A SINGULAR REPRESENTATION OF $E_6$

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Abstract. Algebraic properties of a singular representation of $E_6$ are studied. This representation has the Joseph ideal as its annihilator and it remains irreducible when restricted to $F_4$.

It is of interest in the representation theory of semisimple Lie groups to understand singular representations. For example, a detailed understanding of the metaplectic representation has been quite fruitful. There is also a particularly interesting representation of $SO(4, 4)$ studied by Kostant [11]. These are examples of "unipotent representations" which should in some sense form the building blocks for the unitary dual of a semisimple Lie group. In this paper we study a singular representation of $E_6$ and show that it has many of the same algebraic properties as the metaplectic representation.

We study an irreducible unitary representation, call it $V$, of the real form of $E_6$ with Hermitian symmetric space. We see that $V$ is a ladder representation, has minimal Gelfand-Kirillov dimension and its annihilator is the Joseph ideal. Thus $V$ is in some sense associated to a minimal nilpotent orbit in the dual of the real Lie algebra. We also restrict $V$ to the rank one $F_4$ and see that it remains irreducible. This reflects the fact that each minimal nilpotent real orbit in $E_6$ contains a dense $F_4$ orbit.

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1

We will first set down some notation and specify the representation we will be studying.

Let $G$ be the real form of $E_6$ with an associated Hermitian symmetric space (denoted by $E III$ in Helgason [7, p. 534]). Assume $G$ is contained in the simply connected complex group $E_6$. The maximal compact subgroup of $G$ is $K = SO(2) \times Spin(10)$. We fix a compact Cartan subalgebra $t$ of $g = \text{Lie}(G)_c$ and let $\Delta = \Delta(g, t)$ be the corresponding set of roots. We choose a positive system of roots as follows: since $G/K$ is Hermitian symmetric there is an element $\zeta$ in the center of $K$ so that $J = \text{Ad}(\zeta)$ gives the complex structure at $eK$. Let $p_\pm$ be the $\pm i$ eigenspace of $J$. Let $\Delta^+$ be any positive system of roots as follows: since $G/K$ is Hermitian symmetric there is an element $\zeta$ in the center of $K$ so that $J = \text{Ad}(\zeta)$ gives the complex structure at $eK$. Let $p_\pm$ be the $\pm i$ eigenspace of $J$. Let $\Delta^+$ be any positive system so
that $\Delta(p_+) \subset \Delta^+$ (where $\Delta(p_+)$ is the set of $t$ roots in $p_+$). Such positive systems are exactly those for which (nontrivial) irreducible unitary highest weight representations exist.

We label the simple roots of the Dynkin diagram of $E_6$ as follows:

\[
\begin{array}{cccccccc}
& & & & & & \alpha_6 & \\
& & & & \alpha_1 & & & \\
& & & \alpha_2 & & & & \\
& & \alpha_3 & & & & & \\
& \alpha_4 & & & & & & \\
& & \alpha_5 & & \\
\end{array}
\]

with $\alpha_1$ the unique noncompact simple root. The simple roots may be written in coordinates as

\[
\begin{align*}
\alpha_1 &= \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \\
\alpha_2 &= (0, 0, 0, 0, 1, -1), \\
\alpha_3 &= (0, 0, 0, 1, -1, 0), \\
\alpha_4 &= (0, 0, 1, -1, 0, 0), \\
\alpha_5 &= (0, 1, -1, 0, 0, 0), \\
\alpha_6 &= (0, 0, 0, 0, 1, 1),
\end{align*}
\]

with the usual Euclidean inner product. If $\{e_1, \ldots, e_6\}$ is the standard basis of $\mathbb{R}^6$, then the compact and noncompact positive roots are:

\[
\begin{align*}
\Delta^+(e) &= \{e_i \pm e_j \mid 2 \leq i < j \leq 6\}, \\
\Delta^+(p) &= \Delta(p_+) = \left\{\left(\frac{\sqrt{3}}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \mid \text{an even number of } -\frac{1}{2}'s \text{ occur}\right\}.
\end{align*}
\]

If we let $\lambda_1, \ldots, \lambda_6$ denote the corresponding fundamental weights

\[
(2\langle \lambda_i, \alpha_j \rangle/||\alpha_j||^2 = \delta_{ij}),
\]

then $\lambda_1 = \left(\frac{\sqrt{3}}{2}, 0, 0, 0, 0, 0\right)$. Also,

\[
\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = (4\sqrt{3}, 4, 3, 2, 1, 0).
\]

It is clear from the form of the noncompact roots that the representations of $Spin(10)$ on $p_\pm$ are the representations with highest weights $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}\right)$; these are the two "spinor representations" of $Spin(10)$.

We will study the irreducible representation of $G$ with highest weight $-3\lambda_1$ and we will call its $(\mathfrak{g}, K)$ module $V$. If $\lambda \in \mathfrak{t}^*$ is $\Delta^+$-dominant, we let

\[
M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{t}+\mathfrak{p}_+)} F_{\lambda}
\]

where $F_{\lambda}$ is the irreducible representation of $\mathfrak{t}$ with highest weight $\lambda$, and let $L(\lambda)$ denote the unique irreducible quotient. Both $M(\lambda)$ and $L(\lambda)$ have highest weight $\lambda$ and $V = L(-3\lambda_1)$. The infinitesimal character of $V$ is $-3\lambda + \rho$ which is conjugate under the Weyl group to $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6$.

Our first observation is that $V$ occurs in the parameterization of unitary highest weight modules given by Enright, Howe, and Wallach (see [4]) as the first reduction point in the series with 1-dimensional highest $K$-type. Specifically, the family has highest weights $(t-11)\lambda_1$ and the modules $L((t-11)\lambda_1)$ are unitary for $t \leq 8$ and $t = 11$. The $L((t-11)\lambda_1)$ are holomorphic discrete representations for $t < 0$. $V$ occurs when $t = 8$ (the first reduction point) and the trivial representation corresponds to $t = 11$. 

Theorem 1.1. \( V \) is a ladder representation with \( K \)-types \( \{-3\lambda_1 - n\alpha_1 | n = 0, 1, 2, \ldots\} \).

Proof. We first write down the decomposition of \( U(p_-) \approx S(p_-) \) under \( K \) by applying Schmid's Theorem [12]. Let \( \mu_1, \mu_2 \in \Delta(p_-) \) be defined by setting \( \mu_1 = -\alpha_1 \) and taking \(-\mu_2\) to be the lowest root in \( \Delta(p_+) \) that is strongly orthogonal to \( \alpha_1 \). Then

\[
S(p_-) = \bigoplus_{n \geq m \geq 0} F_{n\mu_1 + m\mu_2}
\]

where \( F_\mu \) is the finite dimensional representation of \( K \) with highest weight \( \mu \).

For \( \alpha \in \Delta \), let \( X_\alpha \) denote a nontrivial vector in the root space \( g_\alpha \). By a result of Vogan (Lemma 3.4 in [15]) we know that either \( X_\alpha \) or \( X_{-\alpha} \) acts injectively on any infinite dimensional irreducible \((g, K)\)-module. Since \( V \) has a highest weight it must be \( X_{-\alpha} \) that acts injectively. If \( \mu \) is the highest weight of a \( K \)-type in \( V \) and \( v_\mu \) is the corresponding weight vector, then \((X_{-\alpha})^n v_\mu\) is the highest weight vector of another \( K \)-type \((\approx F_{\mu - n\alpha})\); for if \( \alpha \in \Delta^+ \), then \([X_\alpha, X_{-\alpha}] = 0\) because \( \alpha - \alpha_1 \notin \Delta \), so \( X_\alpha (X_{-\alpha})^n v_\mu = (X_{-\alpha})^n X_\alpha v_\mu = 0 \). We conclude that the \( n\mu_1 + m\mu_2 \) are \( K \)-types occurring in \( V \).

