then the $B_{p}^{p,q}(F)$-norm is equivalent to expression (1).

**Proof.** The $B_{p}^{p,q}(F)$ norm equals, if $p = q$, $\|f\|_{p, \mu}$ plus

$$\left(\sum_{\nu=0}^{\infty} 2^{\nu(ap-n)} \sum_{\tau=1}^{\infty} \int_{2^{-\tau-1} \leq |x-y| < 2^{-\tau}} \frac{|f(x) - f(y)|^p}{m_{\nu}(x)m_{\nu}(y)} \, d\mu(x) \, d\mu(y)\right)^{1/p}.$$

Since $\mu$ satisfies $(L_d)$ we have $m_{\nu}(x) = \mu(B(x, 2^\tau - 2^{-\tau})) \geq c 2^{(\tau - \nu)d} m_{\tau}(x)$. Using this in (6) and reversing the order of the summations one obtains that (6) is less than

$$\left(\sum_{\nu=0}^{\infty} 2^{2\nu d} \sum_{\tau=0}^{\infty} 2^\nu (ap-n+2d) \int_{2^{-\tau-1} \leq |x-y| < 2^{-\tau}} \frac{|f(x) - f(y)|^p}{m_{\nu}(x)m_{\nu}(y)} \, d\mu(x) \, d\mu(y)\right)^{1/p},$$

where the sum with respect to $\nu$ may be replaced by $2^{\nu(ap-n+2d)}$. Since, by (2), $m_{\nu}(y) \geq cm_{\nu-1}(y) \geq c\mu(B(x, |x-y|))$ if $|x-y| < 2^{-\tau}$, it follows that the $B_{p}^{p,q}(F)$-norm is less than (1).

The converse follows immediately if we write the double integral in the latter norm as a sum of integrals over $2^{-\nu-1} \leq |x-y| < 2^{-\nu}$, and use that then we have $\mu(B(x, |x-y|)) \geq cm_{\nu}(x), cm_{\nu}(y)$.

3. **The extension theorem**

3.1. The proof of part (b) in Theorem 1 is similar to the proof of the extension theorem in [2], and we shall now and then refer to [2] for details. To define the extension operator, we need the Whitney decomposition of the complement of $F$, which has the following properties (cf. [2, p. 23]). It consists of a collection of closed cubes $Q_i$, $i = 1, 2, \ldots$, with mutually disjoint interiors and sides
parallel to the axes such that $\mathcal{C}F = \bigcup Q_i$. Denote the center of $Q_i$ by $x_i$, its diameter by $l_i$, and its sidelength, which we assume is of the form $2^{-M}$, $M$ integer, by $s_i$. Then

$$(7) \quad l_i \leq d(Q_i, F) \leq 4l_i,$$

where $d(Q_i, F)$ is the distance from $Q_i$ to $F$, and if $Q_i$ and $Q_j$ touch, then

$$(8) \quad 1/4l_i \leq l_j \leq 4l_i.$$

Let $0 < \varepsilon < 1/4$, and put $Q^* = (1 + \varepsilon)Q_i$. Then each point in $\mathcal{C}F$ is contained in at most $N_0 = N_0(n)$ cubes $Q^*_i$, and, furthermore, $Q^*_i$ intersects a cube $Q_j$ only if $Q_i$ touches $Q_j$. To this decomposition we associate a partition of unity, consisting of nonnegative functions $\varphi_i$ such that $\varphi_i(x) = 0$ if $x \notin Q^*_i$, $\sum \varphi_i(x) = 1$, $x \in \mathcal{C}F$, and

$$(9) \quad |D^j \varphi_i(x)| \leq A_j(l_i)^{-|j|}.$$

We are now prepared to define the extension operator $\mathcal{E}$, and assume first that $\mu$ satisfies $(D_\eta)$ for some $\eta < n$; the remaining case will be taken care of in §3.5. Putting $c_i = \mu(B(x_i, 6l_i))^{-1}$, we define $\mathcal{E}f$ on $B^{n+q}(F)$ by

$$\mathcal{E}f(x) = \sum_{i \in I} \varphi_i(x)c_i \int_{|t-x_i| \leq 6l_i} f(t) \, d\mu(t), \quad x \in \mathcal{C}F,$$

where $I$ means those $i$ such that $s_i \leq 1$. Note that since $s < n$, this defines $\mathcal{E}f$ a.e. on $\mathbb{R}^n$, by the discussion in the preceding section. We prove in §3.3 that $\mathcal{E}f \in L^q_\eta(\mathbb{R}^n)$ and in §3.4 that $\mathcal{E}f|F = f$, but give first some preliminary estimates related to the above construction.

3.2. Assume that $Q_i$ touches $Q_x$. By $(7)$, there is a point $p_i \in F$ with $|p_i - x_i| \leq 5l_i$, and since, if $|t - x_i| \leq 6l_x$, we have by $(8)$ that

$$|t - x_i| \leq |t - x_x| + |x_x - x_i| \leq 6l_x + l_i + l_x \leq 30l_i,$$

it follows from the doubling condition that $\mu(B(x_i, 6l_i)) \geq \mu(B(p_i, l_i)) \geq c\mu(B(p_i, 30l_i)) \geq c\mu(B(x_x, 6l_x))$, so

$$(10) \quad c_i \leq c c_x.$$

As we saw, we have

$$(11) \quad |t - x_x| \leq 30l_x \quad \text{if } |t - x_i| \leq 6l_i$$

and by $(9)$ and $(8)$ we have

$$(12) \quad |D^j \varphi_i(x)| \leq c l_x^{-|j|}.$$

Note that $(10)$, $(11)$, and $(12)$ hold if $x \in Q_x$ and $\varphi_i(x) \neq 0$, since then $Q_i$ and $Q_x$ touch. Using $(8)$ it also follows that $\mathcal{E}f(x) = 0$ if $x \in Q_x$, $s_x > 4$ and $\sum_{i \in I} \varphi_i(x) = \sum \varphi_i(x)$ if $x \in Q_x$, $s_x \leq 1/4$.

