

## THE $H^p$ -CORONA THEOREM FOR THE POLYDISC

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**ABSTRACT.** Let  $H^p = H^p(D^n)$  denote the usual Hardy spaces on the polydisc  $D^n$ . We prove in this paper the following theorem: Suppose  $f_1, f_2, \dots, f_n \in H^\infty$ ,  $\|f_j\|_{H^\infty} \leq 1$ , and  $\sum_{j=1}^m |f_j(z)| \geq \delta > 0$ . Then for every  $g$  in  $H^p$ ,  $1 < p < \infty$ , there are  $H^p$  functions  $g_1, g_2, \dots, g_m$  such that  $\sum_{j=1}^m f_j(z)g_j(z) = g(z)$ . Moreover, we have  $\|g_j\|_{H^p} \leq c(m, n, \delta, p)\|g\|_{H^p}$ . (When  $p = 2$ ,  $n = 1$ , this theorem is known to be equivalent to Carleson's corona theorem.)

Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $H^\infty = H^\infty(\Omega)$  the algebra of bounded analytic functions on  $\Omega$  equipped with the sup norm,  $\|f\|_{H^\infty} = \sup_{z \in \Omega} |f(z)|$ . The corona problem is the following: Given  $f_1, f_2, \dots, f_m \in H^\infty$ ,  $\|f_j\|_{H^\infty} \leq 1$ , and  $\sum_{j=1}^m |f_j(z)| \geq \delta > 0$  on  $\Omega$ , are there  $H^\infty$  functions  $g_1, g_2, \dots, g_n$  such that  $\sum_{j=1}^m f_j g_j \equiv 1$ ? When  $n = 1$  and  $\Omega = D =$  the unit disc in  $\mathbb{C}$ , the answer is affirmative and that is Carleson's famous corona theorem [3]. For other planar domains see the extensive bibliography [9]. The deepest result is due to Garnett and Jones [6] who proved the corona theorem for Denjoy domains. In higher dimensions ( $n > 1$ ), only negative results are known, see N. Sibony's articles [11, 12] and B. Cole's example in [7]. However, there are some partial results. Henkin [8] and Varopoulos [13] (with  $m = 2$ ) showed that when  $\Omega = B^n =$  the unit ball in  $\mathbb{C}^n$ , there are corona solutions  $g_1, g_2, \dots, g_m$  in all  $H^p$ ,  $0 < p < \infty$ . The similar result for the polydisc was proved by Varopoulos [14], Chang [4], and Lin [10]. Moreover, E. Amar [1] (with  $m = 2$ ) and M. Andersson [2] (with  $p = 2$ ) have recently proved the following  $H^p$ -corona theorem for the ball  $B^n$ : Suppose  $f_1, f_2, \dots, f_m \in H^\infty = H^\infty(B^n)$ ,  $\|f_j\|_{H^\infty} \leq 1$ , and  $\sum_{j=1}^m |f_j(z)| \geq \delta > 0$  on  $B^n$ . Then for every  $g$  in  $H^p(B^n)$ ,  $1 \leq p < \infty$ , there are  $H^p$  functions  $g_1, g_2, \dots, g_n$  such that  $\sum f_j g_j \equiv g$ . This theorem can be viewed as an evidence that the corona problem may be true for the ball, since when  $n = 1$ ,  $p = 2$ , it is equivalent to Carleson's original theorem. The purpose of this paper is to prove the polydisc version of the above theorem (except for the ranges of  $p$ ):

**Theorem.** Suppose  $f_1, f_2, \dots, f_m \in H^\infty(D^n)$ ,  $\|f_j\|_{H^\infty} \leq 1$ , and  $\sum_{j=1}^m |f_j(z)| \geq \delta > 0$  on  $D^n$ . Then for every  $g$  in  $H^p(D^n)$ ,  $1 < p < \infty$ , there are  $H^p$  functions  $g_1, g_2, \dots, g_m$  such that  $\sum_{j=1}^m f_j g_j \equiv g$ . Moreover, we have  $\|g_j\|_{H^p} \leq c(m, n, \delta, p)\|g\|_{H^p}$ .

