0. Introduction

Let $D$ be a domain in $\mathbb{C}^n$ and $g_i \in H^\infty(D)$ such that

$$\sum_1^k |g_i|^2 \geq \delta^2 > 0. \quad (1)$$

The problem whether there are $u_j \in H^\infty(D)$ such that $\sum_1^k g_i u_i = 1$ is known as the corona problem. The answer is affirmative in e.g. all finitely connected domains in $\mathbb{C}$, but unknown even in the ball if $n > 1$. In [11 and 5] are constructed smooth domains in $\mathbb{C}^3$ and $\mathbb{C}^2$ which have strictly pseudoconvex boundary in all but one point, but in which the corona theorem fails. However it follows from [13] that in case of two generators, i.e. $k = 2$, in a strictly pseudoconvex domain there is a solution in BMO.

It is clear that if the corona problem is solvable, then to any $q \in H^2(D)$ there are $u_j \in H^2(D)$ such that $\sum_1^k g_i u_i = q$. In this paper we prove such a theorem in a pseudoconvex domain $D$ admitting a $C^2$ plurisubharmonic defining function $\rho$, i.e. $\rho$ be of class $C^2$ in a neighborhood of $\overline{D}$, $D = \{\rho < 0\}$, $d\rho \neq 0$ on $\partial D$ and $i\partial\overline{\partial}\rho \geq 0$ in $D$. In particular, the domains in [11 and 5] are of this kind. However, there are examples of pseudoconvex $C^2$ domains without plurisubharmonic defining function, see [3].

Theorem 1. Suppose $D$ is a pseudoconvex domain in $\mathbb{C}^n$ with a $C^2$ plurisubharmonic defining function. Let $g$ be a $j \times k$-matrix of functions in $H^\infty(D)$ such that

$$\det g g^* \geq \delta^2 > 0. \quad (2)$$
Then to every \( j \)-column \( q \) of functions in \( H^2(D) \) there is a \( k \)-column \( u \) in \( H^2(D) \) such that

\[ gu = q \]

and \( \|u\|_{H^2} \leq C_\delta \|q\|_{H^2} \). If \( j = 1 \) one can take \( C_\delta = C_\epsilon / \delta^{1+\epsilon+\min(n,k-1)} \) where \( C_\epsilon \) only depends on \( \|g\|_\infty, n, k \) and \( \epsilon > 0 \).

Note that (2) implies that \( g \) is surjective and hence that (3) is pointwise solvable. In fact it is enough to assume that \( g \) has constant rank, the product of the nonzero eigenvalues of \( g g^* \) are bounded by \( \delta^2 \) from below and that (3) is pointwise solvable to get the conclusion of Theorem 1, cf. [1].

Since any smooth pluriharmonic \( \chi \) on \( \overline{D} \) has the form \( \chi = \log |f|^2 \) for some nonvanishing holomorphic \( f \) (at least if \( H^1(\overline{D}, \mathbb{C}) = 0 \)), we can apply Theorem 1 to \( q/f \) instead of \( q \) and obtain a solution to (3) such that

\[ \int_{\partial D} |u|^2 e^{-\chi} \leq C_\delta^2 \int_{\partial D} |q|^2 e^{-\chi}. \]

If \( n = 1 \), \( \chi \) can be freely chosen in (4) and this implies, see [2] or [1], that there is a bounded solution \( u \) such that \( \|u\|_\infty \leq C_\delta \|q\|_\infty \) if \( q \) is bounded. Hence Theorem 1 is equivalent to the corona theorem in the unit disc.

**Theorem 1'.** Let \( D \) be as in Theorem 1. If \( q \) and \( g = (g_1, \ldots, g_k) \) in \( H^\infty(D) \) satisfy

\[ |q| \leq |g|^{1+\epsilon+\min(n,k-1)}, \]

then there is a solution \( u \) to (3) in \( H^2(D) \) such that

\[ \int_{\partial D} |u|^2 e^{-\chi} \, dS \leq C_\epsilon \int_{\partial D} e^{-\chi} \, dS. \]

Again, for \( n = 1 \), this implies that there is a bounded solution if \( |q| \leq |g|^{2+\epsilon} \). This also follows from Wolff's proof of the corona theorem, see [7].

**Corollary.** If \( q = q_1 q_2 \) where \( q_1 \) satisfies (5) and \( q_2 \in H^2(D) \) is nonvanishing, then there is a solution \( u \) to (3) such that

\[ \int_{\partial D} |u|^2 \, dS \leq C_\epsilon \int_{\partial D} |q_2|^2 \, dS. \]

Our proof of Theorem 1 (and 1'), as most proofs of the corona theorem, is based on an estimate of solutions of a \( \overline{\partial}_b \)-equation. This approach was first introduced by Hörmander in [10]. Using the Koszul complex one can, in principle, reduce the theorem to systems of (scalar-valued) \( \overline{\partial} \)- (or \( \overline{\partial}_b \)-) equations, at least if \( j = 1 \). However, unless \( n = 1 \) or \( k \leq 2 \), one has to solve, successively, a sequence of \( \overline{\partial} \)-equations (involving forms of higher bidegree) and this seems to lead to considerable difficulties. Instead we reformulate the theorem as a \( \overline{\partial}_b \)-problem in a holomorphic vector bundle, following the lines in [12] (but with \( \overline{\partial} \) replaced by \( \overline{\partial}_b \), in which \( L^2 \)-estimates for division problems are treated in a very general setting.

The \( \overline{\partial}_b \)-equation is treated by a generalization of a variant due to Berndtsson [2], of the Morrey-Kohn-Hörmander identity (see §3). In [2], this new identity was used to get \( L^2(\partial D) \)-estimates for the (scalar-valued) \( \overline{\partial}_b \)-equation. Since we
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need estimates for $\partial_b$ in a vector bundle, our first aim is to generalize to this case. This leads to a $L^2(\partial D)$-theorem for $\partial_b$ in a holomorphic vector bundle over a pseudoconvex domain in a Kähler manifold, see Theorem 2 in §4.

However, in order to prove Theorem 1 (and Theorem 1') we cannot use Theorem 2 directly since it just deals with a size estimate of the right-hand side. One also has to take into account an appropriate estimate of derivatives. This is the trick introduced by Wolff in his proof of the corona theorem, see [7].

The paper is divided into six sections. After some necessary preliminaries in §1, we discuss the $\partial_b$-equation in §2. In §3 we derive the above-mentioned equality and show its connection to the Morrey-Kohn-Hörmander equality as well as the Bochner-Kodaira-Nakano equality. In §4 we prove our $L^2$-estimate for $\partial_b$ (Theorem 2) and in the remaining two sections we prove Theorem 1 and Theorem 1'.

