

GRAPHS WITH THE CIRCUIT COVER PROPERTY

BRIAN ALSPACH, LUIS GODDYN, AND CUN-QUAN ZHANG

ABSTRACT. A *circuit cover* of an edge-weighted graph (G, p) is a multiset of circuits in G such that every edge e is contained in exactly $p(e)$ circuits in the multiset. A nonnegative integer valued weight vector p is *admissible* if the total weight of any edge-cut is even, and no edge has more than half the total weight of any edge-cut containing it. A graph G has the *circuit cover property* if (G, p) has a circuit cover for every admissible weight vector p . We prove that a graph has the circuit cover property if and only if it contains no subgraph homeomorphic to Petersen's graph. In particular, every 2-edge-connected graph with no subgraph homeomorphic to Petersen's graph has a cycle double cover.

1. INTRODUCTION

Let (G, p) be an edge-weighted graph (with loops and multiple edges allowed) where $p : E(G) \rightarrow \mathbb{Z}$. The following question, which we shall call the *circuit cover problem*, has attracted considerable interest since it was posed and solved for planar graphs by P. D. Seymour in 1979 [Sey1]: "Find conditions on (G, p) for there to exist a multiset (or list) \mathbf{L} of circuits in G such that each edge e is 'covered' exactly $p(e)$ times by circuits in \mathbf{L} ." More precisely, we say that (G, p) has a *circuit cover* (or that G has a *circuit p -cover*) provided the following holds:

- (1.1) There exists a vector of nonnegative integer coefficients $(\lambda_C : C \in \mathbf{C})$ such that $\sum_{C \in \mathbf{C}} \lambda_C \chi^C = p$.

(Here, \mathbf{C} denotes the collection of circuits in G , and (λ_C) is the multiplicity vector for the circuit cover \mathbf{L} , and for any subgraph H of G , χ^H denotes the $\{0, 1\}$ -characteristic vector of the edge set of H .)

The circuit cover problem is related to problems involving graph embeddings [Arc, Hag, Lit, Tut], flow theory [Cel, Fan2, Jae1, You], short circuit covers [Alo, Ber, Fan1, Gua, Jac, Jam2, Jam3, Tar1, Zha1], the Chinese Postman Problem [Edm, Gua, Ita, Jac], perfect matchings [Ful, God2, p. 22] and decompositions of eulerian graphs [Fle1, Fle2, Sey3]. When p is the constant vector $\mathbf{1}$ (or any odd number), we are characterizing eulerian graphs. When $p = \mathbf{2}$ we have the well known *cycle double cover conjecture*. The cases $p = \mathbf{4}$ and $p = \mathbf{6}$ have been settled for graphs using the 8- and 6-flow theorems by Jaeger [Jae1] and

Received by the editors June 4, 1990 and, in revised form, April 1, 1992 and September 9, 1992.
1991 *Mathematics Subject Classification*. Primary 05C38; Secondary 05C70.

The first author was partially funded by Natural Sciences and Engineering Research Council grant # A-4792. The second author was partially funded by Natural Sciences and Engineering Research Council grant # A-5499. The third author was partially funded by National Science Foundation grant # DMS-8906973.

©1994 American Mathematical Society
0002-9947/94 \$1.00 + \$.25 per page

Fan [Fan2] respectively. Tarsi [Tar1] generalizes the case where p is constant to the class of binary matroids.

Seymour [Sey1] gave three necessary conditions for an arbitrary weighted graph (G, p) to have a circuit cover:

- (1.2) (i) p is nonnegative integer valued;
- (ii) for every edge-cut B and $e \in B$, $p(e) \leq p(B \setminus e)$, (that is, p is *balanced*);
- (iii) for every edge-cut B , $p(B)$ is even, (that is, p is *eulerian*).

(We use the convention that $p(F)$ means $\sum_{e \in F} p(e)$, for any $F \subseteq E$.) These conditions follow easily from the observation that any circuit in a graph intersects any edge-cut in an even number of edges. The conditions in (1.2) are collectively called *admissibility* conditions, and p is said to be *admissible* if it satisfies (1.2).

Our main result characterizes the graphs for which (1.1) and (1.2) are equivalent. We say that a graph G has the *circuit cover property* if (G, p) has a circuit cover for every admissible weight p .

Not every graph has the circuit cover property. Let P_{10} denote Petersen's graph and let p_{10} take the value 1 on some 2-factor of P_{10} , and the value 2 on the complementary 1-factor. Then (P_{10}, p_{10}) is admissible, but has no circuit cover, as has been observed by several authors [Sey1, Sze, Zel]. Clearly, no graph homeomorphic to P_{10} has the circuit cover property. By assigning weight zero to deleted edges, it is easy to see that no graph which has a P_{10} -minor (a minor isomorphic to P_{10}) has the circuit cover property. (Since P_{10} is cubic, a graph has a P_{10} -minor if and only if some subgraph of G is homeomorphic to P_{10} .)

The pertinence of P_{10} to the circuit cover problem for cubic graphs was established by Alspach and Zhang [Als] where they showed that a cubic graph G has a circuit p -cover for every admissible $\{0, 1, 2\}$ -valued weight vector p if and only if G has no P_{10} -minor. Our main result generalizes this result to arbitrary weighted graphs.

Theorem 1. *A graph has the circuit cover property if and only if it has no P_{10} -minor.*

We give some terminology: In this paper, graphs are finite, undirected with loops and multiple edges allowed. Because of a strong connection to matroid theory, we borrow some terms from that area (see [Wel] for an introduction to matroid theory). Most notably, a *cycle* (or *even* subgraph) in a graph $G = (V, E)$ is a subset of edges $F \subseteq E$ such that each vertex of G is incident with an even number of edges in F . A *circuit* is a minimal nonempty cycle. Since any cycle is an edge-disjoint union of circuits, (G, p) has a *cycle cover* if and only if it has a circuit cover. Where no confusion arises, we identify a cycle with the subgraph of G induced by the cycle. For example, we use $V(C)$ to denote the set of vertices of a circuit C . A graph is *eulerian* if it is connected and its edge set forms a cycle. If $e \in E(G)$ then $G \setminus e$ denotes the graph obtained from G by deleting e , and G/e denotes the graph obtained from G by *contracting* e (that is, we identify the endvertices of e , then delete e). Loops and multiple edges (other than e) which arise from a contraction are not deleted. Any graph obtained from G by successive deletions and contractions is called a *minor* of G . The order in which edges are deleted and contracted is irrelevant, so any

minor of G may be written as $G \setminus E_1/E_2$ where E_1 and E_2 are disjoint subsets of $E(G)$. If H is a cubic graph then H is a minor of G if and only if some subgraph of G is homeomorphic to H . For any subset S of vertices of G , the set of edges $\delta(S) = [S, V - S]$ which have exactly one endvertex in S is called an *edge-cut* (or *cocycle*) of G . A *bond* (or *cocircuit*) is a minimal nonempty edge-cut. A *bridge* (or *coloop*) is an edge-cut of cardinality 1. A graph with no bridges is said to be *bridgeless*.

There are several consequences of Theorem 1. Since P_{10} is nonplanar, we obtain the following classic result of P. D. Seymour.

Corollary 1 [Sey1]. *Every planar graph has the circuit cover property.*

Since $P_{10} - v$ is nonplanar for any vertex v we have the following sharpening.

Corollary 2. *If $G - v$ is planar for some vertex v , then G has the circuit cover property.*

The well-known *cycle double cover conjecture* asserts that every bridgeless graph has a circuit 2-cover. This conjecture has been the subject of numerous papers [Bon, Cat, God1, God2, Jae2, Sey1, Sze, Tar2], and has been verified for various classes of graphs. The following corollary is new, although it was previously known to hold for cubic graphs [Als].

Corollary 3. *Every bridgeless graph with no P_{10} -minor has a cycle double cover.*

A graph G is said to have a nowhere zero 4-flow if $E(G) = E_1 \cup E_2$ where each E_i is a cycle in G [Jae1, Mat]. Every graph with a nowhere zero 4-flow has a cycle double cover, namely $\{E_1, E_2, E_1 \Delta E_2\}$. Corollary 3 thus lends support to the following well-known conjecture of Tutte.

Conjecture 1 [Tut]. *Every bridgeless graph with no P_{10} -minor has a nowhere zero 4-flow.*

Several authors have investigated a relationship between the *Chinese Postman Problem* and the *Shortest Circuit Cover Problem*. Let G be a graph. Using our notation, the Chinese Postman Problem [Ber, Edm, Gua, Ita, Jac] essentially is to find the smallest integer c_G such that there exists an eulerian weight vector $p \geq 1$ satisfying $p(G) = c_G$. The Shortest Circuit Cover Problem [Ber, Fan, Gua, Jac, Jam2, Tar1] is to find the smallest integer s_G such that (G, p) has a circuit cover for some (admissible) weight vector $p \geq 1$ satisfying $p(G) = s_G$ (s_G is not defined if G has a bridge). It is immediate from the definitions that for any bridgeless graph G ,

$$(1.3) \quad c_G \leq s_G.$$

In general we do not have equality since $c_{P_{10}} = 20$ while $s_{P_{10}} = 21$ (see [Ita]). For bridgeless graphs we have the general upper bounds $c_G \leq 4|E(G)|/3$ and $s_G \leq 5|E(G)|/3$ [Ber], though it is conjectured that $s_G \leq 7|E(G)|/5$. (Jamshy and Tarsi [Jam3] have shown that this last inequality actually implies the cycle double cover conjecture.)

Although there is a polynomial-time algorithm for determining c_G [Edm], the determination of s_G is considered to be a very difficult problem [Ber, Gua, Jam2, Tar1]. Hence there is considerable interest in determining classes of graphs for which equality holds in (1.3). It is known [Gua, Ber] that equality

holds for all bridgeless planar graphs. This class was extended by Alspach and Zhang [Als] to include all bridgeless cubic graphs which have no P_{10} -minor. Both of these results follow from the fact that equality holds in (1.3) for any bridgeless graph G which has the circuit cover property. (This fact follows easily from Proposition 6 and by observing [Edm] that any eulerian vector $p \geq 1$ with $p(G) = c_G$ is $\{1, 2\}$ -valued.) From Theorem 1 we have the following generalization.

Corollary 4. *If G is a bridgeless graph with no P_{10} -minor, then $s_G = c_G$.*

It is known [Ber, Jac, Zha1] that $s_G = c_G$ whenever G has a nowhere zero 4-flow. This fact, together with Corollary 4, indirectly lends further support to Conjecture 1.