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(Here,  $H^p(D^n)$  are the usual Hardy spaces, namely,  $f$  is in  $H^p(D^n)$  if and only if  $f$  is analytic and

$$\|f\|_{H^p}^p = \sup_{0 < r < 1} \frac{1}{(2\pi)^n} \int_{T^n} |f(rz)|^p d\sigma(z) < \infty,$$

$d\sigma$  is the Lebesgue measure on the distinguished boundary  $T^n$ .) The methods used here are quite different from those in the ball case. We will make a detailed study of the Koszul complex and  $\bar{\partial}$ -equations, as in the articles of Chang [4] and Lin [10]. As a matter of fact, the analysis here is quite similar to those papers. We will prove the theorem for the case  $n = 2$  only. The cases  $n \geq 3$  are not much more difficult, see [10]. We also remark that the proof below does not apply to the case  $p = 1$ , since the Hilbert transform fails to be bounded on  $L^1$ .

*Proof of the theorem  $n = 2$ .* We begin with the case  $m = 2$ , i.e. two generators  $f_1$  and  $f_2$ , and may assume  $f_1, f_2$  and  $g$  are analytic across  $T^2$ , by a normal families argument. Let

$$\varphi_j = \frac{g\bar{f}_j}{|f_1|^2 + |f_2|^2}.$$

Then  $f_1\varphi_1 + f_2\varphi_2 = g$ , and  $\|\varphi_j\|_{L^p(T^2)} \leq c(\delta)\|g\|_{H^p}$ . But  $\varphi_j$  need not be analytic. To rectify that, we put

$$(1) \quad g_1 = \varphi_1 + bf_2, \quad g_2 = \varphi_2 - bf_1.$$

Then we still have  $f_1g_1 + f_2g_2 = g$ . The requirement on the analyticity of  $g_j$  imposes a  $\bar{\partial}$ -equation on  $b$ , namely,

$$(2) \quad \begin{cases} (2.1) & \frac{\partial b}{\partial \bar{z}_1} = \frac{1}{g} \left[ \varphi_1 \frac{\partial \varphi_2}{\partial \bar{z}_1} - \varphi_2 \frac{\partial \varphi_1}{\partial \bar{z}_1} \right] = \frac{g \left( \bar{f}_1 \frac{\partial \bar{f}_2}{\partial \bar{z}_1} - \bar{f}_2 \frac{\partial \bar{f}_1}{\partial \bar{z}_1} \right)}{|f_1|^2 + |f_2|^2} = G_1, \\ (2.2) & \frac{\partial b}{\partial \bar{z}_2} = \frac{1}{g} \left[ \varphi_1 \frac{\partial \varphi_2}{\partial \bar{z}_2} - \varphi_2 \frac{\partial \varphi_1}{\partial \bar{z}_2} \right] = \frac{g \left( \bar{f}_1 \frac{\partial \bar{f}_2}{\partial \bar{z}_2} - \bar{f}_2 \frac{\partial \bar{f}_1}{\partial \bar{z}_2} \right)}{(|f_1|^2 + |f_2|^2)^2} = G_2. \end{cases}$$

The advantage of dealing with two generators is that the  $\bar{\partial}$ -system above is automatically  $\bar{\partial}$ -closed, i.e.,  $\partial G_1 / \partial \bar{z}_2 = \partial G_2 / \partial \bar{z}_1$ . To have  $g_j \in H^p(D^2)$  we need to solve (2) with  $\|b\|_{L^p(T^2)} \leq C(p, \delta)\|g\|_{H^p}$ .

We consider the equation (2.1) first, and will show, by one-variable technique, that it has a solution  $b_1$  so that  $\|b_1\|_{L^p(T^2)} \leq C\|g\|_{H^p}$ . To do this, we fix  $z_2$  in  $D_2$  (we think of  $D^2 = D_1 \times D_2$ ), and take a  $C^\infty$  solution to (2.1), say, for example,

$$b_0(z_1, z_2) = c \int_{D_1} \frac{G_1(\xi_1, z_2)}{\xi_1 - z_1} d\xi_1 \wedge d\bar{\xi}_1.$$

Then functions of the form  $b_0 + k$ , where  $k$  is analytic in  $z_1$ , solve (2.1). We are concerned with  $\|b_0 + k\|_{L^p(\partial D_1)}$ . By duality (one-variable),  $L^p/H^p = (H_0^q)^*$ ,

$\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$\begin{aligned} \inf_{k \in H^p} \|b_0 + k\|_{H^p(D_1)} &= \sup \left\{ c \left| \int_{\partial D_1} b_0 h \, dz_1 \right| : h \in H_0^q(D_1), \|h\|_{H^q(D_1)} \leq 1 \right\} \\ &= \sup \left\{ c \left| \int_{D_1} G_1 h \, dz_1 \wedge d\bar{z}_1 \right| : h \in H_0^q(D_1), \|h\|_{H^q(D_1)} \leq 1 \right\}, \end{aligned}$$

by Stokes' theorem. But Wolff's proof of the corona theorem [5] has shown that

$$\begin{aligned} \left| \int_{D_1} G_1 h \, dz_1 \wedge d\bar{z}_2 \right| &\leq c(\delta) \|g^h\|_{H^1(D_1)} \leq c(\delta, p) \|g\|_{H^p(D_1)} \|h\|_{H^q(D_1)} \\ &\leq c(\delta, p) \|g\|_{H^p(D_1)}. \end{aligned}$$