The following two lemmas will be needed. The first is a variation of Lemma D in [2, p. 111], and we omit its proof since it is similar to the proof in [2], the only essential difference being that we use $(10)$ as a replacement for [2, formula (6)].
The second lemma is the same as Lemma 2 in [2]. Put
\[
J_p(x, \tau) = \left( c_x c_\tau \int_{|t-x| \leq 30l_x} \int_{|s-x| \leq 30l_x} |f(t) - f(s)|^p \, d\mu(t) \, d\mu(s) \right)^{1/p}.
\]

**Lemma 1.** Let \(1 < p < \infty\) and let \(x \in Q_x\) and \(y \in Q_\tau\), \(s_x, s_\tau \leq 1/4\). Then the following hold:

(a) \(|E f(x) - E f(y)| \leq cJ_p(x, \tau)\).
(b) \(|D^j(E f)(x)| \leq c\chi^{|j|} J_p(x, \tau), |j| > 0\).
(c) \(|E f(x) - b| \leq c(c_x \int_{|t-x| \leq 30l_x} |f(t) - b|^p \, d\mu(t))^{1/p}\) where \(b\) is a constant, and
(d) for any \(Q_x\) and \(|j| \geq 0\)
\[
|D^j(E f)(x)| \leq c\chi^{|j|} \left( c_x \int_{|t-x| \leq 30l_x} |f(t)|^p \, d\mu(t) \right)^{1/p}.
\]

Let \(\Delta_x\) denote the union of those cubes \(Q_x\) which have sides of length \(s_x = 2^{-\nu}\).

**Lemma 2.** Let \(a > 0\), let \(h\) be a nonnegative function defined on a closed set \(F \subset \mathbb{R}^n\), and let \(\mu\) be supported by \(F\). Let \(g\) be given by
\[
g(x) = \int_{|t-x| \leq a2^{-\nu}} h(t) \, d\mu(t), \quad x \in \text{int} Q_x, s_x = 2^{-\nu}.
\]

Then, for \(x_0 \in \mathbb{R}^n\) and \(0 < r \leq \infty\)
\[
\int_{x \in \Delta_x \atop |x-x_0| \leq r} g(x) \, dx \leq c 2^{-\nu n} \int_{|t-x_0| \leq r+c_0 2^{-\nu}} h(t) \, d\mu(t);
\]
in particular for \(r = \infty\)
\[
\int_{x \in \Delta_x} g(x) \, dx \leq c 2^{-\nu n} \int h(t) \, d\mu(t).
\]

Here the constants \(c\) and \(c_0\) depend only on \(a\) and \(n\).

Finally we note that it is shown in [2, p. 110], that
\[
y \in \Delta'_\nu = \bigcup_{i=\nu-2}^{\nu+3} \Delta_i \quad \text{if} \quad x \in \Delta_\nu \quad \text{and} \quad |x - y| < 2^{-\nu}/2.
\]

3.3. We are now ready to prove that \(E f \in \Lambda^{\alpha, q}_\nu(\mathbb{R}^n)\). Explicitly, we treat the case \(p < \infty\) and \(q < \infty\) only. Recall that the norm of \(E f\) in \(\Lambda^{\alpha, q}_\nu(\mathbb{R}^n)\) can be given by
\[
||E f||_p + \left( \int_{|h| < H} \frac{||\Delta_h(E f)||_p^q}{|h|^{n+aq}} \, dh \right)^{1/q}
\]
where \(s\) is the smallest integer greater than \(\alpha\) and \(H\) is a positive number.
In order to estimate $\|E f\|_p$, note first that, with $p_x$ as in the argument
leading to (10), it follows from the doubling condition that if $|t - x_\chi| \leq 30 l_\chi$ ,
then
$$
\mu(B(x_\chi, 6l_\chi)) \geq \mu(B(p_x, l_\chi)) \geq c \mu(B(p_x, 36l_\chi)) \geq c \mu(B(t, l_\chi)) \geq c \mu(B(t, s_\chi)) ,
$$
so
(17) \hspace{1cm} c_\chi \leq c / m_\nu(t) \hspace{0.5cm} \text{if} \hspace{0.5cm} |t - x_\chi| \leq 30 l_\chi , s_\chi = 2^{-\nu} .

If $x$ belongs to some $Q_x$ with $s_x > 4$ then $E f(x) = 0$, and otherwise it
follows from (d) in Lemma 1 and (17) that

$$
|E f(x)|^p \leq c \int_{|t - x_\chi| \leq 30 l_\chi} \frac{|f(t)|^p}{m_\nu(t)} \, d\mu(t) , \hspace{0.5cm} x \in Q_x , s_x = 2^{-\nu} .
$$

Using (14) in Lemma 2 with $h(t)$ equal to $h_\nu(t) = |f(t)|^p / m_\nu(t)$ and $g(x)$
equal to $g_\nu(x) = \int_{|t - x_\chi| \leq 30 l_\chi} h_\nu(t) \, d\mu(t) , x \in \text{int}\, Q_x , s_x = 2^{-\nu} ,$ one obtains

$$
\|E f\|_p^p = \sum_{\nu = -2}^{\infty} \int_{\Delta_\nu} |E f(x)|^p \, dx \leq c \sum_{\nu = -2}^{\infty} \int_{\Delta_\nu} g_\nu(x) \, dx
\leq c \sum_{\nu = -2}^{\infty} 2^{-\nu n} \int \frac{|f(t)|^p}{m_\nu(t)} \, d\mu(t) .
$$

Now, if $1 > r_1 > r_0 > 0$ and $\mu$ satisfies $(D_\nu)$, then by writing $B(t, r_1) = B(t, (r_1/r_0)r_0)$ we see that

(18) \hspace{1cm} \mu(B(t, r_1)) \leq c (r_1/r_0)^{\nu} \mu(B(t, r_0)) , \hspace{0.5cm} t \in F .