1. Notational preliminaries

Let $X$ be a Kähler manifold with fundamental form $\omega$, so that $dV = \omega^n/n!$ is the volume measure on $X$, and let $D = \{p < 0\}$ be a relatively compact domain in $X$, where $\rho$ is smooth and $d\rho \neq 0$ on $\partial D$. We give $\partial D$ the orientation so that a $(2n-1)$-form $\alpha$ is oriented on $\partial D$ if and only if $d\rho \wedge \alpha$ is oriented on $X$. Then the surface measure $dS$ on $\partial D$ is given by $dS = d\rho/d\rho|\int dV$ on $\partial D$, where inner multiplication $\int$ for forms $\alpha$ and $\beta$ is defined by

$$\langle \alpha \int \beta, \gamma \rangle = \langle \beta, \bar{\alpha} \wedge \gamma \rangle.$$ 

Here $\langle , \rangle$ is the induced inner product for forms, i.e. if $*$ is the complex-linear Hodge star operator, then $\langle \alpha, \beta \rangle dV = *(\alpha \wedge *\beta)$. Note that $** = (-1)^{p+q}$ on $(p, q)$-forms (since $X$ has even real dimension) and that $\alpha \int \beta = *(\alpha \wedge *\beta)$ for any $1$-form $\alpha$. If $\alpha$ is a $(2n-1)$-form, we have that

$$\int_{\partial D} \alpha = \int_{\partial D} *(d\rho \wedge \alpha) dS/|d\rho|.$$ 

To see (2), notice that $d\rho \wedge *(d\rho \wedge \alpha) dS/|d\rho| = *(d\rho \wedge \alpha) d\rho \wedge dS/|d\rho| = *(d\rho \wedge \alpha) dV = d\rho \wedge \alpha$, and hence both integrands in (2) are equal, considered as forms on $\partial D$.

If $E \to X$ is a holomorphic vector bundle over $X$ with hermitian metric $\langle , \rangle_E$, then we get a metric for $\xi, \eta \in \mathcal{E}_{p, q}(X, E)$ $= \mathcal{E}(X, \Lambda^p, \Lambda^q T^* \otimes E)$ i.e. smooth $E$-valued $(p, q)$-forms by putting $\langle \alpha \cdot s, \alpha' \cdot s' \rangle = \langle \alpha, \alpha' \rangle \langle s, s' \rangle_E$ on composite elements. If $s \to s'$ is the conjugate linear mapping from $E$ to its dual bundle $E^*$, such that $s' \cdot s = \langle s, s \rangle_E$, and $\bar{s}: \mathcal{E}_{p, q}(X, E) \to \mathcal{E}_{q, p}(X, E^*)$ by putting $\bar{s}(\alpha \cdot s) = \bar{s} \cdot \bar{s}$ on composite elements, then

$$\langle \xi, \eta \rangle dV = \xi \wedge \bar{s} \eta, \quad \xi, \eta \in \mathcal{E}_{p, q}(X, E).$$

Moreover, $\alpha \int \xi = \bar{s}(\bar{\alpha} \wedge \bar{s} \xi)$, cf. (5), if $\alpha$ is a $1$-form and $\xi$ any $E$-valued form.

For $\xi, \eta \in \mathcal{E}_{p, q}(X, E)$, one of which has compact support, we have the inner product

$$\langle \xi, \eta \rangle = \int_{\partial D} \alpha \int \bar{s} \cdot \alpha \wedge \bar{s} \eta.$$
Let \( D = D' + D'' \) denote the Chern connection on \( E \) as well as on \( E^* \) (with respect to \( \langle , \rangle_E \)). Then the formal adjoints \((D')^*\) and \((D'')^*\) are given by \[(D'')^* = -\bar{\partial} D'' \bar{\partial}, \quad (D')^* = -\bar{\partial} D' \bar{\partial}.

If \( s \) is a holomorphic section to \( E \) then \( D''(\alpha \cdot s) = (\bar{\partial} \alpha) \cdot s \) and therefore we sometimes write \( \bar{\partial} \) instead of \( D'' \).

2. The \( \bar{\partial}_b \)-Equation

Using the notation from §1 we have

**Proposition 1.** Suppose \( \eta \in \mathcal{E}_{p,q+1}(\bar{D}, E) \) and \( \xi \in \mathcal{E}_{p,q}(\bar{D}, E) \). Then

\[
\int_D (D'' \xi, \eta) \, dV - \int_D \langle \xi, (D'')^* \eta \rangle \, dV = \int_{\partial D} \langle \xi, \partial \rho \perp \eta \rangle \, dS/|d\rho|.
\]

**Remark.** Throughout this paper \((D'')^*\) denotes the formal adjoint of \( D'' \). When dealing with the \( D'' \)-Neumann problem \((D'')^*\) is an operator with a specified domain \( \text{dom}(D'')^* \). For instance, (1) implies that \( \eta \in \mathcal{E}_{p,q}(\bar{D}, E) \) is in \( \text{dom}(D'')^* \) if and only if \( \partial \rho \perp \eta \mid_{\partial D} = 0 \).

**Sketch of proof.** By (1), (3), and (2) in §1 we have that

\[
\int_{\partial D} \langle \xi, \partial \rho \perp \eta \rangle \, dS/|d\rho| = \int_{\partial D} \langle \bar{\partial} \rho \wedge \xi \wedge \bar{\partial} \eta \rangle \, dS/|d\rho| = \int_{\partial D} \xi \wedge \bar{\partial} \eta,
\]

so by Stokes’ theorem and bidegree reasons the right-hand side of (1) equals \( \int_D d(\xi \wedge \bar{\partial} \eta) = (D'' \xi, \eta) - (\xi, (D'')^* \eta) \). \( \square \)

Let \( f \in \mathcal{E}_{n,q+1}(\bar{D}, E) \). We say that an \( E \)-valued form (current) \( u \) on \( \partial D \) solves \( \bar{\partial}_b u = f \) if

\[
\int_{\partial D} \langle u, \partial \rho \perp \alpha \rangle \, dS/|d\rho| = \int_D \langle f, \alpha \rangle \, dV
\]

for all \( \alpha \in \mathcal{E}_{n,q+1}(\bar{D}, E) \) such that \((D'')^* \alpha = 0\).

Hence, by (1), \( u \) must have bidegree \((n, q)\) in the sense that \( \alpha \wedge \bar{\partial} \rho \) has bidegree \((n, q + 1)\), and \( \bar{\partial} f = 0 \) (since one can take \( \alpha = \bar{\partial} \psi \) for any \( \bar{\partial} \)-closed \( \psi \)). Notice that

\[
\|u\|_{\partial D}^2 = \int_{\partial D} |u \wedge \bar{\partial} \rho|^2 \, dS/|d\rho|_3
\]

defines a norm for the space of smooth \((n, q)\)-forms on \( \partial D \), and let \( L^2_{n,q}(\partial D, E) \) be its completion with respect to this norm.

