TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 343, Number 1, May 1994

ON TRANSFORMATION GROUP C^*-ALGEBRAS WITH CONTINUOUS TRACE

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ABSTRACT. In this paper we answer some questions posed by Dana Williams in [19] concerning the problem under which conditions the transformation group C^*-algebra $C^*(G, \Omega)$ of a locally compact transformation group (G, Ω) has continuous trace. One consequence will be, for compact G, that $C^*(G, \Omega)$ has continuous trace if and only if the stabilizer map is continuous. We also give a complete solution to the problem if G is discrete.

1. INTRODUCTION

If (G, Ω) is a locally compact transformation group, then the transformation group C^*-algebra $C^*(G, \Omega)$ is defined to be the crossed product algebra of the covariant system $(G, C_0(\Omega))$, with G acting on $C_0(\Omega)$, the algebra of all continuous functions on Ω which vanish at infinity, in the obvious way. In [19] Williams investigated the problem under which conditions $C^*(G, \Omega)$ has continuous trace, and it is the purpose of this paper to answer some open questions stated on page 68 of [19]. In order to explain our results, let us first recall the main theorem of [19]. For any transformation group (G, Ω), there is a canonical map $S: \Omega \to \mathscr{K}(G)$, the space of all closed subgroups of G [7], which maps each $\omega \in \Omega$ to its stability group S_ω. If this map is continuous, one can form the stabilizer algebra $C^*(\Omega^S)$ (or $C^*(\mathscr{S})$ in the setting of [19]), which can be thought of as a fiber algebra with base space Ω and fibers $C^*(S_\omega)$, with $C^*(S_\omega)$ denoting the group C^*-algebra of S_ω. Furthermore, if we define the space Ω_S as the quotient space of $\Omega \times G$ under the equivalence relation

$$(\omega, x) \sim (\omega', x') \Leftrightarrow \omega = \omega' \quad \text{and} \quad x \in x' S_\omega,$$

then the continuity of S implies that Ω_S is a locally compact Hausdorff space [19, Lemma 2.3]. In this case G is said to act σ-properly on Ω if the map $p: \Omega_S \to \Omega \times \Omega; [(\omega, x)] \to (\omega, x\omega)$ is proper in the usual sense that the inverse image of any compact set is compact. We will call a transformation group (G, Ω) weakly regular if G and Ω are second countable, or if the canonical map from G/S_ω onto the G-orbit $G(\omega)$ is a homeomorphism for all $\omega \in \Omega$. Note that (G, Ω) is always weakly regular if the action of G on Ω

Received by the editors April 17, 1991 and, in revised form, April 15, 1992.
1991 *Mathematics Subject Classification*. Primary 46L05; Secondary 22D25, 22D30, 54H15.
Key words and phrases. C^*-algebra, transformation group, continuous trace, subgroup algebra.
Supported by a fellowship from the German Research Foundation (DFG).

is σ-proper or if G is compact. The main result in [19] can now be formulated as follows (see also [16, p. 406]).

Theorem (Williams). *Let (G, Ω) be a weakly regular transformation group such that the stabilizer map S is continuous. Then $C^*(G, \Omega)$ has continuous trace if and only if $C^*(\Omega^S)$ has continuous trace and G acts σ-properly on Ω.*

In order to see how this result can be applied in certain situations Williams asked the following three questions [19, p. 68].

Q1: Is it necessary for $C^*(G, \Omega)$ to have continuous trace that the stabilizer map is continuous?

Q2: Suppose that the stabilizer map S is continuous. Does $C^*(\Omega^S)$ have continuous trace if $C^*(S_\omega)$ has continuous trace for each $\omega \in \Omega$?

And a restricted version of question Q2, namely

Q3: If the stabilizer map S is continuous and G is compact, then does $C^*(\Omega^S)$ have continuous trace?

In §2 we will show that the answer to Q1 is positive provided that every stability group is amenable. Since there is good evidence that every group with Hausdorff dual space has to be amenable, this could also be the answer in general. As a first consequence we will observe that in the case that all stability groups are abelian, $C^*(G, \Omega)$ has continuous trace if and only if S is continuous and the action of G on Ω is σ-proper. This generalizes [19, Theorem 5.1]. In §3 we are dealing with the case that all stability groups are compact. As a main result it turns out that $C^*(\Omega^S)$ has continuous trace if G is compact, giving a positive answer to question Q3. This shows, together with Williams' theorem and the result of §2, that the transformation group C^*-algebra of a compact transformation group has continuous trace if and only if the stabilizer map S is continuous. But we will also obtain some interesting results for the case that G is a Lie group and the stability groups are compact. A complete discussion whether $C^*(\Omega^S)$ has continuous trace is given in §4 for the special situations when G is discrete or G is a Lie group and all stability groups are finite. It turns out that even in these special cases there are many examples such that $C^*(\Omega^S)$ does not have continuous trace, although $C^*(S_\omega)$ has continuous trace for all $\omega \in \Omega$. This shows that the answer to question Q2 is definitely negative. All these examples are presented in §5.

This paper was written while the auther was visiting the Department of Mathematics at the University of Maryland at College Park. The author would like to take this opportunity to thank Jonathan Rosenberg for some helpful conversations.

2. The stabilizer map

Let us first recall from [4] the basic construction of general subgroup algebras. For this let Ω be any locally compact space, (G, A) a covariant system, and

$H: \Omega \to \mathscr{K}(G); \omega \to H_\omega$ a continuous map. Then $\Omega^H = \{(\omega, x) \in \Omega \times G; x \in H_\omega\}$ is a closed subset of $\Omega \times G$ and hence locally compact. The space $C_c(\Omega^H, A)$ of all continuous A-valued functions with compact support becomes a normed $*$-algebra if we define multiplication, involution and norm by

$$f * g(\omega, x) = \int_{H_\omega} f(\omega, x)^y (g(\omega, y^{-1}x))\, d_{H_\omega} y,$$

$$f^*(\omega, x) = \Delta_{H_\omega}(x^{-1})^x (f(\omega, x^{-1})^*),$$

and

$$\|f\|_1 = \sup_{\omega \in \Omega} \int_{H_\omega} \|f(\omega, x)\|\, d_{H_\omega} x.$$

Here Δ_{H_ω} denotes the modular function of H_ω for each $\omega \in \Omega$, and $(d_H)_{H \in \mathscr{K}(G)}$ denotes a smooth choice of Haar measures on $\mathscr{K}(G)$. The Ω^H-subgroup algebra $C^*(\Omega^H, A)$ is then defined to be the enveloping C^*-algebra of the completion of $C_c(\Omega^H, A)$ with respect to $\|\cdot\|_1$. Note that the set of all equivalence classes of irreducible representations of $C^*(\Omega^H, A)$ is given by the collection of all pairs (ω, ρ), where $\omega \in \Omega$ and ρ is an irreducible covariant representations of the covariant systems (H_ω, A). If f_ω is defined by $f_\omega(x) = f(\omega, x)$, $f \in C_c(\Omega^H, A)$, then $(\omega, \rho)(f) = \rho(f_\omega)$.

The stabilizer algebra $C^*(\Omega^S)$ is the special case of the construction above with $A = \mathbb{C}$ and stabilizer map S. It was first defined by Glimm in [10]. But in this paper we will also deal with the algebra $C^*(\mathscr{K}(G)^{\mathrm{Id}}, A)$, taking $\Omega = \mathscr{K}(G)$ and $H = \mathrm{Id}$ the identity on $\mathscr{K}(G)$, and the algebra $C^*(\mathscr{K}(G)^{\mathrm{Id}})$ where in addition $A = \mathbb{C}$. The latter algebra coincides with Fell's subgroup algebra as defined in [9]. If A is any C^*-algebra, then, as usual, $\hat{A}$ denotes the space of all equivalence classes of irreducible representations of A, and $\mathrm{Rep}(A)$ the space of all equivalence classes of representations with dimensions bounded by a fixed cardinal number. The basic properties of the topology on $\mathrm{Rep}(A)$ can be found in [9]. Note that by [4, Example 2] $C^*(\Omega^H, A)$ can be viewed canonically as a quotient of $C_0(\Omega) \otimes C^*(\mathscr{K}(G)^{\mathrm{Id}}, A)$ such that a representation (ω, ρ) of $C^*(\Omega^H, A)$ is identified with the representation (ω, H_ω, ρ) of $C_0(\Omega) \otimes C^*(\mathscr{K}(G)^{\mathrm{Id}}, A)$. This implies in particular that the map $\mathrm{Rep}(C^*(\Omega^H, A)) \to \mathrm{Rep}(C^*(\mathscr{K}(G)^{\mathrm{Id}}, A))$; $(\omega, \rho) \to (H_\omega, \rho)$ is continuous, which will be used frequently in this paper.