With $r_1 = 1$ and $r_0 = 2^{-\nu}$ we get $c_1 \leq m_0(t) \leq c 2^{\nu s} m_\nu(t) , t \in F ,$ where the
first inequality is due to assumption (3). Using this estimate for $m_\nu$ above and
performing the summation we get, since $s < n ,$ that $\|E f\|_p \leq c\|f\|_{p, \mu}$.

To treat the second term in (16), we for convenience choose $H = 2^{-6}/2s$
and put $h_m = 2^{-m/2s}$. Then the second term in (16) is less than

(19) \hspace{1cm} c \left( \sum_{m = 6}^{\infty} 2^{m(n + q a)} \int_{|h| < h_m} \left( \int_{\Delta_\nu} |\Delta_{h}^\nu f(x)|^p \, dx \right)^{q/p} \, dh \right)^{1/q} .

Since $E f(x) = 0$ if $x \in \Delta_\nu , \nu \leq -3 , it follows from (7) that $\Delta_{h}^\nu (E f)(x) = 0$
if $x \in \Delta_\nu , \nu \leq -5$ and $|h| < h_6 , and putting $G_m = \bigcup_{\nu = -4}^{m} \Delta_\nu$ and $F_m+1 = \bigcup_{\nu = m+1}^{\infty} \Delta_\nu , we see that (19) is less than $c(\sum A_m)^{1/q} + c(\sum B_m)^{1/q} , where $A_m$ and $B_m$ are the terms in (19), but with the $x$-integration taken over $x \in G_m$
and $x \in F_{m+1}$, respectively.

In [2, pp. 115–116] it is shown (cf. (15)) that

$$
\int_{\Delta_\nu} |\Delta_{h}^\nu (E f)(x)|^p \, dx \leq c |h|^{sp} \sum_{|j| = s} \int_{\Delta_\nu} |D_j (E f)(x)|^p \, dx
$$

if $|h| < h_\nu , where $\Delta_\nu = \bigcup_{\nu = -2}^{\nu+3} \Delta_i , this does not depend on the form of $E f$ .

Thus, if $|h| < h_m$,

$$
\int_{G_m} |\Delta_{h}^\nu (E f)(x)|^p \, dx \leq c 2^{-msp} \sum_{|j| = s} \sum_{\nu = -2}^{m+3} \int_{\Delta_\nu} |D_j (E f)(x)|^p \, dx
$$
since $D_j(\mathcal{B} f)(x) = 0$ if $x \in \Delta_\nu$, $\nu \leq -3$, and putting this into $A_m$, and performing the $h$-integration we get

$$\sum_{m=6}^{\infty} A_m \leq c \sum_{m=6}^{\infty} 2^{m(\alpha-s)q} \left( \sum_{|j|=s}^{m+3} \int_{\Delta_\nu} |D_j(\mathcal{B} f)(x)|^p \, dx \right)^{q/p}$$

$$\leq c \sum_{\nu=-2}^{\infty} 2^{\nu(\alpha-s)q} \left( \sum_{|j|=s}^{\nu} \int_{\Delta_\nu} |D_j(\mathcal{B} f)(x)|^p \, dx \right)^{q/p} = c \sum_{\nu=-2}^{\infty} A'_\nu,$$

where the latter inequality is a consequence of Hardy's inequality, [2, p. 121], since $\alpha < s$.

Now we use (b) in Lemma 1 with $\chi = \tau$ and get for $x \in Q_\chi \subset \Delta_\nu$, $\nu \geq 2$, and $|j| = s$ that

$$|D_j(\mathcal{B} f)(x)|^p \leq c 2^{\nu p} \int_{|t-x_\chi| \leq 30l_\chi} \int_{|t-s| \leq c^{-\nu}} |f(t) - f(s)|^p \, d\mu_1,$$

where we temporarily put $d\mu_1 = d\mu(s) \, d\mu(t)/(m_\nu(t)m_\nu(s))$. To obtain this we put $c_\chi$ under the integral sign in $J_p$, applied (17), and used that $|s-x_\chi| \leq 30l_\chi$ and $|t-x_\chi| \leq 30l_\chi$ imply $|s-t| \leq 60l_\chi < c^{-\nu}$. Using (14) in Lemma 2 with the integral with respect to $s$ above as $h(t)$, we obtain

$$\sum_{\nu=2}^{\infty} A'_\nu \leq c \sum_{\nu=2}^{\infty} 2^{\nu(\alpha-s)q} \left( 2^{\nu p} 2^{2\nu n} \int_{|t-s| < c^{-\nu}} |f(t) - f(s)|^p \, d\mu_1 \right)^{q/p},$$

which is less than a constant times the $B_\alpha^{\nu, q}(F)$-norm of $f$ raised to the power $q$ by the remarks given after the definition of $B_\alpha^{\nu, q}(F)$. The estimation of $A'_\nu$, $\nu < 2$, is simpler; by means of (d) in Lemma 1 and (14) in Lemma 2 one easily obtains that $A'_\nu \leq c\|f\|_{p, \nu}^q$, $-2 \leq \nu \leq 1$.