**Remark.** To be more precise, any \( u \in \mathcal{E}(\partial D, \Lambda^{n,q} T(X) \otimes E) \) has an orthogonal decomposition \( u = u_1 + u_2 \), where \( \bar{\partial} \rho \wedge u_1 = 0 \) and \( \bar{\partial} \rho \perp u_2 = 0 \). Thus \((n, q)\)-forms (currents) \( V \) such that \( \partial \rho \perp V = 0 \) can be isometrically identified with intrinsic \((n, q)\)-forms (currents) on \( \partial D \).

**Proposition 2.** Let \( f \in \mathcal{E}_{n,q+1}(\bar{D}, E) \) be \( D'' \)-closed. Then \( \bar{\partial}_b u = f \) has a solution in \( L^2_{n,q}(\partial D, E) \) with norm \( C \) if and only if

\[
\left| \int_D \langle f, \alpha \rangle \, dV \right|^2 \leq C^2 \int_{\partial D} |\partial \rho \perp \alpha|^2 \, dS/|d\rho|^2
\]

for all \((D'')^*\)-closed \( \alpha \in \mathcal{E}_{n,q+1}(\bar{D}, E) \).
Proof. First suppose there is such a solution $u$ considered as a current such that $\partial \rho \wedge u = 0$, cf., the remark above. Then $\int_D \langle f, \alpha \rangle \, dV = \int_{\partial D} \langle u, \partial \rho \wedge \alpha \rangle \, dS / |d\rho|$ so by Schwarz inequality,
\[
\left| \int_D \langle f, \alpha \rangle \, dV \right|^2 \leq \int_{\partial D} |u|^2 \, dS / |d\rho| \int_{\partial D} |\partial \rho \wedge \alpha|^2 \, dS / |d\rho|
\]
which implies (4). For the converse, assume that (4) holds. Define the linear functional $A(\partial \rho \wedge \alpha) = \int_D \langle f, \alpha \rangle \, dV$ on $E$-valued $(n, q)$-forms on $\partial D$ of the form $\partial \rho \wedge \alpha$ where $(D'\alpha)^* \alpha = 0$ in $D$. By (4) it is well defined and $L^2$-bounded, so there is a $u$ such that $\int_{\partial D} |u|^2 / |d\rho| \, dS \leq C^2$ and (2) holds. □

In the next paragraph we shall derive an equality which gives a possibility to obtain estimates like (4).


In the notation from §1, the Bochner-Kodaira-Nokano identity is
\[
(D')^* D'' + D''(D'')^* = (D')^* D' + D'(D')^* + i[\Theta, \Lambda]
\]
where $\Lambda$ is inner multiplication with the fundamental form $\omega$, $\Theta$ is the curvature tensor on $E$, i.e. $\Theta = D^2$, and $[\ , \ ]$ denotes commutator.

If $X$ is compact (so that no boundary terms occur) (1) implies the estimate $\|D''\xi\|^2 + \|(D'')^* \xi\|^2 \geq (i[\Theta, \Lambda] \xi, \xi)$. If $a = i[\Theta, \Lambda]$ happens to be nonnegative on $E$-valued $(p, q)$-forms and $f$ is a $D''$-closed $(p, q)$-form one gets [also using a local regularity result for the elliptic operator $\Box = D''(D'')^* + (D'')^* D''$] the estimate
\[
\left| \int \langle f, \xi \rangle \, dV \right|^2 \leq \int \langle a^{-1} f, f \rangle \, dV \int \|(D'')^* \xi\|^2 \, dV
\]
for all $\xi \in \mathcal{E}_{p, q}(X, E)$, which means that there is a solution to $D''u = f$ with $\|u\|^2 \leq \int \langle a^{-1} f, f \rangle \, dV$, provided the right-hand side is finite.

In a domain $D$ with boundary, one leads to study the $D''$-Neumann problem and here the starting point is the Morrey-Kohn-Hörmander identity. We will derive it below from (1). To deal with the $\overline{\partial}_b$-equation on $D$ we need still another equality (Proposition 7) first found and used in [2] in the case of $(0, 1)$-forms (see the remark below) and trivial bundle. This one too will be derived from (1).

We first note how the various geometrical objects are affected if our original metric $\langle \ , \ \rangle$ on $E$ is modified.

Proposition 3. If $\langle \ , \ \rangle$ is changed to $\langle \ , \ \rangle e^{-\varphi}$, then by obvious use of the index $\varphi$,
\[
(D'')^* \varphi = (D'')^* + \partial \varphi \wedge,
\]
\[
D'_\varphi = D' - \partial \varphi \wedge,
\]
\[
\Theta_\varphi = \Theta + \partial \overline{\partial} \varphi,
\]
and
\[
(D'_\varphi)^* \varphi = (D')^*.
\]
Any of these follows from well-known identities, see e.g. [8], or by simple computations.

Now put \( \varphi = t \log(-1/\rho) \) in \( D = \{ \rho < 0 \} \) so that \( \exp(-\varphi) = (-\rho)^t \) and \( \partial \varphi = O(-1/\rho) \). Also put \( (\ , \ )_p = \int_D (\ , \ ) e^{-\varphi} \, dV \). If \( t > 2 \), we can, cf. (2), \ldots, (5), integrate by parts and obtain

\[
\|D''\alpha\|^2_\varphi + \|((D'')^\varphi \alpha\|^2_\varphi = \|D'_\varphi \alpha\|^2_\varphi + \|((D'_\varphi)^* \alpha\|^2_\varphi + i(\Theta_\varphi, \Lambda_\alpha, \alpha)_\varphi
\]

from (1). Our next task is to compute the various terms in (6). We assume that \( \alpha \in \mathcal{E}_{n,q}(D, E) \) so that \( D'_\varphi \alpha = 0 \) and \( [\Theta_\varphi, \Lambda]_\alpha = \Theta_\varphi \Lambda_\alpha \). Since, by (2),

\[
(D'')^\varphi = (D'')^* + \partial \varphi \perp = (D'')^* - t(\partial \rho/\rho) \perp
\]

we get

\[
\|((D'')^\varphi \alpha\|^2_\varphi = \int_D (-\rho)^t|((D'')^* \alpha\|^2 \, dV
\]

\[
+ 2t \text{Re} \int_D (-\rho)^{t-1}((D'')^* \alpha, \partial \rho \perp \alpha) \, dV + t^2 \int_D (-\rho)^{t-2}|\partial \rho \perp \alpha|^2 \, dV.
\]

By (4),

\[
(8) \quad \Theta_\varphi = \Theta - t \partial \overline{\partial} \rho/\rho + t \partial \rho \wedge \overline{\partial} \rho/\rho^2.
\]

We need also

**Lemma 4.** If \( \alpha \in \mathcal{E}_{n,q}(D, E) \), then

\[
(9) \quad i(\partial \rho \wedge \overline{\partial} \rho \wedge \Lambda_\alpha, \alpha) = |\partial \rho \perp \alpha|^2.
\]

**Lemma 5.** If \( \psi \in \mathcal{E}(D) \), then

\[
\int_D (-\rho)^{t-1} \psi \, dV \to \int_{\partial D} \psi \, dS/|d\rho|
\]

when \( t \searrow 0 \).