Thus there is a solution  $b_1$  to (2.1) such that  $\|b_1\|_{L^p(\partial D_1)} \leq c(\delta, P) \|g\|_{H^p(D_1)}$ . Notice that the argument above is done in such a way that the constant  $c(\delta, p)$  is independent of  $z_2$  in  $D_2$ . Therefore, after integrating over  $z_2 \in \partial D_2$ , we obtain  $\|b_1\|_{L^p(T^2)} \leq c \|g\|_{H^p(D^2)}$ . Similarly, there is a solution  $b_2$  to (2.2) so that  $\|b_2\|_{L^p(T^2)} \leq c(p, \delta) \|g\|_{H^p}$ . However, we need a common solution to (2.1) and (2.2). To this end, we have to find  $h_1, h_2$  in  $L^p(T^2)$  with  $h_j$  analytic in  $z_j$ , so that  $b_1 + h_1 = b_2 + h_2$ . Put

$$h_1(z_1, z_2) = c \int_{\partial D_1} \frac{(b_2 - b_1)(\xi_1, z_2)}{\xi_1 - z_1} d\xi_1,$$

and  $h_2 = h_1 + b_1 - b_2$ . Clearly,  $h_1$  is analytic in  $z_1$ , and  $h_2$  is analytic in  $z_2$ , for

$$\begin{aligned} \frac{\partial h_2}{\partial \bar{z}_2} &= \frac{\partial h_1}{\partial \bar{z}_2} + \frac{\partial}{\partial \bar{z}_2}(b_1 - b_2) \\ &= c \int_{\partial D_1} \frac{\partial (b_2 - b_1)(\xi_1, z_2) / \partial \bar{z}_2}{\xi_1 - z_1} d\xi_1 + \frac{\partial}{\partial \bar{z}_2}(b_1 - b_2) \\ &= \frac{\partial}{\partial \bar{z}_2}(b_2 - b_1) + \frac{\partial}{\partial \bar{z}_2}(b_1 - b_2) \\ &= 0 \end{aligned}$$

since  $\frac{\partial}{\partial \bar{z}_2}(b_2 - b_1)$  is analytic in  $z_1$ , by the  $\bar{\partial}$ -closedness of (2).

Moreover,  $\|h_j\|_{L^p(T^2)} \leq c(\|b_1\|_{L^p(T^2)} + \|b_2\|_{L^p(T^2)}) \leq c \|g\|_{H^p}$ . Hence there is a solution  $b$  to (2) with  $\|b\|_{L^p(T^2)} \leq c \|g\|_{H^p}$ , and the proof of the case  $m = 2$  is complete.

When the number  $m$  of generators is greater than two, the algebraic trick (1) is not available, and higher-order  $\bar{\partial}$ -equations are needed. We refer the readers to the paper [10] for the details of the Koszul complex which reduces the corona problem to a set of  $\bar{\partial}$ -equations. The system of  $\bar{\partial}$ -equations we need to solve here is

$$\begin{aligned} (3) \quad \frac{\partial^2 d_{j,k,l}}{\partial \bar{z}_1 \partial \bar{z}_2} &= \frac{1}{g^2} \left[ \varphi_j \frac{\partial \varphi_k}{\partial \bar{z}_1} \frac{\partial \varphi_l}{\partial \bar{z}_2} - \varphi_j \frac{\partial \varphi_l}{\partial \bar{z}_1} \frac{\partial \varphi_k}{\partial \bar{z}_2} + \varphi_1 \frac{\partial \varphi_j}{\partial \bar{z}_1} \frac{\partial \varphi_k}{\partial \bar{z}_2} \right. \\ &\quad \left. - \varphi_l \frac{\partial \varphi_k}{\partial \bar{z}_1} \frac{\partial \varphi_j}{\partial \bar{z}_2} + \varphi_k \frac{\partial \varphi_l}{\partial \bar{z}_1} \frac{\partial \varphi_j}{\partial \bar{z}_2} - \varphi_k \frac{\partial \varphi_j}{\partial \bar{z}_1} \frac{\partial \varphi_l}{\partial \bar{z}_2} \right] \equiv \Phi \end{aligned}$$