We must check that these are all the \( K \)-types of \( V \). By Proposition 3.9 of [4], if \( F_\mu \) is a \( K \)-type in \( L(\lambda), \mu \neq \lambda \), and \( L(\lambda) \) is unitary, then \( ||\mu + \rho|| > ||\lambda + \rho|| \). An easy calculation shows that this is not the case for \( \mu = -3\lambda_1 + \mu_1 + \mu_2 \); so \( F_{3\lambda_1 + \mu_1 + \mu_2} \) cannot occur in \( V \). Applying \((X_{-\alpha})^n\) as above we conclude that the \( K \)-types \( F_{-3\lambda_1 + n\mu_1 + m\mu_2} \) do not occur in \( V \) (since they occur in the maximal submodule of \( M(-3\lambda_1) \) and the multiplicity is one for all the \( K \)-types of \( M(-3\lambda_1) \)). The picture for the \( K \)-types in \( M(-3\lambda_1) \) is

\[
\begin{array}{c}
m \\
\hline
n
\end{array}
\]

To see that the first row gives exactly the \( K \)-types in \( V \) it is enough to check that \( p_\pm \) sends the first row only to the first or second row. This is an easy calculation which we omit. \( \square \)

Remark 1. There are several other proofs of Theorem 1. (a) By the above, \( F_{-3\lambda_1 + \mu_1 + \mu_2} \) does not occur. One shows that the irreducible representation with highest weight \( -3\lambda_1 + \mu_1 + \mu_2 \) has exactly the remaining \( K \)-types \( \{F_{-3\lambda_1 + n\mu_1 + m\mu_2} | n \geq m \geq 1\} \). One can see that this highest weight representation is an \( A_9(\lambda) \) where \( q \) is the \( \theta \)-stable parabolic defined by \(-\lambda_5\). For \( \lambda = 3\lambda_5 \), \( A_9(\lambda) \) is irreducible and has highest weight \(-3\lambda_1 + \mu_1 + \mu_2 \) (see [21, p. 49]). One can use the Blattner formula to calculate the \( K \)-types of \( V \) (b) From results of Enright and Joseph (see [5, Theorem 5.2]) one can show that the unique maximal subrepresentation of \( M(-3\lambda_1) \) is \( U(p_-)F_{-3\lambda_1 + \mu_1 + \mu_2} \) and then calculate the \( K \)-types.

Remark 2. It follows from (a) above that the maximal submodule is \( L(-3\lambda_1 + \mu_1 + \mu_2) \). This occurs in the series \( M((t - 12)\lambda_1 + \lambda_3) \) for \( t = 7 \), the isolated unitary point \( t \leq 4 \) and \( t = 7 \) are the unitaries in this series.)
Let $q = 1 + u$ be a $\theta$-stable parabolic subalgebra of $g$. For $\lambda \in t^*$ and $\lambda \perp \Delta(l)$, let $R_q^i(C_\lambda)$ be the derived functor modules as defined in [16]. Let $s = \dim(u \cap \mathfrak{t})$. If the positivity condition $\langle \lambda + \rho(u), \alpha \rangle > 0$, $\alpha \in \Delta(u)$, is satisfied, then $R_q^i(C_\lambda) = 0$ for $i \neq s$ and $R_q^s(C_\lambda)$ is an irreducible unitarizable representation with infinitesimal character $\lambda + \rho(u)$. When the parameter $\lambda$ becomes arbitrary the vanishing, irreducibility, and unitarity do not in general hold. We set $A_q(\lambda) = R_q^s(C_\lambda)$ for $\lambda$ arbitrary (subject of course to $\lambda \perp \Delta(l)$, so that a 1-dimensional representation $C_\lambda$ of $L$ with weight $\lambda$ exists).

**Theorem 1.2.** $V$ is not isomorphic to $A_q(\lambda)$, $\lambda$ arbitrary.

**Remark.** We show a little more than this: $V$ is not a quotient of any $A_q(\lambda)$. In particular, if $\lambda$ is in the “unitary range” (i.e., $A_q(\lambda)$ is unitary) then $V$ does not occur as a constituent in $A_q(\lambda)$. However, we do not exclude the possibility of $V$ being some constituent (other than a quotient) of some $A_q(\lambda)$ with $\lambda$ so bad that $A_q(\lambda)$ is not irreducible or unitary.

**Proof.** We can calculate the Gelfand-Kirillov dimension of $V$ using Theorem 1.2 of [14] because we know the $K$-types. We omit the calculation but note that $\dim(F_{-\lambda_1 - \lambda_\alpha})$ is a degree 10 polynomial in $n$. Thus, the Gelfand-Kirillov dimension of $V$ is 11.

Since 11 is also the minimal Gelfand-Kirillov dimension for infinite dimensional representations of $G$, and because the two minimal $K_C$ orbits in $\mathfrak{p}$ are $K_C \cdot X_{\beta} \subset \mathfrak{p}_+$ and $K_C \cdot X_{-\beta} \subset \mathfrak{p}_-$ (where $\beta$ is the highest root), we may conclude that the characteristic variety $\mathcal{V}(V)$ (see Appendix A) is contained in $\mathfrak{p}_+$ or $\mathfrak{p}_-$. When $\langle \lambda + \rho(u), \alpha \rangle > 0$, $\alpha \in \Delta(u)$, the characteristic variety of $A_q(\lambda)$ is $K_C \cdot (u \cap \mathfrak{p})$ (see the theorem in Appendix A). The proposition in Appendix A shows that for arbitrary $\lambda$, as long as $A_q(\lambda) \neq 0$, every quotient of $A_q(\lambda)$ has the same characteristic variety (and so the same Gelfand-Kirillov dimension) as an $A_q(\lambda)$ with $\langle \lambda + \rho(u), \alpha \rangle > 0$, $\alpha \in \Delta(u)$. Thus, we need only prove that no $q = 1 + u$ with $u \cap \mathfrak{p} \subset \mathfrak{p}_+$ (or $u \cap \mathfrak{p} \subset \mathfrak{p}_-$) has $\dim K_C \cdot (u \cap \mathfrak{p}) = 11$.

Consider the fundamental weight $\lambda_6$, this defines a $\theta$-stable parabolic subalgebra (by $\Delta(l + u) = \{ \alpha \in \Delta | \langle \alpha, \lambda_6 \rangle \geq 0 \}$). We consider the roots $\beta_1 = (\sqrt{2} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), \beta_2 = (\sqrt{2} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ and the corresponding root vectors $X_1, X_2$. Both are in $u \cap \mathfrak{p} = u \cap \mathfrak{p}_+$. Let $Y_1, Y_2$ be root vectors for, respectively, $-\beta_1, -\beta_2$. There are two corresponding 3-dimensional subalgebras $\{X_1, Y_1, H_1\}, \{X_2, Y_2, H_2\}$. Since $\beta_1$ and $\beta_2$ are strongly orthogonal these two subalgebras commute and so $\{X = X_1 + X_2, Y = Y_1 + Y_2, H = H_1 + H_2\}$ is a 3-dimensional subalgebra. We claim $K_C \cdot X \subset u \cap \mathfrak{p}$ has dimension 16. This is seen as follows. Since $H$ corresponds to $\lambda_1 + \lambda_5$, the decomposition of $\mathfrak{g}$ under $\{X, Y, H\}$ shows that the stabilizer of $X \in \mathfrak{t}$ has dimension 30. So

$$\dim_K(K_C \cdot X) = \dim_K K_C - \dim_K(\text{stab}_\mathfrak{t}(X)) = 46 - 30 = 16.$$ 

For other parabolics $1 + u$, the argument is the same because $X$ always lies in $u \cap \mathfrak{p}$ (as is easily seen by writing $\beta_1, \beta_2$ as sums of simple roots). \qed

2

We will show that the annihilator of $V$ is the Joseph ideal. This ideal is a maximal primitive ideal in $U(\mathfrak{g})$ associated with the minimal nilpotent complex
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coadjoint orbit and was defined by Joseph in [9]. The characterization we use was formulated by Garfinkle [6] and is as follows.