Taking these for granted for the moment we get from (6), (7), (8) and (9) that

\[
\int_D (-\rho)^t|D'' \alpha|^2 \, dV + \int_D (-\rho)^t|((D'')^* \alpha\|^2 \, dV
\]

\[
+ 2t \text{Re} \int_D (-\rho)^{t-1}((D'')^* \alpha, \partial \rho \perp \alpha) \, dV
\]

\[
+ t^2 \int_D (-\rho)^{t-2}|\partial \rho \perp \alpha|^2 \, dV = \int_D (-\rho)^t|D' \alpha|^2 \, dV - \int_D (-\rho)^t i(\Theta \Lambda_\alpha, \alpha) \, dV
\]

\[
+ t \int_D (-\rho)^{t-1}(i\partial \overline{\partial} \rho \wedge \Lambda_\alpha, \alpha) \, dV + t \int_D (-\rho)^{t-2}|\partial \rho \perp \alpha|^2 \, dV
\]

for \( t > 2 \) and \( \alpha \in \mathcal{E}_{n,q}(D, E) \).

If we now assume that \( D'' \alpha = (D'')^* \alpha = 0 \), combine the last terms on each side of equality (10) and let \( t \searrow 1 \) we get
Proposition 6. Suppose $\alpha \in \mathcal{D}_{n,q}(\overline{D}, E)$ and $D''\alpha = (D'')^*\alpha = 0$. Then

$$i \int_{D} (-\rho)(\Theta \alpha, \alpha) dV + i \int_{D} \langle \partial \overline{\partial} \rho \Lambda \alpha, \alpha \rangle dV$$

$$+ \int_{D} (-\rho)(D')^* \alpha^2 dV = \int_{\partial D} |\partial \rho \perp \alpha|^2 dS/|d\rho|.$$

Remark. Suppose $D \subset \mathbb{C}^n$. If $(\cdot, \cdot)$ is the metric $e^{-\psi}$ on the trivial line bundle (so that $\Theta = \partial \overline{\partial} \psi$) and $(n, 1)$-forms are identified with $(0, 1)$-forms in the obvious way, then (11) is exactly Proposition 5 in [2].

In a similar way we can also obtain the Morrey-Kohn-Hörmander identity.

Proposition 7. If $\alpha \in \mathcal{D}_{n,q}(\overline{D}, E)$ and $\partial \rho \perp \alpha|_{\partial D} = 0$, then

$$\int_{D} |D''\alpha|^2 dV + \int_{D} |(D')^*\alpha|^2 dV = \int_{D} |(D')^*\alpha^2 dV$$

$$+ i \int_{\partial D} \langle \partial \overline{\partial} \rho \Lambda \alpha, \alpha \rangle dS/|d\rho| + i \int_{D} \langle \Theta \alpha, \alpha \rangle dV.$$

Proof. By assumption $\partial \rho \perp \alpha = O(-\rho)$ so (12) follows from (10) when $t \searrow 0$. □

We conclude this paragraph with proofs of the lemmas.

Proof of Lemma 4. Fix a point and $(1, 0)$-forms $\omega_1, \ldots, \omega_n$ such that $\omega = \sum \omega_j \wedge \overline{\omega}_j$ and $\partial \rho = \omega_1$ at this point. We can write $\alpha = \alpha' + \alpha'' = \omega_1 \wedge \overline{\omega}_1 \wedge \gamma + \alpha''$, such that $\gamma$ and $\alpha''$ do not contain $\overline{\omega}_1$. Then $\omega_1 \wedge \overline{\omega}_1 \wedge \alpha = \alpha'$ and $(\alpha', \alpha) = (\alpha', \alpha')$. On the other hand also, $|\omega_1 \perp \alpha|^2 = |\omega_1 \perp \omega_1 \wedge \overline{\omega}_1 \wedge \gamma|^2 = |\omega_1 \gamma|^2 = |\overline{\omega}_1 \wedge \omega_1 \wedge \gamma|^2 = |\alpha'|^2$ since $\overline{\omega}_1 \wedge \gamma$ does not contain $\omega_1$. This proves the lemma. □

Proof of Lemma 5. We may assume that $\psi$ has support in some small neighborhood of a boundary point and we let $\alpha$ be a $(2n-1)$-form such that $dV = d\rho \wedge \alpha/|d\rho|$ there. Then

$$t \int_{D} (-\rho)t^{-1} \psi dV = t \int_{D} (-\rho)t^{-1} d\rho \wedge \psi \alpha/|d\rho|$$

$$= - \int_{D} d(-\rho)t \wedge \psi \alpha/|d\rho| = \int_{D} (-\rho)t \wedge d(\psi \alpha/|d\rho|)$$

$$\rightarrow \int_{D} d(\psi \alpha/|d\rho|) = \int_{\partial D} \psi \alpha/|d\rho| = \int_{\partial D} \psi dS/|d\rho|$$

where we have used Stokes' theorem twice. □

4. A SOLUTION OF THE $\overline{\partial}_b$-EQUATION

In this paragraph $D = \{ \rho < 0 \}$ is pseudoconvex and $\rho$ is a $C^2$ plurisubharmonic defining function. Suppose that the hermitian operator (see [8])

$$A = i(-\rho)\Theta \Lambda + i \partial \overline{\partial} \rho \Lambda$$

is semipositive on $E$-valued $(n, q + 1)$-forms, i.e.

$$\langle A \alpha, \alpha \rangle \geq 0, \quad \alpha \in \mathcal{D}_{n,q+1}(\overline{D}, E).$$
Then Proposition 6 in §3 provides the estimate
\[
\int_D \langle A\alpha, \alpha \rangle \, dV \leq \int_{\partial D} |\partial \rho \perp \alpha|^2 \, dS/|\partial \rho|
\]
for \( \alpha \in \mathcal{E}_{p,q+1}(\overline{D}, E) \) such that \( D''\alpha = (D'')^*\alpha = 0 \).