To proceed we have also to recall the following basic facts about the representation theory of transformation groups (G, Ω). If $\omega \in \Omega$ and π is a unitary representation of S_ω, then the pair (π, ω) defines a covariant representation of $(S_\omega, C_0(\Omega))$ by identifying ω with the $*$-representation of $C_0(\Omega)$ on the Hilbert space $\mathscr{H}_\pi$ of π given by $\varphi \to \varphi(\omega)\,\mathrm{Id}_{\mathscr{H}_\pi}$. It is a classical result that the induced representation $\mathrm{ind}_{S_\omega}^G(\pi, \omega)$ of $(G, C_0(\Omega))$ is always irreducible if π is irreducible. Moreover, if π and ρ are unitary representations of S_ω, then the approximation arguments given in the proof of [18, Proposition 4.2] show that any intertwining operator of $\mathrm{ind}_{S_\omega}^G(\pi, \omega)$ and $\mathrm{ind}_{S_\omega}^G(\rho, \omega)$ also intertwines the representations $(\mathrm{ind}_{S_\omega}^G \pi, M)$ and $(\mathrm{ind}_{S_\omega}^G \rho, M)$ of the imprimitivity algebra $C^*(G, G/S_\omega)$ (in the setting of [18, Lemma 4.1]), from which follows by

the imprimitivity theorem that $\operatorname{ind}_{S_\omega}^G(\pi, \omega)$ is equivalent to $\operatorname{ind}_{S_\omega}^G(\rho, \omega)$ if and only if π is equivalent to ρ. Since a C^*-algebra A with continuous trace always has a Hausdorff spectrum $\hat{A}$, the answer to Q1 follows from

Proposition 1. *Let (G, Ω) be a transformation group such that $C^*(G, \Omega)^\wedge$ is Hausdorff and every stability group is amenable. Then the map $S: \Omega \to \mathscr{K}(G)$; $\omega \to S_\omega$ is continuous.*

Proof. Assume that S fails to be continuous. Then there exists an $\omega_0 \in \Omega$ and a net $(\omega_i)_{i \in I}$ such that $\omega_i \to \omega_0$ but $S_{\omega_i} \not\to S_{\omega_0}$. Since $\mathscr{K}(G)$ is compact, we can find, by passing to a subnet if necessary, an element $K \in \mathscr{K}(G)$, $K \neq S_{\omega_0}$, such that $S_{\omega_i} \to K$ in $\mathscr{K}(G)$. As Ω is Hausdorff it follows easily that $K \subseteq S_{\omega_0}$ (see [1, Proposition 2.2-B]). Hence the pair $(1_K, \omega_0)$ defines a covariant representation of $(K, C_0(\Omega))$.

For convenience let us denote the trivial representation $1_{S_{\omega_i}}$ of S_{ω_i} by 1_i. We claim that the net $(1_i, \omega_i)$ converges to $(1_K, \omega_0)$ in $C^*(\mathscr{K}(G)^{\mathrm{Id}}, C_0(\Omega))$. To see this let $f \in C_0(\mathscr{K}(G)^{\mathrm{Id}})$ and $\varphi \in C_0(\Omega)$. Then it follows from the properties of a smooth choice of Haar measures that

$$\begin{aligned}(1_i, \omega_i)(f \otimes \varphi) &= \int_{S_{\omega_i}} \varphi(\omega_i) f(S_{\omega_i}, x)\, d_{S_{\omega_i}} x \\ &\to \int_K \varphi(\omega_0) f(K, x)\, d_K x = (1_K, \omega_0)(f \otimes \varphi).\end{aligned}$$

This proves the claim, since the set $\{f \otimes \varphi;\ f \in C_0(\mathscr{K}(G)^{\mathrm{Id}}), \varphi \in C_0(\Omega)\}$ generates a dense subset of $C^*(\mathscr{K}(G)^{\mathrm{Id}}, C_0(\Omega))$. Now it follows from [4, Corollary 2] that $\operatorname{ind}_{S_{\omega_i}}^G(1_i, \omega_i)$ converges to $\operatorname{ind}_K^G(1_K, \omega_0)$ in $\operatorname{Rep}(C^*(G, \Omega))$. By inducing in steps we notice that $\operatorname{ind}_K^G(1_K, \omega_0)$ is equivalent to $\operatorname{ind}_{S_{\omega_0}}^G(\operatorname{ind}_K^{S_{\omega_0}}(1_K, \omega_0))$. Furthermore, it is easily seen that $\operatorname{ind}_K^{S_{\omega_0}}(1_K, \omega_0)$ is equivalent to the pair $(\operatorname{ind}_K^{S_{\omega_0}} 1_K, \omega_0)$. Since S_{ω_0} is amenable it follows from [11, Theorem 5.1] that $\operatorname{ind}_K^{S_{\omega_0}} 1_K$ weakly contains the trivial representation $1_{S_{\omega_0}}$ of S_{ω_0}. But it is clear that $\operatorname{ind}_K^{S_{\omega_0}} 1_K$ is not a multiple of the trivial representation. Thus there are at least two nonequivalent irreducible representations of S_ω, say π and ρ, which are weakly contained in $\operatorname{ind}_K^{S_{\omega_0}} 1_K$. This shows that $\operatorname{ind}_{S_{\omega_i}}^G(1_i, \omega_i)$ converges to both, $\operatorname{ind}_{S_{\omega_0}}^G(\pi, \omega_0)$ and $\operatorname{ind}_{S_{\omega_0}}^G(\rho, \omega_0)$. Hence $C^*(G, \Omega)^\wedge$ cannot be a Hausdorff space. □

As $C^*(\Omega^S)$ trivially has continuous trace if all stability groups are abelian, Proposition 1 and Williams' theorem imply the following generalization of [19, Theorem 5.1].

Corollary 1. *Suppose that (G, Ω) is weakly regular and that each stability group is abelian. Then $C^*(G, \Omega)$ has continuous trace if and only if the stabilizer map S is continuous and G acts σ-properly on Ω.*

Remark 1. It follows from the remarks preceding Proposition 1 and the continuity of inducing representations that for every stability group S_ω there exists a continuous injective map from $\hat{S}_\omega$ into $C^*(G, \Omega)^\wedge$. Hence it follows that for $C^*(G, \Omega)^\wedge$ being Hausdorff it is necessary that every stability group has a

Hausdorff dual space. The author does not know any nonamenable group having this property. In fact, it was shown in [2] that all connected or countable discrete groups with Hausdorff spectrum must have relatively compact commutator groups, which implies that they are amenable. Hence there are good reasons to expect that every locally compact group with Hausdorff dual space has to be amenable. If this is true the proposition above clearly gives a general answer to Q1.

3. The compact case

In this section we are going to answer question Q3 of Williams by showing that a subgroup algebra $C^*(\Omega^H)$ always has continuous trace if G is compact. But we will also obtain some interesting results for the case that G is a Lie group and all H_ω are compact. To simplify notations let us denote by $\mathscr{K}_C(G)$ the set of compact subgroups of the locally compact group G. We start with

Definition 1. Let G be a locally compact group, Ω a locally compact space and $H : \Omega \to \mathscr{K}_C(G)$ a continuous map. We say

(1) H is *bounded* if $\bigcup_{\omega\in\Omega} H_\omega$ is relatively compact in G.
(2) H is called *locally bounded* if each $\omega \in \Omega$ has a neighborhood U in Ω such that $H|U$ is bounded, where $H|U$ denotes the restriction of H to U.

The following lemma characterizes convergence of "bounded nets" in the space of compact subgroups of a Lie group. It is a consequence of a well known result of Montgomery and Zippin and plays a crucial role in the proofs of most results in this section. Recall that a basis for the topology on $\mathscr{K}(G)$ is given by the sets

$$U(V_1, \dots, V_n; C) = \{L \in \mathscr{K}(G);\, L \cap V_i \neq \varnothing \text{ for all } 1 \leq i \leq n,\, L \cap C = \varnothing\},$$

where $V_1, \dots, V_n$ denotes any finite family of open subsets of G and $C \subseteq G$ is compact.

