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A PROOF OF C^1 STABILITY CONJECTURE FOR THREE-DIMENSIONAL FLOWS

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Dedicated to Professor Liao Shan-tao on his 70th birthday

ABSTRACT. We give a proof of the C^1 stability conjecture for three-dimensional flows, i.e., prove that there exists a hyperbolic structure over the Ω set for the structurally stable three-dimensional flows. Mañé's proof for the discrete case motivates our proof and we find his perturbation techniques crucial. In proving this conjecture we have overcome several new difficulties, e.g., the change of period after perturbation, the ergodic closing lemma for flows, the existence of dominated splitting over $\Omega\backslash\mathscr{P}$ where $\mathscr{P}$ is the set of singularities for the flow, the discontinuity of the contracting rate function on singularities, etc. Based on these we finally succeed in separating the singularities from the other periodic orbits for the structurally stable systems, i.e., we create unstable saddle connections if there are accumulations of periodic orbits on the singularities.

1. INTRODUCTION

The C^1 Stability Conjecture was stated in [18] by Palis and Smale. The aim is to characterize the structurally stable systems.

Let M^n be an n-dimensional compact Riemannian manifold. Let $\aleph(M^n)$ denote all C^1 vector fields on M^n. Any $S \in \aleph(M^n)$ generates a flow $\phi_t\colon M^n \times (-\infty, \infty) \to M^n$. We say such a flow is structurally stable if the C^1 small perturbations of S generate a flow with equivalent orbital structure. More precisely, there exists a C^1 neighborhood $\mathscr{N}$ of S in $\aleph(M^n)$ such that for any $X \in \aleph(M^n)$, we have a homeomorphism $h\colon M^n \to M^n$ which maps orbits of S bijectively to orbits of X.

In [18] Palis and Smale conjectured that a vector field S is structurally stable if and only if it satisfies Axiom A and the Strong Transversality Conditions. The 'if' part was proved by Robbin [21] and Robinson [22]. The other part was reduced to proving that C^1 structural stable implies Axiom A. After long efforts by many mathematicians, Mañé [14] solved this problem for the discrete case, i.e., for the diffeomorphisms of M^n. After this, it becomes realistic to attack the problem for the flow case. In this paper we will give a proof for three dimensional flows.

Main Theorem. *Let $S \in \aleph(M^3)$, if S is structurally stable, then S satisfies Axiom* A.

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Before continuing on, we give some definitions about Axiom A.

Let S be a C^1 vector field, ϕ_t be the flow generated by S. Given a closed invariant set Λ, we say Λ has hyperbolic structure, if there exists an invariant continuous splitting of $TM|\Lambda = E^s + S(x) + F^u$ and there exist constants $C, \lambda > 0$ such that

$$\|d\phi_t|E^s(x)\| \leq C\exp(-\lambda t), \quad \|d\phi_{-t}|F^u(x)\| \leq C\exp(-\lambda t)$$

for all $t > 0$ and $x \in \Lambda$.

We define the Ω set for S on which one has asymptotically the essential dynamics. This set is defined as the set of points $x \in M$ such that for every neighborhood U of x there exist $y \in U$ and a neighborhood V of y, with $V \subset U$, and $|t| \geq 1$ satisfying $\phi_t(V) \cap V \neq \varnothing$.

Then Axiom A states:

1. The periodic orbits of S are dense in Ω.
2. There exists a hyperbolic structure on the Ω set of S.

The first condition is necessary for a structurally stable system due to the C^1 closing lemma which was proved by Pugh [20] (also see [7]).

The second condition is generally known as the Stability Conjecture. Mañé proved this for the discrete case [14]. In the proof, he developed two powerful perturbation techniques. It is realistic to use these for the flow case.

If we work on the three-dimensional case, the situation becomes simple and the difficulty becomes clear. The difficulties are ergodic closing lemma for flows (§3); change of period under perturbation (Scaling Lemma in §3); existence and uniqueness of the dominated splitting over $\Omega\backslash\mathscr{P}$ (Theorem 4.1); discontinuity of the functions of the contracting and expanding rate at singularities (Lemma 2 in §6); separation of the singularities from the periodic (Theorem 6.1).

In this paper we overcome all of these difficulties and complete the proof for the three-dimensional case.

In §2 the idea is to study the weakest systems, i.e., $\aleph^*(M^3)$ for which the periodic orbits and the singularities stay hyperbolic under C^1 small perturbations. It is known that for such systems there are uniform bounds of the characteristic exponents for the union of the periodic orbits and there is a uniform bound on the combination of contracting rate and expanding rate. We call this *weak uniformity*. Based on the fact and work on differential equations, Liao [9] and Pliss [19] showed that there are only finitely many contracting and expanding periodic orbits for the systems in $\aleph^*(M^3)$. Now in the three-dimensional case there can only be infinitely many periodic orbits of saddle type. The closure of the periodic orbits is often highly irregular and it is to be called *chaos*. We must establish a hyperbolic structure over the closure of periodic orbits for the structurally stable systems.

Section 3 shows that ergodic closing lemma for flows and separation implies Axiom A. Liao [8] showed that if the singularities are bounded away from the periodic orbits then the system must be hyperbolic. From this we see that we only need separate the singularities from the other periodic orbits for the three-dimensional structurally stable systems. However, Liao's proof of that theorem involves his obstruction set technique. In §3 we give a direct proof based on the *ergodic closing lemma*. The proof itself will be used in the final proof of the main theorem. Also in §3, we deal with the change of period after perturbations through the *Scaling Lemma*.

In §4 the construction of dominated splitting on $\Omega\backslash\mathscr{P}$ for $\aleph^*(M^3)$ turns out to be one of the main technical difficulties in carrying out the proof of the main theorem. The dominated splitting is weaker than the hyperbolic splitting but still a good candidate. For the definition, see §4. To apply Mañé's perturbation techniques, we need to establish the dominated splitting first. Fortunately, I found that can be done by using Doering's argument [1]. Originally (10 years ago) he did it for the other type of systems, however it is straightforward to do that for $\aleph^*(M^3)$. We will construct the dominated splitting in great detail in §4. In the final proof we shall use the result of Doering about the limiting behavior of dominated splitting around the singularities.

Section 5 discusses the creation of homoclinic orbit and attainability of contracting sequences. Mañé's remarkable proof of the stability conjecture relies on his two perturbation techniques (Theorems 5.1 and 5.2). This section is devoted to explaining these methods. The problem is if we have some loose connections between the hyperbolic set, can we create some saddle connections between them by arbitrarily C^1 small perturbation? Under the condition that the accumulations are tense on the hyperbolic set (positive measure on it), Mañé solved it with perturbation technique 1. When this condition fails, by the property of the dominated splitting, we have the so called contracting sequences for which the perturbations are amplified on some directions after iteration. By using this property, Mañé created some other real connections by small perturbations. The novelty for the proof of the stability conjecture is that those two cover all cases.

Finally, in §6 we apply Mañé's techniques to prove the main theorem (Theorem 6.1), i.e., to create an unstable saddle connection of the singularities by arbitrarily small perturbation if the periodic orbits accumulate on the singularities. After all the preparations above it is quite routine to do that except for two difficulties. The first is the discontinuity of the contract rate function ($\eta_-(t, x)$, see §2) at the singularities. Fortunately, Mañé's pertubation technique 1 can be used to take care of that. The appearance of singularities forced us to check the dimension and transversality carefully to get an unstable saddle connection.

The proof given here has the advantage that it may very likely be generalized for the higher dimensional case. Certainly there are some new difficulties. For example, contracting implies expanding for the dominated splitting, the existence and uniqueness of the dominated splitting over $\Omega\backslash\mathscr{P}$, etc. We hope these will be overcome in the future. Another problem is for the Ω-Stability Conjecture. It seems there is not much difference between these two. But there is a difference between diffeomorphisms and flows. There are not enough dimensions of stable and unstable manifolds for flow to form a cycle so one has to look for the unstable saddle connections.