We compare Theorem 1 to a theorem of Fleischner and Frank [Fle2] regarding decompositions of eulerian graphs into circuits which avoid certain “forbidden” sets of edges. For each vertex v of an eulerian graph G a partition $P(v)$ of the edges incident with v is specified. We set $P = \bigcup_{v \in V(G)} P(v)$ and call each member of P a *forbidden part*. A decomposition of $E(G)$ into circuits is *good* (with respect to P) if no circuit contains two edges from a single forbidden part. The problem is to establish conditions on (G, P) under which there exists a good decomposition of G with respect to P .

Theorem 2 [Fle2]. *A planar eulerian graph has a good decomposition with respect to P if and only if no edge-cut B contains more than $|B|/2$ edges belonging to the same forbidden set.*

Suppose (G, p) is a planar graph with an admissible edge-weight vector. Let H be the planar eulerian graph obtained from G by replacing each $e \in E(G)$ with $p(e)$ parallel edges. Let these sets of parallel edges constitute a collection P of forbidden parts for H . Since (G, p) is balanced, the pair (H, P) satisfies the hypothesis of Theorem 2. One easily sees that a good decomposition of H with respect to P corresponds to a circuit cover of (G, p) . It follows that Theorem 2 implies Corollary 1. Theorem 2 appears to be a strict generalization of Corollary 1 since there does not seem to be a reverse transformation $(H, P) \rightarrow (G, p)$ which preserves planarity. Theorems 1 and 2 appear to generalize Corollary 1 in very different ways since it is not at all clear that Corollaries 2 or 3 can be derived from Theorem 2. Seymour [Sey3] uses Corollary 1 to prove his Even Circuit Decomposition Theorem. One of the present authors [Zha2, Zha3] has used a strong form of the main theorem of this paper (see Theorem 4) to generalize both Theorem 2 and Seymour’s Even Circuit Decomposition Theorem to the class of graphs with no K_5 -minor.

A further consequence of Theorem 1 involves the natural generalization of circuit covers to weighted matroids (M, p) . For binary matroids, the conditions in (1.2) are still necessary for (M, p) to have a circuit cover, where “edge-cut” is replaced by “cocircuit”. The penultimate conjecture in [Sey2] proposes a forbidden minor characterization of binary matroids with the circuit cover property:

Conjecture 2 [Sey2]. *A binary matroid M has the circuit cover property if and only if no minor of M is isomorphic to either $M(P_{10})$, $M^*(K_5)$, F_7^* , or R_{10} .*

(See [Sey2] for definitions.) By using Theorem 1 together with Seymour’s

matroid decomposition theorems, (see [Sey2]), Fu and Goddyn have settled this conjecture affirmatively [Fu].

The relaxation of (1.1) to nonnegative rational coefficients has been studied for both graphs [Sey1] and matroids [Sey2]. A weight vector p is in the *cone of circuits* of M if there is a nonnegative rational vector $(\alpha_C)_{C \in \mathcal{C}}$ satisfying $\sum_{C \in \mathcal{C}} \alpha_C \chi^C = p$. There are two natural necessary conditions for a vector p to be in the cone of circuits of a matroid; p must be nonnegative and balanced. We say that a matroid M has the *sums of circuits property* if every balanced nonnegative rational weight vector p is in the cone of circuits of M . A forbidden-minor characterization of those matroids with the sums of circuits property is given by Seymour.

Theorem 3 [Sey2]. *A matroid M has the sums of circuits property if and only if M is binary and no minor of M is isomorphic to either $M^*(K_5)$, F_7^* or R_{10} .*

In particular, every graph has the sums of circuits property. We note that this list of forbidden minors is the same as that for Conjecture 2 except for Petersen's graph. Conjecture 2 might be considered to be an "integer analog" of Theorem 3.

2. A STRONGER THEOREM

We shall prove something slightly stronger than Theorem 1. This stronger version (Theorem 4 below) is needed for some applications of Zhang [Zha2, Zha3]. We say that a weighted graph (G, p) is *contra-weighted* if (G, p) is admissible, but has no circuit cover.

First we note three trivial operations that can yield contra-weighted graphs other than (P_{10}, p_{10}) :

- (2.1) (i) new vertices and edges of weight zero may be added,
- (ii) any edge may be subdivided into a path of edges of the same weight,
- (iii) if some vertex of degree 2 is adjacent to two edges of weight 2, then one of these edges may be replaced by two parallel edges of weight 1.

Any weighted graph obtainable from (P_{10}, p_{10}) by repeated application of these three operations is called a *blistered P_{10}* . A typical blistered P_{10} appears in Figure 1 (edges of weight 0 are not shown). Note that (P_{10}, p_{10}) is the only 3-connected blistered P_{10} which has no edges of weight 0.

We write $p \leq q$, if $p(e) \leq q(e)$ for all $e \in E$.

Theorem 4. *If (G, p) is a contra-weighted graph, then there exists $q \leq p$ such that (G, q) is a blistered P_{10} .*

Theorem 4 follows from two lemmas whose proofs comprise the next three sections of this paper.

Lemma 1. *If (G, p) is a contra-weighted graph, then there exists $q \leq p$ such that (G, q) is a $\{0, 1, 2\}$ -valued contra-weighted graph.*

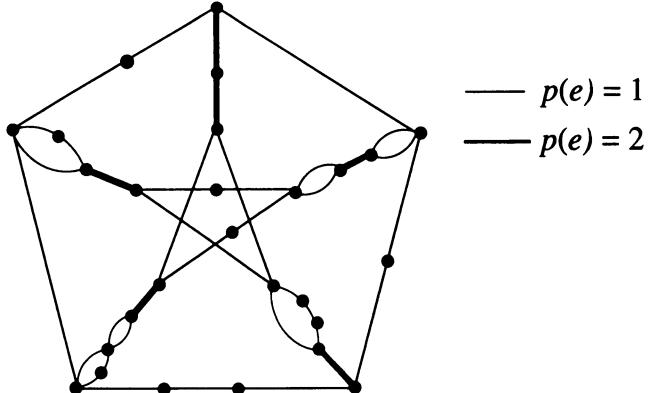


FIGURE 1

Lemma 2. *If (G, p) is a $\{1, 2\}$ -valued contra-weighted graph, then there exists $q \leq p$ such that (G, q) is a blistered P_{10} .*

As we shall see, the proof of Lemma 1 has the flavour of Seymour's proof of Corollary 1, while the proof of Lemma 2 is essentially an extension of that given by Alspach and Zhang in [Als].

3. PREPARATION FOR LEMMA 1

We say that p is *positive* if $p \geq 1$. We lose no generality when we assume p is positive since edges of weight zero simply may be deleted.

We first study special edge-cuts. Let (G, p) be a positive, balanced weighted graph. An edge-cut B is called a *tight cut* if $p(e) = p(B \setminus e)$ for some $e \in B$. In this case, e is called a *tight cut leader* for B , and any other edge in B is called a *tight cut follower* for B . Since p is positive and balanced, any tight cut must be a bond. Furthermore, B has a unique tight cut leader provided $|B| \geq 3$. If $|B| = 2$, then each edge in B is both a leader and a follower. The importance of tight cuts is manifested in the following observation of Seymour [Sey1].

Proposition 1. *In any circuit cover of (G, p) , every circuit which intersects with a tight cut B contains a tight cut leader for B , and exactly one other edge in B .*

Let $e, f \in E(G)$. We say that e follows f in (G, p) if either $e = f$ or some tight cut B has e as a follower and f as a leader.

Proposition 2. *If p is balanced, then “follows” is a transitive relation on $E(G)$.*

Proof. Let B be a tight cut in which e follows f and let C be a tight cut in which f follows g . If $e = g$, then $B = C$ and $|B| = 2$ follows from the paragraph preceding Proposition 1, in which case Proposition 2 holds. Thus we assume $e \neq g$. By definition, $f \in B \cap C$. If either e or g is in $B \cap C$, then we are done since both B and C are tight. Hence we assume that $e \in B \setminus C$, $f \in B \cap C$, and $g \in C \setminus B$. Thus $e, g \in B \Delta C$. We have the following sequence

of inequalities:

$$\begin{aligned}
 2p(g) &= p(C \setminus B) + p(C \cap B) && (\text{since } C \text{ is tight}) \\
 &\geq p(C \setminus B) + p(f) && (\text{since } f \in B \cap C) \\
 &= p(C \setminus B) + p(B \setminus f) && (\text{since } B \text{ is tight}) \\
 &\geq p(C \setminus B) + p(B \setminus C) && (\text{since } f \in C) \\
 &\geq 2p(g) && (\text{since } B \Delta C \text{ is an edge-cut, and hence} \\
 &&& \text{is balanced}).
 \end{aligned}$$

We have equality throughout. In particular, $B \Delta C$ is a tight cut in which $2p(g) = p(B \Delta C)$ and therefore e follows g . \square

For any collection $\mathbf{H} = \{H_1, \dots, H_m\}$ of subgraphs of G , denote by $\chi^{\mathbf{H}}$ the sum of the characteristic vectors χ^{H_i} , $i = 1, 2, \dots, m$.

Proposition 3. *If \mathbf{H} is a collection of circuits in G , then $(G, \chi^{\mathbf{H}})$ is admissible.*

Proposition 4. *Let $F \subseteq E(G)$ and let p_F denote the restriction of p to $E(G) \setminus F$. If (G, p) has a circuit cover, then so does the contracted graph $(G/F, p_F)$.*

A *minimal contra-weighted graph* is a contra-weighted graph (G, p) such that (G, q) is not contra-weighted for any $q < p$. Although it is possible that minimal contra-weighted graphs have tight cuts, such tight cuts must be “well behaved”.

Proposition 5. *Let (G, p) be a positive minimal contra-weighted graph and let B be a tight cut in (G, p) with a leader $e_1 \in B$. Then there exists a sequence $X = (x_1, x_2, \dots, x_k)$ of distinct vertices such that $B = \delta(X)$ and, for $i = 1, 2, \dots, k$, $\delta(\{x_1, x_2, \dots, x_i\})$ is a tight cut having e_1 as a leader.*

Proof. In a minimal contra-weighted graph (G, p) , let $B = [X_1, X_2]$ be a tight cut with a leader $e_1 = x_1y_1$, $x_1 \in X_1$, $y_1 \in X_2$. Let (G_i, p_i) be the weighted graph obtained by contracting the edges with neither endvertex in X_i , $i = 1, 2$. That is, $G_i = G/E(G[X_{3-i}])$.