In order to treat $\sum B_m$, we start from a simple estimate given in [2, p. 117], which gives that if $|h| < h_m$, then

$$\int_{F_{m+1}} |\Delta_h(\mathcal{B} f)(x)|^p \, dx \leq c \int_{F_{m-2}} |\mathcal{B} f(x) - \mathcal{B} f(x + h)|^p \, dx.$$

It is easy to show (we perform it at the end of this paragraph), that the right-hand side is, if $|h| < h_m$, less than

$$c|h|^{-n} \int_{|t| < 2|h|} \int_{x, x+t \in F_{m-4}} |\mathcal{B} f(x) - \mathcal{B} f(x + t)|^p \, dx \, dt.$$

Thus

$$\sum_{m=6}^{\infty} B_m \leq c \sum_{m=6}^{\infty} 2^{m(\alpha-q)} \left( 2^{mn} \int_{x, y \in F_{m-4}} |\mathcal{B} f(x) - \mathcal{B} f(y)|^p \, dx \, dy \right)^{q/p}.$$
The calculation involves starting with the estimate (a) in Lemma 1 for $|\mathcal{E} f(x) - \mathcal{E} f(y)|^p$, then replacing $c_x$, as we may by (17), by $1/m_{\nu}(t)$ if $s_x = 2^{-\nu}$ and integrating over $x \in \Delta_{\nu}$, $|x - y| < 2h_{\nu}$ by means of (13) in Lemma 2, replacing then $c_x$ by $1/m_N(s)$ and integrating over $y \in \Delta_{\nu}$ applying (14). From (18) we get that $m_{\nu}(t) \geq c_{2^{(m-\nu)s}m_{m}(t)}$, $\nu \geq m$, an inequality which is obviously valid for $m - 4 \leq \nu < m$, also, and using this and the analogous estimate for $m_{N}(s)$ one obtains

$$
\sum_{m=6}^{\infty} B_m \leq c \sum_{m=6}^{\infty} 2^{maq} \left( \sum_{\nu, N=m-4}^{\infty} 2^{mn2-2ms2-\nu(n-s)2-N(n-s)s} I_m \right)^{q/p},
$$

where $I_m$ is the double integral with respect to $t$ and $s$ above with $m_{\nu}$ and $m_N$ replaced by $m_m$. Since we assume $s < n$, we can perform the inner summations and get the desired estimate, which completes the proof of the result $\mathcal{E} f \in L^{p,q}_{\nu}(\mathbb{R}^n)$.

We conclude this section by showing how (20) follows. For $M \subset \mathbb{R}^n$ we have for a certain constant $c_1$

$$
c_1 |h|^n \int_{x \in M} |f(x + h) - f(x)|^p dx = \int_{|t| < |h|} dt \int_{x \in M} |f(x + h) - f(x)|^p dx
\leq c \int_{|t| < |h|} \int_{x \in M} |f(x + t) - f(x)|^p dx
+ c \int_{|t| < |h|} \int_{x \in M} |f(x + h) - f(x + t)|^p dx,
$$

where the latter term equals $c \int_{|t| < |h|} \int_{x \in M_h} |f(x) - f(x + t - h)|^p dx$, where $M_h$ is a certain translation of $M$. Thus the right-hand side of the inequality above is less than $c \int_{|t| < 2|h|} \int_{x \in M \cup M_h} |f(x + t) - f(x)|^p dx$. In our case we have $M = F_{m-2}$ and $|h| < h_m$, and (20) follows with the aid of (7).

3.4. It remains to prove that $\mathcal{G} f = f$. From (c) in Lemma 1 it follows (cf. [2, p. 119]) with the aid of (17) and (13) in Lemma 2 that

$$
\int_{|x - t_0| \leq r} |\mathcal{E} f(x) - f(t_0)|^p dx \leq c2^{-\nu n} \int_{|t - t_0| \leq r + c2^{-\nu}} \frac{|f(t) - f(t_0)|^p}{m_{\nu}(t)} d\mu(t).
$$

If $|x - t_0| \leq r$ and $x \in \Delta_{\nu}$, then (see [2]) $\sqrt{n}2^{-\nu} \leq r$, so if $\tau$ is the integer such that $2^{-\tau} \leq r/\sqrt{n} < 2^{-(\tau-1)}$, we get integration over $|x - t_0| \leq r$ in the left-hand side if we perform summation from $\tau$ to $\infty$, and using again that, by (18), $m_{\nu}(t) \geq c_{2^{(m-\nu)s}m_{m}(t)}$ and that $s < n$, we get

$$
\int_{|x - t_0| \leq r} |\mathcal{E} f(x) - f(t_0)|^p dx \leq c2^{-\nu n} \mu(B(t_0, 2^{-\tau})) g_{\tau}(t_0),
$$

where

$$
g_{\tau}(t_0) = \int_{|t - t_0| < 2^{2^{-\tau}}} \frac{|f(t) - f(t_0)|^p}{m_{\tau}(t)m_{\tau}(t_0)} d\mu(t)
$$

and where we also extended the fraction by $m_{\tau}(t_0)$. By the definition of $B^{p,q}_{\nu}(F)$ and the remarks given after it, we have

$$
\sum_{\tau=0}^{\infty} \left( 2^{(\alpha - n/p)} \left( \int g_{\tau}(t_0) d\mu(t_0) \right)^{1/p} \right)^q < \infty,
$$
so \( \int g_\tau(t_0) \, d\mu(t_0) \leq c 2^{-\tau(\alpha p - n)} \). Now, if we divide both sides in (21) by \( m(B(t_0, r)) \) and use (15), we obtain on the right-hand side

\[
2^{-\tau d} g_\tau(t_0) = 2^{-\tau d} 2^{-\tau(d - \varepsilon)} g_\tau(t_0).
\]