2. Weak uniformity and finiteness for $\aleph^*(M^n)$

We start with some definitions. Let $\aleph(M^n)$ be the set of all C^1 vector fields X on M^n for a given $n \geq 2$, endowed with C^1 norm $\|X\|_1$. The topology is induced from the norm.

Let $\aleph^*(M^n)$ be the vector fields such that $X \in \aleph(M^n)$ if and only if there exists a C^1 neighborhood $\mathscr{N}$ in $\aleph(M^n)$, s.t. if $Y \in \mathscr{N}$, for the flow generated by Y, all the singularities and periodic orbits are hyperbolic. Or equivalently,

each Y in $\mathscr{U}$ has only a finite number of singularities and at most a countable number of periodic orbits.

Let $S \in \aleph(M^n)$. S induces a C^1 one-parameter group

$$\phi_t \colon M^n \to M^n (-\infty < t < \infty),$$

and induces a one-parameter group on the tangent bundle $\mathscr{C}$ of M^n,

$$\Phi_t = d\phi_t \colon \mathscr{C} \to \mathscr{C} (-\infty < t < \infty).$$

Denote respectively by $\mathscr{P}$ and Λ the set of all singularities of S and the set of all points on periodic orbits of S. Write $M = M^n \backslash \mathscr{P}$.

Consider the normal bundle $N = \bigcup_{x \in M} N_x$ of S, which is the bundle with base space M and with fiber N_x over $x \in M$ consisting of all tangent vectors at x orthogonal to $S(x)$. For any $(t, u) \in (-\infty, \infty) \times N_x$, take $\Psi_t(u)$ as the orthogonal projection of $\Phi_t(u)$ on $N_{\phi_t(x)}$. This gives a one parameter group $\Psi_t \colon N \to N(-\infty < t < \infty)$.

Note that Ψ_t maps N_x linearly onto $N_{\phi_t(x)}$. Extend Ψ_t linearly to be a flow from TM to itself so that $\Psi_t(S(x)) = S(\phi_t(x))$.

For any given $x \in \Lambda$ write

$$N_-(x) = \left\{ u \in N_x \middle| \lim_{t \to \infty} \|\Psi_t(u)\| = 0 \right\},$$

and

$$N_+(x) = \left\{ u \in N_x \middle| \lim_{t \to -\infty} \|\Psi_t(u)\| = 0 \right\}.$$

These are linear subspaces of N_x, and

$$\Psi_t(N_-(x)) = N_-(\phi_t(x)), \qquad \Psi_t(N_+(x)) = N_+(\phi_t(x)).$$

For any given $x \in \Lambda$ and $0 \le t < \infty$, write

$$\eta_-(t, x) = \begin{cases} \sup_{u \in N_-(x), \|u\|=1} \{\log \|\Psi_t(u)\|\}, & \text{if } \dim N_-(x) \ge 1, \\ -\infty \quad \text{if } \dim N_-(x) = 0, \end{cases}$$

$$\eta_+(t, x) = \begin{cases} \inf_{u \in N_+(x), \|u\|=1} \{\log \|\Psi_t(u)\|\}, & \text{if } \dim N_+(x) \ge 1, \\ +\infty \quad \text{if } \dim N_+(x) = 0. \end{cases}$$

Geometrically $\eta_-(t, x)$ is the contracting rate in the contract directions after time t.

Clearly, $N_x = N_-(x) \oplus N_+(x)$ for $S \in \aleph^*(M^n)$, is a good candidate for the hyperbolic splitting. Using the notations above, hyperbolicity over an invariant closed set Λ would simply mean there exist two constants $\overline{\eta} > 0$, $\overline{T} > 0$, such that

$$\eta_-(t, x) \le -\overline{\eta} t, \quad \text{and} \quad \eta_+(t, x) \ge \overline{\eta} t,$$

for all $x \in \Lambda$ and $t \ge \overline{T}$.

It is known by Franks [3] that structurally stable systems and Ω-stable systems are necessarily in $\aleph^*(M^n)$. We can see that the systems in $\aleph^*(M^n)$ are really the weakest in the sense of global stability from its definition. However, it turns out that in the discrete case such systems are very close to Axiom A systems. Palis [16] has shown that Axiom A systems are dense in $\aleph^*(M^n)$ in the discrete case. It is conjectured that these two are the same and this conjecture has been recently confirmed by Aoki [1].

However, we cannot prove the uniform hyperbolicity for the system in $\aleph^*(M^n)$ for the flow case since there are counterexamples such as the geometric Lorenz attractor [4]. It seems the problem is caused by the saddle connections of the singularities. We may pose a conjecture here.

Conjecture. In $\aleph(M^n)$, Axiom A systems are dense in the interior of Kupka-Smale systems, i.e., dense in the interior of systems with not only hyperbolic periodic orbits, but also with transverse stable and unstable manifolds.

For $\aleph^*(M^n)$ we can prove some kind of weak uniformness as in the following theorem. The next result can be found in [6].

Theorem 2.1. *There exists an open covering $\mathscr{B}$ of $\aleph^*(M^n)$, and corresponding to each V in $\mathscr{B}$, there exist numbers $\eta_V > 0$ and $T_V > 0$ such that, if $S \in V$, then*

1. *Whenever x is a point on a periodic orbit of S and $T_V \leq T < \infty$, we have*
$$\eta_+(T, x) - \eta_-(T, x) \leq -\eta_V.$$
2. *Whenever $\mathscr{O}$ is a periodic orbit of S with period T_0, $x \in \mathscr{O}$, and $0 = t_0 < t_1 < \cdots < t_l = T_0$ is a division of $[0, T_0]$ satisfying $t_k - t_{k-1} \geq T_V$, $k = 1, 2, \ldots, l$, we have*
$$\frac{1}{T_0}\sum_{k=1}^{l} \eta_-(t_k - t_{k-1}, \phi_{t_{k-1}}(x)) \leq -\eta_V$$
and
$$\frac{1}{T_0}\sum_{k=1}^{l} \eta_+(t_k - t_{k-1}, \phi_{t_{k-1}}(x)) \geq \eta_V.$$

Remark. For the three-dimensional case, property 2 shows nothing more than that the characteristic exponents of all the periodic orbits are uniformly bounded away from zero.

This fundamental theorem serves for several purposes.

- To construct a dominated splitting, i.e., to extend the bundle splitting over the union of periodic orbits to its closure. We will do this in detail in §4.
- To show the binding is tense, i.e., in the three-dimensional case, if the periodic orbits accumulate on the singularities, the periodic orbits have to spend considerable amounts of time around the singularities. For the precise statement, see Lemma 2 in §6.
- Finiteness of contracting and expanding periodic orbits for the system in $\aleph^*(M^n)$. We state that as a theorem.

Theorem 2.2 (Finiteness) (see [9]). *If $S \in \aleph^*(M^n)$, then for S there are only finitely many attracting and expanding periodic orbits.*

Remark. Liao proved the above theorem by using Theorem 1.1. and his work on standard differential equations which copies the vector field on a tubular neighborhood of an orbit in a canonical way. Pliss [19] proved it independently.

3. Ergodic closing lemma and separation implies Axiom A

By combining Theorem 2.1 with the ergodic closing lemma for flows, we can prove another important theorem, i.e., the separation of the periodic orbits of different indices implies Axiom A.

Definition ((C^1, ε) Closing). Let $L_0 = \{\phi_t(x_0) | 0 \le t \le T_0\}$ be an arc of S starting at x_0. For any $\varepsilon > 0$, we say the arc L_0 can be (C^1, ε) closed to a periodic orbit, if there exists $X \in \aleph(M^n)$ for which there is a periodic orbit L, and a map $f: L_0 \to L$ satisfying

(i) $\|X - S\|_1 < \varepsilon$,
(ii) $f(L_0) = L$,
(iii) $f(x_0) = f(\phi_{T_0}(x_0))$, and
(iv) $\operatorname{dist}(x, f(x)) < \varepsilon$ for all $x \in L_0$.