We claim that either (G_1, p_1) or (G_2, p_2) is contra-weighted. Since edge contraction introduces no new edges or edge-cuts, each (G_i, p_i) is positive and admissible, just as (G, p) is. If neither (G_1, p_1) nor (G_2, p_2) were contra-weighted, then they would both have circuit covers. By Proposition 1, we could then pair off the circuits which contain e_1 in G_1 with those in G_2 , obtaining a circuit cover of (G, p) . This establishes the claim.

Assume that (G_2, p_2) is contra-weighted. Let $e_1 = x_1y_2 \in E(G_1)$, where $x_1 \in X_1$ and y_2 is the new vertex created in the definition of G_1 . As (G_1, p_1) is balanced, no edge-cut in $G_1 \setminus e_1$ separating x_1 and y_2 has weight less than $p(e_1)$. Thus, by an undirected version of the Max-Flow Min-Cut theorem [For] there is an integer-valued (x_1, y_2) -flow f of value $p(e_1)$ in some acyclic orientation D of $G_1 \setminus e_1$ such that $0 \leq f(e) \leq p_1(e)$, $e \in E(G_1 \setminus e_1)$. It is well known that f can be “decomposed” into a collection \mathbf{P} of $p(e_1)$ directed (x_1, y_2) -paths in D . That is, $f = \chi^{\mathbf{P}} \leq p_1$. As B is tight, $\chi^{\mathbf{P}}(e) = p(e)$ for every $e \in B \setminus e_1$.

Consider the new weight vector q on $E(G)$ defined by

$$q(e) = \begin{cases} \chi^{\mathbf{P}}(e), & \text{if } e \in E(G[X_1]), \\ p(e), & \text{otherwise.} \end{cases}$$

We claim that (G, q) is admissible. That q is eulerian follows from the fact that (G_2, p_2) is eulerian and that each path in \mathbf{P} has exactly two odd-degree vertices. To show q is balanced, we first argue as in the previous paragraph, deducing that there exists a collection \mathbf{Q} of exactly $q(e_1) = p(e_1)$ circuits in G , each containing e_1 , such that $\chi^{\mathbf{Q}} \leq q$ and $\chi^{\mathbf{Q}}(e) = q(e)$ for every $e \in E(G_1)$ (in fact, we can arrange for each circuit in \mathbf{Q} to be an extension of a corresponding path in \mathbf{P}). Let B' be an edge-cut in G and let $e' \in B'$. If $e' \in E(G_1) = E(G[X_1]) \cup B$, then each of the $q(e')$ circuits in \mathbf{Q} which contain e' contains at least one edge in $B' \setminus e'$. Thus $q(B' \setminus e') \geq q(e')$. We assume that $e' \in E(G[X_2])$. Suppose B' contains the leader e_1 of the tight cut B . Then $B' \Delta B$ is an edge cut with $e' \in B' \Delta B$, $e_1 \notin B' \Delta B$, and $q(B' \setminus e') \geq q((B' \Delta B) \setminus e')$. So to show that $q(B' \setminus e') \geq q(e')$ it suffices to show that $q((B' \Delta B) \setminus e') \geq q(e')$. Hence we can assume $e_1 \notin B'$. Let $B' = \delta(X')$ where the set of vertices X' is chosen to contain neither endvertex of e_1 . Consider the edge-cut $B'' = \delta(X' \cap X_2)$. Note that $e' \in B' \cap B''$. Let $e \in B'' \setminus B'$. Since $B'' \setminus B'$ consists entirely of followers in the tight cut B , each of the $q(e) = p(e)$ circuits $C \in \mathbf{Q}$ which contain e contains no other edge in $B'' \setminus B'$. Since $B' \Delta B'' = (B'' \setminus B') \cup (B' \setminus B'')$ is an edge-cut and $|C \cap (B'' \setminus B')| = 1$, C contains at least one edge in $B' \setminus B''$. This implies $q(B' \setminus B'') \geq q(B'' \setminus B')$, whence $q(B' \setminus e') \geq q(B'' \setminus e')$. Since p is balanced and coincides with q on B'' , $q(B'' \setminus e') = p(B'' \setminus e') \geq p(e') = q(e')$. The last two inequalities establish that q is balanced and hence that q is admissible as claimed.

Were (G, q) to have a circuit cover then so would the contra-weighted graph (G_2, p_2) (by Proposition 4), a contradiction. Thus (G, q) is contra-weighted. Since $q \leq p$ and p is minimal, we must have $q = p$.

As D is acyclic and x_1 is a source and y_2 is a sink, there is an ordering $(x_1, x_2, \dots, x_{k+1} = y_2)$ of the vertices in $V(G_1)$ such that all directed arcs (x_i, x_j) in D have $i < j$. Thus, $X_1 = \{x_1, x_2, \dots, x_k\}$ and, since q agrees with $\chi^{\mathbf{P}}$ on $E(G[X_1])$, all edge-cuts of the form $\delta(\{x_1, x_2, \dots, x_i\})$, $i = 1, 2, \dots, k$, are tight, with e_1 as their common leader. \square

4. PROOF OF LEMMA 1

As mentioned above, part of this proof is essentially the same as a large part of Seymour's proof of Corollary 1. Unfortunately, Seymour's proof cannot be directly modified into a proof of Lemma 1. The main obstacle is that Seymour relies on a reduction method (vertex "splitting") which, although preserving planarity, can inadvertently introduce P_{10} -minors. We use instead a more involved "circuit cover-splicing" argument. An edge-cut $[X, Y]$ is *trivial* if $|X| = 1$ or $|Y| = 1$, and is *nontrivial* otherwise.

Let (G, p) be a minimal positive contra-weighted graph. Our aim is to show that p is $\{1, 2\}$ -valued.

(4.1) Suppose that (G, p) has a nontrivial tight cut. By Proposition 5 there exist two tight cuts $\delta(\{x_1, x_2\}), \delta(\{x_1\})$ with a common tight cut leader $e_1 = x_1 y_1$. Let S be the set of edges joining x_1 to x_2 and let $T = \delta(\{x_2\}) \setminus S$. Since $\delta(\{x_1\})$ and $\delta(\{x_1, x_2\})$ are both tight, we have $p(S) = p(T)$. Let (G', p') be obtained from (G, p) by contracting S . As (G, p) is admissible, so is (G', p') . Furthermore, (G', p') has no circuit cover since, by the fact that $p(S) = p(T)$, such a circuit cover is easily modified to be one of (G, p) .

By induction on $|E(G)|$, we can assume there exists a $\{0, 1, 2\}$ -valued weight vector $q' \leq p'$ such that (G', q') is contra-weighted. We now extend q' into a $\{0, 1, 2\}$ -valued weight vector q for G by defining $q(e) = q'(e)$ for $e \in E(G) \setminus S$ and by specifying $q(e)$ for $e \in S$ as follows. If $q(T)$ is odd, then we define $q(e_2) = 1$ for some $e_2 \in S$ and $q(e) = 0$ for $e \in S \setminus e_2$. If $q(T)$ is even and $|S| \geq 2$, then we define $q(e_2) = q(e_3) = 1$ for some $e_2, e_3 \in S$ and $q(e) = 0$ for $e \in S \setminus \{e_2, e_3\}$. Finally, if $q(T)$ is even and $S = \{e_2\}$, then we define either $q(e_2) = 0$ or $q(e_2) = 2$ depending on whether or not there exists an edge cut B in G such that $e_2 \in B$ and $q'(B \setminus e_2) = 0$. In each case, we have $q(S) \equiv q(T) \pmod{2}$ ensuring that (G, q) is eulerian. Since q is $\{0, 1, 2\}$ -valued and eulerian, (G, q) is balanced provided that no edge cut B in G contains an edge e with $q(e) = 2$ and $q(B \setminus e) = 0$ (see Proposition 6). That no such edge cut exists follows from the definition of q on S and the fact that (G', q') is balanced. Thus (G, q) is admissible. Since p is positive and eulerian, and since $q' \leq p'$, one easily checks that $q \leq p$. Furthermore, (G, q) has no circuit cover otherwise contracting S would yield a circuit cover of (G', q') . Thus (G, q) is contra-weighted and, by minimality of p , we have $q = p$. In this case there is nothing to prove. Hence, we can assume that every tight cut is trivial.

(4.2) Similarly, by contracting one edge of any 2-edge-cut (such a cut must be tight) and using the induction hypothesis, we can assume that G is 3-edge-connected.

Any edge which is not a follower in any tight cut of (G, p) is called a *nonfollower*. Let e be an edge in $E(G)$ of maximum weight. We may assume $p(e) \geq 2$ since otherwise $p = 1$ (recall that $p \geq 1$) and G is eulerian, whence it has a circuit decomposition. By (4.2) and the fact that p is positive, e is a nonfollower.

Let $e_0 = xy$ be any nonfollower of weight at least 2 such that $p(e_0)$ is as small as possible. Let $r = p(e_0)$. By (4.1), any edge which is a tight cut follower is adjacent to a tight cut leader. This leader must itself be a nonfollower since otherwise, as in the proof of Proposition 2, the symmetric difference of the two tight cuts would be a nontrivial tight cut, contradicting (4.1). Thus any edge of weight at least 2 is either a nonfollower, or is adjacent to a nonfollower (of greater weight). By choice of e_0 we have the following.

(4.3) Every edge of weight at least 2 either has weight at least r or is a follower in a trivial tight cut whose leader has weight at least r .

Define a new weight vector p' by $p' = p - 2\chi^{e_0}$. We claim that (G, p') is admissible. Since $p(e_0) \geq 2$, p' is nonnegative. As p is eulerian, so is p' . We now show that p' is balanced. Let B be an edge-cut and let $e \in B$. Since $p' \leq p$ and p is balanced, then $p'(e) \leq p(e) \leq p(B \setminus e)$. We can assume $e_0 \in B \setminus e$ since otherwise $p(B \setminus e) = p'(B \setminus e)$ and we are done. As e_0 is a nonfollower, we have $p(B \setminus e) - p(e) > 0$. Since $p(B)$ is even, this implies $p(B \setminus e) - p(e) \geq 2$. Hence $p'(e) = p(e) \leq p(B \setminus e) - 2 = p'(B \setminus e)$. Thus (G, p') is balanced and hence admissible as claimed.

By minimality of p , there exists a circuit cover \mathbf{L} of (G, p') . We write $\mathbf{L} = \mathbf{L}_1 \cup \mathbf{L}_2$ where the circuits in \mathbf{L}_1 do not contain $e_0 = xy$ and those in \mathbf{L}_2 do. Each of the $r - 2$ circuits in \mathbf{L}_2 is endowed with an orientation such that e_0 is traversed from y to x .

The next paragraph closely follows Seymour's argument in [Sey1], starting

with the fifth paragraph of p. 349. For completeness, we reiterate the main points, omitting some details.