If \( f \in \mathcal{E}_{p,q+1}(\overline{D}, E) \) is \( \overline{\partial} \)-closed, we thus get
\[
(1) \quad \left\| \int_D \langle f, \alpha \rangle \, dV \right\|^2 \leq \int_D \langle A^{-1} f, f \rangle \, dV \int_{\partial D} |\partial \rho \perp \alpha|^2 \, dS/|\partial \rho|
\]
for \( \alpha \in \mathcal{E}_{p,q+1}(\overline{D}, E) \) such that \( D''\alpha = (D'')^*\alpha = 0 \). In order to ease the condition that \( D''\alpha = 0 \) in (1), we first, for simplicity, assume that \( \partial D \) is strictly pseudoconvex. Then, \( |\alpha|^2 \leq C(|i\partial \overline{\partial} \rho \Lambda \alpha|, \alpha) \), for \( \alpha \) such that \( \partial \rho \perp \alpha = 0 \) on \( \partial D \) and since \( |(\Theta \Lambda \alpha, \alpha)| \leq C|\alpha|^2 \) (recall that \( E \) is assumed to be a bundle over \( X \)) we get from Proposition 7 the Basic Estimate
\[
\int_D |(D')^*\alpha|^2 \, dV + \int_{\partial D} |\alpha|^2 \, dS \\
\leq C \left[ \int_D |D''\alpha|^2 \, dV + \int_D |(D'')^*\alpha|^2 \, dV + \int_D |\alpha|^2 \, dV \right]
\]
if \( \partial \rho \perp |\alpha|_{\partial D} = 0 \). This ensures, see [6], regularity for the \( D''\)-Neumann problem and then any \( \alpha \in \mathcal{E}_{p,q+1}(\overline{D}, E) \) has a smooth orthogonal decomposition \( \alpha = \alpha' + \alpha'' \) where \( \partial \alpha' = 0 \) and \( \alpha'' \) is orthogonal to \( \overline{\partial} \)-closed \( E \)-valued forms. In particular, \( (D'')^*\alpha'' = 0 \) and \( \partial \rho \perp \alpha''_{\partial D} = 0 \), cf. the remark after Proposition 1. Thus, if \( (D'')^*\alpha = 0 \) then \( (D'')^*\alpha' = D''\alpha' = 0 \) so (1) applies to \( \alpha' \) and we hence obtain (1) for \( \alpha \) as well. If \( \partial D \) is just pseudoconvex, we can still decompose \( \alpha = \alpha' + \alpha'' \) as before. Since then \( \alpha'' \in \text{Dom}(D'') \cap \text{Dom} D'' \) (in the densely defined operator sense) there are, by Proposition 2.1.1 in [9], \( \alpha'' \in \mathcal{E}_{n,q}(\overline{D}, E) \cap \text{Dom}(D'')^* \) such that \( \alpha'' \to \alpha'' \) in graph norm. In particular \( \partial \rho \perp \alpha''|_{\partial D} = 0 \). If \( \alpha_j' = \alpha - \alpha'' \), then \( \alpha_j' \to \alpha' \) in graph norm and \( \partial \rho \perp \alpha_j|_{\partial D} = \partial \rho \perp \alpha_j|_{\partial D} \). Since also \( (D'')^*\alpha_j' \to 0 \), \( D''\alpha_j' \to 0 \) and \( \int \langle f, \alpha_j' \rangle \to \int \langle f, \alpha \rangle \) one can proceed as before, but instead using the variant of (11) in which \( (D'')^*\alpha \) and \( D''\alpha \) are not supposed to vanish, cf. (10). By Proposition 2 we then have proved

**Theorem 2.** Let \( E \to X \) be a hermitian holomorphic vector bundle over the Kähler manifold \( X \), and let \( D = \{ \rho < 0 \} \) be a pseudoconvex relatively compact domain and \( \rho \) a \( C^2 \) plurisubharmonic defining function. Also suppose that \( A = i(-\rho)\Theta \Lambda + i\partial \overline{\partial} \rho \Lambda \) is semipositive on \( E \)-valued \((n, q + 1)\)-forms. If \( f \in \mathcal{E}_{n,q+1}(\overline{D}, E) \) is \( D'' \)-closed, then there is a solution to \( \overline{\partial} u = f \) in \( L^2_{n,q}(\partial D, E) \) such that
\[
\int_{\partial D} |\partial \rho \perp u|^2 \, dS/|\partial \rho|^3 \leq \int_D \langle A^{-1} f, f \rangle \, dV.
\]

We recall that a bundle \( E \) is called Nakano semipositive if \( \langle \Theta \Lambda \alpha, \alpha \rangle \geq 0 \) for all \( \alpha \in \mathcal{E}_{n,1}(X, E) \).

**Corollary.** Suppose \( E \to X \) is Nakano semipositive, \( i\partial \overline{\partial} \rho \geq \delta I \) in \( D \), and \( \psi \) is smooth and plurisubharmonic. Then if \( f \in \mathcal{E}_{n,1}(\overline{D}, E) \) is \( \overline{\partial} \)-closed, there is
a solution $u$ to $\overline{\partial}_b u = f$ such that

$$\int_{\partial D} |u|^2 e^{-\psi} \ dS/|d\rho| \leq \frac{1}{\delta} \int_D |f|^2 e^{-\psi} \ dV.$$  

In particular, one can let $E$ be the trivial bundle over a domain $D$ in $\mathbb{C}^n$ and thus get (2) for $(0,1)$-forms $f$.

5. The Division Problem

When proving Theorem 1, we assume that $q$ and $g$ are holomorphic in a neighborhood of $\overline{D}$. The general case then follows by a normal family argument on the solutions in $D_{e} = \{ \rho < -\varepsilon \}$, since it turns out that the occurring constants only depend on derivatives up to second order of $\rho$ near $\partial D$.

Remark. It is proved in [4] that any $C^2$ pseudoconvex domain admits a $C^2$-defining function $\rho$ such that $-(-\rho)^\eta$ is (strictly) plurisubharmonic in $D$ for some $\eta > 0$. Unfortunately, by our method the constants belonging to $D_{e} = \{ \rho < -\varepsilon \}$ seem to be unbounded when $\varepsilon \to 0$ so we cannot prove Theorem 1 in this general case. □

We thus have to consider the following situation. An exact sequence $0 \to S \to E \xrightarrow{\xi} Q \to 0$ of hermitian holomorphic vector bundles over $X$, such that $j$ and $g$ are holomorphic, $S$ is equipped with the metric induced from $E$, $|g| \leq 1$ and $\det g g^* \geq \delta^2 > 0$. The problem then is if for any holomorphic $q \in \mathcal{E}_n,0(\overline{D},Q)$ there is a holomorphic solution $u \in \mathcal{E}_n,0(D,S)$ to

$$gu = q$$  

such that

$$\int_{\partial D} |u|^2 \ dS \leq C^2 \int_{\partial D} |q|^2 \ dS.$$  

To find such holomorphic solutions, we proceed as follows. First we note that the pointwise minimal solution $\gamma q = g^*(g g^*)^{-1} q$ satisfies (2). Moreover, $\overline{\partial}(\gamma q) = (\overline{\partial}\gamma)q$ is a $\overline{\partial}$-closed $(n,1)$-form with values in $S$, since $g \overline{\partial} \gamma q = \overline{\partial}(g \gamma q) = \overline{\partial}q = 0$. The hard step then is to find a $v \in L^2_{n,0}(\partial D,S)$ satisfying (2) such that $\overline{\partial}_b v = (\overline{\partial}\gamma)q$ in $S$, i.e. such that

$$\int_{\partial D} \langle v, \partial \rho \perp \xi \rangle \ dS/|d\rho| = \int_D \langle (\overline{\partial}\gamma)q, \xi \rangle \ dV$$

for all $\xi \in \mathcal{E}_{n,1}(\overline{D},S)$ such that $(D^\delta)^\perp \xi = 0$.