Theorem 3.1 ((C^1, ε) Closing Lemma) (see [7]). *Let $a \in \Omega_S$. Then for any $\varepsilon > 0$, there exists a $\delta > 0$, s.t. for any arc of S, say $\overline{L} = \{\phi_t(\overline{x}) | t \in [0, \overline{t}]\}$, $(1 \le \overline{t} \le \infty)$, if $\operatorname{dist}(a, \overline{x}) < \delta$, and $\operatorname{dist}(a, \phi_{\overline{t}}(\overline{x})) < \delta$, then there exists a subarc L_0 of $\overline{L}$, i.e., $L_0 = \{\phi_t(\overline{x}) | t \in [t_0, t_0 + T_0]\}$ $(0 \le t_0 < t_0 + T_0 \le \overline{T})$. This L_0 can be (C^1, ε) closed to a periodic orbit satisfying $\operatorname{dist}(a, \phi_{t_0}(\overline{x})) < \varepsilon$ and $\operatorname{dist}(a, \phi_{t_0+T_0}(\overline{x})) < \varepsilon$.*

Theorem 3.2 (Separation implies Axiom A). *Let $S \in \aleph^*(M^n)$, and the periodic orbits of S are dense in Ω_S then S satisfies Axiom A if and only if there exist mutually disjoint open neighborhoods $W_0, W_1, \dots, W_n$ of $\mathscr{P}$, $\Lambda_1, \dots, \Lambda_n$, respectively, where $\mathscr{P}$ is the set of fixed points and Λ_i is the set of the union of the periodic orbits of $\dim i$ of the stable manifold, and a neighborhood $\mathscr{N}$ of S in $\aleph^*(M^n)$, such that if $X \in \mathscr{N}$, then $\mathscr{P}(X) \subset W_0$, $\Lambda_i(X) \subset W_i$, for $1 \le i \le n$.*

Corollary (Singularity bounded away implies Axiom A in three dimensions). *Let $S \in \aleph^*(M^3)$, and the periodic orbits of S are dense in Ω_S, then if the periodic orbits of S are bounded away from the singularities then S satisfies Axiom A.*

Proof. From Theorem 2.2 we know that there are only finitely many attracting and expanding periodic orbits. So for the three-dimensional flow case, there can only be infinitely many periodic orbits of saddle type. It is easy to see that the periodic orbits of different indices (the dimension of the stable manifold) are separated. From the hypothesis that the periodic orbits are bounded away from the singularities, we see that all the critical elements, i.e., singularities and periodic orbits, are separated if the indices are different. The hyperbolicity follows from Theorem 3.2.

Remark. Using his obstruction set technique, Liao proved the corollary without the density hypothesis. The problem that if $S \in \aleph^*(M^n)$, whether the periodic orbits in Ω_S are dense or not, has been solved by Hong-yu Ding by constructing a counterexample.

For the proof of Theorem 3.2, Liao used his obstruction set technique [8]. Here we will give a different proof for the three-dimensional case based on weak uniformity and the ergodic closing lemma. The proof itself will be used in our final proof. The proof is based on two lemmas. We only need to prove one part, i.e., separation implies Axiom A.

Note that by Theorem 1.1 and the hypothesis that the critical elements are separated, it is not difficult to have a dominated splitting on Ω, i.e., $TM^n|\Omega = E \oplus S \oplus F$. For the definition, see next section.

Let T_S, η_S be given as in Theorem 2.1. Choose T, $\gamma > 0$, such that $T = T_S$, $0 < \gamma < \eta_S$.

Lemma 1. *Suppose that the periodic orbits are bounded away from the singularities. Let Λ be a compact invariant set such that, $\Omega(\phi_t|\Lambda) = \Lambda$. If we have*

$$\int \eta_-(T, x)\, d\mu(x) \le -\gamma, \qquad \int \eta_+(T, x)\, d\mu(x) \ge \gamma,$$

for any probabilistic invariant measure μ supported on Λ, then Λ must be hyperbolic.

Proof. Let x be any point in Λ. Consider the uniform measure μ_n supported on the arcs $\langle x, \phi_{nT}(x)\rangle$ for $n > 0$. The set of all these measures is a compact set. So we can choose a subsequence $\mu_{n_i} \to \mu$ in the weak topology. Choose one of its limit points μ, and note it is a probabilistic invariant measure on Λ. Now we know that

$$\int \eta_-(T, y)\, d\mu(y) \le -\gamma.$$

In the three-dimensional case, $\eta_-(T, x)$ can be written as

$$\eta_-(T, x) = \int_0^T \omega(\phi_s(x))\, ds,$$

where the function ω on the manifold is defined as

$$\omega(x) = \frac{d}{dt}\|\Psi_t(u)\|\,|_{t=0,\ u\in N_-(x),\ \|u\|=1},$$

ω is continuous.

Let μ_i be the uniform measure supported on $\langle x, \phi_{n_iT}(x)\rangle$, for $0 < \gamma' < \gamma$ for all large n_i, then

$$\int \eta_-(T, y)\, d\mu_{n_i}(y) \le -\gamma'.$$

Now,

$$\begin{aligned}\int \eta_-(T, y)\, d\mu_{n_i}(y) &= \frac{1}{n_iT}\int_0^{n_iT} \eta_-(T, \phi_t(x))\, dt \\ &= \frac{1}{n_iT}\int_0^{n_iT}\int_0^T \omega(\phi_{s+t}(x))\, ds\, dt \\ &= \int_0^T \frac{1}{n_iT}\int_0^{n_iT} \omega(\phi_{s+t}(x))\, dt\, ds < -\gamma',\end{aligned}$$

so for some number $0 \le s_0 \le t$, we have

$$\frac{1}{n_iT}\int_0^{n_iT} \omega(\phi_{s_0+t}(x))\, dt \le -\frac{\gamma'}{T}.$$

The left side is

$$\frac{1}{n_iT}\int_0^{n_iT} \omega(\phi_{s_0+t}(x))\, dt = \frac{1}{n_iT}\int_{s_0}^{s_0+n_iT} \omega(\phi_t(x))\, dt.$$

Because s_0 is bounded, this is very close to

$$\frac{1}{n_i T}\int_0^{n_i T} \omega(\phi_t(x))\,dt,$$

if n_i is large.

In conclusion, for any x, we can find a $t > 0$, such that $\eta_-(t, x) \le -t\gamma''$, for some number $0 < \gamma'' < \gamma'$.

For different x, the t may be different. By the continuity property of η_-, for a small neighborhood of x, we can choose the same t.

Since the set Ω is compact, and $\eta_-(t, x) \to 0$ as $t \to 0$, we see that there exist $0 < T'' < T'$ and $\lambda > 0$, such that, for any $x \in \Lambda$, there exists a t_x, $0 < T'' < t_x < T'$ such that $\eta_-(t_x, x) \le -\lambda$.

Now it is easy to see the hyperbolicity over Λ from this. For any arc $\langle x, \phi_t(x)\rangle$ in Λ, we can divide it into segments such that on each segment the above inequality is valid. Notice that the number of segments is comparable to the length t, and the hyperbolicity over that arc follows from this immediately. It is easy to see the bound is uniform over Λ. And the proof for Lemma 1 is complete.

Lemma 2. *Let T, γ be given as before, then for any invariant measure μ, supported on Λ, we have*

$$\int \eta_-(T, x)\,d\mu(x) \le -\gamma.$$

Proof. Suppose on the contrary, for some μ, we have

$$\int \eta_-(T, x)\,d\mu(x) \ge -\gamma.$$

Then it would be true for some ergodic component μ_e. According to the Birkhoff Ergodic Theorem [15], we have that the space average equals the space average of the time averages. More specifically, let μ_x be the measure that is the limit as t goes to infinity of the uniform measure along the orbit segment $\langle x, \phi_t(x)\rangle$. Then

$$\int \eta_-(T, x)\,d\mu(x) = \int_{U_T} \mu(dx) \int \eta_-(T, y)\mu_x(dy).$$

Here the set U_T means the set of quasi-regular points under the flow ϕ_t. A point x is said to be quasi-regular, if the invariant measure supported on the orbit of x exists and the Birkhoff Ergodic Theorem is true with respect to this measure, i.e., $\lim_{t\to\infty} t^{-1}\int_0^t f(\phi_s(x))\,ds$ exists for any continuous function f.