We define an auxiliary directed graph $G_{\mathbf{L}}$ with $V(G_{\mathbf{L}}) = V(G)$. For each $C \in \mathbf{L}_1$ and each pair $u, v \in V(C)$ we have an arc $u \rightarrow v$ (this arc is labelled with “ C ”). For each $C \in \mathbf{L}_2$ and each pair $u, v \in V(C)$ which are distinct from x, y , we have an arc $u \rightarrow v$ (labelled with “ C ”) provided that C passes through y, x, v, u in that order (this arc goes the “wrong way” with respect to the orientation of C). As in [Sey1], the fact that (G, p) is balanced implies that there is a directed path from x to y in $G_{\mathbf{L}}$. Let $x = v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_k = y$ be a shortest such path, and let (C_1, C_2, \dots, C_k) be the sequence of arc labels along this path. Let $\mathbf{L}' \subseteq \mathbf{L}$ denote the underlying set of circuits $\{C_1, \dots, C_k\}$ which appear in the sequence (C_1, C_2, \dots, C_k) (repetitions eliminated), and consider the weight vector $\chi^{\mathbf{L}'} + 2\chi^{e_0}$. Using Proposition 3 and the definition of $G_{\mathbf{L}}$, one can check that $(G, \chi^{\mathbf{L}'} + 2\chi^{e_0})$ is admissible. Suppose that \mathbf{L}' is a proper subset of \mathbf{L} so that $\chi^{\mathbf{L}'} + 2\chi^{e_0} < \chi^{\mathbf{L}} + 2\chi^{e_0} = p$. Then $(G, \chi^{\mathbf{L}'} + 2\chi^{e_0})$ has a circuit cover by minimality of p . Adjoining the circuits in $\mathbf{L} \setminus \mathbf{L}'$ to this circuit cover yields a circuit cover of $(G, \chi^{\mathbf{L}} + 2\chi^{e_0}) = (G, p)$, a contradiction. We conclude that $\mathbf{L} = \mathbf{L}' = \{C_1, C_2, \dots, C_k\}$.

We now focus on the sequence $(v_0, C_1, v_1, C_2, \dots, C_k, v_k)$ of vertices and circuits to determine some structural characteristics of (G, p) and the circuit cover $\mathbf{L} = \mathbf{L}_1 \cup \mathbf{L}_2$ of $(G, p - 2\chi^{e_0})$.

Let $e \in E(G)$. It follows from the minimality of the length of $x = v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_k = y$ that there are at most two circuits in \mathbf{L}_1 passing through e . Since $|\mathbf{L}_2| = r - 2$ we have $p(e) \leq 2 + (r - 2) = r$. This fact, along with (4.3), implies that every edge in (G, p) of weight at least 2 either has weight exactly r or is adjacent to an edge of weight exactly r . This implies the following:

(4.4) Each edge of weight at least 2 has an endvertex w such that either $w = x$, or $w = y$, or w is contained in each of the $r - 2$ circuits in \mathbf{L}_2 , as well as two adjacent circuits $C_{i-1}, C_i \in \mathbf{L}_1$.

Consider the sequence (C_1, C_2, \dots, C_k) of circuits which label the arcs of the above (x, y) -path in $G_{\mathbf{L}}$. Each circuit in \mathbf{L}_2 may occur more than once in this sequence. For each $D \in \mathbf{L}_2$ we define the nonempty set of indices $I(D) := \{i : D = C_i\}$.

Recall that each $D \in \mathbf{L}_2$ is endowed with an orientation. For each $i \in I(D)$, D meets the two vertices and v_i and v_{i-1} in that order (starting from x). (Note that $v_i \in V(D)$ does not imply $i \in I(D)$; in particular, $\{0, k\} \cap I(D) = \emptyset$.) The set of vertices $\{x, y\} \cup \{v_{i-1}, v_i : i \in I(D)\}$ partitions D into $2|I(D)| + 2$ subpaths which are called the *segments* of D . These segments inherit a natural orientation from D . A segment of D starting at v_i and ending at v_j is denoted by $D[v_i, v_j]$. Segments of the form $D[v_i, v_{i-1}]$ where $i \in I(D)$ are called *reverse segments* of D ; segments of the form $D[v_i, v_j]$ where $0 \leq i < j \leq k$ are called *forward segments* of D ; the remaining segment, $D[v_k, v_0] = e_0$ is called the *root segment* of D .

Let $C_s, C_t \in \mathbf{L}_1$. By the definition of $G_{\mathbf{L}}$ (any two vertices $u, v \in V(C)$, $C \in \mathbf{L}_1$, are joined by the two arcs $u \rightarrow v$, $v \rightarrow u$), and by the choice of $v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_k$, C_s and C_t are distinct circuits if $s \neq t$, and are vertex-disjoint if $|t - s| > 1$. Thus $\chi^{\mathbf{L}_1}$ is $\{0, 1, 2\}$ -valued and the circuits in \mathbf{L}_1 form connected “chains” of circuits in G . More precisely, a *chain* is a maximal

nonempty consecutive sequence $\mathbf{Q} = (C_{i+1}, C_{i+2}, \dots, C_j)$ of circuits in \mathbf{L}_1 . The vertices v_i and v_j are called the *initial* and *terminal* vertices of \mathbf{Q} , and \mathbf{Q} is called a (v_i, v_j) -chain. Let $\bigcup \mathbf{Q}$ denote the 2-edge-connected subgraph of G which is the union of the circuits in \mathbf{Q} . For each (v_i, v_j) -chain \mathbf{Q} we arbitrarily choose two (v_i, v_j) -paths, say S_1 and S_2 , in $\bigcup \mathbf{Q}$ such that $\chi^{\{S_1, S_2\}} \leq \chi^{\mathbf{Q}}$. The paths S_1, S_2 are called *chain segments* associated with \mathbf{Q} .

We have much flexibility in the choice of chain segments for \mathbf{Q} . In fact, for any (v_i, v_j) -path S in $\bigcup \mathbf{Q}$, there exists a pair $\{S_1, S_2\}$ of chain segments for \mathbf{Q} with $S_1 = S$. (This follows from the Max-flow Min-cut Theorem and the fact that any $\{v_i, v_j\}$ -separating cut B has even weight in $(\bigcup \mathbf{Q}, \chi^{\mathbf{Q}})$ whereas B has strictly smaller odd weight in $(\bigcup \mathbf{Q}, \chi^S)$.) Since the block-graph of $\bigcup \mathbf{Q}$ is a path, we have the following:

(4.5) For any edge $e \in \bigcup \mathbf{Q}$, there exists a pair $\{S_1, S_2\}$ of chain segments for \mathbf{Q} such that $e \in S_1$.

Any vertex v_s , where $i < s < j$, is called an *internal* vertex of the (v_i, v_j) -chain \mathbf{Q} . Thus, a chain of one circuit has no internal vertices. Any vertex v_s , $1 \leq s \leq k$, which is not an internal vertex of some chain is called an *external* vertex of G . Thus every external vertex v_s either is an initial or terminal vertex of some chain, or each of C_s, C_{s+1} belongs to \mathbf{L}_2 . The set of external vertices is exactly the set of initial and terminal vertices of all forward segments, reverse segments and chain segments (collectively called *segments*).

We define an auxiliary directed graph H . The vertices of H are the set of external vertices in G . There are three *types* of arcs in $E(H)$, corresponding to the three types of segments.

- (i) For each (v_i, f_j) -chain \mathbf{Q} in G we have exactly two parallel arcs in H from v_i to v_j . These two arcs correspond to the two chain segments associated with \mathbf{Q} .
- (ii) For each circuit D in \mathbf{L}_2 and each forward segment $D[v_i, v_j]$, we have a corresponding arc (v_i, v_j) in H .
- (iii) For each circuit D in \mathbf{L}_2 and each reverse segment $D[v_i, v_{i-1}]$, we have a corresponding arc (v_{i-1}, v_i) in H .

We note that all arcs (v_i, v_j) in $E(H)$ have $i < j$ and that there is no arc in H joining $y = v_k$ to $x = v_0$ (we ignore the root segment). Figure 2 depicts a typical example of a circuit cover \mathbf{L} of (G, p') and the associated directed graph H .

For $s = 1, 2, \dots, k$, let $K(s)$ denote the set of those arcs (v_i, v_j) with $i < s \leq j$ (this definition makes sense even if v_s is not a vertex of H). We claim that each $K(s)$ is an arc-cut in H of cardinality r . As all arcs (v_i, v_j) in H have $i < j$, $K(s)$ is indeed an arc-cut in H . Each of the $r - 2$ circuits D in \mathbf{L}_2 contributes exactly one arc (having type (ii)) to $K(s)$, unless $D = C_s$, in which case D contributes exactly three arcs to $K(s)$ (one arc of type (iii) and two arcs of type (ii)). Thus $|K(s)| = r$ if $C_s \in \mathbf{L}_2$. If $C_s \in \mathbf{L}_1$ then $K(s)$ contains two arcs of type (i) (corresponding to the chain containing C_s) in addition to the $r - 2$ arcs of type (ii) contributed by \mathbf{L}_2 . Thus $|K(s)| = r$ if $C_s \in \mathbf{L}_1$, proving our claim.

It follows from the Max-flow Min-cut theorem [For] that the arcs of H can be partitioned into a set of r arc-disjoint directed (x, y) -paths $\mathbf{P} = \{P_1, \dots, P_r\}$. We also have the following:

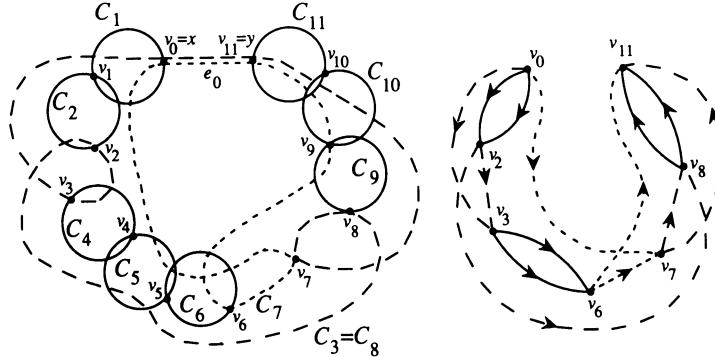


FIGURE 2

(4.6) Each $P_i \in \mathbf{P}$ intersects each cut $K(s)$ in exactly one arc.