We now claim that (3) implies that actually $\overline{\partial}_b v = (\overline{\partial}\gamma)q$ in $E$. Taking this for granted for the moment we conclude that $\overline{\partial}_b (\gamma q - v) = 0$ in (the trivial bundle) $E$ which means that $u = \gamma q - v$ satisfies the tangential Cauchy-Riemann equation weakly.

This implies that $u$ is the boundary values of a $U \in H^2(D)$ with norm

$$\|U\|^2_{H^2} = \int_{\partial D} |u|^2 \ dS = \int_{\partial D} |v|^2 \ dS + \int_{\partial D} |\gamma q|^2 \ dS$$

($v$ and $\gamma q$ being orthogonal) and since $gU = q$ on $\partial D$, it must hold in $D$.  

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Thus our problem is solvable if (and only if) we can obtain (3) such that the $L^2(\partial D)$-norm of $v$ is controlled by $(\int_{\partial D} |q|^2 dS)^{1/2}$. By Proposition 2 this amounts to verifying the inequality

$$
\left| \int_D (\bar{\partial} q, \xi) dV \right|^2 \leq C^2 \int_{\partial D} |q|^2 dS \int_{\partial D} |\bar{\partial} \rho \cdot \xi|^2 dS/|\partial \rho|^2
$$

for all $\xi \in \mathcal{E}_{n,1}(D, S)$ such that $(D'')^* \xi = 0$.

When trying to prove the estimate (4) one encounters two main difficulties. Firstly, although $E$ and $Q$ are trivial bundles with trivial metrics in our case, $S$ acquires negative curvature which must be taken care of. However, it turns out that the curvature on $S$ becomes nonnegative if the original metric is modified by a factor $e^{-\varphi}$, where $\varphi$ is a bounded plurisubharmonic function. Since $e^\varphi$ is bounded, it does not affect the estimates in any essential way. Secondly, even if we forget about the curvature problems, i.e. consider the scalar-valued case, an essential difficulty remains. As was mentioned in the introduction, one cannot use Theorem 2 directly since we must use more information about the right-hand side in our $\bar{\partial}_b$-equation than just a size estimate. Here the Wolff trick comes into play. Restricted to the scalar-valued case, our "Wolff theorem" can be stated

**Proposition 8.** Suppose $f \in \mathcal{E}_{n,1}(\bar{D})$ is $\bar{\partial}$-closed, and that there is a bounded plurisubharmonic $\varphi$ on $\bar{D}$ such that $||(f, \alpha)||^2 \leq (i\bar{\partial}\varphi \Lambda \alpha, \alpha)$ and

$$
\sum_{k=1}^n \left| \left\langle \frac{\partial f}{\partial z_k}, \alpha \right\rangle \right|^2 \leq \Delta \varphi (i\bar{\partial}\varphi \Lambda \alpha, \alpha)
$$

for all $(n, 1)$-forms $\alpha$. Then for any holomorphic $q$ there is a solution $v \in L^2_{n,0}(\partial D)$ to $\bar{\partial}_b v = \bar{\partial} q$ such that

$$
\int_{\partial D} |v|^2 dS \leq C^2 \int_{\partial D} |q|^2 dS,
$$

where $C$ only depends on $||\varphi||_\infty$ and $D$.

This proposition will be proved implicitly in the next paragraph. In the unit disc the assumption in the proposition is essentially, see [2], that $(1 - |\xi|^2)|f|^2$ and $(1 - |\xi|^2)|\partial f/\partial z|$ be Carleson measures, and the conclusion of the proposition implies, cf. the introduction, that there is a bounded solution.

**Proof of the claim above.** We actually have to verify that if $v \in L^2_{n,0}(\partial D, S)$ and (3) holds for all $\xi \in \mathcal{E}_{n,1}(\bar{D}, S)$ such that $(D'')^* \xi = 0$, then it also holds for all $\alpha \in \mathcal{E}_{n,1}(\bar{D}, E)$ such that $(D'')^*_E \alpha = 0$. However, if $p: E \to S$ is the orthogonal projection, then clearly (3) holds for $\xi = \alpha - p \alpha$. Moreover, if $(D'')^*_E \alpha = 0$, then for any compactly supported $\eta \in \mathcal{E}_{n,1}(\bar{D}, S)$,

$$
0 = \int_D (\langle (D'')^*_E \alpha, \eta \rangle) = \int_D (\langle \alpha, \bar{\partial} \eta \rangle) = \int_D (\langle p \alpha, \bar{\partial} \eta \rangle) = \int_D (\langle (D'')^*_E p \alpha, \eta \rangle),
$$

so that $(D'')^*_E p \alpha = 0$. Hence (3) holds for $\alpha = p \alpha + (\alpha - p \alpha)$.
6. Proofs of Theorems 1 and 1′

Let $\beta$ be the element in $\mathcal{E}_{1,0}(\overline{D}, \text{Hom}(S, Q))$ such that its adjoint, with respect to the quotient metric on $Q$, $\beta^* \in \mathcal{E}_{0,1}(\overline{D}, \text{Hom}(Q, S))$ equals $-\overline{\partial} \gamma$. Thus our equation to be solved ((3) in §5) becomes $\overline{\partial}_b v = -\beta^* q$. Moreover, since $E$ has no curvature,

$$\Theta_S = \beta^* \wedge \beta$$

and since $Q$ has no curvature,

$$-i(\beta^* \wedge \beta \Lambda \xi, \xi) \leq r(i \partial \overline{\partial} \psi \Lambda \xi, \xi), \quad \xi \in \mathcal{E}_{n,1}(\overline{D}, S),$$

where $r = \min(n, \text{rank} S)$ and $\psi = \log \text{det} g^*$ (note that $g^*$ depends on the metric on $Q$). We also need the estimate

$$|\langle \beta^* q, \xi \rangle|^2 \leq -|g^*(g g^*)^{-1} q||^2 i(\beta^* \wedge \beta \Lambda \xi, \xi)$$

for $q \in \mathcal{E}_{n,0}(\overline{D}, Q)$ and $\xi \in \mathcal{E}_{n,1}(\overline{D}, S)$. For proofs of (1), (2) and (3) we refer to [12].