By Kryloff-Bogoliuboff theory [15] we know that the set $x \in U_T$ such that the μ_x are transitive is of full measure. So we can choose a point x_0, such that the orbit of x_0 is transitive and

$$\int \eta_-(T, y)\mu_{x_0}(dy) \ge -\gamma,$$

i.e.,

$$\lim_{|T'|\to+\infty} \frac{1}{T'}\int_0^{T'} \eta_-(T, \phi_t(x_0))\,dt \ge -\gamma.$$

Because the measure μ_{x_0} is transitive, for any $\varepsilon > 0$, we can find a $T_0 > 1$, such that $\operatorname{dist}(x_0, \phi_{T_0}(x_0)) < \varepsilon$. Now we can apply Theorem 3.1 to (C^1, ε)-close a subarc of $(x_0, \phi_{T_0}(x_0))$ and for the perturbed system we have a periodic orbit $\mathscr{O}$.

There is a technical difficulty here. We do not know which subarc is being closed with the closing lemma. For some arcs we may lose the above inequality. Given the orbit $\langle x_0, \phi_{T'}(x_0)\rangle$, suppose we close the subarc corresponding to $\langle t_0, t_0 + T_0\rangle$. We would have trouble when either T_0 is small or t_0 is large. However, we can choose the tubular neighborhood to be very small, forcing T_0 to be large. So, the essential difficulty is that of t_0 being unbounded when the perturbation is arbitrarily small. The ergodic closing lemma for flows [12], done by the same techniques as in the discrete case, take care of this.

Given $\mathscr{U}$ a neighborhood of S in $\aleph(M^3)$ and $\varepsilon > 0$. Let $\Sigma(\mathscr{U}, \varepsilon)$ be the set of points $x \in M^3$ such that there exists $X \in \mathscr{U}$, $y \in M^3$, $T' \in R^+$ and a continuous strictly increasing function h on $[0, T]$ satisfying $\psi_{h(T')}(y) = y$, $X = S$ on $M^3 - B_\varepsilon(S, x)$, and $d(\phi_t(x), \psi_{h(t)}(y)) \le \varepsilon$ for all $0 \le t \le T'$. Here ϕ_t, ψ_t are the flows generated by S and X respectively. $B_\varepsilon(S, x)$ is the tubular neighborhood of radius ε of the orbit of x for S. Let $\{\mathscr{U}_n, n \ge 0\}, \{\varepsilon_n, n \ge 0\}$ be the basis of S in $\aleph(M^3)$ and a sequence converging to 0, we form a set

$$\Sigma(S) = \bigcap_{n \ge 0} \Sigma(\mathscr{U}_n, \varepsilon_n).$$

Then $x \in \Sigma(S)$ has the closing property for every neighborhood $\mathscr{U}$ of S and every $\varepsilon > 0$. By the same argument as in [12], we can show the following lemma.

Ergodic Closing Lemma. *For every ergodic $\mu \in \mathscr{M}(S)$ (i.e., the set of all probability measures invariant under the flow generated by S), $\mu(\Sigma(S)) = 1$.*

Or we can say that for the points of full measure, the arc can be closed even if we choose $t_0 = 0$. The proof is omitted.

Now we come back to the proof of Theorem 3.2. If we apply the Ergodic Closing Lemma to μ_{x_0}, we can choose at least one point x in the orbit of x_0 for x that has the desired closing property. The measure μ_{x_0} is invariant, so the inequality is still true with respect to the measure μ_x, i.e., we have

$$\lim_{T' \to \infty} \frac{1}{T'} \int_0^{T'} \eta_-(T, \phi_t(x))\, dt \ge -\gamma.$$

This implies that for any positive number γ' slightly smaller than γ there exists a $T' > 0$, such that whenever $T'' > T'$, we have

$$\frac{1}{T''} \int_0^{T''} \eta_-(T, \phi_t(x))\, dt \ge -\gamma'.$$

Then we know that $x \in \Sigma(S)$. Then for every neighborhood $\mathscr{U}$ of S and every $\varepsilon > 0$, we have the perturbed system X and nearby periodic orbit $\mathscr{O}$ through y as described above.

Since $h(t)$ is the scaling function of the periodic orbit, $h(T'') = pT_0$ for a certain integer $p \ge 1$, where T_0 is the period of $\mathscr{O}$. We need to show that

$$\left| \frac{1}{T''} \int_0^{T''} \eta_{S_-}(T, \phi_t(x))\, dt - \frac{1}{T_0} \int_0^{T_0} \eta_{X_-}(T, \psi_t(y))\, dt \right|$$

is arbitrarily small when ε goes to zero. Here $\eta_{X_-}(t, x)$ denotes the function $\eta_-(t, x)$ for the specific vector field X. It is easy to see that

$$\begin{aligned}
&\left|\frac{1}{T''}\int_0^{T''} \eta_{S_-}(T, \phi_t(x))\,dt - \frac{1}{pT_0}\int_0^{pT_0} \eta_{X_-}(T, \psi_t(y))\,dt\right|, \\
&\quad \le \left|\frac{1}{T''}\int_0^{T''} (\eta_{S_-}(T, \phi_t(x)) - \eta_{X_-}(T, \psi_{h(t)}(y)))\,dt\right| \\
&\qquad + \left|\frac{1}{T''}\int_0^{T''} \eta_{X_-}(T, \psi_{h(t)}(y))\,dt - \frac{1}{T''}\int_0^{pT_0} \eta_{X_-}(T, \psi_t(y))\,dt\right| \\
&\qquad + \left|\left(\frac{pT_0}{T''} - 1\right)\frac{1}{pT_0}\int_0^{pT_0} \eta_{X_-}(T, \psi_t(y))\,dt\right| \\
&\quad = \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Part I clearly can be arbitrarily small due to the continuity property of $\eta_{X_-}(T, y)$ with respect to y and X. To prove Part II and Part III to be arbitrarily small we need to deal with the rescaling of the periodic orbits under small perturbation and prove the following lemma.

Scaling Lemma. *There exists a constant $K > 0$, that depends only on the norm of S, such that $|h'(t) - 1| \le K\varepsilon$ for all $t \in [0, T'']$.*

For example, assume $|h'(t) - 1| \le K\varepsilon$, holds for all $t \in [0, T'']$. Now integrate from 0 to pT_0 and we have

$$\left|\int_0^{pT_0} (h'(t) - 1)\,dt\right| = |T'' - pT_0| \le KpT_0\varepsilon,$$

which clearly implies Part III is small.

Part II is small can be seen directly by changing the variable of the first integral, i.e., $s = h(t)$.

Proof of Scaling Lemma. To show this lemma, we observed that the orbits we are working on are bounded away from the singularities. This has two consequences:

- There is a lower bound of the C^0 norm of the vector fields and its small perturbations.
- For S and a subset of M^3 which is bounded away from the singularities, the set can be covered by finitely many flow boxes, i.e., on each chart the flow can be diffeomorphically transformed to the flow of unit straight lines. The norm of the transformation only depends on the norm of S.