But by the estimate above we have

\[
\int \sum_{\tau=0}^{\infty} 2^{-\tau(d - \varepsilon)} g_\tau(t_0) \, d\mu(t_0) \leq \sum_{\tau=0}^{\infty} 2^{-\tau(\alpha p - n + d - \varepsilon)},
\]

so if \( \alpha p - (n - d) - \varepsilon > 0 \), then \( 2^{-\tau(d - \varepsilon)} g_\tau(t_0) \) is uniformly bounded in \( \tau \) for \( \mu \)-almost all \( t_0 \). From our assumption \( \alpha > (n - d)/p \) it follows that \( \varepsilon > 0 \) may be chosen so that \( \alpha p - (n - d) - \varepsilon > 0 \), and thus we get for \( \mu \)-almost all \( t_0 \)

\[
\frac{1}{m(B(t_0, r))} \int_{|x-t_0|\leq r} |\mathcal{E} f(x) - f(t_0)|^p \, dx \to 0, \quad r \to 0,
\]

which gives the desired result (cf. [2]).

3.5. The case \( s = n \). In this case the assumption on \( \alpha \) is \( (n - d)/p < \alpha < 1 \). We shall reduce this case to the case \( s < n \) by considering \( \mathbb{R}^n \) as a subset of \( \mathbb{R}^{n+1} \). It is clear from the definition, that the space \( B_{p',q}^s(F) \), \( F \subset \mathbb{R}^n \), is the same as \( B_{p',q}^{s+1/p}(F) \) when \( F \) is considered as a subset of \( \mathbb{R}^{n+1} \). Let \( f \in B_{p',q}^s(F) \). Since \( (n+1-d)/p < \alpha + 1/p < 1+(n+1-s)/p \) and \( s = n < n+1 \), the already proved part of the extension theorem gives us a function \( E f \in \Lambda_{a+1/p}^p(\mathbb{R}^{n+1}) \) such that \( E f \mid F = f \) \( \mu \)-a.e. The classical restriction theorem for Besov spaces tells us that \( E f \mid \mathbb{R}^n \in \Lambda_{a+1/p}^p(\mathbb{R}^n) \), and we take \( \mathcal{E} f = E f \mid \mathbb{R}^n \) as our extension of \( f \). Possibly, \( \mathcal{E} f \mid F \) could be different from \( E f \mid F \), since the restriction operators are defined in different ways, but at least \( \mathcal{E} f \mid F \in B_{p',q}^s(F) \) by the restriction part of Theorem 1, which will be proved in the next section. However, using that both \( g \to g \mid F \) and \( g \to (g \mid \mathbb{R}^n) \mid F \) are continuous operators from \( \Lambda_{a+1/p}^p(\mathbb{R}^{n+1}) \) to \( B_{p',q}^s(F) \), and that \( C_{0}^\infty(\mathbb{R}^{n+1}) \) is dense in \( \Lambda_{a+1/p}^p(\mathbb{R}^{n+1}) \), it is easy to see that the two ways of taking restrictions coincide (cf. [2, p. 211]). Thus, \( \mathcal{E} f \mid F = f \) \( \alpha \), and since all operators involved in the above argument are continuous, the extension theorem in the case \( s = n \) follows.

To see that different measures \( \mu \) on \( F \) give rise to equivalent spaces \( B_{p',q}^s(F) \), we argue in a similar way. Let \( f \) belong to the space \( B_{p',q}^s(F) \) defined by means of \( \mu_1 \). By the extension theorem, there is a function \( \mathcal{E} f \in \Lambda_{a+1/p}^p(\mathbb{R}^n) \) such that \( \mathcal{E} f \mid F = f \) except on a set \( E_1 \) with \( \mu(E_1) = 0 \). By the restriction theorem, \( \mathcal{E} f \mid F \) exists except on \( E_2 \), \( \mu_2(E_2) = 0 \), and belongs to \( B_{p',q}^s(F) \), the space defined by means of \( \mu_2 \). Thus, it is possible to redefine \( f \) (as \( \mathcal{E} f \mid F \)) on a set with \( \mu_1 \)-measure zero \( (E_1 \setminus E_2) \), so that \( f \) is defined \( \mu_2 \)-a.e. and belongs to \( B_{p',q}^s(F) \).

Finally we mention that the equivalence of \( \Lambda_{a+1/p}^p(\mathbb{R}^n) \) and \( B_{p',q}^s(\mathbb{R}^n) \) also follows by means of an argument similar to the one used above to prove the extension theorem when \( s = n \).

4. The restriction theorem

Following [2], we shall prove the restriction part of Theorem 1 by first proving two lemmas (Lemmas 6 and 7) on Bessel potentials, and then use the real
interpolation method to obtain the result. We start by giving some preliminary estimates in a sequence of lemmas. Recall that by the comments in the beginning of §2, the conditions \( (D_s) \), \( (L_d) \), and \( (3) \), are valid in a slightly more general setting; we will use this without further comments.

**Lemma 3.** Let \( r \leq 1 \), \( \gamma \geq 0 \), and \( t \in \mathbb{R}^n \), and let \( \mu \) satisfy \( (2) \) and \( (3) \).

(i) If \( 0 \leq d \leq n \) and \( \mu \) satisfies \( (L_d) \), then

\[
\int_{|t-y|<r} \frac{|t-y|^{-\gamma}}{\mu(B(y,r))} \, d\mu(y) \leq cr^{-\gamma} \quad \text{for} \quad \gamma < d.
\]

The inequality also holds for \( \gamma = d = 0 \).