Recall (Proposition 6 in §3) that

$$\int_D (-\rho)(i(\Theta_S + \partial \overline{\partial} \phi) \Lambda \alpha, \alpha) e^{-\psi} dV + \int_D (i \partial \overline{\partial} \rho \Lambda \alpha, \alpha) e^{-\psi} dV$$

$$+ \int_D (-\rho)(D')^* \alpha|^2 e^{-\psi} dV = \int_{\partial D} |\partial \rho \perp \alpha|^2 e^{-\psi} dS/|d\rho|$$

for $\alpha \in \mathcal{E}_{n,1}(\overline{D}, S)$ if $\overline{\partial} \alpha = (D'')^* \alpha = 0$.

Hence if $\phi = (r + \varepsilon) \psi$ and $\rho$ is plurisubharmonic, we get by (1) and (2)

$$\int_D (-\rho)(D')^* \alpha|^2 e^{-\psi} dV \leq \int_{\partial D} |\partial \rho \perp \alpha|^2 e^{-\psi} dS/|d\rho|$$

and

$$\int_D (-\rho)(i \partial \overline{\partial} \phi \Lambda \alpha, \alpha) e^{-\psi} dV \leq \frac{r + \varepsilon}{r} \int_{\partial D} |\partial \rho \perp \alpha|^2 e^{-\psi} dS/|d\rho|.$$}

From (3) we also get

$$|\langle \beta^* q, \xi \rangle|^2 \leq |g^*(g g^*)^{-1} q||^2 \frac{r}{r + \varepsilon} (i \partial \overline{\partial} \phi \Lambda \xi, \xi),$$

for $\xi \in \mathcal{E}_{n,1}(\overline{D}, S)$ and $q \in \mathcal{E}_{n,0}(\overline{D}, Q)$. We now claim that Theorem 1 follows from

**Proposition 9.** If $D$ and $\rho$ are as in Theorem 1, then

$$\left| \int_D (\beta^* q, \xi) dV \right|^2 \leq C_\delta^2 \int_{\partial D} |q|^2 dS \int_{\partial D} |\partial \rho \perp \xi|^2 e^\phi dS$$

for any holomorphic $q \in \mathcal{E}_{n,0}(\overline{D}, Q)$ and $\xi \in \mathcal{E}_{n,1}(\overline{D}, S)$ such that $(D'')^*(e^\phi \xi) = \overline{\partial}(e^\phi \xi) = 0$.

Here $C_\delta$ is the constant described in Theorem 1.

**Proof of Theorem 1.** By the discussion in §5 it is enough to verify (4) in §5 for $\xi \in \mathcal{E}_{n,1}(\overline{D}, S)$ such that $(D'')^* \xi = 0$. Note that

$$(D'')^* \xi = 0 \text{ iff } (D'')^*(e^\phi \xi) = 0.$$  

Putting $\alpha = e^\phi \xi$, (7) then says that
for all $\alpha$ such that $\overline{\partial}\alpha = (D'')^*\overline{\alpha} = 0$. As in §4, we can obtain (8) for all $\alpha$ with $(D'')^*\overline{\alpha} = 0$. But this means that (7) holds for all $\xi$ with $(D'')^*\xi = 0$. Finally $e^\varphi = (\det g g^*)^{-1} \leq 1$ by assumption, and hence we have verified (4) in §5.

**Proof of Proposition 9.** We consider $g$ as a $j \times k$-matrix of holomorphic functions on $\overline{D}$ and use the norms

$$
\|g\|^2 = \sum_{tu} |g_{tu}|^2, \quad |g'|^2 = \sum_{itv} |\partial g_{tu}/\partial \zeta_i|^2.
$$

The assumptions on $g$ imply that

$$
|g (g^*)^{-1}| \leq 1/\delta^2, \quad |g^* (g g^*)^{-1} q| \leq (1/\delta^2)|q|^2.
$$

If $\Delta$ is the $\mathbb{R}^{2n}$-Laplacian, then

$$
|g'|^2/|g|^2 - 2\varepsilon \leq C \varepsilon |g|^{2\varepsilon}, \quad \varepsilon > 0.
$$

We also need the inequalities

$$
\int_D (-\rho)|f'|^2 \, dV \leq \int_{\partial D} |f|^2 \, dS
$$

and

$$
\int_D (-\rho)|f|^2 \Delta \psi e^\psi \, dV \leq \int_{\partial D} |f|^2 e^\psi \, dS
$$

for holomorphic $f$ and subharmonic $\psi$. They follow from Green's formula. Recall that $-\overline{\beta*} = \overline{\partial} \gamma = \overline{\partial}[g (g g^*)^{-1}]$ so that, for an $S$-valued $\xi$,

$$
-\langle \beta^* q, \xi \rangle = \langle (\partial g)^* (g g^*)^{-1} q, \xi \rangle.
$$

We have to estimate

$$
I = \int_D (\beta^* q, \xi) \, d\lambda + (D'')^* (e^\varphi \xi) = \overline{\partial}(e^\varphi \xi) = 0.
$$

Let $\chi$ be a smooth function which is identically 1 near $\partial D$ and such that $\partial \rho$ is nonvanishing on its support. If $L = \chi|\partial \rho|^{-2} \sum_j (\partial \rho/\partial \overline{\zeta}_j) \partial/\partial \zeta_j$, then we can write

$$
I = \int_D (1 - \chi) (\beta^* q, \xi) \, dV - \int_D L (-\rho) (\beta^* q, \xi) \, dV,
$$

and an integration by parts in the second integral gives us

$$
I = \int_D (1 - \chi) (\beta^* q, \xi) \, dV + \int_D (-\rho) O(1) (\beta^* q, \xi) \, dV
$$

$$
+ \int_D (-\rho) ((L \beta^*) q, \xi) \, dV + \int_D (-\rho) (\beta^* L q, \xi) \, dV
$$

$$
+ \int_D (-\rho) (\beta^* q, (L \varphi) \xi + \overline{L \xi}) \, dV - \int_D (-\rho) (\beta^* q, \overline{L \varphi} \xi) \, dV
$$

$$
= I_0 + I_1 + I_2 + I_3 + I_4 + I_5.
$$
where $O(1)$ only depends on derivatives up to second order of $\rho$ and $\chi$, and where $\phi = (r + \varepsilon) \log \det gg^*$. The proof is concluded by estimating each $I_i$. By (6) and Schwarz inequality we have that, for $i = 0, 1$,

$$|I_i|^2 \lesssim \int_D (-\rho)|g^*(gg^*)^{-1}q|^2 e^{-\rho} dV \int_D (-\rho)(i\partial \bar{\partial} \phi \Lambda \xi, \xi)e^\rho dV.$$  