Notice that what we want to prove is a local property. So we can assume we are working in a flow box of S. For any small $\delta > 0$, we know that

$$\operatorname{dist}(\phi_t(x), \psi_{h(t)}(y)) < \varepsilon \quad \text{and} \quad \operatorname{dist}(\phi_{t+\delta}(x), \psi_{h(t+\delta)}(y)) < \varepsilon.$$

Because S generates flow ϕ_t and X generates flow ψ_t, we know that

$$|\operatorname{dist}(\phi_t(x), \phi_{t+\delta}(x)) - \delta| \le K_1\delta^2,$$

and

$$|\operatorname{dist}(\psi_{h(t)}(y), \psi_{h(t+\delta)}(y)) - ||X_{\phi_{h(t)}}(y)||(h(t+\delta) - h(t))| \le K_2\delta^2$$

where K_1, K_2, and in the following, K_3, K_4, K_5 only depend on the norm of S. Now we assume that $h(t)$ is a C^2 function and $h'(t)$ is bounded. We can see this by observing that $h(t)$ is an integral of a C^1 function. Then we have $|h(t+\delta) - h(t) - h'(t)\delta| \leq K_3\delta^2$. So we have

$$|\mathrm{dist}(\psi_{h(t)}(y), \psi_{h(t+\delta)}(y)) - |\,|X_{\psi_{h(t)}(y)}|\,|h'(t)\delta| \leq K_3\delta^2.$$

It is easy to see that

$$|\mathrm{dist}(\psi_{h(t)}(y), \psi_{h(t+\delta)}(y)) - \mathrm{dist}(\phi_t(x), \psi_{t+\delta}(x))|$$

is approximately the area of the rectangle formed by the four vertices $\phi_t(x)$, $\phi_{t+\delta}(x)$, $\psi_{h(t)}(y)$, and $\psi_{h(t+\delta)}(y)$ which is bounded by $K_4\varepsilon\delta$. So if we choose $\varepsilon = \delta$, then we would have

$$|\,|X_{\psi_{h(t)}(y)}|\,|h'(t) - 1|\,| \leq K_5\varepsilon.$$

We know that $\|X - 1\| \leq \varepsilon$ in this box, so finally we have

$$|h'(t) - 1| \leq K\varepsilon$$

where K only depends on the norm of S. This completes the proof of the Scaling Lemma.

Now let us return to the proof of Lemma 2. The inequality on μ can be continuously deformed to the one with respect to the measure supported on the periodic orbit $\mathscr{O}$, and so we have

$$\int \eta_-(T, x)\, d\mu_{x_0} \geq -\gamma'.$$

Here $0 < \gamma < \gamma' < \eta_S$.

This means the characteristic exponents of $\mathscr{O}$ are within γ' of 0, which contradicts weak uniformity. The proof is complete.

4. Construction of dominated splitting on $\Omega\backslash\mathscr{P}$ for $\aleph^*(M^3)$

The strategy in proving the stability conjecture is to prove existence of the so-called dominated splitting first. The dominated splitting is weaker than the hyperbolic splitting and is easier to be established.

Definition (Dominated Splitting). Let Λ be an invariant set for the flow on ϕ_t of a vector field X, with $\Lambda \cap \mathscr{P} = \varnothing$. Let Θ_t be a linear bundle map flow on $TM|_\Lambda$ which covers ϕ_t. (In applications Θ_t is either $d\phi_t$ or the extended normal flow Ψ_t.) Given constants $C, \lambda \geq 0$,

$$TM|_\Lambda = E \oplus X \oplus F,$$

is a (C, λ)-dominated splitting for Θ_t, if it is a continuous splitting which is invariant by Θ_t and for all $t \geq 0$ we have the following

$$\|\Theta_t|_{E(x)}\|\,\|\Theta_{-t}|_{F\phi_t(x)}\| \leq C\exp(-\lambda t).$$

The geometric interpretation of the dominated splitting is that for every one-dimensional subspace $L \subset E_x \oplus F_x$, $x \in \Lambda$, not contained in $E(x)$, the angle between $\Theta_t(L)$ and $F(\phi_t(x))$ converge exponentially to zero as $n \to \infty$. This interpretation is crucial in our arguments.

In this section, we prove the following theorem that solves one of the main technical difficulties in the proof of the stability conjecture. Its statement concerns the extended normal bundle bundle flow Ψ_t which is defined in §2. Remember that N is the normal bundle in terms of some Riemannian metric, so N_x is the set of tangent vectors at x which are orthogonal to X_x. Ψ_t preserves N and also preserves X.

Theorem 4.1 (Bundle extension). *Let $X \in \aleph^*(M^3)$, then there exists a dominated splitting $E \oplus X \oplus F$ for the extended normal bundle flow Ψ_t on $\Omega_X \backslash \mathscr{P}$, where $\mathscr{P}$ is the set of singularities. This splitting is unique if we assume that $E(x) \oplus F(x) = N_x$ at each point x.*

The idea is to extend the bundle splitting over periodic orbits to its closure. It is easy to give a formal extension by using Theorem 1.1. To prove that it is well defined and continuous we have to show that such an extension is unique. This can be done in two steps.

1. Show that there is a unique dominated splitting over

$$\Omega - \bigcup (W^u(\mathscr{P}) \cup W^s(\mathscr{P})).$$

This can be carried out by using Theorem 1.1 and a trick in [11].

2. Show that there is a unique dominated splitting around the saddle singularities. This is essentially due to C. Doering. Originally he did it for another kind of system. I observed that his idea can be applied to $\aleph^*(M^3)$.

Problem. Generalize Theorem 4.1 to higher dimensions.

Let us describe in detail how to construct the dominated splitting. Let Λ be the set of periodic orbits of saddle type. We define the splitting

$$TM_x^3 = G_x^s \oplus X_x \oplus G_x^u \quad \text{for } x \in \Lambda,$$

where the splitting is the canonical invariant splitting for Ψ_t over the hyperbolic periodic orbits. We would like to extend the splitting to closure of Λ minus the singularities, $\overline{\Lambda} \backslash \mathscr{P}$, in a manner so it is still invariant. For an orbit $\mathscr{O}$ in $\overline{\Lambda} \backslash \mathscr{P}$, take one point $x \in \mathscr{O}$, then take $x_n \in \Lambda$ such that $x = \lim_{n\to\infty} x_n$. Because the space of one-dimensional spaces in the tangent space is compact we can choose a subsequence of x_n such that $G_{x_n}^s$ and $G_{x_n}^u$ converge. Then define

$$G_x^s = \lim_{n\to\infty} G_{x_n}^s, \qquad G_x^u = \lim_{n\to\infty} G_{x_n}^u.$$

For any other $y \in \mathscr{O}$, with $y = \phi_t(x)$, we define

$$G_y^s = \Psi_t(G_x^s), \qquad G_y^u = \Psi_t(G_x^u).$$

We see the following is still true: $\|\Psi_t|_{G^s}\| \, \|\Psi_{-t}|_{d\phi_t(G^u)}\| \le C \exp(-\lambda t)$.

From the inequality above we see that $G_x^s \cap G_x^u = \{0\}$, and we know that $\dim G_x^s + \dim G_x^u = 2$. So, we have defined $TM_x^3 = G_x^s \oplus X_x \oplus G_x^u$ for all $x \in \overline{\Lambda} \backslash \mathscr{P}$.

This gives an invariant splitting of $TM^3|_{\overline{\Lambda}\backslash\mathscr{P}}$. But it may not be well defined and may not be continuous. By adapting the argument in [11], we can prove the uniqueness over $\Omega - (W^s(\mathscr{P}) \cup W^u(\mathscr{P}))$.

Suppose for $x \in \Omega - (W^s(\mathscr{P}) \cup W^u(\mathscr{P}))$ there is another dominated splitting $TM_x = L_x^s \oplus X_x \oplus L_x^u$ with $L_x^s \oplus L_x^u = N_x$. Assume that $L_x^s \ne G_x^s$. We know that L_x^s, G_x^s are both one dimensional, so $L_x^s \cap G_x^s = \{0\}$. By the property of

dominated splitting, we have $\operatorname{dist}(\Psi_t(L_x^s), G_{\phi_t(x)}^u) \to 0$ as $t \to +\infty$, where the distance of two lines in the tangent space is the angle between them.

Let y be an ω-limit point of x. By the hypotheses that $x \notin W^s(\mathscr{P})$ we can choose y to be a regular point with $y \notin W^u(\mathscr{P})$. Then there will be a sequence of numbers $t_k \to +\infty$, $k \in Z^+$, such that $\lim \phi_{t_k}(x) = y$. By taking a subsequence if necessary, we have $\Psi_{t_k}(L_x^s)$, $\Psi_{t_k}(L_x^u)$, $\Psi_{t_k}(G_x^s)$, $\Psi_{t_k}(G_x^u)$ converging to L_s', L_u', G_s', and G_u' respectively. By the geometric properties of dominated splitting noted above we see that $L_s' = G_u'$. Because the angle between $\Psi_{t_k}(L_x^s)$ and $\Psi_{t_k}(L_x^u)$ cannot go to zero, $L_u' \cap G_u' = \{0\}$.