(ii) If \( 0 < s \leq n \) and \( \mu \) satisfies \( (D_s) \) and \( (3) \), then

\[
\int_{r\leq|t-y|<1} \frac{|t-y|^{-\gamma}}{\mu(B(y,r))} \, d\mu(y) \leq cr^{-\gamma} \quad \text{for} \quad \gamma > s.
\]

**Proof.** (i) We assume that \( r = 2^{-\nu_0} \) for some integer \( \nu_0 \geq 0 \). For \( |t-y| < r \) we have by \( (2) \) that \( \mu(B(y,r)) \geq c\mu(B(y,2r)) \geq c\mu(B(t,2r)) \) which gives the inequality for \( \gamma = d = 0 \), and that the integral in statement (i) is less than \( c \sum_{\nu=\nu_0}^{\infty} 2^{\nu d} \mu(B(t,2^{-\nu})) / \mu(B(t,2^{-\nu_0})) \) for \( d > 0 \). Here we used dyadic rings and the fact that \( \mu \) has no point mass if \( d > 0 \). From \( (L_d) \) we have

\[
\mu(B(t,2^{-\nu_0})) \geq c 2^{\nu(\nu_0)} \mu(B(t,2^{-\nu_0})),
\]

so the sum is less than \( c \sum_{\nu=\nu_0}^{\infty} 2^{\nu d} 2^{\nu(\nu-d)} = c 2^{\nu_0 \gamma} = c r^{-\gamma} \) if \( \gamma < d \).

(ii) We assume that \( r = 2^{-\nu_0} \) for some \( \nu_0 \geq 1 \). With \( k = 3|y-t|/r \) we get from \( (D_s) \) that \( \mu(B(t,2|y-t|)) \leq \mu(B(y,kr)) \leq c k^s \mu(B(y,r)) \). Putting this estimate into the integral in (ii) and using again dyadic rings, one obtains that it is less than

\[
cr^{-s} \sum_{\nu=0}^{\nu_0-1} 2^{\nu d} 2^{\nu(\nu-d)} \frac{\mu(B(t,2^{-\nu}))}{\mu(B(t,2^{-\nu_0}))} \leq cr^{-s} 2^{\nu_0 (s-d)} = cr^s \quad \text{if} \quad \gamma > s.
\]

For the definition of the Bessel kernel \( G_\alpha \), see, e.g., [2, p. 6]. We also need some estimates on it which may be found in [2, p. 104], for example we have

\[
|D^n G_\alpha(x)| \leq c|x|^{\alpha-n}|x|^{-n}, \quad x \neq 0, \, \alpha < n + |j|.
\]

In the remaining part of this section, we will use the shorthand notation

\[
d\nu_r = d\mu(x) d\mu(y) / \mu(B(x,r)) \mu(B(y,r)).
\]

**Lemma 4.** Let \( 0 < \alpha < n + 1 \), \( \alpha \neq n \), \( 0 < d \leq s \leq n \), \( 0 < r \leq 1 \), \( \theta > 0 \), \( t \in \mathbb{R}^n \), and let \( \mu \) satisfy \( (D_s) \), \( (L_d) \), and \( (3) \). Then

\[
J_1 = \int \int_{|x-y|<r \wedge |y-t|<2r} |G_\alpha(x-t) - G_\alpha(y-t)|^\theta d\nu_r \leq c r^{(\alpha-n)\theta} \quad \text{if} \quad \frac{d}{\theta} - (n - \alpha) > 0.
\]

The same estimate holds for \( s/\theta -(n-\alpha) < 1 \) if in the domain of integration \( |y-t| < 2r \) is replaced by \( |y-t| \geq 2r \) (we denote the corresponding integral by \( J_2 \)).

**Proof.** Starting with \( J_1 \), we consider the cases \( \alpha < n \) and \( n < \alpha < n + 1 \) separately. If \( n < \alpha < n + 1 \), the Bessel kernel is Lipschitz continuous with
exponent $\alpha - n$, so

$$J_1 \leq c r^{(\alpha - n)\theta} \int_{|x-y|<r, |y-t|<2r} \frac{1}{\mu(B(x, r))\mu(B(y, r))} d\mu(x) d\mu(y).$$

If $|x - y| < r$ we have $B(y, r) \subset B(x, 2r)$ and (2) gives $\mu(B(y, r)) \leq c\mu(B(x, r))$, so replacing $\mu(B(x, r))$ by $c\mu(B(y, r))$ in the denominator and performing the $x$-integration we get

$$J_1 \leq c r^{(\alpha - n)\theta} \int_{|y-t|<2r} \frac{1}{\mu(B(y, r))} d\mu(y) \leq c r^{(\alpha - n)\theta},$$

where we used (i) in Lemma 3 with $\gamma = 0$. (To apply Lemma 1 here and below, we have to make some small adjustments: Replace $\mu(B(y, r))$ by $\mu(B(y, 2r))$, which is allowed because of (2), and use that the limitation $r < 1$ in Lemma 3 may be replaced by $r \leq r_0$.) If $\alpha < n$ we use the triangle inequality to split $J_1$ into two parts with $|G_\alpha(x - t)|^\theta$ and $|G_\alpha(y - t)|^\theta$ in the integrands. To treat the second part we replace as above $\mu(B(x, r))$ by $c\mu(B(y, r))$ and perform the $x$-integration which gives $c \int_{|y-t|<2r} |G_\alpha(y - t)|^\theta/\mu(B(y, r)) d\mu(y)$ which is less than $r^{(\alpha - n)\theta}$ by (22) and Lemma 3 if $(n - \alpha)\theta < \delta$. The first part may be treated in the same way if one first replaces the domain of integration by $\{(x, y); |x-\gamma| < 3r, |x-y| < r\}$.