Lemma 1. *Let G be a Lie group and $(H_i)_{i\in I}$ a net in $\mathscr{K}_C(G)$ which converges to the compact subgroup H. Suppose further that $\bigcup_{i\in I} H_i$ is relatively compact. Then there exists a subnet $(H_j)_{j\in J}$ of $(H_i)_{i\in I}$ and a net $(x_j)_{j\in J} \subseteq G$ such that $x_j \to e$ in G and $H_j^{x_j} = x_j H_j x_j^{-1} \subseteq H$ for all $j \in J$. Furthermore, $H_j^{x_j} \to H$ in $\mathscr{K}(H)$.*

Proof. Let V be a compact neighborhood of e in G. By the theorem of Montgomery and Zippin (see page 216 of [15]), there exists an open set $O \subseteq G$ containing H such that for each compact subgroup K of G contained in O there is an $x \in V$ satisfying $xKx^{-1} \subseteq H$. Now let $C \subseteq G$ be a compact set containing H_i for all $i \in I$. Since $H_i \to H$ in $\mathscr{K}(G)$ we can assume that for each $i \in I$, H_i is an element of the neighborhood $U(O; C \setminus O)$ of H in $\mathscr{K}(G)$. But this implies that $H_i \subseteq O$ for all $i \in I$. Hence, for each $i \in I$ we can find an $y_i \in V$ such that $H_i^{y_i} \subseteq H$. Let $(y_j)_{j\in J}$ be a subnet of $(y_i)_{i\in I}$ such that $y_j \to y$ for some $y \in V$. As G acts continuously on $\mathscr{K}(G)$ by conjugation, this yields $H_j^{y_j} \to H^y$ in $\mathscr{K}(G)$, and since $\mathscr{K}(H)$ is closed in $\mathscr{K}(G)$ it follows that $H^y \subseteq H$ which implies that y belongs to the

normalizer of H. Hence, if we define $x_j = y_j y^{-1}$, then $(x_j)_{j\in J}$ has all the desired properties. □

We will now show that $C^*(\Omega^H)$ always has continuous trace if H is locally bounded and G is a Lie group. Note that any C^*-algebra A having only finite dimensional irreducible representations automatically has continuous trace if the dimension map $\dim\colon \hat{A} \to \mathbb{N};\ \rho \to \dim\rho$ is continuous. This is an easy consequence of [8, Theorem 4.3]. The converse is true if A has a unit, but not in general.

Lemma 2. *Let G be a Lie group, Ω a locally compact space, and $H\colon \Omega \to \mathscr{K}_C(G)$ a locally bounded continuous map. Then the map*

$$\dim\colon C^*(\Omega^H)^\frown \to \mathbb{N};\ (\omega, \rho) \to \dim\rho$$

is continuous. In particular, $C^*(\Omega^H)$ has continuous trace.

Proof. Let $((\omega_i, \rho_i))_{i\in I}$ be a net in $C^*(\Omega^H)^\frown$ converging to some $(\omega, \rho) \in C^*(\Omega^H)^\frown$. Then $(H_{\omega_i}, \rho_i) \to (H_\omega, \rho)$ in $C^*(\mathscr{K}(G)^{\mathrm{Id}})^\frown$. Since H is locally bounded, we can assume that there is a compact set $C \subseteq G$ such that $H_{\omega_i} \subseteq C$ for all $i \in I$. Hence, by Lemma 1 and by passing to a subnet if necessary, we can find a net $(x_i)_{i\in I} \subseteq G$ converging to e in G such that $H_{\omega_i}^{x_i} \subseteq H_\omega$ for all $i \in I$. Now for each $i \in I$ let us denote by $\rho_i^{x_i}$ the conjugate of ρ_i by x_i, i.e., $\rho_i(h) = \rho(x_i^{-1}hx_i)$ for all $h \in H_i^{x_i}$. Since conjugation is continuous on $C^*(\mathscr{K}(G)^{\mathrm{Id}})^\frown$ (see [9, Lemma 2.6] or [4, §4] for general subgroup algebras), it follows that $(H_{\omega_i}^{x_i}, \rho_i^{x_i}) \to (H_\omega, \rho)$. Thus $\operatorname{ind}_{H_{\omega_i}^{x_i}}^{H_\omega} \rho_i^{x_i} \to \rho$ in $\operatorname{Rep}(H_\omega)$ by the continuity of inducing representations. As $\hat{H}_\omega$ is discrete this implies, by passing to another subnet if necessary, that ρ is a subrepresentation of $\operatorname{ind}_{H_{\omega_i}^{x_i}}^{H_\omega} \rho_i^{x_i}$ for all $i \in I$. Hence it follows from the Frobenius reciprocity theorem that $\rho_i^{x_i}$ is a subrepresentation of $\rho|H_{\omega_i}^{x_i}$. This shows that $\dim\rho_i \le \dim\rho$ for all $i \in I$. But a finite dimensional irreducible representation can never be approximated by representations of lower dimension [3, Proposition 3.6.3]. Hence we conclude that there is an $i_0 \in I$ such that $\dim\rho_i = \dim\rho$ for all $i \ge i_0$ and the lemma is proved. □

Clearly, this lemma already gives an answer to question Q3 in the case where G is a Lie group. At the end of this section we will also state some consequences of Lemma 2 for transformation groups (G, Ω) where G is a Lie group. But let us first show how to avoid the Lie group assumption for compact G.

Theorem 1. *Let G be a locally compact group, Ω a locally compact space, and $H\colon \Omega \to \mathscr{K}_C(G)$ a continuous map. Suppose that each $\omega_0 \in \Omega$ has a neighborhood V such that H_ω is contained in a fixed compact subgroup, say K, of G for all $\omega \in V$. Then $C^*(\Omega^H)$ has continuous trace.*

Proof. We again show that the dimension map $\dim\colon C^*(\Omega^H)^\frown \to \mathbb{N}$ is continuous. So let $(\omega_i, \rho_i)_{i\in I}$ be a net in $C^*(\Omega^H)^\frown$ converging to some (ω, ρ). We may assume that H_{ω_i} is contained in the compact subgroup K of G for all $i \in I$. Hence we get $(H_{\omega_i}, \rho_i) \to (H_\omega, \rho)$ in $C^*(\mathscr{K}(K)^{\mathrm{Id}})^\frown$. This implies

$\operatorname{ind}_{H_{\omega_i}}^K \rho_i \to \operatorname{ind}_{H_\omega}^K \rho$ in $\operatorname{Rep}(K)$. Now let π be any irreducible subrepresentation of $\operatorname{ind}_{H_\omega}^K \rho$. Then $\operatorname{ind}_{H_{\omega_i}}^K \rho_i$ also converges to π in $\operatorname{Rep}(K)$ by [9, Proposition 1.3]. Since $\hat{K}$ is discrete we get, by passing to a subnet if necessary, that π is also contained in $\operatorname{ind}_{H_{\omega_i}}^K \rho_i$ for all $i \in I$. Hence it follows from the Frobenius reciprocity theorem that ρ_i is contained in $\pi|H_{\omega_i}$ for all $i \in I$ and that ρ is contained in $\pi|H_\omega$. Thus, if $N = \ker \pi$ denotes the group kernel of π in K, then $N \cap H_{\omega_i}$ is contained in the kernel of ρ_i for all $i \in I$.

We define $\dot{H}_i := H_{\omega_i}N/N$ and $\dot{\rho}_i$ by $\dot{\rho}_i(hN) = \rho(h)$ for all $h \in H_{\omega_i}$ and $i \in I$. We claim that $(\dot{H}_i, \dot{\rho}_i) \to (\dot{H}, \dot{\rho})$, where $(\dot{H}, \dot{\rho})$ is defined similarly from (H_ω, ρ). This will finish the proof because K/N is a Lie group and we can apply Lemma 2 to see that $\dim \rho_i \to \dim \rho$.