Now take an α-limit point z of y which can be taken to be a regular point since y does not belong to $W^n(\mathscr{P})$ and a subsequence $s_k \to \infty$, as $k \to \infty$, such that $\Psi_{-s_k}(L_s')$, $\Psi_{-s_k}(L_u')$, $\Psi_{-s_k}(G_s')$, $\Psi_{-s_k}(G_u')$ converge to subspaces L_s'', L_u'', G_s'', and G_u'', then since $L_s' = G_u'$, $L_s'' = G_u''$, by an argument as above, since $L_u' \cap G_u'$ it follows that $L_u'' = G_s''$. Now take $v_1 \in L_u'' = G_s''$ and $v_2 \in L_s'' = G_u''$.

By the property of the dominated splitting and since $u_s \in G_s''$, $v_u \in G_u''$, we see $\|\Psi_t(v_s)\| \, \|\Psi_{-t}(\Psi_t(v_u))\| \le C \exp(-\lambda t)\|v_s\|/\|\Psi_t(v_u)\|$.

This would imply that

$$\frac{\|\Psi_t(v_s)\|}{\|\Psi_t(v_u)\|} = \frac{\|\Psi_t(v_s)\| \, \|\Psi_{-t}(\Psi_t(v_u))\|}{\|\Psi_t(v_u)\| \, \|\Psi_{-t}(\Psi_t(v_u))\|} \le C \exp(-\lambda t) \frac{\|v_s\|}{\|v_u\|}.$$

Here $\|\Psi_t(v_s)\| \, \|\Psi_{-t}(\Psi_t(v_u))\| \ge 1$ because $\Psi_{-t} \circ \Psi_t$ is really the identity map. Since $v_s = G_u''$ and $v_u = G_s''$, we also have

$$\|\Psi_t(v_u)\|/\|\Psi_t(v_s)\| \le C \exp(-\lambda t)\|v_u\|/\|v_s\|,$$

for all $t \ge 0$. This is clearly impossible. Multiplying these two inequalities would imply $1 \le C^2 \exp(-2\lambda t)$ for all $t \ge 0$. So the proof of uniqueness of the dominated splitting on $\Omega - (W^s(\mathscr{P}) \cup W^u(\mathscr{P})$ is complete.

How about the splitting around the singularities? We know that for $X \in \aleph^*(M^3)$, the singularities are necessarily hyperbolic. They also have the local hyperbolic splitting. The main idea is that around the singularities the dominated splitting is compatible with that of the splitting of the singularities. The following two propositions express the fact. The proposition will be used in our final proof for the main theorem.

Notation. We denote the orthogonal plane of X_x in the tangent bundle TM_x as N_x, and the space of one-dimensional spaces of N_x as $G(N_x)$.

Proposition 4.2 (see [2]). *Let x_0 be a singularity. Let $\sigma = s, u$ be such that* $\dim W^\sigma(x_0) = 2$. *Suppose that $x \in W^\sigma(x_0)$ and that N^s, $N^u \in G(N_x)$. If N^s dominates N^u, then $N^\sigma = N_x \cap T_x W^\sigma(x_0)$. See Figure* 1.

Another result of Doering gives information on the dominated splitting for the case of the one-dimensional stable manifolds. See Figure 2. Let $D = DX(x_0)\colon T_{x_0}M \hookleftarrow$ be the derivative of X at x_0, $E^s \oplus E^u = TM_{x_0}^3$ be the hyperbolic splitting for D and $e^{tD} = (\Psi_t)_0$ for all $t \in R$.

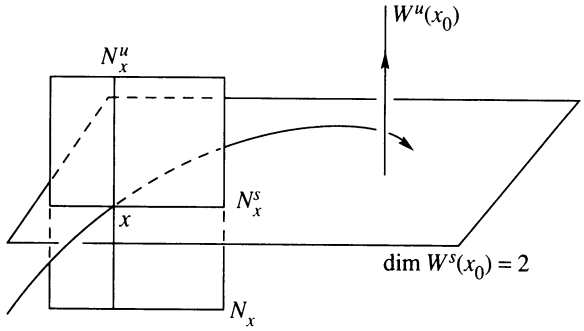

FIGURE 1. Two-dimensional case

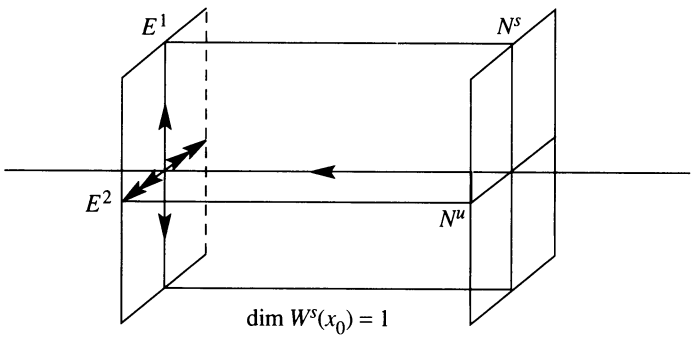

FIGURE 2. One-dimensional case

Proposition 4.3 (see [2]). *Suppose that* $\dim W^s(x_0) = 1$ *and let* $x \in W^s_{x_0}$, $c > 0$, $0 < \lambda < 1$, *and* $N^s_x, N^u_x \in G(N_x)$ *be given. If* N^s_x (c, λ) *dominates* N^u_x *for* Ψ_t *and for all* $t \geq 0$, *then there exists* $C_0 > 0$, *such that* $d\phi_t(N^s)$ $(d\phi; C_0; \lambda)$-*dominates* $d\phi_t(N^u)$ *for all* $t \geq 0$. *In particular, the two eigenvalues of* $D|E^u$ *are real, positive, and distinct; say* $0 < \lambda_1 < \lambda_2$. *Moreover, if* $E^1 \oplus E^2 = E^u$ *is the associated eigenspace splitting with* E^i *the eigenspace for* λ_i *then*

$$\lim_{t \to +\infty} \Psi_t(N^s) = E^1 \quad \textit{and} \quad \lim_{t \to +\infty} \Psi_t(N^u) = E^2.$$

We include the proof of these two propositions for further use, i.e., for the generalization to higher dimensional case.

Proof of Proposition 4.2. We assume that $\dim W^s(x_0) = 2$, the case of $\dim W^u(x_0) = 2$ is similar. Let $x \in W^s(x_0)$ be given and set $E = N_x \cap T_x W^s(x_0)$. From Palis' Inclination Lemma [17] we know that

$$\lim_{t \to +\infty} d\phi_t(L) = E^u$$

for any $L \in G(N_x) - E$. By compactness of $G(TM)$ we may choose $t_n \to +\infty$ and $F \in G(T_{x_0}M)$ such that

$$\lim_{n \to \infty} d\phi_{t_n}(E) = F,$$

but $TW^s(x_0)$ is $d\phi$-invariant, hence $F \subseteq T_{x_0}W^s(x_0) = E^s$. Now let $N^s_x, N^u_x \in G(N_x)$ be such that N^s_x dominates N^u_x. First we prove that $N^u_x \neq E$. Assume, on the contrary, that $N^u_x = E$, then $N^s_x \neq E$. Choosing $L \in G(N_x) - \{E, N^s_x\}$, we have

$$E^u = \lim_{n\to\infty} d\phi_{t_n}(L) = \lim_{n\to\infty} d\phi_{t_n}(N_x^u) = \lim_{n\to\infty} d\phi_{t_n}(E) = F \subseteq E^s,$$

which is impossible. Thus $N_x^u \neq E$. Now suppose that $N_x^s \neq E$. Then we have

$$E^s \supseteq F = \lim_{n\to\infty} d\phi_{t_n}(E) = \lim_{n\to\infty} d\phi_{t_n}(N_x^u) = E^u.$$

This is impossible too. The only choice left is $N_x^s = E$, thus proving Proposition 4.2.