Let $\mathfrak{g}$ be a simple, complex Lie algebra not of type $A_n$. Let $\beta$ be the highest root with respect to any positive system of roots. The space of degree 2 elements in the symmetric algebra of $\mathfrak{g}$ decomposes as $S^2(\mathfrak{g}) \approx E_{2\beta} + S$, where $E_{2\beta}$ is the irreducible representation of $\mathfrak{g}$ with highest weight $2\beta$ and $S$ is the $\mathfrak{g}$-invariant complement of $E_{2\beta}$ in $S^2(\mathfrak{g})$. If $J \subset U(\mathfrak{g})$ is an ideal, consider the graded ideal $\text{gr}(J) \subset \text{gr}(U(\mathfrak{g})) \approx S(\mathfrak{g})$.

**Theorem 2.1 (Garfinkle).** If $J$ is an ideal of infinite codimension in $U(\mathfrak{g})$, then $J$ is the Joseph ideal if and only if $\text{gr}(J) \cap S^2(\mathfrak{g}) = S$.

Let us return to the case where $\mathfrak{g} = \mathfrak{e}_6$.

**Lemma 2.2.** \( S^2(\mathfrak{g}) \approx E_{2\beta} \oplus E_{(\sqrt{3}, 1, 0, 0, 0, 0)} \oplus \mathbb{C} \).

**Proof.** $\beta = (\sqrt{3}/2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the highest root. The possible highest weights of subrepresentations occurring in $\mathfrak{g} \otimes \mathfrak{g}$ are of the form $\beta + \alpha$ where $\alpha \in \Delta \cup \{0\}$ and the multiplicity of such a subrepresentation is one (or six in case $\alpha = 0$). The possible subrepresentations along with their dimensions are:

- $E_{2\beta} = E_{(\sqrt{3}, 1, 1, 1, 1, 1)}; \quad 2430,$
- $E_{(\sqrt{3}, 1, 1, 1, 0, 0)}; \quad 2925,$
- $E_{(\sqrt{3}, 1, 0, 0, 0, 0)}; \quad 650,$
- $E_{\beta} = E_{(\sqrt{3}/2, 1/2, 1/2, 1/2, 1/2, 1/2)}; \quad 78,$
- $E_{(0, 0, 0, 0, 0, 0)}; \quad 1.$

Also $\dim S^2(\mathfrak{g}) = \frac{78 + 79}{2} = 3081$. The only way to get the dimensions to add up correctly is to take $3081 = 2430 + 650 + 1$. \( \square \)

**Remark.** Although it is not necessary, one can check using Helgason's theorem (see [8, p. 535]) that $E_{(\sqrt{3}, 1, 0, 0, 0, 0)} = E_{\lambda_1 + \lambda_5}$ is spherical.

Let $\Omega_{\mathfrak{e}_6}$, $\Omega_K$ be the Casimirs of, respectively, $\mathfrak{e}_6$ and $\mathfrak{k} = SO(2) \times \text{Spin}(10)$, and let $H_I$ correspond to $\lambda_1$ by the Killing form of $\mathfrak{e}_6$.

**Lemma 2.3.** $\Omega_K - 2(H_1)^2 - \frac{5}{3} \Omega_{\mathfrak{e}_6}$ annihilates $V$.

**Proof.** $(H_1)^2$ acts on $F_{-3\lambda_1 - n\alpha_1}$ by

\[
((-3\lambda_1 - n\alpha_1)(H_1))^2 = (-3(\lambda_1, \lambda_1) - n(\alpha_1, \lambda_1)) = (n + 4)^2,
\]

$\Omega_K$ acts on $F_{-3\lambda_1 - n\alpha_1}$ by

\[
\| - 3\lambda_1 - n\alpha_1 + \rho_c \|^2 - \| \rho_c \|^2 = 2n^2 + 16 + 12,
\]

$\Omega_{\mathfrak{e}_6}$ acts on $F_{-3\lambda_1 - n\alpha_1}$ by

\[
\| - 3\lambda_1 + \rho \| ^2 - \| \rho \|^2 = -36.
\]

Thus,

\[
(\Omega_K - 2(H_1)^2 - \frac{5}{3} \Omega_{\mathfrak{e}_6})F_{-3\lambda_1 - n\alpha_1} = 0.
\]

**Theorem 2.4.** The annihilator, $J$, of $V$ in $U(\mathfrak{g})$ is the Joseph ideal.

**Proof.** $(X_{-\beta})^2 \in E_{-2\beta}$ and $X_{-\beta}$ acts injectively on $V$ so $\text{gr}(J) \cap S^2(\mathfrak{g}) \subset S$. However, there are two linearly independent $K$-fixed vectors in $S$ which kill
V : (Ω_{E_6} + 36), and (Ω_K - 2(H_1)^2 - \frac{8}{3}Ω_{E_6}). Since the multiplicity of a K-fixed vector in an irreducible finite dimensional representation of G is at most one, we see that each summand in S meets the annihilator. Since the annihilator is g-invariant, S = gr(J) ∩ S^2(g). □

Remark. Kostant used this argument when studying a special representation of SO(4, 4) (see [11, §3]).

3

The real form of E_6 under consideration contains the rank one real form of F_4 as the fixed points of an outer automorphism. We will restrict V to F_4 and see that it remains irreducible. We also identify V|_{F_4}.

First we describe the automorphism which has F_4 as the fixed points. Recall that there is a decomposition g = ℋ ⊕ p_+ ⊕ p_- and that p_± are the 16-dimensional spin representations. Let σ_0 be the automorphism of K ≈ SO(2) × Spin(10) defined by multiplying by -I_2 on the so(2) and conjugating by

\[
\left( \begin{array}{cc}
I_9 & 0 \\
0 & -1
\end{array} \right)
\]
on so(10).

Lemma 3.1. If (τ, F) is a finite dimensional irreducible representation of K with highest weight (λ_1, ..., λ_6) then the representation (τ^{σ_0}, F^{σ_0}) defined by τ^{σ_0}(k) = τ(σ_0(k)) on F^{σ_0} = F is an irreducible representation with highest weight

σ_0(λ) = (-λ_1, λ_2, λ_3, λ_4, λ_5, -λ_6).

In particular, (p_+)^{σ_0} ≈ p_- and (p_-)^{σ_0} ≈ p_+.

Proof. This is easy and is omitted. □

We conclude from this lemma that there is an intertwining map f : p_+ → (p_-)^{σ_0}; that is, a linear map f : p_+ → p_- such that

\[
f(Ad(k) · Y) = Ad(σ_0(k)) · f(Y), \quad \text{for all } k ∈ K \text{ and all } Y ∈ p_+.
\]

We now extend σ_0 : ℋ → ℋ to a map σ : g → g by σ(X + Y + Z) = σ_0(X) + f(Y) + f^{-1}(Z) for all X ∈ ℋ, all Y ∈ p_+ and all Z ∈ p_-.