Each $P_i \in \mathbf{P}$ naturally corresponds to an (undirected) (x, y) -walk in $G \setminus e_0$; traversing an arc in P_i corresponds to traversing the corresponding segment in G . (Note that the reverse segments are traversed in the “wrong” direction.) Adding the root segment (y, x) to this walk gives a closed walk in G denoted by W_i . Let $\mathbf{W} = \{W_1, W_2, \dots, W_r\}$. We claim the following:

(4.7) No edge is traversed twice along W_i . Thus each W_i is a cycle.

(Recall that a cycle is any edge-disjoint union of circuits.) To prove (4.7), suppose that some edge $e \in E(G)$ is contained in two of the segments, say S_1 and S_2 , constituting two subwalks in W_i . Let s_1 and s_2 denote the arcs in H corresponding to S_1 and S_2 . As S_1 and S_2 each contain e , $p(e) \geq 2$. Neither x nor y can be an endvertex of e for this would imply that either $K(1)$ or $K(k)$ contains both s_1 and s_2 , contradicting (4.6). Thus by (4.4), some endvertex v of e is contained in two adjacent circuits $C_{s-1}, C_s \in \mathbf{L}_1$. These two circuits belong to some chain \mathbf{Q} . Each S_j , $j = 1, 2$, is either

- (i) a chain segment associated with \mathbf{Q} , or
- (ii) a segment of the form $D[v_m, v_n]$ for some $D \in \mathbf{L}_2$.

In case (ii), $v \neq v_m, v_n$ since v is not an external vertex. Thus in $G_{\mathbf{L}}$ we have $v_n \rightarrow v \rightarrow v_m$. By the minimality of the sequence $v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_k$, this implies $m < s < n$. In either case, s_j belongs to $K(s)$, $j = 1, 2$, contradicting (4.6), and proving (4.7).

The cycles $W_i \in \mathbf{W}$ might not be circuits since consecutive segments in W_i might have many vertices in common. However, only “nearby” segments can overlap as the following attests.

(4.8) Let (v_a, v_b) and (v_c, v_d) be two arcs in H such that $a < b < c < d$. Then the two corresponding segments $S_{a,b}, S_{c,d} \subseteq G$ are vertex-disjoint.

Let $v \in V(S_{a,b})$. If $S_{a,b}$ is not a chain segment, then since $b < k$, $S_{a,b}$ is either the forward segment $C_b[v_a, v_b]$ or the reverse segment $C_b[v_b, v_{b-1}]$. In each case, either $v = v_{b-1}$ or $G_{\mathbf{L}}$ contains the arc $v_{b-1} \rightarrow v$. If $S_{a,b}$ is a chain segment, then v lies on some circuit $C_\alpha \in \mathbf{L}_1$ where $a+1 \leq \alpha \leq b$. Here, either $v = v_{\alpha-1}$ or the arc $v_{\alpha-1} \rightarrow v$ is contained in $G_{\mathbf{L}}$. In any case, there exists an $s < b$, such that either $v = v_s$ or the arc $v_s \rightarrow v$ is contained in $G_{\mathbf{L}}$. Similarly, if $u \in V(S_{c,d})$, then there exists a $t > c$ such that either $u = v_t$ or the arc $u \rightarrow v_t$ is contained in $G_{\mathbf{L}}$. If $v = u$, then in $G_{\mathbf{L}}$ we have either

$v_s = v = u \rightarrow v_t$ or $v_s \rightarrow v = u = v_t$ or $v_s \rightarrow v = u \rightarrow v_t$, where $s \leq t - 3$, contradicting the minimality of the sequence $v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_k$. Thus $S_{a,b}$ and $S_{c,d}$ are vertex-disjoint, proving (4.8).

By construction of \mathbf{W} , $\chi^{\mathbf{W}}(e_0) = r$. Furthermore, $\chi^{\mathbf{W}}(e) \leq \chi^{\mathbf{L}}(e) = p(e)$ for every edge e belonging to some circuit in \mathbf{L}_1 . Since \mathbf{W} constitutes a partition of all forward, reverse, root, and chain segments, the following is true.

(4.9) We have $\chi^{\mathbf{W}} \leq p$, with equality on all edges not belonging to some circuit in \mathbf{L}_1 .

We now foreshadow the completion of this proof. For each chain \mathbf{Q} in G , we shall define an admissible $\{0, 1, 2\}$ -valued vector $q_{\mathbf{Q}}$ such that $\chi^{\mathbf{Q}} \leq q_{\mathbf{Q}} \leq p$. If $(G, q_{\mathbf{Q}})$ has a circuit cover for each chain \mathbf{Q} , then we can obtain a circuit cover of (G, p) by “splicing” these circuit covers together (using \mathbf{W}), a contradiction. Thus $(G, q_{\mathbf{Q}})$ is contra-weighted for some chain \mathbf{Q} , implying, by minimality of p , that $p = q_{\mathbf{Q}}$, whence we shall have proven Lemma 1.

Let \mathbf{Q} be a (v_i, v_j) -chain and let S_1, S_2 be the two chain segments associated with \mathbf{Q} . Exactly two cycles in \mathbf{W} , say W_1 and W_2 , contain the chain segments S_1 and S_2 , respectively. We define a weight vector $q_{\mathbf{Q}}$ on $E(G)$ as follows:

$$q_{\mathbf{Q}} = \chi^{\mathbf{Q}} + \chi^{\{W_1 \setminus S_1, W_2 \setminus S_2\}}.$$

The path $W_s \setminus S_s$ is edge-disjoint from the chain segment S_s , for $s = 1, 2$, by (4.7). This statement holds true regardless of which particular pair $\{S_1, S_2\}$ of chain segments were initially chosen for \mathbf{Q} (just prior to (4.5)). Because of the flexibility in our choice of $\{S_1, S_2\}$ described in (4.5) and because of (4.8), we may conclude that the entire subgraph $\bigcup \mathbf{Q}$ is edge-disjoint from $W_s \setminus S_s$, $s = 1, 2$. By the definition of \mathbf{Q} and the facts $\mathbf{L}_1 \subseteq \mathbf{L}$ and $p' \leq p$ and by (4.9) we have that $\chi^{\mathbf{Q}} \leq p$, $\chi^{\mathbf{W}} \leq p$ and both $\chi^{\mathbf{Q}}$ and $\chi^{\{W_1, W_2\}}$ are $\{0, 1, 2\}$ -valued. Hence $q_{\mathbf{Q}} \leq p$ and is $\{0, 1, 2\}$ -valued. Note that $q_{\mathbf{Q}}(e_0) = 2$. Since W_1 and W_2 are cycles, v_i and v_j are the only vertices of odd degree in each of the subgraphs $W_1 \setminus S_1$ and $W_2 \setminus S_2$. It follows that $\chi^{\{W_1 \setminus S_1, W_2 \setminus S_2\}}$ is eulerian. By Proposition 3, $\chi^{\mathbf{Q}}$ is eulerian, so $q_{\mathbf{Q}}$ is eulerian. As $q_{\mathbf{Q}}$ is eulerian, $\{0, 1, 2\}$ -valued and has as support the 2-edge-connected subgraph $\bigcup \mathbf{Q} \cup W_1 \cup W_2$, $q_{\mathbf{Q}}$ is admissible by Proposition 6.

Suppose $(G, q_{\mathbf{Q}})$ has a circuit cover $\mathbf{X}_{\mathbf{Q}}$ for each chain \mathbf{Q} . It remains to show that we can splice these circuit covers together and obtain a circuit cover \mathbf{X} of (G, p) . Roughly, \mathbf{X} shall consist of a modification of the cycles in \mathbf{W} together with a subset $\mathbf{Y}_{\mathbf{Q}}$ of each circuit cover $\mathbf{X}_{\mathbf{Q}}$.

Let $\mathbf{Q} = (C_{i+1}, C_{i+2}, \dots, C_j)$ be any (v_i, v_j) -chain and let $W_1, W_2 \in \mathbf{W}$ be as above. For $s = 1, 2$, let v_i^s and v_j^s denote the first and last vertices, respectively, of $\bigcup \mathbf{Q}$ encountered when W_s is traversed (in the usual direction) starting at x . The three vertices v_i^1, v_i^2 , and v_i might not be distinct (and similarly for v_j^1, v_j^2 , and v_j). For example, we know that $v_0^1 = v_0^2 = v_0 = x$. The vertices in $\{v_i^1, v_i^2, v_j^1, v_j^2\}$ are called the *connector vertices* of \mathbf{Q} . If $i > 0$, then the last edge in the subtrail $W_s[x, v_i^s]$ is denoted by e_i^s , $s = 1, 2$. If $i = 0$, then we define $e_i^s = e_0$, $s = 1, 2$. Similarly, e_j^s is defined to be either e_0 (if $j = k$) or the first edge in the subtrail $W_s[v_j^s, y]$, $s = 1, 2$. The edges in $\{e_i^1, e_i^2, e_j^1, e_j^2\}$ are called the *connector edges* of \mathbf{Q} .

Let $i > 0$ and let S_s^- denote the segment in $W_s[x, v_i]$ which terminates

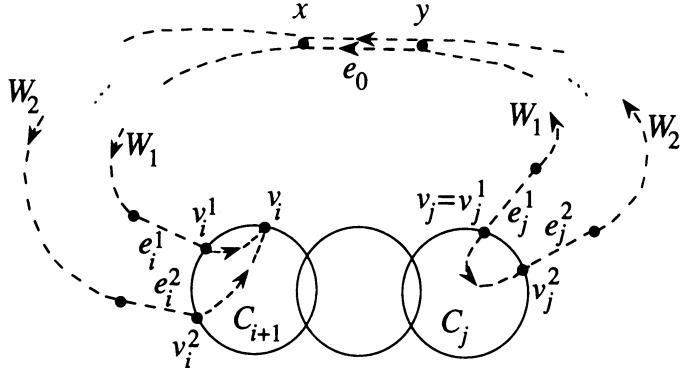


FIGURE 3

at v_i , $s = 1, 2$. Applying (4.8), all segments in $W_s[x, v_i] \setminus S_s^-$ are vertex-disjoint from the chain segments S_1 and S_2 . Because of the arbitrary choice of chain segments for \mathbf{Q} , and by (4.5), $W_s[x, v_i] \setminus S_s^-$ is vertex-disjoint from all of $\bigcup \mathbf{Q}$, $i = 1, 2$. Also, by (4.8), $W_1[x, v_i] \cup W_2[x, v_i]$ is vertex-disjoint from $W_1[v_j, y] \cup W_2[v_j, y]$. Thus, by definition of connector edges, we have $e_i^s \in S_s^-$, $s = 1, 2$, and that $\{e_i^1, e_i^2\}$ is an edge-cut in $\bigcup \mathbf{Q} \cup W_1[x, y] \cup W_2[x, y]$ separating the vertices in $W_1[x, v_i^1] \cup W_2[x, v_i^2] - \{v_i^1, v_i^2\}$ from those in $\bigcup \mathbf{Q} \cup W_1[v_i^1, y] \cup W_2[v_i^2, y]$. Since S_1^- and S_2^- are distinct segments contained in $C_i \in \mathbf{L}_2$ we have $e_i^1 \neq e_i^2$. We summarize as follows.