Proof of Proposition 4.3. Here is an outline of the proof. There is a uniform lower bound of the angle between N_x^s and N_x^u which implies $d\phi_t(N_x^s)$ $(d\phi; C_0; \lambda)$-dominates $d\phi_t(N_x^u)$ for some C_0 and all $t \geq 0$.

For the other parts, again by Palis' Inclination lemma, $\lim_{t\to\infty} d\phi_t(N_x^\sigma) = E^u$. So, we can choose a subsequence $t_n \to \infty$ such that $\lim_{n\to\infty} d\phi_{t_n}(N^\sigma) = N_0^\sigma$, $\sigma = s, u$ that is invariant under the flow. In the limit, N_0^s still dominates N_0^u, which implies the desired property of the eigenvalues, thus proving Proposition 4.3.

Proof of Theorem 4.1. The discussion above proved that there is a unique dominated splitting away from the stable and unstable manifolds of singularities. We will treat the case for $x \in W^s(x_0)$ where x_0 is a singularity. The other case of $x \in W^u(x_0)$ can be proved similarly. From the discussion above we only need to show the uniqueness of the dominated splitting at x.

Suppose (N_x^s, N_x^u) and (L_x^s, L_x^u) are both dominated splitting for X at the orbit of x. We will show that $N_x^s = L_x^s$. The proof for $N_x^u = L_x^u$ is similar (or consider $-X$ instead).

If $\dim W^s(x_0) = 2$, Proposition 4.2 implies $N_x^s = N_x \cap T_x W^s(x_0) = L_x^s$.

If $\dim W^s(x_0) = 1$, Proposition 4.3 assumes the existence of a splitting $E^1 \oplus E^2 = T_{x_0} W^u(x_0)$ such that

$$\lim_{t\to\infty} d\phi_t(N_x^s) = E^1 \quad \text{and} \quad \lim_{t\to\infty} d\phi_t(L_x^u) = E^2;$$

moreover L_x^s $d\phi$-dominates L_x^u. Now if $N_x^s \neq L_x^s$, it follows that

$$\lim_{t\to\infty} d\phi_t(N_x^s) = \lim_{t\to\infty} d\phi_t(L_x^u) = E^2.$$

Since this limit is also E^1, this is impossible. Thus $N_x^s = L_x^s$, and we have completed the proof of Theorem 4.1.

5. Creation of a homoclinic orbit and attainability of contracting sequences

In this section we state and explain Mañé's two perturbation techniques. Originally the theorems are stated for the diffeomorphisms. It is no problem to adapt the proof to give the analogous result for flows (as privately communicated by Mañé). In the following we will state the theorems for flows with apparent modifications.

Perturbation technique 1: Creation of a homoclinic orbit.

Theorem 5.1. *Let X be a C^1 vector field and $\phi_t\colon M^n \to M^n$ be the flow generated by X. Let Λ be an isolated hyperbolic set of X such that $\Omega(\phi_t|_\Lambda) = \Lambda$. Suppose there exists a point $x \in W^u(\Lambda) - \Lambda$ and a sequence of numbers $t_1 < t_2 < \cdots \to \infty$ such that the probabilities $\mu_n(x)$, i.e., the uniform measure on the*

arc $\langle x, \phi_{t_n}(x)\rangle$, *converge to a probability* μ *such that* $\mu(\Lambda) > 0$. *Then in every* C^1 *neighborhood of* X *there exist* Y *that coincides with* X *in a neighborhood of* Λ *and has a homoclinic orbit associated to* Λ.

Perturbation technique 2: Contracting sequences and attainability. First of all, we need some definitions.

Definition (strings and uniform strings). Let Λ be a compact invariant set under ϕ_t having a dominated splitting $TM|\Lambda = E \oplus X \oplus F$. Given $T > 0$, and $\gamma \in (0, 1)$, the functions $\eta_-(T, x)$ and $\eta_+(T, x)$ can be defined in a similar way as in §2. We say that an arc $\langle x, \phi_{-nT}(x)\rangle$ in Λ, $n > 0$, is a (T, γ)-*string*, if

$$\int \eta_-(T, y)\, d\mu_n(y) \leq -\gamma$$

where $\mu_n(y)$ are the uniform measures supported on the arc $\langle x, \phi_{-nT}(x)\rangle$. We say that $\langle x, \phi_{-nT}(x)\rangle$ is a *uniform* (T, γ)-*string*, when $\langle \phi_{-jT}(x), \phi_{-nT}(x)\rangle$ are (T, γ)-*string* for all $0 \leq j \leq n$.

Definition (contracting sequence). Given $T \to 0$ and $\gamma \in (0, 1)$, a pair (S, ν) is called a *contracting sequence* if $S = \{x_1, x_2, \dots\} \subset \Lambda$ is a sequence in Λ and $\mu\colon S \to R^+$ is a function satisfying $\lim_{n\to\infty} \mu(x_0) = +\infty$, and if there exists a $n' > 0$ such that $(x, \phi_{-jT}(x))$ is a (T, γ)-*string* for all $n' < j < \nu(x)$ and $x \in S$. The sequence

$$\overline{S} = \{\phi_{-T\nu(x_n)}(x_n))|n > 1\}$$

is called the sequence of *endpoints* of (S, μ).

Remark. For the contracting sequences, one observation is crucial. If one produces some perturbations along the F direction, they will be amplified exponentially under backward iteration. Then for some perturbations, it will hit some prescribed set. The following is the precise definition for hitting.

Definition (attainability). Given a sequence $S = \{x_1, x_2, \dots\}$ converging to a point x_0 and a set $\Sigma \subset M$, we say that Σ is *attainable* from S if given $\delta > 0$, a neighborhood U of x_0 and a C^1 neighborhood $\mathscr{U}$ of X, there exist $Y \in \mathscr{U}$ and numbers $k > 0$ and $T_0 > 0$ such that

(a) $x_k \in U$ and $\psi_{T_0}(x_k) \in \Sigma$,
(b) $X = Y$ on $M^3 - (\phi_{-1}(U) \cup \psi_{-1}(U))$,
(c) $d(\phi_{-t}(x_k), \psi_{-t}(x_k)) \leq \delta$ for all $0 \leq t \leq T_0$.

By similar methods as in [5], the domination property of the splitting $TM^3|\Lambda = E \oplus X \oplus F$ implies that there exists a family of embedded C^1 disks $D(y)$, $y \in \Lambda$, such that

(1) $y \in D(y)$ and $T_yD(y) = F(y)$;
(2) $\psi_t(D(y))$ contains a nbd of $\phi_t(y)$ in the disk $D(\phi_t(y))$ for $t \geq 0$;
(3) $D(y)$ depends continuously on y.

Define $D_r(y)$ as the set of points in $D(y)$ whose distance in $D(y)$ to y is $< r$.

From the observation above, the following theorem is reasonable. Certainly there are some technical difficulties. For example, the backward iteration can be very near to the starting point. All these have been treated by Mañé in [14].

Theorem 5.2. *Given $r > 0$, $T_0 \in R^+$, and $0 < \gamma < 1$, there exists $\varepsilon = \varepsilon(r, T_0, \gamma)$ such that if (S, ν) is strongly contracting sequence and S converges to a nonperiodic point x_0, then, if $y \in \Lambda$ is ε-near to an accumulation point of the sequence of endpoints of (S, ν), $D_r(y)$ is attainable from S.*

6. Final proof of the main theorem

Now we come to the final part of the proof. Let $S \in \aleph^*(M^3)$, so it generates a flow $\phi_t\colon M^3 \to M^3$. Let $\mathscr{P} = \{p_1, p_2, \dots, p_k\}$ be the set of singularities and Λ the set of periodic orbits of dim 1 of the stable manifold. The following is the main theorem.