To treat $J_2$, we first note that it is less than $J'_2 + J''_2$, where in $J'_2$ and $J''_2$ the domains of integration are as in $J_2$, but with $2r \leq |y-t| < 2$ and $|y-t| \geq 2$, respectively. Using the mean value theorem on $G_\alpha(x - t) - G_\alpha(y - t)$ and estimating in the same way as in [2, p. 106], using (22), we get

$$J'_{2} \leq c r^\theta \int_{2r \leq |y-t|} |y-t|^{(\alpha - n - 1)\theta} d\nu_r,$$

the integral taken over $|x-y| < r$, $2r \leq |y-t| < 2$. Replacing as above $\mu(B(x, r))$ by $c\mu(B(y, r))$, performing the $x$-integration and using part (ii) in Lemma 3 we get $J'_2 \leq c r^{(\alpha - n)\theta}$ if $(n + 1 - \alpha)\theta > s$. Finally a similar calculation (cf. [2]) gives, using the exponential decrease of $G_\alpha$, (4), and (5), that $J''_2 \leq c r^{\theta - s}$, so $J'_2 \leq c r^{(\alpha - n)\theta}$ if $s/\theta - (n - \alpha) < 1$.

**Lemma 5.** Let $0 < \alpha < n + 1$, $\alpha \neq n$, $\theta > 0$, $\gamma = n/\theta - (n - \alpha)$, $h \in \mathbb{R}^n$, $|h| \leq 1$. Then

$$\int_{|t|<2|h|} |G_\alpha(t + h) - G_\alpha(t)|^\theta \, dt \leq c |h|^\gamma \theta \quad \text{if } \gamma > 0.$$

The same estimate holds for $\gamma < 1$ if in the domain of integration $|t| < 2|h|$ is replaced by $|t| \geq 2|h|$.

The proof of this lemma, which is essentially the same as Lemma B in [2, p. 106], is similar to but simpler than the proof of Lemma 4. From now on, $u$ will denote the Bessel potential defined by $u(x) = \int G_\alpha(x - t)f(t) \, dt$, $f \in L^p(\mathbb{R}^n)$; cf. also [2, p. 107].

**Lemma 6.** Let $1 \leq p \leq \infty$, $0 \leq d \leq n$, $d \leq s \leq n$, $0 < r < 1$, $s > 0$, $\alpha > 0$, $\alpha \neq n$, and $(n - d)/p < \alpha < 1 + (n - s)/p$. Let furthermore $\mu$ satisfy $(D_s)$,
(Ld), and (3), and let \( u = G_\alpha * f, f \in L^p(\mathbb{R}^n) \). Then

\[
\left( \iint_{|x-y|<r} |u(x) - u(y)|^p \, dv_r \right)^{1/p} \leq cr^{\alpha-n/p} \|f\|_p,
\]

where \( c \) is independent of \( r \) and \( f \).

**Proof.** The left-hand side of the inequality to be proved is less than or equal to \( A + B \), where

\[
A = \left( \iint_{|x-y|<r} \left( \int_{|y-t|<2r} |G_\alpha(x-t) - G_\alpha(y-t)| \, |f(t)| \, dt \right)^p \, dv_r \right)^{1/p},
\]

and \( B \) is given by the same expression but with \(|y-t| \geq 2r\). We first estimate \( A \). Let \( 0 < a < 1 \). From Hölder's inequality it follows that the integral with respect to \( t \) in \( A \) raised to the power \( p \) is less than or equal to

\[
\int_{|y-t|<2r} |G_\alpha(x-t) - G_\alpha(y-t)|^{ap} \, |f(t)| \, dt \, I_A(x, y),
\]

where, with \( 1/p' = 1 - 1/p \),

\[ I_A(x, y) = \left( \int_{|y-t|<2r} |G_\alpha(x-t) - G_\alpha(y-t)|^{(1-a)p'} \, dt \right)^{p/p'}.
\]

If \(|x-y| < r\) and

\[
\gamma = n/(1-a)p' - (n-\alpha) > 0
\]

then, by Lemma 5, \( I_A(x, y) \leq cr^{\gamma(1-a)p} \), and after a change of order in the integration we get

\[
A \leq cr^{\gamma(1-a)} \left( \int |f(t)|^p J_A(t) \, dt \right)^{1/p},
\]

where

\[ J_A(t) = \iint_{|x-y|<r} |G_\alpha(x-t) - G_\alpha(y-t)|^{ap} \, dv_r.
\]

To deal with \( J_A \), we separate two cases. If \( d = 0 \) and so \( \alpha p > n \) we take \( a = 0 \). Then clearly (23) holds, and estimates performed in the beginning of the proof of Lemma 4 show that \( J_A \leq c \). Consequently, \( A \leq cr^{\gamma} = cr^{\alpha-n/p} \). If \( d > 0 \) we obtain from Lemma 4 that \( J_A \leq cr^{(\alpha-n)p} \) if

\[
d - (n-\alpha)p > 0,
\]

and putting this estimate into (24) we get \( A \leq cr^{\alpha-n/p} \|f\|_p \). It remains to see that \( a \) with \( 0 < a < 1 \) may be chosen so that (23) and (25) hold. But these inequalities can be simplified to, if \( \alpha < n \) (the case \( \alpha > n \) is obvious), \( a > (n-\alpha p)/(n-\alpha)p \) and \( a < d/(n-\alpha)p \), respectively, so \( a \) can be chosen since we assume \( n-\alpha p < d \) and, in this calculation, \( p > 1 \) and \( d > 0 \).