As a first step to prove the claim, note that it follows easily from the definition of the topologies on $\mathscr{K}(K)$ and $\mathscr{K}(K/N)$ that $\dot{H}_i \to \dot{H}$ in $\mathscr{K}(K/N)$. Now recall that by [9, Theorem 3.1′] for any locally compact group G the topology on $C^*(\mathscr{K}(G)^{\mathrm{Id}})^\frown$ can be described in terms of positive definite functions on the subgroups. To use this description let ϕ be a positive definite function on H associated to ρ, i.e., $\phi(h) = \langle \rho(h)\xi, \xi\rangle$ for all $h \in H$ and some $\xi \in \mathscr{H}_\rho$. Assume further that $(H_j, \rho_j)_{j\in J}$ is a subnet of $(H_{\omega_i}, \rho_i)_{i\in I}$ such that for each $j \in J$ there exist functions ϕ_j such that

(1) H_j is the domain of ϕ_j for all $j \in J$.

(2) $\phi_j(h) = \langle \rho_j(h)\xi_j, \xi_j\rangle$ for all $h \in H_j$ and some $\xi_j \in \mathscr{H}_{\rho_j}$.

(3) If $(H_l)_{l\in L}$ is a subnet of $(H_j)_{j\in J}$ and $h_l \in H_l$ for all $l \in L$ such that $h_l \to h$ in K for some $h \in H$, then $\phi_l(h_l) \to \phi(h)$ in $\mathbb{C}$.

Furthermore, let $\dot{\phi}(hN) = \phi(h)$ for all $hN \in \dot{H}$, and define similarly $\dot{\phi}_j$, for each $j \in J$, by $\dot{\phi}_j(hN) = \phi_j(h)$, $hN \in \dot{H}_j$. Using [9, Theorem 3.1′] and the remark following it, it is enough to show that $(\dot{\phi}_j)_{j\in J}$ also satisfies condition (3) relative to $\dot{\phi}$. To see this let $(\dot{H}_l)_{l\in L}$ be a subnet of $(\dot{H}_j)_{j\in J}$ and $h_lN \in \dot{H}_l$ such that $h_lN \to hN$ in K/N for some $h \in H$. By passing to a subnet if necessary, we can find $n_l \in N$ such that $h_ln_l \to h$ in K. Since N is compact, again by passing to a subnet if necessary, we can further assume that $n_l \to n$ for some $n \in N$. Therefore, $h_l \to hn^{-1}$ in K. It follows that $hn^{-1} \in H$, since $H_l \to H$ in $\mathscr{K}(K)$. Hence we get

$$\dot{\phi}_l(h_lN) = \phi_l(h_l) \to \phi(h) = \dot{\phi}(hN),$$

and the theorem is proved. □

Corollary 2. *Let G be a compact group acting on the locally compact space Ω. Then the following conditions are equivalent*:

(1) *The stabilizer map S is continuous.*

(2) *$C^*(G, \Omega)$ has continuous trace.*

(3) *$C^*(G, \Omega)^\frown$ is Hausdorff.*

Proof. Since the action of compact groups is always σ-proper if the stabilizer map S is continuous, the corollary follows easily from Theorem 1, Proposition 1, and Williams' theorem. □

Clearly, it follows from Theorem 2 that the corollary remains to be true under the slightly more general assumptions that G acts σ-properly on Ω and

that for each $\omega_0 \in \Omega$ there exists a neighborhood V of ω_0 such that S_ω is contained in a fixed compact subgroup of G for all $\omega \in V$.

Recall now that for any locally compact transformation group (G, Ω) the action of G on Ω is called proper, if the map

$$P : \Omega \times G \to \Omega \times \Omega; \qquad (\omega, x) \to (\omega, x\omega)$$

is a proper map. It is clear that the action of a compact group is always proper. In fact, a straightforward proof shows that, if S is continuous, G acts properly on Ω if and only if G acts σ-properly on Ω and S is locally bounded. Hence in the Lie group case we get the following generalization of Corollary 2.

Corollary 3. *Let G be a Lie group acting properly on the locally compact space Ω. Then $C^*(G, \Omega)$ has continuous trace if and only if the stabilizer map S is continuous.*

The proof follows directly from Proposition 1, Lemma 2, and Williams' theorem. In fact, these results also imply

Corollary 4. *Suppose (G, Ω) is weakly regular, G is a Lie group, and the stabilizer map S is continuous and locally bounded. Then $C^*(G, \Omega)$ has continuous trace if and only if G acts properly on Ω.*

We now look for situations in which a given map $H: \Omega \to \mathscr{K}_C(G)$ is automatically locally bounded.

Definition 2. Let G be a locally compact group, Ω a locally compact space, and $H: \Omega \to \mathscr{K}_C(G)$ a continuous map. Furthermore, let $\omega_0 \in \Omega$, W a neighborhood of ω_0 in Ω, and $L : W \to \mathscr{K}_C(G)$ a continuous map such that

(1) L is bounded and $L_{\omega_0} = H_{\omega_0}$.
(2) There exists an open set $O \subseteq G$ such that $L_\omega = H_\omega \cap O$ for all $\omega \in W$.

Then the pair (W, L) is called a *bounded part of H in ω_0*. Furthermore (W, L) is called *proper*, if $H|V \neq L|V$ for all neighborhoods V of ω_0 contained in W.

The following lemma shows that bounded parts always exist. The result will be used extensively in the next section.

Lemma 3. *Let $H: \Omega \to \mathscr{K}_C(G)$ be a continuous map and let $\omega_0 \in \Omega$. Then there exists a bounded part (W, L) of H in ω_0. If O is an open subset of G containing H_{ω_0} such that the closure $\bar{O}$ of O is compact, then there exists a neighborhood V of ω_0 in Ω such that $L_\omega = H_\omega \cap O$ for all $\omega \in V$. In particular, (W, L) is locally unique in the sense that for any other bounded part (W', L') of H in ω_0 there exists a neighborhood W'' of ω_0 contained in $W \cap W'$ such that $L|W'' = L'|W''$.*

Proof. Let O be an open subset of G such that $H_{\omega_0} \subseteq O$ and $\bar{O}$, the closure of O, is compact. We claim that there exists a neighborhood W of ω_0 such that the pair (W, L), with L defined by $L_\omega = H_\omega \cap O$, is a bounded part of H in ω_0.

To prove the claim let V be an open neighborhood of e in G such that $H_{\omega_0} V^2 \subseteq O$ and let U be another neighborhood of E in G such that $xUx^{-1} \subseteq$

V for all $x \in H_{\omega_0}$. Since H is continuous we can find a neighborhood W of ω_0 such that $H_\omega \in U(H_{\omega_0}U; \bar{O} \setminus H_{\omega_0}U)$ for all $\omega \in W$. Now, for $\omega \in W$, let $L_\omega = H_\omega \cap O$. It is clear that $L_{\omega_0} = H_{\omega_0}$. We have to show that L_ω is a subgroup of H_ω for $\omega \in W$. For this let $x, y \in L_\omega$. Then $x, y \in H_{\omega_0}U$. Hence we can find $x', y' \in H_{\omega_0}$ and $u, v \in U$ such that $x = x'u$ and $y = y'v$. Thus $xy = x'uy'v = x'y'y'^{-1}uy'v \in H_{\omega_0}V^2 \subseteq O$. Hence $xy \in L_\omega$ which shows in fact that L_ω is an open subgroup of H_ω. The continuity of L follows from the fact that L_ω is contained in an open set $U(V_1, \dots, V_n; C) \subseteq \mathscr{K}(G)$ if and only if $H_\omega \in U(V_1 \cap O, \dots, V_n \cap O; C \cap H_{\omega_0}\bar{U})$, and it follows directly from the construction of L that L is bounded.

Suppose now that (W', L') is another bounded part of H in ω_0. Then, by definition, there exists a relative compact open set $O' \subseteq G$ such that $L'_\omega = H_\omega \cap O'$ for all $\omega \in W'$. Let $V = H^{-1}(U(O \cap O'; \bar{O}' \setminus O))$. Then $V \subseteq W \cap W'$ and $L_\omega = H_\omega \cap O = L'_\omega$ for all $\omega \in V$. This proves the lemma. □

It follows from Lemma 3 that a continuous map $H: \Omega \to \mathscr{K}(G)$ is locally bounded if and only if it has no proper bounded parts. Since for a bounded part (W, L) of H, L_ω is always open in H_ω for each $\omega \in W$, we get

Corollary 5. *Let $H: \Omega \to \mathscr{K}_C(G)$ be a continuous map such that H_ω is connected for all $\omega \in \Omega$. Then H is locally bounded.*

We close this section with a proposition about the important case of a transformation group where all stabilizers are contained in the G-orbit of some compact subgroup of G.