(4.10) If $i > 0$, then $\{e_i^1, e_i^2\}$ is a 2-edge-cut in $(\bigcup \mathbf{Q} \cup W_1 \cup W_2) \setminus e_0$; similarly, if $j < k$, then $\{e_j^1, e_j^2\}$ is a 2-edge-cut in $(\bigcup \mathbf{Q} \cup W_1 \cup W_2) \setminus e_0$ (see Figure 3).

Let $\bigcup \mathbf{Q}^+$ denote the connected component of

$$\left(\bigcup \mathbf{Q} \cup W_1 \cup W_2 \right) \setminus \{e_i^1, e_i^2, e_j^1, e_j^2\}$$

which contains the connector vertices. Thus $\bigcup \mathbf{Q}^+$ is the union of $\bigcup \mathbf{Q}$ and the (v_i^1, v_i, v_i^2) -subpath of C_i and the (v_j^1, v_j, v_j^2) -subpath of C_{j+1} (if $i = 0$ or $j = k$, then we use the empty path).

Note that $\mathbf{X}_{\mathbf{Q}}$ is a circuit cover of $(\bigcup \mathbf{Q} \cup W_1 \cup W_2, q_{\mathbf{Q}})$ (see Figure 3). Let A_1, A_2 be the two circuits in $\mathbf{X}_{\mathbf{Q}}$ which contain e_0 . By (4.10), A_1 contains exactly one edge from each of $\{e_i^1, e_i^2\}$ and $\{e_j^1, e_j^2\}$; A_2 contains the remaining two connector edges. We relabel A_1, A_2 so that $e_i^1 \in E(A_1)$ and $e_i^2 \in E(A_2)$. Every circuit in $\mathbf{X}_{\mathbf{Q}} \setminus \{A_1, A_2\}$ is either contained wholly in $\bigcup \mathbf{Q}^+$ or is vertex-disjoint from $\bigcup \mathbf{Q}^+$. We denote the subset of circuits of the former type by $\mathbf{Y}_{\mathbf{Q}}$.

We recall that a *cycle cover* of (G, p) is a multiset \mathbf{A} of cycles in G such that $\chi^{\mathbf{A}} = p$. For example, any circuit cover of (G, p) is also a cycle cover of (G, p) . Conversely, by decomposing the cycles in a cycle cover of (G, p) , one obtains a circuit cover of (G, p) . We aim to produce a cycle cover of (G, p) . Let \mathbf{Y} denote the union of $\mathbf{Y}_{\mathbf{Q}}$ over all chains \mathbf{Q} . Although \mathbf{W} is a cycle cover of $(G, \chi^{\mathbf{W}})$, $\mathbf{Y} \cup \mathbf{W}$ is not quite a cycle cover of (G, p) . We must still modify the cycles in \mathbf{W} so that they “mesh” correctly with $\mathbf{Y}_{\mathbf{Q}}$ within each chain \mathbf{Q} .

For each chain \mathbf{Q} , $\mathbf{Y}_{\mathbf{Q}}$ is a circuit cover of $(G, \chi^{\mathbf{Q}} + \chi^{\{W_1^+ \setminus S_1, W_2^+ \setminus S_2\}} - \chi^{\{A_1^+, A_2^+\}})$, where W_s^+ and A_s^+ denote $W_s \cap \bigcup \mathbf{Q}^+$ and $A_s \cap \bigcup \mathbf{Q}^+$, respectively

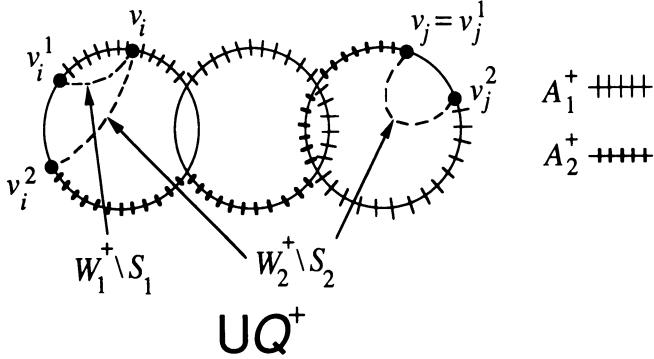


FIGURE 4

(see Figure 4). We modify the two cycles $W_1, W_2 \in \mathbf{W}$ in one of two ways, depending on which of e_j^1, e_j^2 is an edge of A_1 (Figure 4 depicts the second possibility). (Recall that $e_j^1 \in E(A_1)$.) If $e_j^1 \in E(A_1)$, then $e_j^2 \in E(A_2)$ and we modify W_s by replacing the (v_i^s, v_j^s) -subpath W_s^+ of W_s with the (v_i^s, v_j^s) -path A_s^+ , $s = 1, 2$. If $e_j^2 \in E(A_1)$, then $e_j^1 \in E(A_2)$ and we modify W_s by replacing the (v_i^s, v_j^s) -subpath W_s^+ of W_s with the (v_i^s, v_j^{3-s}) -path A_s^+ , $s = 1, 2$, and then interchanging the (v_j^1, y) -subpath of W_1 with the (v_j^2, y) -subpath of W_2 . In either case, each of the resulting two subgraphs are cycles that can take the places of W_1 and W_2 in \mathbf{W} . After this modification we have $\chi^{\mathbf{W} \cup \mathbf{Y}_\mathbf{Q}}(e) = \chi^\mathbf{L}(e) = p(e)$ for all $e \in E(\bigcup \mathbf{Q}^+)$.

We perform the modification of \mathbf{W} as described in the previous paragraph for every chain \mathbf{Q} in G (in any order). By (4.9) and the observations of the previous paragraph, $\mathbf{W} \cup \mathbf{Y}$ is a cycle cover of (G, p) , as required.

5. PROOF OF LEMMA 2

We shall need the following lemma which was essentially proved by Ellingham [Ell].

Lemma 3. *Let H be a simple cubic graph which has a perfect matching M such that the 2-factor $H \setminus M$ has exactly two components (which are circuits), and every edge in M has one endvertex in each of these circuits. If H does not have a proper 3-edge-coloring, then there exists a subset $S \subseteq M$ such that $H \setminus S$ is a subdivision of Petersen's graph.*

When a weight vector p is $\{0, 1, 2\}$ -valued, the admissibility conditions (1.2) degenerate slightly. The set of edges of weight i in (G, p) is denoted E_i .

Proposition 6. *A weight vector $p : E(G) \rightarrow \{0, 1, 2\}$ is admissible if and only if both of the following hold:*

- (1) (balance) G has no edge cut containing exactly one positive-weight edge, and
- (2) (eulericity) E_1 is a cycle in G .

We note that if (2) holds and (1) fails then the positive-weight edge has weight 2.

The following proof of Lemma 2 is a generalization of that given by Alspach and Zhang [Als], which was for cubic graphs only. Since there does not appear to be an analog of Lemma 3 for graphs with higher-degree vertices, it is critical that we reduce to the cubic graph case. The main difficulty here turns out to be the elimination of vertices of degree 4 in minimal contra-weighted graphs.

Let (G, p) be a $\{1, 2\}$ -valued minimal contra-weighted graph. We aim to show that (G, p) is a blistered Petersen graph. As in the proof of Lemma 1, our first step is to eliminate 2-edge-cuts and nontrivial tight cuts.

(5.1) We can assume G has no vertices of degree 2. If x is such a vertex then we contract one of its incident edges, obtaining (G', p') . By induction on $|E(G)|$, there exists a blistered $P_{10}, (G', q')$, with $q' \leq p'$. By applying (ii) of (2.1) to (G', p') , we can obtain a blistered $P_{10}, (G, q)$, with $q \leq p$, and we are done.

(5.2) We can assume (G, p) has no nontrivial tight cuts. Suppose G has a nontrivial tight cut. There exist two tight cuts $\delta(\{x_1, x_2\}), \delta(\{x_1\})$ with a common tight cut leader $e_1 = x_1y_1$ by Proposition 5. By (5.1), x_1 and x_2 have degree at least 3. Since p is $\{1, 2\}$ -valued, it must be the case that $p(e_1) = 2$, and that there are two parallel edges of weight 1, e_2 and e_3 , joining x_1 to x_2 , and that no other edges meet x_1 . We now replace e_2 and e_3 with a single edge of weight 2, and argue as in (5.1), applying either (ii) or (iii) of (2.1).

It follows from (5.1), (5.2), and Proposition (6.1) that G is 3-edge-connected. We define E_1 and E_2 as above. By Proposition 6, E_1 is a cycle.

The next two paragraphs are specializations of arguments presented in the proof of Lemma 1. We include them for completeness. If $p = 1$, then (G, p) is not contra-weighted, since G is eulerian. Let e_0 be an arbitrary edge of weight 2 and let $p' = p - 2\chi^{e_0}$. By Proposition 6 and since G is 3-edge-connected, (G, p') is admissible. By minimality of p , (G, p') has a circuit cover.

Let \mathbf{L} be any circuit cover of (G, p') , and let \mathbf{L}' be a minimal subset of \mathbf{L} such that $(G, \chi^{\mathbf{L}'} + 2\chi^{e_0})$ is admissible. (By Proposition 6, this is equivalent to requiring that $\bigcup \mathbf{L}' + e_0$ be a bridgeless subgraph of G .) If $(G, \chi^{\mathbf{L}'} + 2\chi^{e_0})$ were to have a circuit cover, then adjoining $\mathbf{L} - \mathbf{L}'$ to this circuit cover would yield a circuit cover of (G, p) , a contradiction. Thus $(G, \chi^{\mathbf{L}'} + 2\chi^{e_0})$ is a contra-weighted graph. By minimality of p , we have $\mathbf{L} = \mathbf{L}'$. It follows that $\mathbf{L} = \{C_1, C_2, \dots, C_k\}$ where C_i and C_j intersect (in at least one vertex) if and only if $|i - j| \leq 1$. Furthermore, C_i intersects with e_0 (at a vertex) if and only if $i = 1$ or $i = k$. Using terminology from the proof of Lemma 1, we have the following.