Proposition 3.2. When f is scaled properly, σ is an involution of g which commutes with the conjugation of g over g_0 and commutes with Cartan involution θ. The fixed point set of σ in g is the rank 1 real form of F_4.

Proof. We must show that σ is a Lie algebra homomorphism and σ commutes with the complex conjugation. Let \langle , \rangle be the Killing form (or any other nondegenerate K-invariant pairing between p_+ and p_-).

We first prove that for Y ∈ p_+ , Z ∈ p_-,

\[
\langle f(Y), f^{-1}(Z) \rangle = \langle Y, Z \rangle.
\]

Since Y, Z → \langle f(Y), f^{-1}(Z) \rangle is a nondegenerate K-invariant pairing between the irreducible K-modules p_+ with p_- , we know that \langle f(Y), f^{-1}(Z) \rangle = c\langle Y, Z \rangle for some nonzero constant c. It is enough to show that \langle f(Y), f^{-1}(Z) \rangle = \langle Y, Z \rangle \neq 0 for some choice of Y ∈ p_+ and Z ∈ p_-.

Let \beta = (\sqrt{3}_2 , \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) denote the highest root and set \gamma = -σ_0(β) =
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$(\sqrt{3}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, and let $X_\beta, X_\gamma$ be corresponding root vectors. Consider also the compact roots $\xi_1 = (0, 1, 1, 0, 0, 0)$ and $\xi_2 = (0, 0, 0, 1, 1, 0)$. We see that $\beta = \gamma + \xi_1 + \xi_2$, so for appropriately chosen root vectors, $X_{\xi_1}, X_{\xi_2}$,

(i) $\text{ad}(X_{\xi_1}) \text{ad}(X_{\xi_2}) X_\gamma = X_\beta$,
(ii) $\sigma_0(X_{\xi_1}) = X_{\xi_1}$ and $\sigma_0(X_{\xi_2}) = X_{\xi_2}$,
(iii) $[X_{\xi_1}, X_{\xi_2}] = 0$.

Note that $f(X_\beta)$ is a root vector for $\sigma_0(\beta) = -\gamma$. Thus,

$$0 \neq \langle f(X_\beta), X_\gamma \rangle = \langle f(\text{ad}(X_{\xi_1}) \text{ad}(X_{\xi_2}) X_\gamma), X_\gamma \rangle$$
$$= \langle \text{ad}(\sigma_0(X_{\xi_1})) \text{ad}(\sigma_0(X_{\xi_2})) f(X_\gamma), X_\gamma \rangle \quad \text{(by (3.1))}$$
$$= \langle \text{ad}(X_{\xi_1}) \text{ad}(X_{\xi_2}) f(X_\gamma), X_\gamma \rangle \quad \text{(by (ii))}$$
$$= \langle f(X_\gamma), \text{ad}(X_{\xi_1}) \text{ad}(X_{\xi_2}) X_\gamma \rangle \quad \text{(by K-inv.)}$$
$$= \langle f(X_\gamma), X_{\xi_1} \rangle \quad \text{(by (iii))}$$
$$= \langle f(X_\gamma), X_{\xi_2} \rangle \quad \text{(by (i))}$$
$$= \langle X_\beta, f(X_\gamma) \rangle$$

Thus, (3.2) is proved.

We check that $\sigma$ is a Lie algebra homomorphism. Let $X, X' \in \mathfrak{k}$, $Y \in \mathfrak{p}^+$, and $Z \in \mathfrak{p}^-$. Now,

$$\sigma([X, X']) = \sigma_0([X, X']) = [\sigma_0(X), \sigma_0(X')]$$
$$\sigma([X, Y]) = f([X, Y]) = [\sigma_0(X), f(Y)] = [\sigma(X), \sigma(Y)]$$
$$\sigma([X, Z]) = f^{-1}([X, Z]) = [\sigma_0(X), f^{-1}(Z)] = [\sigma(X), \sigma(Z)]$$.

Finally, we need to confirm

$$\sigma([Y, Z]) = [\sigma(Y), \sigma(Z)]. \quad (3.3)$$

For any $X \in \mathfrak{k}$,

$$\langle X, \sigma([Y, Z]) \rangle = \langle X, \sigma_0([Y, Z]) \rangle = \langle \sigma_0(X), [Y, Z] \rangle = ([\sigma_0(X), Y], Z)$$
$$= \langle f([\sigma_0(X), Y]), f^{-1}(Z) \rangle \quad \text{(by (3.2))}$$
$$= \langle [X, f(Y)], f^{-1}(Z) \rangle = \langle X, [f(Y), f^{-1}(Z)] \rangle.$$

Since the Killing form is nondegenerate on $\mathfrak{k}$, (3.3) follows.

Let $\sigma$ denote conjugation of $g$ over $g_0$. We now show that $\sigma(U) = \overline{\sigma(U)}$, for all $U \in g$. Since $\langle Y', Y \rangle$ and $\langle Y', f(f(Y)) \rangle$ are both $\mathfrak{k}$-invariant non-degenerate Hermitian forms on $\mathfrak{p}^+$,

$$\langle Y', f(Y) \rangle = \sigma(Y', f(f(Y)))$$

for some $r \in \mathbb{R}^+$. Replacing $f$ by $\frac{1}{\sqrt{r}} f$, we get $\langle Y', Y \rangle = \langle Y', f(f(Y)) \rangle$ for all $Y' \in \mathfrak{p}^+$. Since the pairing is nondegenerate, it follows that $Y = f(f(Y))$ or $\overline{f(Y)} = f^{-1}(Y)$. Thus, if $f$ is properly scaled $f(Y) = f^{-1}(Y)$ for all $Y \in \mathfrak{p}^+$. For this $f$, one also has $f^{-1}(Z) = f(Z)$ for all $Z \in \mathfrak{p}^-$. Let $X \in \mathfrak{k}$, $Y \in \mathfrak{p}^+$, and $Z \in \mathfrak{p}^-$. Then

$$\sigma(X + Y + Z) = \sigma_0(X) + f^{-1}(Y) + f(Z)$$
$$= \overline{\sigma_0(X) + f(Y) + f^{-1}(Z)}$$
$$= \overline{\sigma(X + Y + Z)},$$

so (3.3) is proved.
Set 
\[ g_1 = \{ X \in g \mid \sigma(X) = X \}, \quad \ell_1 = \ell \cap g_1, \quad p_1 = p \cap g_1. \]
So \( g_1 = \ell_1 \oplus p_1 \), \( \ell_1 = \text{so}(9) \), and \( p_1 = \{ Y + f(Y) : Y \in p_+ \} = \{ f^{-1}(Z) + Z : Z \in p_- \} \). The roots are as follows:
\[ \Delta(\ell_1) = \{ \pm e_i \pm e_j, \pm e_k \mid 1 \leq i < j \leq 4, \ 1 \leq k \leq 4 \}, \]
\[ \Delta(p_1) = \{ (\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}) \}. \]

\( g_1 \) is thus of type \( \text{F}_4 \).

As \( \sigma \) commutes with conjugation of \( g \) with respect to \( g_0 \), \( g_1 \cap g_0 \) is a real form of \( \text{F}_4 \) and \( \theta|_{\text{F}_4} \) is a Cartan involution. It follows, for example, from the tables in [7] that \( g_1 \cap g_0 \) is the rank one real form of \( \text{F}_4 \).