Now we turn to the term \( B \), and take an \( a \) with \( 0 < a < 1 \), which need not be the same as above. We make the same calculations as above, and get analogous factors \( I_B(x, y) \) and \( J_B(t) \) with \(|y-t| \geq 2r\) in the domains of
introduction. By the second parts of Lemmas 5 and 4 we have with $\gamma$ as in (23) that $I_B(x, y) \leq cr^{(1-a)p}$ if $\gamma < 1$, and that (now we have no case $a = 0$) $J_B(x, y) \leq cr^{(\alpha-n)ap}$ if $s - (n - \alpha)ap < ap$. Thus we get $B \leq cr^{(n-a)p}$ if $a$ may be chosen. The limitations may be simplified to $a < (n + p - \alpha p)/(n + 1 - \alpha)p$ and $a > s/(n + 1 - \alpha)p$, respectively, and thus $a$ may be chosen since we assume $s < n + p - \alpha p$ and $p > 1$.

**Lemma 7.** Let $1 \leq p \leq \infty$, $0 \leq d \leq n$, $\alpha > 0$, $\alpha \neq n$, and $(n-d)/p < \alpha$, let $\mu$ satisfy $\mu(d)$ and (3), and let $u = G_\alpha * f$, $f \in L^p(\mathbb{R}^n)$. Then $\|u\|_{L^p(\mathbb{R}^n)} \leq c\|f\|_{L^p(\mathbb{R}^n)}$, where $c$ is independent of $f$.

**Proof.** We have (cf. the preceding proof) that $\|u\|_{L^p(\mathbb{R}^n)} \leq A + B$, where

$$A = \left( \int \left( \int_{|x-t|<1} |G_\alpha(x-t)f(t)|^p \, dt \right)^{1/p} \, d\mu(x) \right)^{1/p}$$

and $B$ is the same but with $|x-t| \geq 1$. To treat $A$, let $0 \leq a < 1$. By means of Hölder’s inequality we get

$$A \leq I_A^{1/p'} \left( \int |f(t)|^p J_A(t) \, dt \right)^{1/p},$$

where $I_A = \int_{|t|<1} |G_\alpha(t)|^{(1-a)p'} \, dt$ and $J_A(t) = \int_{|x-t|<1} |G_\alpha(x-t)|^{a p} \, d\mu(x)$. Thus we get the desired estimate for $A$ if $I_A$ and $J_A$ converge, the latter with a bound independent of $t$. If $\alpha > n$ they converge since the integrands are continuous. If $\alpha < n$, we get from (22) that $I_A$ converges if

$$(26) \quad (n - \alpha)(1 - a)p' < n.$$ 

To treat $J_A$, we consider first the case $d = 0$, so that $\alpha p > n$, and take $a = 0$. Then (26) is fulfilled so $I_A$ converges, and we have $J_A(t) \leq c$ as a consequence of (3). If $d > 0$ then (22) and the estimate $\mu(B(t, r)) \leq cr^d$, $t \in \mathbb{R}^n$, easily give (cf. the proof of (i) in Lemma 3) that $J_A(t) \leq c$ if $(n - \alpha)ap < d$. As we saw in the proof of Lemma 6, $a$ may be chosen so that this inequality and (26) hold simultaneously, so the desired estimate follows. To deal with $B$ we take $0 < a < 1$ and use Hölder’s inequality as above. Then it is easily seen that the correspondence factors $I_B$ and $J_B(t)$ are convergent, due to the exponential decrease of $G_\alpha$ and (4).

Now we prove the restriction part of Theorem 1. The proof consists of an application of the real interpolation method to the estimates in Lemmas 6 and 7. We need the space $L^p_\alpha(\mathbb{R}^n)$ of Bessel potentials. This consists of all functions $u$ of the form $u = G_\alpha * f$, $f \in L^p(\mathbb{R}^n)$, and the norm of $u$ in $L^p_\alpha(\mathbb{R}^n)$ is defined as $\|u\|_p$.

Let $\alpha_0$ and $\alpha_1$ satisfy $0 < \alpha_0 < \alpha < \alpha_1$, $\alpha_0, \alpha_1 \neq n$, $(n-d)/p < \alpha_i < 1+(n-s)/p$, $i = 0, 1$, and define $\theta$ by $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Using interpolation together with Lemma 7 in the same way as in [2], we get that the restriction operator is bounded from $L^p_\theta(\mathbb{R}^n)$ to $L^p(\mu)$. To take care of the other part of the $B^\theta_{p,q}(F)$-norm of $f$ we proceed as follows. Define $T^\nu \nu y$ by

$$(T^\nu \nu y)(x, y) = (f(x) - f(y))/(m_\nu(x)m_\nu(y))^{1/p}$$

if $|x - y| < 2^{-\nu}$ and $(T^\nu \nu y)(x, y) = 0$ otherwise, and denote by $l^2_\nu(A)$ the space of all sequences $(a_\nu)_{\nu=0}^{\infty}$, $a_\nu \in A$, with norm $(\sum_{\nu=0}^{\infty} (2^\nu s \|a_\nu\|_A)^q)^{1/q} < \infty$, where
We must prove that $Tf = (T∂f)^\infty_0$ is bounded from $Λ^p_q(\mathbb{R}^n)$ to $l^{α-n/p}_q(\mathcal{A})$. Lemma 6 shows that $T$ is bounded from $\mathcal{L}^p_0(\mathbb{R}^n)$ to $l^{α-1}_{∞}(\mathcal{A})$, $i = 0, 1$, and thus by interpolation $T$ is bounded from the intermediate space $(\mathcal{L}^p_0(\mathbb{R}^n), \mathcal{L}^p_1(\mathbb{R}^n))_{θ_2} = Λ^p_q(\mathbb{R}^n)$ to the intermediate space $(l^{α-1}_{∞}(\mathcal{A}), l^{α-1-n/p}_1(\mathcal{A}))_{θ_2} = l^{α-n/p}_q(\mathcal{A})$.

**References**


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