Proposition 2. *Suppose that G is a Lie group and (G, Ω) is weakly regular. Assume further that the stabilizers are compact and conjugate to each other. Then $C^*(G, \Omega)$ has continuous trace if and only if G acts properly on Ω.*

Proof. Assume first that G acts properly on Ω. We show that the stabilizer map is continuous. For this let $\omega_i \to \omega$ in Ω. By passing to a subnet if necessary, we can assume that there exists a subgroup $H \subseteq S_\omega$ such that $S_{\omega_i} \to H$ in $\mathscr{K}(G)$. Since G acts properly on Ω, we can further assume that $\bigcup_{i \in I} S_{\omega_i}$ is relatively compact in G. Hence by Lemma 1 we can pass to another subnet to find a net $x_i \to e$ in G such that $S_{\omega_i}^{x_i} \subseteq H$. But this implies already that $H = S_\omega$ since S_ω itself is a conjugate of S_{ω_i}. Hence the "if part" follows from Corollary 4.

Assume now that $C^*(G, \Omega)$ has continuous trace. Then Proposition 1 implies that S is continuous. We show that this also yields that S is locally bounded. For this purpose assume that there is an $\omega_0 \in \Omega$ such that there exists a proper bounded part (W, L) of S in ω_0. Then we can find a net $\omega_i \to \omega_0$ in W such that $L_{\omega_i} \neq S_{\omega_i}$ for all $i \in I$. By Lemma 1 and passing to a subnet, we can find a net $x_i \to e$ in G such that $L_{\omega_i}^{x_i} \subseteq S_{\omega_0}$ for all $i \in I$. Since L_{ω_i} is an open subgroup of a conjugate of S_{ω_0}, it follows easily that the connected component, say N, of S_{ω_0} is contained in $L_{\omega_i}^{x_i}$ for all $i \in I$. Hence we get $L_{\omega_i}^{x_i}/N \to S_{\omega_0}/N$ in $\mathscr{K}(S_{\omega_0}/N)$. But since the latter space is finite we may as well assume that $L_{\omega_i}^{x_i} = S_{\omega_0}$ for all $i \in I$. But this shows that L_{ω_i} is also a conjugate of S_{ω_0} which contradicts the assumption that $L_{\omega_i} \neq S_{\omega_i}$. Hence there is no proper bounded part for S, which implies that S is locally bounded. The proof now follows from Corollary 4. □

4. The finite and the discrete case

In this section we give a complete description of subgroup algebras $C^*(\Omega^H)$ with continuous trace in the special cases that G is a Lie group and all H_ω are finite or that G is discrete. As we will see in the next section, this gives rise to many examples of transformation groups (G, Ω) such that $C^*(\Omega^S)$ does not have continuous trace although the stabilizer map S is continuous and $C^*(S_\omega)$ has continuous trace for all $\omega \in \Omega$, thus giving a negative answer to question Q2. We start with the result about finite subgroups.

Theorem 2. *Let G be a Lie group, Ω a locally compact space, and $H: \Omega \to \mathscr{K}(G)$ a continuous map such that H_ω is finite for all $\omega \in \Omega$. Then $C^*(\Omega^H)$ has continuous trace if and only if for every $\omega_0 \in \Omega$ there exists a bounded part (W, L) of H in ω_0 such that the restriction $\pi|L_\omega$ is irreducible for all $\omega \in W$ and $\pi \in \hat{H}_\omega$.*

For the proof we need several lemmas. The first one shows that the "if part" of this theorem is also true if H_ω is only assumed to be compact. It follows directly from Lemma 2 by using the continuity of restricting representations.

Lemma 4. *Let G be a Lie group and $H : \Omega \to \mathscr{K}_C(G)$ be a continuous map such that every $\omega_0 \in \Omega$ has a bounded part (W, L) such that $\pi|L_\omega$ is irreducible for every $\omega \in W$ and $\pi \in \hat{H}_\omega$. Then $C^*(\Omega^H)$ has continuous trace.*

Lemma 5. *Let G be arbitrary, $H: \Omega \to \mathscr{K}_C(G)$ be a continuous map and (W, L) a bounded part of H in ω_0. Furthermore, let $(\omega_i)_{i\in I}$ be a net in W converging to ω_0 and for each $i \in I$ let π_i be a representation of H_{ω_i}. Then for any $\pi \in \hat{H}_{\omega_0}$, $(\omega_i, \pi_i) \to (\omega_0, \pi)$ in $\mathrm{Rep}(C^*(\Omega^H))$ if and only if $(\omega_i, \pi_i|L_{\omega_i}) \to (\omega_0, \pi)$ in $\mathrm{Rep}(C^*(W^L))$.*

Proof. The "only if part" follows directly from the continuity of restricting representations. To show the if part, let O be an open neighborhood of the identity such that $L_\omega = H_\omega \cap O$ for all $\omega \in W$, and let f be a nonnegative function on G such that $\operatorname{supp} f \subseteq O$ and $f(e) \neq 0$. Then, by the construction of a smooth choice of Haar measures on $\mathscr{K}(G)$ given in [10], we may assume that $\int_{H_\omega} f(h)\, d_{H_\omega}h = \int_{L_\omega} f(l)\, d_{L_\omega}l = 1$ for all $\omega \in W$. Hence by the choice of f it follows that the Haar measure on L_ω is given as the restriction of the Haar measure of H_ω for all $\omega \in W$.

Now let $\pi \in \mathrm{Rep}(H_{\omega_0})$ and assume that $(\omega_i, \pi_i|L_{\omega_i}) \to (\omega_0, \pi)$ in $\mathrm{Rep}(C^*(W^L))$. Furthermore, let $g \in C_c(\Omega^H)$. Let C denote the compact projection of $\operatorname{supp} g$ to G, and let O be an open subset of G with compact closure $\bar{O}$ such that $C \cup H_{\omega_0} \subseteq O$. By Lemma 3 we can find a neighborhood V of ω_0 in W such that $L_\omega = H_\omega \cap O$ for all $\omega \in V$. Hence, for all $\omega \in V$, it follows that $\{h \in H_\omega;\, g(\omega, h) \neq 0\} \subseteq L_\omega$. Thus, if $\omega_i \in V$ and $\xi \in \mathscr{H}_{\pi_i}$, we can find an element $\tilde{g} \in C_c(W^L)$ such that

$$\int_{H_{\omega_i}} g(\omega_i, h)\langle \pi_i(h)\xi, \xi\rangle\, d_{H_{\omega_i}}h = \int_{L_{\omega_i}} \tilde{g}(\omega_i, l)\langle \pi_i(l)\xi, \xi\rangle\, d_{L_{\omega_i}}l.$$

But this shows easily that a positive functional of $C^*(\Omega^H)$ associated to π is a weak*-limit of positive functionals associated to any subnet of $((\omega_i, \pi_i))_{i\in I}$ if the corresponding positive functional of $C^*(W^L)$ is a weak*-limit of positive

functionals associated to the corresponding subnet of $((\omega_i, \pi_i|L_{\omega_i}))_{i\in I}$. Hence the result follows from [9, Proposition 1.1]. □

Lemma 6. *Suppose that Ω is compact, G is a Lie group, and $H\colon \Omega \to \mathscr{K}(G)$ is a continuous map such that H_ω is finite for all $\omega \in \Omega$. Then $C^*(\Omega^H)$ has a unit.*

Proof. We claim first that there exists a neighborhood V of e in G such that $H_\omega \cap V = \{e\}$ for all $\omega \in \Omega$. In order to prove this let $\omega_0 \in \Omega$ and let (W, L) be a bounded part of H in ω_0. Let $V_1 = V_1^{-1}$ be a compact neighborhood of the identity such that $V_1^3 \cap H_{\omega_0} = \{e\}$. Then we can find a neighborhood $V \subseteq V_1$ of e in G such that for every subgroup H of G contained in $H_{\omega_0}V$ there exists a $y \in V_1$ such that $yHy^{-1} \subseteq H_{\omega_0}$. By the construction of bounded parts we may assume that $L_\omega = H_\omega \cap H_{\omega_0}V$ for all $\omega \in \Omega$. Now, for $\omega \in W$, let $x \in H_\omega \cap V$. Then $x \in L_\omega$ and we can find $y \in V_1$ such that $yxy^{-1} \in H_{\omega_0}$. Hence $yxy^{-1} \in V_1^3 \cap H_{\omega_0}$ from which follows that $x = yxy^{-1} = e$. The proof of the claim now follows from the fact that Ω is compact.