We fix a positive root system as follows:
\[ \Delta^+(g_1) = \Delta^+(\ell_1) \cup \Delta^+(p_1) \]
\[ = \{ e_i \pm e_j, e_k \mid 1 \leq i < j \leq 4, \ 1 \leq k \leq 4 \} \cup \{ (\frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}) \}. \]
The simple roots are \( \gamma_1 = e_2 - e_3, \gamma_2 = e_3 - e_4, \gamma_3 = e_4, \gamma_4 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \) and the corresponding fundamental weights are \( \varpi_1 = (1, 1, 0, 0), \varpi_2 = (2, 1, 1, 0), \varpi_3 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \varpi_4 = (1, 0, 0, 0). \) The Dynkin diagram is

\[ \alpha_1 \alpha_2 \alpha_3 \alpha_4 \]

and \( \rho(g_1) = \rho(\ell_1) + \rho(p_1) = (\frac{11}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}) \). The corresponding Satake diagram is

\[ \bullet \bullet \bullet \bullet \]

Here \( e_1 \) restricts to \( 2t \) and \( e_2, e_3, e_4 \) restrict to \( 0 \). Let \( \Sigma^+(a, g) = \{ t, 2t \} \) where \( a \) is an appropriate maximal abelian subspace of \( p \), \( \rho(a, g) = \frac{23}{2}t \).

**Theorem 3.3.** \( V|_{\text{F}_4} \) is irreducible and has infinitesimal character \( (\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}) \).

**Proof.** Every \( K \)-type remains irreducible when restricted to \( K_1 = \text{Spin}(9) \). This follows either from the branching law (see [Ze, §129]) or from a dimension count using the Weyl dimension formula. The \( K_1 \)-types have highest weights \( (\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}) \) with \( n \in \mathbb{Z}^+ \).

The highest noncompact root in \( g_1 \) is \( \gamma_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \). By [15, Lemma 3.4], \( X_{\gamma_1} \) acts injectively in any irreducible representation of \( \text{F}_4 \). Since \( V \) is unitary, and because the highest weights of the \( K_1 \)-types of \( V \) lie along a single ray, we must have \( V|_{\text{F}_4} \approx E \oplus V_1 \) where \( E \) is finite dimensional and \( V_1 \) is irreducible. By unitarity, we may also conclude that \( E \) must correspond to the trivial representation. If \( E \neq \{0\} \), we may choose a nonzero element \( v_o \) of \( E \). Then \( v_o \) is a highest weight vector of \( V \) (the \( K \)-type is necessarily the one corresponding to \( n = 0 \)). So \( p_+ \cdot v_o = 0 \). Since \( v_o \) is fixed by \( p_1 = \{ f^{-1}(Z) + Z \mid Z \in p_- \} \),
\[ 0 = (f^{-1}(Z) + Z) \cdot v_o = Z \cdot v_o \]
for all \( Z \in p_- \); which contradicts the irreducibility of \( V \) with respect to \( g \). Thus, \( V|_{\text{F}_4} \) is irreducible.
To calculate the infinitesimal character of $V|_{F_4}$ we use the following trick. The diagram automorphism $\phi$ with respect to the positive system of $E_6$ given in §1 is not the automorphism of Proposition 3.5. However, the fixed points of the two automorphisms are conjugate. Thus, it is enough to calculate the infinitesimal character of $V$ restricted to the fixed points of $\phi$. This representation is an irreducible highest weight representation of $F_4$ (not a $(g_1, K_1)$-module, but that is okay). We calculate the highest weight.

Let $\phi$, interchangeably, denote the automorphism of $g$ and the roots $A$. Let $g^\phi$, $t^\phi$ denote the elements of $g$, respectively, $t$ fixed by $\phi$. The roots of $g^\phi$ with respect to $t^\phi$ are as follows:

(i) Short roots: $a \in A | a^\phi = a, \phi(a) \neq a$. The corresponding root vectors are $X_a + \phi(X_a)$. Note that we have listed each root vector twice, once for $a$ and once for $\phi(a)$.

(ii) Long roots: $a \in A | a^\phi = a$. The corresponding root vectors are $X_a$.

Remark. $a^\phi/2$ corresponds to an element in $t^\phi$ (via the Killing form), also $a^\bot \phi(a)$ when $\phi(a) \neq a$. Thus the length of $a^\phi/2$ in $F_4$ is its length in $E_6$ which is $||a||/\sqrt{2}$.

The positive system for which $V|_{F_4}$ is a highest weight representation consists of those roots in (i) and (ii) with $a \in A^+(g, t)$. The simple roots, in terms of the simple roots of $E_6$, are $\gamma_1 = \alpha_6, \gamma_2 = \alpha_3, \gamma_3 = \frac{1}{2}(\alpha_2 + \alpha_4), \gamma_4 = \frac{1}{2}(\alpha_1 + \alpha_5)$ and the connection between the corresponding fundamental weights is $\omega_1 = \lambda_6, \omega_2 = \lambda_3, \omega_3 = \frac{1}{2}(\lambda_2 + \lambda_4), \omega_4 = \frac{1}{2}(\lambda_1 + \lambda_5)$. The restriction of $-3\lambda_1$ (the highest weight of $V$ as an $E_6$ representation) to $t^\phi$ is $-3\omega_4$. In terms of the fundamental weights the infinitesimal character is $\omega_1 + \omega_2 + \omega_3 - 2\omega_4 = (\frac{3}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$ in the coordinates appearing after Lemma 3.2. Now note that, if $w_\gamma, \gamma \in t^*$, is the reflection through the hyperplane orthogonal to $\gamma$,

$$w_\gamma w_\lambda (\frac{3}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}) = (\frac{7}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}) .$$

Thus the infinitesimal character of $V|_{F_4}$ is $(\frac{7}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$. $\Box$

Working in $F_4$ and the coordinates given above, set $\Lambda_0 = (1, 1, 0, 0)$. This defines a $\theta$-stable parabolic $q = I + u$ by

$$\Delta(q) = \{ \alpha \in \Delta | \langle \alpha, \Lambda_0 \rangle \geq 0 \},$$

$$\Delta(t) = \{ \alpha \in \Delta | \langle \alpha, \Lambda_0 \rangle = 0 \},$$

$$\Delta(u) = \{ \alpha \in \Delta | \langle \alpha, \Lambda_0 \rangle > 0 \} .$$

Let $\lambda = a\Lambda_0$. $A_q(\lambda) = \mathcal{H}_q^s(C_2)$, the cohomologically induced representation, has infinitesimal character $\lambda + \rho$.

Theorem 3.4. $V|_{F_4} \approx A_q(\lambda)$ with $\lambda = (-2, -2, 0, 0)$. This is the Langlands quotient of $\text{Ind}_{MAN}^G(1 \otimes \frac{5}{2} t \otimes 1)$, where $P = MAN$ is the minimal parabolic with $n$ corresponding to $\Sigma^+(a, g)$.

Proof. We show that $A_q(\lambda)$ has the right infinitesimal character, the correct $K_1$-types (is spherical) and that this data determines the representation.
The infinitesimal character is \((-2, -2, 0, 0) + (y, \frac{m}{2}, \frac{m}{2}, \frac{m}{2}) = (y, \frac{m}{2}, \frac{m}{2}, \frac{m}{2})\)
which is conjugate to \((\frac{m}{2}, \frac{m}{2}, \frac{m}{2}, \frac{m}{2})\) as in Theorem 3.3. To determine the \(K_1\)-types, we use the Blattner formula (see [16, Theorem 6.3.12]).

**Lemma 3.5.** The degree \(m\) elements of the symmetric algebra \(S(u \cap p)\) form an irreducible \(L \cap K_1\) representation with highest weight \(\left(\frac{m}{2}, \frac{m}{2}, \frac{m}{2}, \frac{m}{2}\right)\).