Now, cf. (9),

$$\int_D (-\rho)|g^*(gg^*)^{-1}q|^2 e^{-\rho} dV \lesssim \frac{1}{\delta^2 \delta^2 (r + \varepsilon)} \int_D (-\rho)|q|^2 dV$$

$$\lesssim \left( \frac{1}{\delta^1 + \rho + \varepsilon} \right)^2 \int_D |q|^2 dS$$

since $q$ is holomorphic (cf. (12) with e.g. $\psi = |\xi|^2$). Also

$$\int_D (-\rho)(i\partial \bar{\partial} \phi \Lambda \xi, \xi)e^\rho dV = \int_D (-\rho)(i\partial \bar{\partial} \phi \Lambda (e^\rho \xi), e^\rho \xi)e^{-\rho} dV$$

and hence by (5), $\lesssim \int_{\partial D} |\partial \rho - 1| |\xi|^2 e^\rho dS$. Thus we have obtained the required estimate for $I_0$ and $I_1$. To handle $I_2$, first note that

$$- \langle (L\beta^*) q, \xi \rangle = \langle (\partial g)^*(gg^*)^{-1}(Lg)g^*(gg^*)^{-1}q, \xi \rangle$$

$$= \langle \beta^*(Lg)g^*(gg^*)^{-1}q, \xi \rangle$$

so by (6) and (9),

$$|\langle (L\beta^*) q, \xi \rangle|^2 \leq |g^*(gg^*)^{-1}(Lg)g^*(gg^*)^{-1}q|^2 e^{-\rho} \langle i\partial \bar{\partial} \phi \Lambda \xi, \xi \rangle e^\rho$$

$$\lesssim \frac{1}{\delta^2 \delta^2 (r + \varepsilon) \delta^2 \delta^2 (\det gg^*)^{-1}} |q|^2 \langle i\partial \bar{\partial} \phi \Lambda \xi, \xi \rangle e^\rho.$$  

If $g$ is a row matrix, i.e. $j = 1$, then $\det gg^* = |g|^2$ so we can use (10) and get

$$|I_2|^2 \lesssim \left( \frac{1}{\delta^1 + \rho + \varepsilon} \right)^2 \int_D (-\rho)|q|^2 \Delta |g|^2 e^{i\rho} dV \int_D (-\rho)(i\partial \bar{\partial} \phi \Lambda \xi, \xi)e^\rho$$

$$\lesssim \left( \frac{1}{\delta^1 + \rho + \varepsilon} \right)^2 \int_{\partial D} |q|^2 dS \int_{\partial D} |\partial \rho - 1| |\xi|^2 e^\rho dS.$$  

If $j > 1$ we estimate $1/\det gg^*$ by $1/\delta^2$ and use the simpler inequality $|g'|^2 \leq \Delta |g|^2$. For simplicity we assume for the rest of the proof that $j = 1$. To handle $I_3$, we note that

$$|\langle \beta^* Lq, \xi \rangle|^2 \lesssim \left( \frac{1}{\delta^1 + \rho + \varepsilon} \right)^2 |q|^2 \langle i\partial \bar{\partial} \phi \Lambda \xi, \xi \rangle e^\rho$$

and here the first factor is treated by (11) and the second one as before. Further, we have

$$|I_4|^2 \lesssim \int_D (-\rho)|\beta^* q|^2 e^{-\rho} dV \int_D (-\rho)|(\bar{L}\phi)\xi + \bar{L}\xi|^2 e^\rho dV$$

$$\lesssim \left( \frac{1}{\delta^1 + \rho + \varepsilon} \right)^2 \int_D (-\rho)\frac{|g|^2}{|g|^2 - 2\varepsilon} |q|^2 dV \int_D (-\rho)\bar{L}(e^\rho \xi)^2 e^{-\rho} dV.$$
The first factor is handled as before and the second one is estimated by (4). Finally,

$$|I_3|^2 \lesssim \left( \frac{1}{\delta^{1+r+e}} \right)^2 \int_D (\rho |L\rho|^2) q^2 dV \int_D (\rho) (i\partial\bar{\partial} \rho \Lambda \xi, \xi) e^\rho dV.$$  

Note that $|\bar{L}\rho|^2 \leq |g|^2/|\rho|^2$ so that the first factor is dominated by

$$\frac{1}{\delta^{2e}} \int_D (\rho) (\Lambda |g|^2 e^\rho)^2 dV \lesssim \frac{1}{\delta^{2e}} \int_{\partial D} |q|^2 dS$$

and hence the proposition is proved. \(\square\)

**Proof of Theorem 1'.** It is enough to show that

$$\left| \int_D (\beta^* q, \xi) dV \right|^2 \leq C_\delta \int_{\partial D} e^{-\chi} dS \int_{\partial D} |\partial \rho \perp \xi|^2 e^{\rho+\chi}$$

for all $\xi$ such that $(D'')^* \psi (e^\psi \xi) = \tilde{\partial} (e^\psi \xi) = 0$, where we have put $\psi = \varphi + \chi = (r + e') \log |g|^2 + \chi$ and $e'$ is less than the $\epsilon$ in the hypothesis of Theorem 1'.

Then most arguments when estimating the left-hand side work as before. We have just two new difficulties. For the term $I_3$ we have

$$|I_3|^2 \lesssim \int_D (\rho) |Lq|^2 |g|^{2(1+r+e')} e^{-\chi} \int_D (\rho) (i\partial\bar{\partial} \rho \Lambda \xi, \xi) e^\rho.$$  

Since $|g|^{2(1+r+e')} \geq |q|^2$, $|g|^{2(1+r+e')} \geq |q|^{2-e''}$ and hence the first factor is

$$\lesssim \int_D (\rho) \Delta |q|^{e''} e^{-\chi} \leq \int_D (\rho) \Delta (|q|^{2''} - \chi) e^{|q|^{2''-x}}$$

$$\lesssim \int_{\partial D} e^{q|^{2''-x}} dS \leq \int_{\partial D} e^{-x} dS.$$  

When estimating $I_5$ we show up with a factor

$$\int_D (\rho) |g|^2 |\nabla \psi|^2 e^{-\psi} \leq \int_D (\rho) |g|^{e''} |\nabla \psi|^2 e^{-\chi}$$

$$\lesssim \int_D (\rho) |g|^{e''} |\nabla (\log |g|^2)|^2 e^{-\chi} + \int_D (\rho) |\nabla \chi|^2 e^{-\chi} = a_1 + a_2.$$  

But

$$a_1 \lesssim \int_D (\rho) \Delta (|g|^{e''} - \chi) e^{|g|^{e''-x}} \lesssim \int_{\partial D} e^{-x}$$

and so is $a_2$. This concludes the proof of Theorem 1'. \(\square\)

**References**


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