Now let V be as in the claim and let f be a nonnegative function with support in V such that $f(e) = 1$. We can assume that $\sum_{x\in H_\omega} f(x) = 1$ for all $\omega \in \Omega$. Now define $\tilde{f} \in C_c(\Omega^H)$ by $\tilde{f}(\omega, x) = f(x)$. Then $\tilde{f}$ is a unit for $C^*(\Omega^H)$. □

Proof of Theorem 2. Assume that $C^*(\Omega^H)$ has continuous trace and that there exists an ω_0 in Ω such that for all bounded parts (W, L) of H in ω_0 there exists an $\omega \in W$ and a $\pi \in \hat{H}_\omega$ such that $\pi|L_\omega$ is reducible. Since all these properties are local, we may assume that Ω is compact. By the assumption we can find a bounded part (W, L) of H in ω_0 and a net $((\omega_i, \pi_i))_{i\in I} \subseteq C^*(\Omega^H)^\wedge$ such that $\omega_i \to \omega_0$ in W and $\pi|L_{\omega_i}$ is reducible for all $i \in I$. By passing to a subnet if necessary and using Lemma 1, we can find a net $(x_i)_{i\in I} \subseteq G$ such that $x_i \to e$ and $L_{\omega_i}^{x_i} \subseteq H_{\omega_0}$. As H_{ω_0} is finite we may therefore assume that $L_{\omega_i}^{x_i} = H_{\omega_0}$ for all $i \in I$. Again by passing to a subnet if necessary, we can find a $\rho \in \hat{H}_{\omega_0}$ such that ρ is a subrepresentation of $(\pi_i|L_{\omega_i})^{x_i}$ for all $i \in I$. But this yields that $(\pi_i|L_{\omega_i})^{x_i} \to \rho$ in $\mathrm{Rep}(H_{\omega_0})$ from which it follows that $(\omega_i, \pi_i|L_{\omega_i}) \to (\omega_0, \rho)$ in $\mathrm{Rep}(C^*(W^L))$. Thus by Lemma 5 we know that $(\omega_i, \pi_i) \to (\omega_0, \rho)$ in $C^*(\Omega^H)^\wedge$. Now observe that $\dim \pi_i > \dim \rho$ for all $i \in I$ since ρ is a proper subrepresentation of $(\pi_i|L_{\omega_i})^{x_i}$. This shows that the dimension map is not continuous on $C^*(\Omega^H)^\wedge$. But this is a contradiction to the assumption that $C^*(\Omega^H)$ has continuous trace because we may assume by Lemma 6 that $C^*(\Omega^H)$ has a unit. □

Before proceeding with the discrete case we would like to state the consequences of Theorem 2 for transformation groups.

Corollary 6. *Suppose that (G, Ω) is a weakly regular transformation group, G is a Lie group, and S_ω is finite for all $\omega \in \Omega$. Then $C^*(G, \Omega)$ has continuous trace if and only if S is continuous, G acts σ-properly on Ω and for all $\omega_0 \in \Omega$ there exists a bounded part (W, L) of S in ω_0 such that $\pi|L_\omega$ is irreducible for all $\pi \in \hat{S}_\omega$ and $\omega \in W$.*

As mentioned earlier, it is well known that G requires a finite commutator subgroup, if G is a countable discrete group with Hausdorff dual space.

However, the countability assumption is only used to ensure that G is of type I which is necessary to use Thoma's result about the structure of discrete groups of type I [17]. Since every group with continuous trace group C^*-algebra is of type I, it follows that every discrete group with continuous trace C^*-algebra must have a finite commutator subgroup. On the other side it is well known that every discrete type I group with finite commutator subgroup has continuous trace. This can easily be obtained by the fact that all irreducible representations of such groups are finite dimensional, and by using the Mackey theory to show that the dimension map on $\hat{G}$ is continuous. More generally, it was shown by Kaniuth [13, Corollary] (see also [6, Lemma 6]) that all type I locally compact groups with relatively compact conjugacy classes have continuous trace group C^*-algebras. Note that for a discrete group G being of type I and having a finite commutator subgroup is equivalent to the statement that the center has finite index in G. We will now state our result for discrete groups.

Theorem 3. *Let G be a discrete group and $H\colon \Omega \to \mathscr{K}(G)$ a continuous map. Then $C^*(\Omega^H)$ has continuous trace if and only if*

(1) *H_ω is of type* I *and has a finite commutator subgroup for all $\omega \in \Omega$.*
(2) *For each $\omega_0 \in \Omega$ there exists a neighborhood V of ω_0 in Ω such that for each $\omega \in V$ and $\pi \in \hat{H}_\omega$ the restriction of π to $H_\omega \cap H_{\omega_0}$ is irreducible.*

Recall that one of the main tools in the proof of the compact case was the Frobenius reciprocity theorem for compact groups. But it was shown by Kaniuth [14] that a weak version of this theorem holds for all locally compact groups with relatively compact conjugacy classes. In the discrete case this was proved by Henrichs [12], and this is all we need in the proof of the following

Lemma 7. *Let G be a type* I *discrete group with finite commutator subgroup. Then the map $C^*(\mathscr{K}(G)^{\mathrm{Id}})^\frown \to \mathbb{N}$; $(H, \pi) \to \dim \pi$ is continuous at (G, ρ) for all $\rho \in \hat{G}$.*

Proof. Since every subgroup of a type I discrete group is also of type I, it follows easily that all irreducible representations of $C^*(\mathscr{K}(G)^{\mathrm{Id}})$ are finite dimensional. Let $\rho \in \hat{G}$ and $(H_i, \rho_i)_{i\in I}$ be a net in $C^*(\mathscr{K}(G)^{\mathrm{Id}})^\frown$ converging to (G, ρ). Then $\mathrm{ind}_{H_i}^G \rho_i \to \mathrm{ind}_H^G \rho$ in $\mathrm{Rep}(C^*(G))$. Hence, by passing to a subnet if necessary, we can find representations $\pi_i \in \hat{G}$ such that π_i is weakly contained in $\mathrm{ind}_{H_i}^G \rho_i$ for all $i \in I$ and $\pi_i \to \rho$ in $\hat{G}$. Since the set of irreducible representations with dimension equal to $\dim \rho$ forms an open set in $\hat{G}$, we can assume that $\dim \pi_i = \dim \rho$ for all $i \in I$. By the weak Frobenius reciprocity theorem it now follows that ρ_i is weakly contained in $\pi_i|H_i$ for all $i \in I$. But since π_i is finite dimensional this implies that ρ_i is in fact contained in $\pi_i|H_i$ for all $i \in I$. Hence $\dim \rho_i \leq \dim \rho$ for all $i \in I$ from which it follows that there exists $i_0 \in I$ such that $\dim \rho_i = \dim \rho$ for all $i \geq i_0$. □

It should be remarked that the dimension map is in general not continuous on all of $C^*(\mathscr{K}(G))^\frown$, even if G is a type I discrete group with finite commutator subgroup. A counterexample will be given in the next section (Example 3).

Proof of Theorem 3. Let us first assume that H satisfies conditions 1 and 2 of the theorem. We show again that the dimension map on $C^*(\Omega^H)^\frown$ is continuous.

So let $(\omega_i, \rho_i) \to (\omega_0, \rho)$ in $C^*(\Omega^H)^\wedge$. Furthermore, let $L_i = H_{\omega_i} \cap H_{\omega_0}$ for all $i \in I$. We claim that $L_i \to H_{\omega_0}$ in $\mathscr{K}(H_{\omega_0})$. To see this let $V_1, \dots, V_n$ be any finite family of sets in H_{ω_0}. Then $L_i \in U(V_1, \dots, V_n; \varnothing)$ if and only if $H_{\omega_i} \in U(V_1, \dots, V_n, \varnothing)$ and the claim follows from the fact that $H_{\omega_i} \to H_{\omega_0}$. By the continuity of restricting representations it follows that $(L_i, \rho_i|L_i) \to (H_{\omega_0}, \rho)$ in $\mathrm{Rep}(C^*(\mathscr{K}(H_{\omega_0})^{\mathrm{Id}}))$. By condition 2 we can assume that $\rho_i|L_i$ is irreducible for all $i \in I$. Hence we can apply Lemma 7 to see that $\dim \rho_i \to \dim \rho$.