$$(4) \quad \begin{cases} (4.1) & \frac{\partial b_{j,k}}{\partial \bar{z}_1} = \frac{1}{g} \left[ \varphi_j \frac{\partial \varphi_k}{\partial \bar{z}_1} - \varphi_k \frac{\partial \varphi_j}{\partial \bar{z}_1} \right] - \sum_1 f_l \frac{\partial d_{j,k,l}}{\partial \bar{z}_1}, \\ (4.2) & \frac{\partial b_{j,k}}{\partial \bar{z}_2} = \frac{1}{g} \left[ \varphi_j \frac{\partial \varphi_k}{\partial \bar{z}_2} - \varphi_k \frac{\partial \varphi_j}{\partial \bar{z}_2} \right] - \sum_1 f_l \frac{\partial d_{j,k,l}}{\partial \bar{z}_2}, \end{cases}$$

where

$$\varphi_j = \frac{g \bar{f}_j}{|f_1|^2 + |f_2|^2 + \dots + |f_m|^2}.$$

Then the functions  $g_j$  defined by  $g_j = \varphi_j - \sum_k f_k b_{j,k}$  will be analytic and satisfy  $f_1 g_1 + f_2 g_2 + \dots + f_m g_m = g$ . To have  $g_j$  in  $H^p$  we will have to solve (3) and (4) so that  $b_{j,k}$  is in  $L^p(T^2)$ .

We start with the equation (3),  $\partial^2 d / \partial \bar{z}_1 \partial \bar{z}_2 = \Phi$ , and will solve it so that  $\|d\|_{L^p(T^2)} \leq c(\delta, P, m) \|g\|_{H^p}$ . (The indices  $j, k, l$  are suppressed.) A solution to (3) is

$$d_0(z_1, z_2) = c \int_{D^2} \frac{\Phi(\xi_1, \xi_2)}{(\xi_1 - z_1)(\xi_2 - z_2)} d\xi_1 \wedge d\bar{\xi}_1 \wedge d\xi_2 \wedge d\bar{\xi}_2,$$

and functions of the form  $d_0 + k$ , where  $\partial^2 k / \partial \bar{z}_1 \partial \bar{z}_2 = 0$ , are also solutions to (3). By duality,

$$\begin{aligned} & \inf \left\{ \|d_0 + k\|_{L^p(T^2)} : \frac{\partial^2 k}{\partial \bar{z}_1 \partial \bar{z}_2} = 0 \right\} \\ &= \sup \left\{ c \left| \int_{T^2} d_0 h dz_1 dz_2 \right| : h \in H_0^q(D^2), \|h\|_{H^q} \leq 1 \right\}, \quad \frac{1}{p} + \frac{1}{q} = 1. \\ &= \sup \left\{ c \left| \int_{D^2} \Phi h \right| : h \in H_0^q(D^2), \|h\|_{H^q} \leq 1 \right\}, \quad \text{by Stokes' theorem.} \end{aligned}$$

But the arguments in the papers of [4 and 10], which involves some difficult estimates on Carleson measures on the bidisc, already showed that

$$\left| \int_{D^2} \Phi h \right| \leq c \|gh\|_{H^1} \leq c \|g\|_{H^p} \|h\|_{H^q} \leq c \|g\|_{H^p}.$$

Hence there is indeed a solution  $d$  to (3) with  $\|d\|_{L^p(T^2)} \leq c \|g\|_{H^p}$ .

We now turn to equation (4). (We will suppress the indices  $j$  and  $k$  again.) For (4.1), one-variable result gives a solution  $b_{1,1}$  with  $\|b_{1,1}\|_{L^p(T^2)} \leq c(p, \delta, m) \|g\|_{H^p}$ , to

$$\frac{\partial b_{1,1}}{\partial \bar{z}_1} = \frac{1}{g} \left[ \varphi_j \frac{\partial \varphi_k}{\partial \bar{z}_1} - \varphi_k \frac{\partial \varphi_j}{\partial \bar{z}_1} \right].$$

Thus the function  $b_1 = b_{1,1} - \sum_1 f_l d_{j,k,l}$  is a solution to (4.1) with the correct  $L^p$  bound. Similarly, we have a solution  $b_2$  to (4.2). Now the argument presented earlier gives a common solution  $b_{j,k}$  to the equation (4) with the desired  $L^p(T^2)$  estimate. We have completed the proof of the theorem.

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