**Proof.** Since
\[
\Delta(u \cap p_1) = \{(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})\},
\Delta(I \cap \xi_1) = \{\pm(e_1 - e_2), \pm e_3 \pm e_4, \pm e_3, \pm e_4\},
\]
we see that \(I \cap \xi_1 \cong so(3) \oplus so(2) \oplus so(5)\). The \(so(3)\) acts trivially on \(u \cap p_1\), while the \(so(2)\) acts by \((\frac{1}{2}, \frac{1}{2}, 0, 0)\). Also
\[
\dim S^m(u \cap p_1) = \dim S^m(C^4) = \left(\frac{m + 3}{3}\right).
\]
The highest \(so(5)\) weight occurring in \(S^m(u \cap p_1)\) is \(\left(\frac{m}{2}, \frac{m}{2}\right)\) and the corresponding irreducible representation of \(so(5)\) also has dimension \(\left(\frac{m + 3}{3}\right)\). 

If \(F_\mu\) denotes the irreducible representation of \(K_1\) with highest weight \(\mu\), then if \(a \geq -2\)
\[
H^k(I \cap \xi_1, F_{(a+2+m/2, a+2+m/2, m/2, m/2)}) = \begin{cases} E_{(a+2+m/2, a+2+m/2, m/2, m/2)} & \text{if } k = 0, \\ 0 & \text{if } k > 0, \end{cases}
\]
where \(E_{(a+2+m/2, a+2+m/2, m/2, m/2)}\) is the irreducible highest weight representation of \(u \cap \xi_1\) with highest weight \((a + 2 + \frac{m}{2}, a + 2 + \frac{m}{2}, \frac{m}{2}, \frac{m}{2})\). This is just Kostant’s Borel-Weil Theorem (see [16, Theorem 3.2.3]). Thus for \(\lambda = a \Lambda_0\), \(\mathcal{R}_q^s(\lambda) = 0, j \neq s = \dim(u \cap \xi_1)\), and \(a \geq -2\) (in fact, \(a \geq -5\)). \(\mathcal{R}_q^s(\lambda)\) is irreducible for \(a \geq -1\) by §5 of [19]. We shall show that \(\mathcal{R}_q^s(\lambda)\) is also irreducible for \(a = -2\) and trivial if \(a < -3\). The \(K_1\)-types of \(\mathcal{R}_q^s(\lambda)\) are \({(a + 2 + \frac{m}{2}, a + 2 + \frac{m}{2}, \frac{m}{2}, \frac{m}{2}) \mid m \in \mathbb{Z}^+}\) (note: \(\lambda + 2 \rho(u \cap \xi_1) = (a+2, a+2, 0, 0)\)). Thus, \(A_q(\lambda)\) has the same \(K_1\)-types as \(V|_{F_4}\) when \(a = -2\).

Since our \(F_4\) has rank one, all irreducible spherical representations occur in some \(Ind_{MAN}^G(1 \otimes \nu \otimes 1)\) with \(\text{Re}(\nu) \geq 0\). The infinitesimal character is \(\rho_m + \nu = (\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \nu)\). This has length \(\frac{35}{4} + \nu^2\), the length of the infinitesimal character of \(V|_{F_4}\) is 15. Therefore, \(\nu = \frac{5}{2}\) and there is just one spherical representation with infinitesimal character \((\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})\). We conclude that \(A_q(\lambda), \lambda = (-2, -2, 0, 0)\), is irreducible and equivalent to \(V|_{F_4}\) (and to the Langlands quotient of \(Ind_{MAN}^G(1 \otimes \frac{5}{2} \otimes 1)\)).

**Remark.** There is a “dual pair” picture here. Let \((\pi, V)\) be our unitary highest weight representation and \(\sigma\) the automorphism of Proposition 3.2. Let \(\tilde{G} = \mathbb{Z}_2 \times \tilde{G}\), where \(\mathbb{Z}_2 = \{1, \sigma\}\). \((\pi, V)\) does not extend to a representation of \(\tilde{G}\). However, \((\pi \otimes \pi^\sigma, V \oplus V^\sigma)\) does extend:
\[
(1, g) \cdot (v_1, v_2) = (\pi(g)v_1, \pi(\sigma(g))v_2),
(\sigma, g) \cdot (v_1, v_2) = (\pi(\sigma(g))v_2, \pi(g)v_1).
\]
Call this representation \(\tilde{V}\). This is an irreducible representation of \(\tilde{G}\). For the connected subgroup \(G_1 \subset \tilde{G}\) with Lie algebra \(g_1\), there is a dual pair \((\mathbb{Z}_2, \tilde{G}_1)\)
in \( \hat{G} \), \( \hat{G}_1 = \mathbb{Z}_2 \times G_1 \). (The centralizer of \( \mathbb{Z}_2 \) is clearly \( \hat{G}_1 \), as \( G_1 \) is the group of fixed points of \( \sigma \). The centralizer of \( \hat{G}_1 \) is \( \mathbb{Z}_2 \times Z_{\hat{G}_1} = \mathbb{Z}_2 \). Note that the center \( Z_{G_1} \) of \( G_1 \) is \( \{ e \} \) as \( \zeta \in Z_{G_1} \) satisfies \( \zeta \in K_1 \) and \( \text{Ad}(\zeta) = \text{Id} \), but \( \text{Ad}: K \to GL(p_1) \) is the spin representation of \( K_1 = \text{Spin}(9) \) and has no kernel.)

\[
\hat{V}|_{\hat{G}_1} = \{(v, v) \mid v \in V\} \oplus \{(v, -v) \mid v \in V\} \approx (1 \otimes \pi_1) \oplus (\text{sign} \otimes \pi_1).
\]

So there is a one-to-one correspondence between representations of \( \mathbb{Z}_2 \) and representations of \( \hat{G}_1 \) occurring in \( \hat{V} \).

4

We now give several additional properties of \( V \).

(a) We determine the Langlands parameters of \( V \) by applying Proposition 2.8 of [20]. We obtain the following:

\[
V \leftarrow \text{Ind}_{MAN}^{G}(\sigma \otimes e^\nu \otimes 1)
\]

where \( MAN \) is a minimal parabolic, \( \sigma \in \hat{M} \) is the 1-dimensional representation of \( M = \mathbb{Z}_2 \cdot U(1) \times SU(4) \) given by \( \sigma(\epsilon, \theta, \kappa) = \text{sign}(\epsilon) e^{-3i\theta} \cdot 1 \), and \( \nu = -8\epsilon_1 - 2\epsilon_2 \) (where \( \Delta(n) = \{ \epsilon_1 \pm \epsilon_2, \epsilon_1, \epsilon_2, 2\epsilon_1, 2\epsilon_2 \} \) as in [20]). Note that \( \nu = -10s + 8t \) where \( s, t \) are the simple roots on the Satake diagram pictured below.

(b) The Harish-Chandra module \( V \) is special unipotent in the sense of Vogan (see [17, Chapter 12]). This can be easily checked by comparing the infinitesimal character of \( V \) with the tables in [3]. We will now confirm for \( V \) the \( K \)-type formula given in Conjecture 12.1 of [18].

Let \( \beta \) denote the maximal root in \( \mathfrak{p}_+ \), and \( K_C(\beta) \) the stabilizer of \( X_\beta \) in \( K_C \). The \( K_C \) orbit \( K_C \cdot X_\beta \approx K_C/\text{Ker}(\beta) \) is admissible in the sense of Definition 7.13 of [18] (see also [13]); i.e.,

\[
\chi(k) = [\det(\text{Ad}(k))]|_{(t/t(\beta))}|^{1/2}
\]

is a well-defined character of \( K_C(\beta) \). To see this, one must check that

\[
1 \leq \begin{pmatrix}
11\sqrt{3} & 3 & 3 & 3 & 3 & 3 \\
2' & 2' & 2' & 2' & 2' & 2'
\end{pmatrix}
\]

integrates to the torus \( HC(\beta) = HC \cap K_C \).