(5.3) Every circuit cover of (G, p') consists of a single (x, y) -chain of circuits, where x and y are the endvertices of e_0 (see Figure 5). A k cycle cover of (G, p) is a multiset of at most k cycles which covers each edge $e \in E$ exactly $p(e)$ times. Let $D_0 = \bigcup \{C_i : i \text{ is even}\}$ and let $D_1 = \bigcup \{C_i : i \text{ is odd}\}$. Each D_i is a cycle in G and $\{D_0, D_1\}$ is a 2 cycle cover of (G, p') . Recall that $E_1 = p^{-1}(1)$ and $E_2 = p^{-1}(2)$. Consider the contracted graph G/E_1 , and let D_i/E_1 denote the cycle in G/E_1 which is induced by the edge set $D_i \cap E_2 = E_2 \setminus e_0$. Then $\{D_0/E_1, D_1/E_1\}$ is a 2 cycle cover of $(G/E_1, 2\chi^{E_2 \setminus e_0})$. Thus, $D_0/E_1 = D_1/E_1 = E(G/E_1 \setminus e_0)$ so $G/E_1 \setminus e_0$ is eulerian. Since e_0 is an arbitrary edge in E_2 , there are exactly two possibilities for G/E_1 :

(5.4) G/E_1 contains exactly one vertex, and every edge of G/E_1 is a loop.

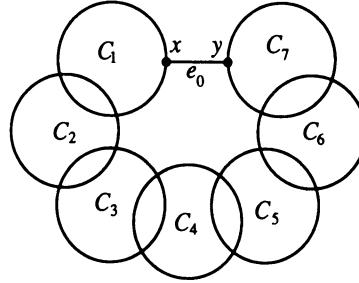


FIGURE 5

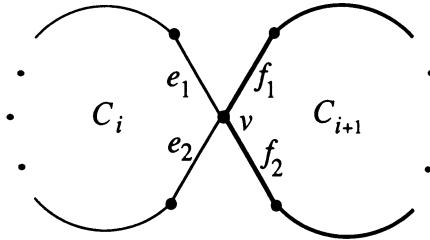


FIGURE 6

(5.5) G/E_1 contains exactly two vertices, both vertices have odd degree, and every edge of G/E_1 joins these two vertices.

Suppose that (5.4) is the case. Then there exists an (x, y) -path P in E_1 . Let C be the circuit $P + e_0$ in G . Then $\{D_0 \Delta C, D_1 \Delta C\}$ is a cycle cover of (G, p) , a contradiction. Thus (5.5) is the case.

Let \mathbf{L} be a circuit cover of (G, p') . By (5.1) and (5.3), every vertex in $V(G) \setminus \{x, y\}$ is contained in exactly two (consecutive) circuits in \mathbf{L} . Thus every vertex in G is either *cubic* (degree 3) or *quartic* (degree 4). Each cubic vertex is adjacent with exactly one edge in E_2 and two edges in E_1 . Each quartic vertex is adjacent with exactly four edges in E_1 . Both x and y are cubic vertices. We write $V(G) = V_3 \cup V_4$ where V_i denotes the set of vertices of degree i in G .

Since p is eulerian, each of the two connected components induced by E_1 is an eulerian subgraph of G which is either a circuit or a subdivision of some connected 4-regular graph.

We intend to establish that $V_4 = \emptyset$ and hence that G is a cubic graph. Suppose that $v \in V_4$. Let $e_0 \in E_2$ be arbitrary and let \mathbf{L} be a circuit cover of $(G, p') = (G, p - 2\chi^{e_0})$ of maximum possible cardinality. By (5.3), \mathbf{L} is an (x, y) -chain $\{C_1, C_2, \dots, C_k\}$. Thus $v \in V(C_i) \cap V(C_{i+1})$ for some unique $i \in \{1, 2, \dots, k-1\}$. Let $\{e_1, e_2\}$ be the two edges in C_i incident with v , and let $\{f_1, f_2\}$ be the two edges in C_{i+1} incident with v (see Figure 6).

Consider the subgraph $J := C_i \cup C_{i+1}$ of G . There must be some vertex in $V(C_i) \cap V(C_{i+1})$ which is different from v for, otherwise, $\{e_0, e_1, e_2\}$ would be a nontrivial tight cut in (G, p) , contradicting (5.2), or else $i = 1$ and C_1 and C_2 are 2-gons, in which case we obtain a contradiction to the choice of (G, p) . Thus J is 2-connected. Suppose that $E(C_i \cap C_{i+1}) = \emptyset$. Then J is eulerian. Since J is 2-connected, there is a circuit C in J such that $(\mathbf{L} \setminus \{C_i, C_{i+1}\}) \cup \{C\}$ is an (x, y) -chain. The union of this chain with e_0 is 2-

connected. Since (G, p) and (G, χ^{J-C}) are eulerian, so is $(G, p - \chi^{J-C})$. By construction, $p - \chi^{J-C}$ is also balanced. Thus, by Proposition 6, $(G, p - \chi^{J-C})$ is admissible so, by minimality of p , $(G, p - \chi^{J-C})$ has a circuit cover. The union of this circuit cover with the cycle $\{J - C\}$ is a cycle cover of (G, p) , a contradiction. Thus $E(C_i) \cap E(C_{i+1}) \neq \emptyset$.

A *subcycle* is a subset of a cycle which is also a cycle. Let $r = \chi^{\{C_i, C_{i+1}\}}$, let $F_1 = r^{-1}(1) = C_i \Delta C_{i+1}$ and let $F_2 = r^{-1}(2) = C_i \cap C_{i+1}$. Let C be any subcycle of the cycle F_1 . Like $\{C_i, C_{i+1}\}$, $\{C_i \Delta C, C_{i+1} \Delta C\}$ is a 2 cycle cover of (G, r) . Hence $\mathbf{L}_C := (\mathbf{L} \setminus \{C_i, C_{i+1}\}) \cup \{C_i \Delta C, C_{i+1} \Delta C\}$ is a cycle cover of (G, p') (in [God2], the transformation $\mathbf{L} \rightarrow \mathbf{L}_C$ is called a *pivot* of $\{C_i, C_{i+1}\}$ on C). Note that if C is the empty cycle, then $\mathbf{L}_C = \mathbf{L}$. Since F_2 is not empty, C is different from both C_i and C_{i+1} , so neither $C_i \Delta C$ nor $C_{i+1} \Delta C$ is the empty cycle. By maximality of $|\mathbf{L}|$, we have $|\mathbf{L}_C| = |\mathbf{L}|$, so each of the cycles $C_i \Delta C$ and $C_{i+1} \Delta C$ is a circuit. Thus \mathbf{L}_C is a circuit cover of (G, p') which, by (5.3), must be an (x, y) -chain of circuits.

A *block* in a graph H is a maximal 2-connected subgraph of H . The blocks of H induce a partition of $E(H)$. In the following two paragraphs we compare the block structures of the cycles F_1 and E_1 . In general these two cycles are different, since any edge in $E(C_{i-1} \cap C_i)$ is in $F_1 \cap E_2$. However, we shall see that all but one of the blocks of F_1 is also a block of E_1 . Furthermore, we shall see that the quartic vertex v is a cut-vertex of F_1 , and hence of E_1 .

Let v_{i-1} be any vertex in $V(C_{i-1}) \cap V(C_i)$, and let v_{i+1} be any vertex in $V(C_{i+1}) \cap V(C_{i+2})$ (here, we temporarily define $C_0 = C_{k+1} = e_0$). Then v_{i-1}, v_{i+1} are vertices of degree 2 in F_1 . Let C be any subcycle of F_1 which contains one of these two vertices, say v_{i-1} . Then C must also contain v_{i+1} , for otherwise the circuit $C_{i+1} \Delta C$ would contain both v_{i+1} and v_{i-1} , contradicting the fact that \mathbf{L}_C is an (x, y) -chain of circuits. Hence every subcycle of F_1 contains either all or none of the vertices in $(V(C_{i-1}) \cap V(C_i)) \cup (V(C_{i+1}) \cap V(C_{i+2}))$. This is true, in particular, when the subcycle C of F_1 is a circuit. Thus all of these vertices belong to a single block B of F_1 . It follows that each block of $F_1 \setminus B$ is vertex-disjoint from each circuit in $\mathbf{L} \setminus \{C_i, C_{i+1}\}$. Thus we have shown the following.

(5.6) There exists a block B in F_1 such that every block of $F_1 \setminus B$ is also a block of E_1 .

Let C be any circuit in F_1 containing the quartic vertex v (see Figure 6). Then C must contain exactly one edge from $\{e_1, e_2\}$ and one edge from $\{f_1, f_2\}$, for otherwise v would be a vertex of degree 4 in either $C_i \Delta C$ or $C_{i+1} \Delta C$, contradicting the fact that they are circuits. Thus e_1 and e_2 belong to distinct blocks of F_1 , and v is a cut-vertex of F_1 . By (5.6) we have the following.

(5.7) The quartic vertex v is a cut-vertex of E_1 . By interchanging the labels of f_1 and f_2 if necessary, we may assume that, for $i = 1, 2$, e_i and f_i belong to the same block of E_1 , whereas e_1 and e_2 (respectively f_1 and f_2) belong to distinct blocks of E_1 .

We define a new weighted graph (G^v, p^v) from (G, p) by replacing v with two new (cubic) vertices, v_1 and v_2 , such that, for $i = 1, 2$, v_i is incident with both e_i and f_i . A new edge e_v of weight 2 joins v_1 and v_2 (see Figure 7). Thus $E(G^v) = E_1 \cup E_2 \cup \{e_v\}$.

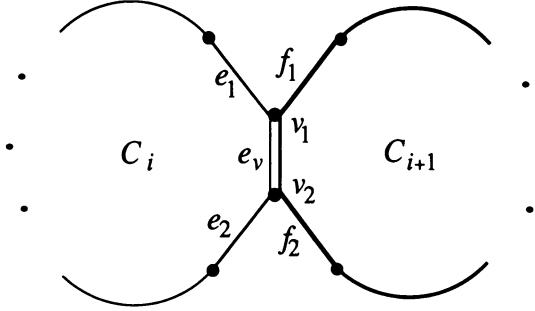


FIGURE 7

The definition of (G^v, p^v) depends only on the block structure of E_1 and the quartic vertex v , and is independent of the choice of \mathbf{L} and, indeed, the choice of e_0 . By (5.5) and (5.7), E_1 induces exactly three connected components in G^v . A minor modification of \mathbf{L} , as depicted in Figure 7, yields a 2-cycle cover $\{D_0^v, D_1^v\}$ of $(G^v, p^v - 2\chi^{e_0})$. As in the derivation of (5.5), $\{D_0^v/E_1, D_1^v/E_1\}$ is a 2-cycle cover of the contracted graph $(G^v/E_1, 2\chi^{E_2 \cup \{e_v\} \setminus \{e_0\}})$. The arbitrary choice of $e_0 \in E_2$ implies that exactly two of the three vertices in the contracted graph G^v/E_1 have odd degree, and that every edge in $E_2 = E(G^v/E_1) \setminus \{e_v\}$ joins these two odd vertices. One easily sees that such a graph cannot exist (unless G^v/E_1 is disconnected, which clearly is not the case). This contradiction establishes that $V_4 = \emptyset$.