Now let us assume conversely that $C^*(\Omega^H)$ has continuous trace. Then it is clear that $C^*(H_\omega)$ must have continuous trace for all $\omega \in \Omega$ since this algebra is a quotient of $C^*(\Omega^H)$. This implies condition 1. To prove condition 2 assume that there is a net $((\omega_i, \rho_i))_{i \in I}$ in $C^*(\Omega^H)^\wedge$ such that $\omega_i \to \omega_0$ for some $\omega_0 \in \Omega$ and such that $\rho_i|H_{\omega_i} \cap H_{\omega_0}$ is reducible for all $i \in I$. Again let $L_i = H_{\omega_i} \cap H_{\omega_0}$ for all $i \in I$. Furthermore, let V be any compact neighborhood of ω_0 in Ω and assume that $\omega_i \in V$ for all $i \in I$. Since V is compact, it follows that $C^*(V^H)$ has a unit, and therefore $C^*(V^H)^\wedge$ is compact. This implies, by passing to a subnet if necessary, that there exists a $\rho \in \hat{H}_{\omega_0}$ such that $(\omega_i, \rho_i) \to (\omega_0, \rho)$ in $C^*(V^H)^\wedge$. It follows that $(L_i, \rho_i|L_i) \to (H_{\omega_0}, \rho)$ in $C^*(\mathscr{K}(H_{\omega_0})^{\mathrm{Id}})^\wedge$. By passing to another subnet if necessary, we find irreducible subreprepresentations π_i of $\rho_i|L_i$ such that $(L_i, \pi_i) \to (H_{\omega_0}, \rho)$. By Lemma 7 we can assume that $\dim \pi_i = \dim \rho$ for all $i \in I$. But this implies that $\dim \rho_i > \dim \rho$ for all $i \in I$ since by assumption $\rho_i|L_i$ is not irreducible for all $i \in I$. This shows that the dimension map is not continuous on $C^*(V^H)^\wedge$, which implies that $C^*(V^H)$ has not continuous trace since $C^*(V^H)$ has a unit. Finally this implies that $C^*(\Omega^H)$ has not continuous trace because it has $C^*(V^H)$ as a quotient. Hence we have a contradiction and the theorem is proved. □

Corollary 7. *Suppose that (G, Ω) is a weakly regular transformation group such that G is discrete. Then $C^*(G, \Omega)$ has continuous trace if and only if*

(1) *Every stability group is of type* I *and has a finite commutator subgroup.*
(2) *The stabilizer map S is continuous.*
(3) *G acts σ-properly on Ω.*
(4) *For each $\omega_0 \in \Omega$ there exists a neighborhood V of ω_0 such that $\pi|S_\omega \cap S_{\omega_0}$ is irreducible for all $\pi \in \hat{S}_\omega$ and $\omega \in V$.*

Proof. It follows from Theorem 3 and Williams' theorem that the four conditions in the corollary imply that $C^*(G, \Omega)$ has continuous trace. Assume now that $C^*(G, \Omega)$ has continuous trace. Since G is discrete it follows that $C^*(S_\omega, \Omega)$ is a subalgebra of $C^*(G, \Omega)$ for $\omega \in \Omega$. But this implies that $C^*(S_\omega, \Omega)$ is of type I for all $\omega \in \Omega$. As $C^*(S_\omega)$ is a quotient of $C^*(S_\omega, \Omega)$ it follows that S_ω must be of type I, too. Now Thoma's theorem implies that S_ω is amenable for all $\omega \in \Omega$. Hence condition 2 follows from Proposition 1 and conditions 1, 3, and 4 now are consequences from Williams' theorem and Theorem 3. □

To complete this section we would like to add a result about the property that the restrictions of all irreducible representations of a group G to a fixed subgroup, say H, are irreducible.

Proposition 3. *Let G be a locally compact group with relatively compact conjugacy classes and let H be a closed subgroup of G such that $\pi|H$ is irreducible for all $\pi \in \hat{G}$. Then H is a normal subgroup of G and G/H is abelian. Furthermore, $[G, G] = [H, H]$, where $[G, G]$ and $[H, H]$ denote the closed commutator subgroups of G and H, respectively. In particular, if H is abelian, then so is G.*

Proof. Let χ be any character (i.e., one-dimensional representation) of H, and suppose that $\pi \in \hat{G}$ is weakly contained in $\operatorname{ind}_H^G \chi$. Then by the weak Frobenius reciprocity theorem it follows that χ is weakly contained in $\pi|H$ and $\pi|H$ is irreducible by assumption. Furthermore $\operatorname{Prim}(H)$, the primitive ideal space of H, is Hausdorff by [13, Theorem 1]. It follows that $\pi|H$ and χ have the same kernel in $C^*(H)$, which shows that $\chi = \pi|H$ since χ is one-dimensional. Applying this to the trivial representation 1_H of H it follows easily that the set Γ of all $\pi \in \hat{G}$ such that π is weakly contained in $\operatorname{ind}_H^G 1_H$ forms a group of characters. We claim that $\bigcap_{\chi \in \Gamma} \ker \chi$, the intersection of the group kernels, is equal to H. Since the intersection of the group kernels of all characters of G is $[G, G]$, this will imply that $H \supseteq [G, G]$, and hence that H is a normal subgroup of G such that G/H is abelian. To prove the claim let $L = \bigcap_{\chi \in \Gamma} \ker \chi$. Then $1_L = \chi|L$ for all $\chi \in \Gamma$. Using the fact that G is amenable, we obtain easily from [11, Theorem 5.1] that, in case $L \neq H$, there exists at least one irreducible representation, say ρ, of L which is not equivalent to 1_L, but which is weakly contained in $\operatorname{ind}_H^L 1_H$ (compare the arguments in the proof of Proposition 1). By induction in stages and continuity of inducing representations, it follows that each representation which is weakly contained in $\operatorname{ind}_L^G \rho$ must also be weakly contained in $\operatorname{ind}_H^G 1_H$, and hence lies in Γ. But the restrictions of those representations to L weakly contain ρ by Frobenius, and therefore are not equal to 1_L. This is a contradiction and the claim is proved.

To see that $[G, G] = [H, H]$ recall that $[H, H]$ is the intersection of the group kernels of all characters of H. But as we have seen above, every character of H can be extended to one of G. Hence the intersection of all group kernels of characters of G is contained in $[H, H]$ which shows $[G, G] = [H, H]$. □

5. Examples

In this final section we will give some examples which show that the answer to question Q3 is negative, and which illustrate what kind of situations may occur. But before we will give our examples we would like to show how any continuous map $H: \Omega \to \mathscr{K}(G)$ gives rise to a transformation group with σ-proper action such that the stabilizer map has almost the same properties as H. Similar to the construction of the space Ω_S in the introduction we define Ω_H to be the quotient space of $\Omega \times G$ by the equivalence relation

$$(\omega, x) \sim (\omega', x') \Leftrightarrow \omega = \omega' \quad \text{and} \quad x' \in xH_\omega.$$

We can look at Ω_H as a fiber space with base space Ω and fibers G/H_ω, and it makes sense to denote the elements of Ω_H by $(\omega, \dot{x})$, where $\dot{x}$ denotes the left H_ω coset of $x \in G$. The proof of [19, Lemma 2.3] shows that Ω_H is a locally compact Hausdorff space, and that the quotient map $q: \Omega \times G \to \Omega_H$ is open.

There is a natural action of G on Ω_H which is given by $y(\omega, \dot{x}) = (\omega, y\dot{x})$, $\omega \in \Omega$, $x, y \in G$.