**Proposition 4.1.** \( \chi \) defines an admissible character of \( K_C(\beta) \) and

\[
V \approx \text{Ind}_{K_C(\beta)}^{K_C}(\chi)
\]
as \( K_C \)-representations.

**Proof.** \( K_C(\beta) \) is connected by Appendix B. \( K_C(\beta) \) is nearly a parabolic subgroup of \( K_C \) in the sense that there is a parabolic subgroup \( K' \) such that \( K_C(\beta) \subset K' \subset K_C \) and

\[
\mathfrak{t}' = \mathfrak{t}(\beta) + (\mathbb{C} \cdot h_\beta) = u(5) \oplus \mathfrak{so}(2) \oplus \mathfrak{n}' = \mathfrak{m}' \oplus \mathfrak{n}'
\]
where $\Delta(u(5)) = \{\epsilon_i - \epsilon_j \mid 2 \leq i, j \leq 6\}$, $\Delta(n') = \{\epsilon_i + \epsilon_j \mid 2 \leq i < j \leq 6\}$, and $h_\beta$ is the element of $t$ corresponding to $\beta$ via the Killing form.

We use induction in stages. $\text{Ind}^{K'}_{K'_{\beta}}(\chi) = \bigoplus_{n \in \mathbb{Z}} \sigma_n$, where $\sigma_n$ is the 1-dimensional representation of $K' = M'N'$ with weight $-3\lambda_1 - n\lambda_6$ on $M'$ and trivial on $N'$. An irreducible finite dimensional $K_C$ representation $F$ occurs in $\text{Ind}^{K'}_{K'_{\beta}}(\sigma_n)$ with multiplicity $\dim \text{Hom}_{K'}(F, \sigma_n) = \dim \text{Hom}_{M'}(F/n'F, \sigma_n)$. This is zero unless $F$ is the representation with lowest weight $-3\lambda_1 - n\lambda_6$, i.e., highest weight $-3\lambda_1 - n\lambda_6$ (which is dominant for $n \geq 0$). These are precisely the $K$-types of $V$.

**APPENDIX A**

The following theorem seems to be well known, but we were unable to find a proof written down in one place. Below we will give a very brief sketch of the proof, indicating references for the key steps.

**Theorem A.** Assume $(\lambda + \rho(u), \alpha) > 0$ for $\alpha \in \Delta(u)$. Then the characteristic variety of $A_q(\lambda)$ is $K_C \cdot (u \cap p)$. Thus, the Gelfand-Kirillov dimension of $A_q(\lambda)$ is $\dim_{c}(K_C \cdot (u \cap p))$.

**Proof.** First note that the Gelfand-Kirillov dimension of a representation $M$ is equal to the complex dimension of its characteristic variety $Z'(M)$ (see, e.g., [14 and 9, Lemma 10.1]).

Consider the generalized flag variety $X$ of all parabolics conjugate to $q = t + u$. The moment map $\mu : T^*X \to g^*$ is defined by identifying $T^*X$ with $G \times_{\lambda} t$ and setting $\mu(g, \eta) = \text{Ad}(g) \eta \in g \cong g^*$. If $M$ is a Harish-Chandra module and $\mathcal{M}$ is the localization of $M$ on $X$, then

$$V(M) = \mu(\text{Ch}(\mathcal{M}))$$

where $\text{Ch}(\mathcal{M})$ is the characteristic variety of $\mathcal{M}$ in $T^*X$. (See [1, §1, particularly, 1.6, 1.8, and 1.9].)

Now consider $M = A_q(\lambda)$. It follows (under the positivity condition) from [2, Theorem 2.4], that $M = \Gamma(X, \mathcal{M})$ where $\mathcal{M}$ is a $G_{\lambda}$-module on $X$ supported in the closed orbit $Z = K_C \cdot q$. The characteristic variety is thus $T_Z^*X$, the conormal bundle. Thus,

$$\text{Ch}(\mathcal{M}) \approx T_Z^*X \approx K_C \times_{Q \cap K_C} (g/(t + q))$$

$$\approx K_C \times_{Q \cap K_C} (u \cap p)^* \approx K_C \times_{Q \cap K_C} (u \cap p).$$

So,

$$V(A_q(\lambda)) = \mu(\text{Ch}(\mathcal{M})) = \mu(K_C \times (u \cap p)) = K_C \cdot (u \cap p). \quad \square$$

For the following proposition we consider arbitrary $A_q(\lambda)$ representations (assuming the parameters $\lambda$ satisfy $\lambda \perp \Delta(I)$).

**Proposition.** Let $\lambda_o$ be any parameter so that $(\lambda_o + \rho(u), \alpha) > 0$, $\alpha \in \Delta(u)$, and let $\lambda$ be arbitrary. Then $V(A_q(\lambda)) = V(A_q(\lambda_o))$ as long as $A_q(\lambda) \neq 0$.

**Proof.** It is clear from the definition of characteristic variety that for any admissible $(g, K)$-module $X$, $V(X \otimes F) \subseteq V(X)$. If $Y$ is a quotient of $X \otimes F$ then $V(Y) \subseteq V(X \otimes F) \subseteq V(X)$. Since $\text{Hom}(X \otimes F, Y) = \text{Hom}(X, Y \otimes F^*)$, for $X$ irreducible, we get $X \subseteq Y \otimes F^*$. Therefore, $V(X) \subseteq V(Y \otimes F^*) \subseteq V(Y)$ and we conclude that $V(Y) = V(X)$. We will show that for appropriate $\lambda_o$ and $F$, $A_q(\lambda)$ is a quotient of $A_q(\lambda_o) \otimes F$; the proposition will follow.
Recall that the derived functor modules are defined by
\[ R_q^i (W) = \Gamma^i \left( \text{Hom}_{U(q)} \left( U(g), W \otimes \bigwedge^{\text{top}} u \right) \right) \]
where \( W \) is an \((q, L \cap K)\)-module and \( \Gamma^i \) is the right derived functor of the "K-finite" functor \( \Gamma \) (see [16, 6.31]).

Let \( \lambda \) be arbitrary, by the lemma below there is a finite dimensional \( g \)-module \( F \), a \( \lambda_0 \) so that \( \langle \lambda_0 + \rho(u), \alpha \rangle > 0 \), \( \alpha \in \Delta(u) \), and an exact sequence
\[ 0 \to \text{Ker} \to C_{\lambda_0} \otimes F \to C_{\lambda} \to 0 \]
(as \( q \)-modules), hence a long exact sequence
\[ \cdots \to R_q^s (C_{\lambda_0}) \otimes F \to R_q^s (C_{\lambda}) \to R_q^{s+1} (\text{Ker}) \to \cdots . \]
Here we have used \( R_q^s (C_{\lambda_0} \otimes F) \approx R_q^s (C_{\lambda_0}) \otimes F \) \((F \text{ is a representation of } g)\). The vanishing theorem for degrees greater than \( s \) [16, 6.3.21] along with a standard filtration argument shows \( R_q^{s+1} (\text{Ker}) = 0 \). Thus we get
\[ \cdots \to A_q(\lambda_0) \otimes F \to A_q(\lambda) \to 0 . \]
The theorem follows by noting that if \( \langle \lambda_0 + \rho(u), \alpha \rangle > 0 \), \( \alpha \in \Delta(u) \), then \( V (A_q(\lambda_0)) = V (A_q(\lambda_0')) \) by the standard tensoring (within a chamber) argument (see, for example, [16, 7.2.9 and 7.2.22]). \( \square \)

**Lemma.** Given \( \lambda \), there is a \( \lambda_0 \) so that \( \langle \lambda_0 + \rho(u), \alpha \rangle > 0 \), \( \alpha \in \Delta(u) \), a finite dimensional \( g \)-module \( F \), and a surjection \( C_{\lambda_0} \otimes F \to C_\lambda \) as \( q \)-modules.