Theorem 6.1. *If Λ accumulates on $\mathscr{P}$, then there exists an arbitrarily C^1 small perturbation Y of X, where Y coincides with X on $\mathscr{P}$ and Λ, such that, for Y, $W^u(\mathscr{P}) \cap W^s(\mathscr{P})$ is not empty. And furthermore there exists an unstable saddle connection of the singularities.*

The proof is based on several lemmas.

Lemma 1. *If Λ accumulates on $\mathscr{P}$, then $\overline{\Lambda}\backslash\mathscr{P}$ is not closed. Without loss of generality we can assume $\overline{\Lambda}\backslash\{p_1\}$ is not closed, then $\{W^s(p_1) - \{p_1\}\} \cap \overline{\Lambda}$ is not empty.*

Proof. See Mañé [14].

Also without loss of generality we can assume that $\dim W^s(p_1) = 1$. Along the lines of [14], the proof is divided into two cases. In the first case there exists $x \in (W^s(\{p_1\})\backslash p_1) \cap \overline{\Lambda}$, such that $p_1 \in \alpha(x)$, i.e., we can take a sequence of integers $n_1 < n_2 < \cdots \to \infty$, such that $\{\phi_{-n_i T}(x) | i \geq 1\}$ converges to $\{p_1\}$.

Now pick some γ and T such that $0 < \gamma < \eta_S$ and $T = T_S$, where η_S, T_S are given in Theorem 2.1.

The next lemma can be considered as a qualitative version of Theorem 3.2, i.e., separation of periodic orbits of different indices implies Axiom A. In its proof there is a difficulty because of the discontinuity of $\eta_-(T, x)$ at the singularities. It can be resolved with the help of perturbation technique 1. Denote $\mathscr{M}(\Lambda)$ as the set of the probability measures on Λ which are invariant under the flow ϕ_t.

Lemma 2. *If there exists a $\mu \in \mathscr{M}(\Lambda)$, such that*

$$\int \eta_-(T, x)\, d\mu(x) \geq -\gamma$$

then $\mu(\mathscr{P}) > 0$.

Proof. The idea is similar to the proof of Theorem 3.2 (separation implies Axiom A) by using the ergodic closing lemma and the weak uniformity of the union of the periodic orbits. Now assume that there is an accumulation point μ of the set $\{\mu_n\}$ with $\mu(\mathscr{P}) = 0$. Because $\mu(\mathscr{P}) = 0$, the given inequality can be continuously deformed to the one of the measure which is uniformly supported on a periodic orbit. Because the only difference here is that the function $\eta_-(T, x)$ is not continuous at $\mathscr{P}$. However, the change of the integral under small perturbation can be controlled because $\mu(\mathscr{P}) = 0$. Then we have a contradiction. Thus we can assume $\mu(\mathscr{P}) > 0$ for all the accumulation points

of $\{\mu_i\}$. Take such a μ. Again applying perturbation technique one to the set of singularities there is a saddle connection between the singularities.

Lemma 3. *If* $(\{x\}, n_1, n_2, \dots)$ *is not a* (T, γ) *contracting sequence, then the theorem is true.*

Proof. Now there exists a sequence $j_1 < j_2 < \cdots$ such that, setting $\overline{n}_i = n_{j_i}$, we have

$$\int \eta_-(T, x)\, d\mu_i(x) \geq -\gamma.$$

Here μ_i denotes the measure uniformly supported on the arc $\langle p, \phi_{-\overline{n}_i T}(p)\rangle$.

Now we use the trick again to treat the case for the discontinuity of $\eta_-(T, x)$. If there is already an accumulation point μ of the set $\{\mu_i\}$ such that $\mu(\mathscr{P}) > 0$, we are done. Otherwise, all the accumulation points have zero support on $\mathscr{P}$. But the set of μ_i are compact with weak topology, so let μ be an accumulation point. Because $\mu(\mathscr{P}) = 0$, we can take a limit

$$\int \eta_-(T, x)\, d\mu(x) \geq -\gamma.$$

And by Lemma 2 we must have $\mu(\mathscr{P}) > 0$, a contradiction.

Now we have a μ supported on Λ and $\mu(\mathscr{P}) > 0$. When we apply perturbation technique 1 directly, we get an unstable saddle connection.

Lemma 4. *If* $(\{x\}, n_1, n_2, \dots)$ *is a* (T, γ) *contracting sequence, the theorem is also true.*

Proof. Apply Theorem 4.1 in [14] and choose $0 < \gamma < \gamma_1 < \eta_S$. In the first case, there exists a subsequence $\{\overline{n}_1, \overline{n}_2, \dots\}$ of $\{n_1, n_2, \dots\}$ such that $(\{x\}, \{\overline{n}_1, \overline{n}_2, \dots\})$ is strongly (T, γ_1)-contracting sequence and $\{\phi_{-\overline{n}_i T}(x) | i \geq 1\}$ converges to a point $y \in \Lambda$ satisfying

$$\int \eta_-(T, z)\, d\mu_i(z) \geq -\gamma_2$$

where $\mu_i(z)$ are the measures uniformly supported on $\langle y, \phi_{-n_i T}(y)\rangle$, for all n_i larger than a certain N and for any $0 < \gamma_2 < \gamma$.

We claim that $y \in W^u(p)$ for some $p \in \mathscr{P}$. This can be proved by a similar argument as in the proof of Lemma 3.

From the property of dominated splitting, we have

$$\eta_+(-T, \phi_T(z)) + \eta_-(T, z) \leq -\gamma,$$

which implies

$$\begin{aligned}\int \eta_+(-T, x)\, d\mu_i(z) &= \int (\eta_+(-T, z) + \eta_-(T, z))\, d\mu_i(x) \\ &\quad - \int \eta_-(T, z)\, d\mu_i(z) \\ &\leq -\gamma + \gamma_2 = \lambda_0 < 0.\end{aligned}$$

Thus, we have

$$\int \eta_+(-T, z)\, d\mu_i(z) \leq \lambda_0 < 0.$$

Let $D_r(x)$, $x \in \Lambda$, be the family of disks tangent at x to $F(x)$, associated with the dominated splitting.

It is easy to see that

$$\lim_{i\to+\infty} \operatorname{diam} \phi_{-\overline{n}_i T}(D_r(y)) = 0$$

when r is small enough. So

$$D_r(y) \subset W^u(\mathscr{P}).$$

We claim that $\dim W^u(p) = 2$.

This follows from Proposition 4.3. Suppose $\dim W^u(p) = 1$. Then we know that the flow direction already occupies the unstable manifold. Now we have $D_r(y) \subset W^u(p)$ and $D_r(y)$ one dimensional, and there is no space in $W^u(p)$ for $D_r(y)$ to stay. So $\dim W^u(p) = 2$. From Proposition 4.2, we know that the F direction intersects $W^u(p)$ transversely.

Now we can apply perturbation technique 2 to the strongly (T, γ_1) *contracting sequence* $(\{x\}, \{\overline{n}_1, \overline{n}_2, \dots\})$ and y. From above we must make sure that there are enough dimensions and the direction of perturbation intersects the unstable manifold of p transversely. So the perturbations really hit the unstable manifold and we have the unstable saddle connection.

For the other case, when applying Theorem IV.1 [14] to $(\{x\}, \{n_1, n_2, \dots\}$, it is property (b) that holds. Then there exists a sequence of positive integers $0 < \overline{n}_j \le n_j$ such that $(\{x\}, \{\overline{n}_1, \overline{n}_2, \dots\})$ is a strongly (T, γ_1)-*contracting sequence* and $\sup_j(n_j - \overline{n}_j) < \infty$. This last relation implies that

$$\lim_{j\to\infty} \phi_{-\overline{n}_j T}(x) = p_1.$$

Using the same argument as before, we have a homoclinic orbit of p_1 after a small perturbation that is also unstable.

For the case $p \notin \alpha(p)$, the proof is similar. Because we have $\Omega(\phi_t|\Lambda) = \Lambda$, one knows that the periodic orbits are dense in Λ, so it eventually returns when perturbed a little bit. Thus the proof of the theorem is finished.

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