For any element $(\omega, \dot{x}) \in \Omega_H$ the stability group $S_{(\omega, \dot{x})}$ is just $xH_\omega x^{-1}$. Hence it follows easily from the continuity of H that the stabilizer map $S: \Omega_H \to \mathscr{K}(G)$ is continuous, too. Furthermore, a proof similar to the proof of [16, Proposition 4.3] shows that the action of G on Ω_H is always σ-proper. It follows from [5, Corollary 1] that $C^*(\Omega^H)$ is always Morita equivalent to $C^*(G, \Omega_H)$ from which particularly follows that $C^*(G, \Omega^H)$ has continuous trace if and only if $C^*(\Omega^H)$ has this property [19, Theorem 2.15]. Hence the following examples give automatically examples of transformation group C^*-algebras with or without continuous trace.

Example 1. For $\lambda \in \mathbb{R} \setminus \{0\}$ let

$$A_\lambda = \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix} \quad \text{and let} \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Then A_λ and J generate a subgroup H_λ of $SL(2, \mathbb{C})$ of order eight, namely the group consisting of the elements $E, -E, J, -J, A_\lambda, -A_\lambda, JA_\lambda, -JA_\lambda$, where E denotes the unit matrix. Let us denote by H_0 the normal subgroup $\{E, -E, J, -J\}$. Then it is easily seen that the map $H: \mathbb{R} \to \mathscr{K}(SL(2, \mathbb{C}))$ is continuous since the norm of A_λ tends to infinity if λ tends to 0. Clearly, H is not locally bounded, but the only element in $\mathbb{R}$ such that there exists a proper bounded part of H is 0. Furthermore, it is clear that each bounded part of H in 0 is given by a neighborhood W of 0 in $\mathbb{R}$ and the constant map $L_\lambda = H_0$ for all $\lambda \in W$. Hence $C^*(\mathbb{R}^H)$ does not have continuous trace since H_0 is abelian, but H_λ is not for all $\lambda \neq 0$.

Clearly, it is not hard to show that $C^*(\mathbb{R}^H)^\frown$ is even not Hausdorff in this example. The following example shows that it is as well possible to construct $H: \Omega \to \mathscr{K}(G)$ with H_ω finite for all $\omega \in \Omega$ such that $C^*(\Omega^H)^\frown$ is Hausdorff but $C^*(\Omega^H)$ does not have continuous trace.

Example 2. Let F be any nonabelian finite group and denote by G the infinite direct sum $\sum_{n \in \mathbb{N}} F$. Hence G consists of all infinite tuples with entries from F such that almost all entries are equal to the identity of F. Let us denote by $\mathbb{N}_\infty$ the one point compactification of $\mathbb{N}$. We define a map $H: \mathbb{N}_\infty \to \mathscr{K}(G)$ by defining H_n to be the nth copy of F in G for $n \in \mathbb{N}$, and by mapping ∞ to the trivial subgroup of G. Then it is easily seen that H is continuous. Furthermore, it is clear that for each neighborhood W of ∞ in $\mathbb{N}_\infty$, (W, L) is a bounded part of H in ∞ if and only if $L_w = \{e\}$ on a neighborhood $W' \subseteq W$ of ∞. But this shows that $C^*(\mathbb{N}_\infty^H)$ does not have continuous trace since H_n is not abelian for all $n \in \mathbb{N}$. But on the other hand it is easy to see that $C^*(\mathbb{N}_\infty^H)^\frown$ is Hausdorff.

The next example handles with the case where the H_ω are infinite subgroups of the discrete group G.

Example 3. Let $\mathbb{Z}_3$ denote the cyclic group $\mathbb{Z}/3\mathbb{Z}$ and for all $n \in \mathbb{Z}$ let $\alpha(n)$ be the nontrivial automorphism of $\mathbb{Z}_3$ if n is odd and the trivial automorphism if n is even. Furthermore, let $G = \mathbb{Z} \times_\alpha \mathbb{Z}_3$ be the semidirect product of $\mathbb{Z}$

with $\mathbb{Z}_3$. For each $n \in \mathbb{N}$ let $H_n = (2n+1)\mathbb{Z} \times_\alpha \mathbb{Z}_3$ and let $H_\infty = \mathbb{Z}_3$. Then it is easily seen that $H\colon \mathbb{N}_\infty \to \mathscr{K}(G)$ is continuous. But $C^*(\mathbb{N}_\infty^H)$ does not have continuous trace by Theorem 3 and Proposition 3 since $H_n \cap H_\infty$ is abelian for all $n \in \mathbb{N}$ but H_n is not. However, since each H_n has a finite commutator subgroup, it follows that $C^*(H_n)$ has continuous trace for all $n \in \mathbb{N}$.

Our final example shows that the description of subgroup algebras with continuous trace as given in Theorem 2 or Theorem 3 are not valid under the more general assumption that G is arbitrary and H_ω is compact for all $\omega \in \Omega$. It also shows that it might be very difficult to give a general description of subgroup algebras with continuous trace.

Example 4. Let $N = \prod_{n\in\mathbb{N}} \mathbb{Z}_2$ be the infinite product of $\mathbb{N}$ copies of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and let $G = S_\infty \times_\alpha N$, where S_∞ denotes the infinite permutation group which acts on N by permuting the order of the product. For each $n \in \mathbb{N}$ let H_n be the subgroup of G which is generated by N and the element $\sigma_n \in S_\infty$ which permutes 1 with n. Then N has index two in H_n for all $n \in \mathbb{N}$ and it is easy to see that $H_n \to N$ in $\mathscr{K}(G)$. Hence, if we define $H_\infty = N$ we get a continuous map $H\colon \mathbb{N}_\infty \to \mathscr{K}_C(G)$. It is easy to see that every bounded part of H in ∞ is constantly equal to N on a neighborhood of ∞. Hence, if Theorem 2 would be true in the more general setting of arbitrary G and compact H_ω, then $C^*(\mathbb{N}_\infty^H)$ cannot have continuous trace since N is abelian but H_n is not for all $n \in \mathbb{N}$.

We will now show that $C^*(\mathbb{N}_\infty^H)$ does have continuous trace. For this let $f \in C_c(\mathbb{N}_\infty^H)$. We define $f_n \in C(H_n)$ by $f_n(x) = f(n, x)$ and $f_\infty \in C(N)$ by $f_\infty(x) = f(\infty, x)$. Then it is not hard to see that $f_n|N \to f_\infty$ uniformly on N. Now let χ be any character of N. Then χ can be identified with an $\mathbb{N}$-tuple with entries 1 or -1 such that almost all entries are 1. Furthermore σ_n acts on χ by permuting the first entry with the nth entry. Hence H_n acts trivially on χ for almost all $n \in \mathbb{N}$ if and only if the first entry of χ is 1.

Now let $(n, \pi_n) \to (\infty, \chi)$ in $C^*(\mathbb{N}_\infty^H)\hat{}$. Then $\pi_n|N \to \chi$ in $\mathrm{Rep}(N)$ which shows that χ has to be a subrepresentation of $\pi_n|N$ for almost all $n \in \mathbb{N}$. If the first entry of χ is equal to 1 it follows in fact that there is an $n_0 \in \mathbb{N}$ such that $\pi_n|N = \chi$ for all $n \geq n_0$. Furthermore, arguments similar to those used in Lemma 5 show that $\pi_n(f_n) = \pi_n|N(f_n|N)$ if n is greater than, say n_1. Hence in this case it is clear that

$$\mathrm{tr}((n, \pi)(f)) = \mathrm{tr}(\pi_n(f_n)) = \chi(f_n|N) \to \chi(f_\infty) = \mathrm{tr}((\infty, \chi)(f)).$$

Now assume that the first entry of χ is -1. Then $\pi_n|N = \chi \oplus \chi_n$ for n sufficiently large, where the tuple of χ_n can be obtained from χ by permuting the first and the nth entry. Now we get

$$\mathrm{tr}((n, \pi_n)(f)) = \chi(f_n|N) + \chi_n(f_n|N) \to \chi(f_\infty) = \mathrm{tr}((\infty, \chi)(f)),$$

since $\chi_n(f_n|N) \to 0$ if $n \to \infty$.

It is clear that the trace map $(\omega, \pi) \to \mathrm{tr}((\omega, \pi)(f))$ is continuous at any other irreducible representation (ω, π) of $C^*(\mathbb{N}_\infty^H)$. Hence $C^*(\mathbb{N}_\infty^H)$ has continuous trace. Note that in this example the dimension map $\dim\colon C^*(\mathbb{N}_\infty^H)\hat{} \to \mathbb{N}$ is not continuous.

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