**Proof.** The \( u \)-invariants \( (C_{\lambda} \otimes F_0)^u = C_{\lambda} \otimes F_0^u \) contains a highest weight vector (for \( l \)). Choose \( \lambda_0 \) so that \( \lambda_0 - \lambda \) is \( \Delta^+(g) \)-dominant then take \( F_0 \) to be the irreducible finite dimensional \( g \)-representation with highest weight \( \lambda_0 - \lambda \). Thus we have arranged for a nonzero map \( C_{\lambda_0} \to C_{\lambda} \otimes F_0 \) as \( q \)-modules. Thus,
\[ \text{Hom}(C_{\lambda_0}, C_{\lambda} \otimes F_0) \approx \text{Hom}(C_{\lambda_0} \otimes F_0^*, C_{\lambda}) \]
is nonzero. Take \( F = F_0^* \) in the lemma. \( \square \)

**Appendix B**

In this appendix we prove a fact about stabilizers for a minimal nilpotent orbit of \( K_C \) in \( p \) and then apply this to our \( E_6 \) example.

Let \( G \) be a complex, connected, simply connected, semisimple group. Let \( \theta \) be the complexification of a Cartan involution for some real form of \( G \), and let \( \mathfrak{k} + p \) be the corresponding Cartan decomposition of the \( g = \text{Lie}(G) \). Assume \( \text{rank } K = \text{rank } G \). In this case, \( \theta \) is an inner automorphism, so \( K_C = G^\theta \) is connected. Fix a Cartan subalgebra \( \mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g} \) (and let \( H \) be the corresponding Cartan subgroup) and a positive system of roots \( \Delta^+ \). Let \( \beta \) be the highest root in \( p \), and let \( x_\beta \) be a root vector for \( \beta \). There is a triple \( \{ x_\beta, h_\beta, y_\beta \} \) spanning a subalgebra of \( g \) isomorphic to \( \text{sl}(2, \mathbb{C}) \) with \( h_\beta \in \mathfrak{t} \) and \( y_\beta \) a root vector for \( -\beta \). \( h_\beta \) is normalized such that \( \beta(h_\beta) = 2 \).

A \( \theta \)-stable parabolic \( q = l + u \) is defined by \( \Delta(l) = \{ \alpha \in \Delta \mid \alpha(h_\beta) = 0 \} \) and \( \Delta(u) = \{ \alpha \in \Delta \mid \alpha(h_\beta) > 0 \} \). Let \( Q = LU \) be the corresponding parabolic subgroup in \( G \). \( Q \) is connected. The stabilizer \( G(\beta) \) of \( x_\beta \) is contained in \( LU \) by Theorem 3.6 of [10]. We may conclude that \( K_C(\beta) \subset K_C \cap LU \subset (K_C \cap L) \cdot (K_C \cap U) \); furthermore, \( K_C \cap L \) is the Levi component of the parabolic \( (K_C \cap LU) \subset K_C \) and is connected.
Proposition. \(K_C(\beta)/K_C(\beta)_e = H(\beta)/H(\beta)_e\), where \(H(\beta) = \text{Stab}_{H}(x_\beta)\).

Proof. Let \(HN\) be a Borel subgroup of \(K_C \cap L\) and let \(W\) be the Weyl group of \(K_C \cap L\). The Bruhat decomposition is

\[K_C \cap L = \bigcup_{w \in W} NHwN.\]

The following lemma shows that \(N\) and \(W\) stabilize \(x_\beta\).

Lemma. The semisimple part of \(K_C \cap L\) is contained in \(K_C(\beta)\) and \(W \subset K_C(\beta)\).

Proof. Since \(K_C \cap L\) is connected, we need only check this for the corresponding Lie algebras. If \(\alpha \in \Delta^+(k \cap l)\) then \((\alpha, \beta) = 0\). But \(\beta + \alpha \notin \Delta\) since \(\beta\) is the highest root; thus also \(\beta - \alpha \notin \Delta\). So all root vectors in \(k \cap l\) kill \(x_\beta\) and thus the semisimple part of \(K_C \cap L\) is contained in \(K_C(\beta)\). It follows that \(W \subset K_C(\beta)\). \(\square\)

Now suppose \(k = n_1 h w n_2 \in K_C \cap L\) fixes \(x_\beta\), then \(x_\beta = k \cdot x_\beta = n_1 h w n_2 x_\beta\) and \(x_\beta = h \cdot x_\beta\), so \(h \in H(\beta)\). Therefore,

\[K_C(\beta) \cap L = \bigcup_{w \in W} NH(\beta)wN.\]

Let \(F\) be a set of representatives of distinct cosets in \(H(\beta)/H(\beta)_e\), such that

\[K_C(\beta) \cap L = F \cdot (K_C(\beta) \cap L)_e.\]

We claim that \(F\) is a set of representatives of distinct cosets in \((K_C(\beta) \cap L)/(K_C(\beta) \cap L)_e\). Suppose \(f \in F\) is in \((K_C(\beta) \cap L)_e\). Then, since \(f\) commutes with \(H(\beta)_e\), \(f\) is in \(H(\beta)_e\) because \(H(\beta)_e\) is a Cartan subgroup of the connected group \((K_C(\beta) \cap L)_e\). The proposition is proved.

For our situation with \(E_6\), we calculate the component group \(H(\beta)/H(\beta)_e\). Let \(L_{\text{root}} = \text{span}_\mathbb{Z}\{h_\alpha | \alpha \in \Delta\}\) where \(h_\alpha\) is the element of \(t\) corresponding to \(\frac{2\alpha}{(\alpha, \alpha)} \in t^*\) via the Killing form. Let \(e(h) = \exp(2\pi ih), h \in t\). Since \(G\) is simply connected, \(L_{\text{root}} = \text{Ker}(e)\). Write \(h \in t\) as \(h_1 + th_\beta\), with \(h_1 \perp h_\beta\). Then

\[\text{Ad}(e(h))x_\beta = e^{2\pi i(2t)}x_\beta\]

so \(e(h) \in H(\beta)\) if \(t \in \frac{1}{2}\mathbb{Z}\). Thus,

\[H(\beta) = \exp(2\pi i(h_\beta^+ + \frac{1}{2}\mathbb{Z}h_\beta))\]

\[= \{e, \exp(\pi ih_\beta)\} \exp(2\pi ih_\beta^+) = \{e, \exp(\pi ih_\beta)\}H(\beta)_e.\]

We now show that \(\exp(\pi ih_\beta) \in H(\beta)_e\); i.e., \(\exp(\pi ih_\beta) \in \exp(h_\beta^+)\). For this it is sufficient to find an element \(h_1 \in h_\beta^+\) such that \(\frac{1}{2}h_\beta - h_1 \in \text{Ker}(e) = L_{\text{root}}\). Choose \(h_1 = \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_5)\). One easily checks that \(\frac{1}{2}h_\beta - h_1 \in L_{\text{root}}\). Thus, in our \(E_6\) example \(K_C(\beta)\) is connected.

References


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