Thus G is cubic, the two components comprising E_1 are circuits in G , and every edge in the 1-factor E_2 has an endvertex in each of these circuits (such graphs are called σ -prisms in [Als]). Suppose that G has a proper 3-edge coloring. Let Z_i be the cycle obtained by deleting the i th color class from G , $i = 1, 2, 3$. Then $\{Z_1, Z_2, Z_3\}$ is a 3 cycle double cover of G , and hence $\{Z_1 \Delta E_1, Z_2 \Delta E_1, Z_3 \Delta E_1\}$ is a cycle cover of (G, p) , a contradiction. Thus G has no proper 3-edge coloring. By Lemma 3, the deletion of some edges $S \subseteq E_2$ yields a subdivision of Petersen's graph. Hence $(G, p - 2\chi^S)$ is a blistered (P_{10}, p_{10}) such that E_1 induces exactly two disjoint circuits. By minimality of p , we must have $S = \emptyset$. Since G is 3-edge-connected, we have $(G, p) = (P_{10}, p_{10})$, and we have proved Lemma 2. \square

6. COMPLEXITY

We do not know the complexity of deciding whether a general weighted graph has a circuit cover (we call this the *circuit cover problem*). The difficulty of the Shortest Circuit Cover Problem and the Cycle Double Cover Conjecture suggests that this problem is \mathcal{NP} -hard. Indeed, we do not even know whether the circuit cover problem belongs to either of the classes \mathcal{NP} or co- \mathcal{NP} (see [Gar] for definitions). It is conceivable that the number of distinct circuits needed in a circuit cover of (G, p) grows linearly with $r := \max\{p(e) | e \in E(G)\}$ rather than a polynomial in the input size $|E(G)| \log(r)$, hence the ambiguity of membership in \mathcal{NP} .

If we restrict the input to graphs with no P_{10} -minor, then the circuit cover problem belongs to the complexity class \mathcal{P} . (Incidentally, determining whether a graph has a P_{10} -minor can be done in polynomial time [Sey4].) Indeed,

testing the admissibility of a weight vector p requires only $|V(G)|$ parity checks and $|E(G)|$ applications of the Max-flow Min-cut algorithm, both of which are polynomial in $|E(G)| \log(r)$.

The following questions, however, warrant further investigation. Suppose that G has no P_{10} -minor and (G, p) is admissible.

(6.1) Does (G, p) have a circuit cover where the number of distinct circuits is bounded by a polynomial in $|E(G)| \log(r)$?

(6.2) Is there a polynomial-time algorithm which will construct a circuit cover of (G, p) ?

Of course (6.2) is stronger than (6.1). From the proof of Theorem 5 below, we shall see that (6.1) holds true. In fact, if (G, p) has a circuit cover and G has no P_{10} -minor, then (G, p) has a circuit cover using fewer than $2|E(G)|$ distinct circuits. The following is a partial answer to (6.2).

Theorem 5. *Question (6.2) holds true if and only if there is a polynomial time algorithm for the following problem.*

*Input : A bridgeless graph H with maximum degree 4 and
(6.3) containing no P_{10} -minor, together with a cycle Z in H .*

Output : A circuit C such that $(H, 2 - \chi^Z - \chi^C)$ is admissible.

Proof. Suppose that (6.2) has a positive answer. By Proposition 6, the $\{1, 2\}$ -weighted graph $(H, 2 - \chi^Z)$ is admissible, and hence has a circuit cover which can be constructed in polynomial time. Any one of the circuits in this cover can be used for C .

Conversely, let (G, p) be an admissible weighted graph where G has no P_{10} -minor, and let \mathbf{O} denote an oracle which can solve (6.3) in polynomial time. We note that by applying oracle \mathbf{O} repeatedly, one can obtain a circuit cover of $(H, 2 - \chi^Z)$. A naive implementation (CirCov1, outlined below) based on the proof of Lemma 1 can find a circuit cover of (G, p) using oracle \mathbf{O} . Unfortunately, CirCov1 is only *pseudo-polynomial* (see [Gar]) since the number of distinct circuits in the circuit cover \mathbf{L}' it produces can be proportional to $|E(G)|r$, where $r = \max\{p(e) | e \in E(G)\}$. We shall subsequently demonstrate, however, the existence of a *strongly polynomial-time* algorithm (CirCov2) which produces a pair (\mathbf{L}, μ) where \mathbf{L} is a list of $t < 2|E(G)|$ circuits in G , and where $\mu = (\mu_1, \dots, \mu_t)$ is a corresponding multiplicity vector (whose entries are bounded by r), such that (\mathbf{L}, μ) describes a circuit cover of (G, p) .

CirCov1 : Input : An admissible edge weighted graph (G, p) where G has no P_{10} -minor.

Output : A circuit cover \mathbf{L}' of (G, p) .

1. Preprocessing: Delete edges of weight 0. Reduce any nontrivial tight cut. Such a tight cut yields two admissible contracted graphs (G_1, p_1) , (G_2, p_2) (see Proposition 5) which are solved separately, then spliced appropriately at the tight cut. We assume from here that (G, p) is 3-edge-connected, positive, admissible and that all tight cuts are trivial.

2. If $p = 1$, then we exit with a circuit decomposition of the eulerian graph G . Otherwise let $e_0 = xy$ be any edge of minimum weight subject to e_0 being a nonfollower (cf. (4.3)) having weight at least 2.

3. Call CirCov1 recursively to find a circuit cover \mathbf{M} of $(G, p - 2\chi^{e_0})$.

4. As in the proof of Lemma 1, we find a shortest (x, y) -path in the auxiliary graph $G_{\mathbf{M}}$ and obtain a subset $\mathbf{M}' = \{C_1, C_2, \dots, C_k\} \subseteq \mathbf{M}$ having the form of Figure 2.

5. Use the Max-flow Min-cut algorithm on the auxiliary graph H to find $p(e_0)$ closed trails $\mathbf{W} = \{W_1, W_2, \dots, W_r\}$ as in (4.7), and use these to define, for each chain $\mathbf{Q} \subseteq \mathbf{M}'$, the $\{0, 1, 2\}$ -valued weight vector $q_{\mathbf{Q}} \leq p$.

6. For each chain \mathbf{Q} we apply oracle \mathbf{O} repeatedly to find a circuit cover of $(G, q_{\mathbf{Q}})$. This can be done since the support of $q_{\mathbf{Q}}$ is an admissible $\{1, 2\}$ -weighted subgraph of G having maximum degree 4 and containing no P_{10} -minor. Finally, we combine these circuit covers as described at the end of the proof of Lemma 1 to obtain a circuit cover of $(G, \chi^{\mathbf{M}'} + 2\chi^{e_0})$. Adjoining the list of circuits $\mathbf{L}' \setminus \mathbf{M}'$ to this circuit cover gives the desired circuit cover of (G, p) . Exit.

Detecting nontrivial tight cuts in Step 1 requires $O(|E|)$ network flow calculations. Nonfollowers are easy to detect in Step 2 as all tight cuts are trivial here. Steps 4 through 6 also involve only network flow, shortest path, and parity check calculations and are easily seen to be polynomial in $|E|$ and the running time of oracle \mathbf{O} . Finally, the total number of invocations of CirCov1 is at most $p(G)/2$ as the total weight of each successive graph is reduced by 2.

A strongly polynomial algorithm for (6.2) can be obtained from CirCov1 by using a trick which first appeared in essence in a paper by Cook, Fonloupt, and Schrijver [Coo] regarding *Hilbert bases*. In the terminology of Hilbert bases, the main result of this paper can be stated as follows.

(6.4) The circuits of a graph form a Hilbert basis if and only if the graph has no P_{10} -minor.

The idea is to polynomially solve a linear program relaxation of the circuit cover problem for (G, p) , and to separate out any fractional part of the resulting solution. We then use CirCov1 to replace the (relatively small) fractional part with an integer solution.

Recall that \mathbf{C} denotes the set of circuits in G . Let M denote the circuit-edge $\{0, 1\}$ -incidence matrix for G , let $\mathbf{1}$ denote the column vector of $|\mathbf{C}|$ ones, and suppose that p is a row vector.

CirCov2 : Input: An admissible edge weighted graph (G, p) where G has no P_{10} minor.

Output: A circuit cover (\mathbf{L}, μ) of (G, p) where \mathbf{L} is a list of at most $2|E(G)| - 1$ circuits and μ is a multiplicity vector whose entries are bounded by $r = \max\{p[(e)|e \in E(G)\}$.

1. Find a basic feasible solution $\lambda = (\lambda_C)_{C \in \mathbf{C}}$ to the following linear program:

$$(6.5) \quad \begin{aligned} & \max \lambda \mathbf{1} \\ & \lambda M = p \\ & \lambda \geq 0 \end{aligned}$$

2. Let $\lfloor \lambda \rfloor := (\lfloor \lambda_C \rfloor)_{C \in \mathbf{C}}$ and $\{\lambda\} := \lambda - \lfloor \lambda \rfloor$ be the integer and fractional parts of λ , and let $p' := \{\lambda\}M = p - \lfloor \lambda \rfloor M$. As p' is a nonnegative combination of circuits, (G, p') is balanced. Furthermore, (G, p') is eulerian since both p and $\lfloor \lambda \rfloor M$ are. Thus (G, p') is admissible.

3. Call CirCov1 with input (G, p') to obtain a circuit cover \mathbf{L}' of (G, p') .
4. Adjoin \mathbf{L}' to the circuit cover $(\Lambda, [\lambda])$ of $(G, p - p')$, where $\Lambda := \{C \in \mathbf{C} \mid [\lambda_C] > 0\}$, and exit with the resulting circuit cover (\mathbf{L}, μ) .

We bound the size of \mathbf{L} as follows. As λ is a basic solution, $|\Lambda| \leq |E|$. Also, by maximality of $\lambda\mathbf{1}$ we have $|\mathbf{L}'| + [\lambda]\mathbf{1} \leq \lambda\mathbf{1} = [\lambda]\mathbf{1} + \{\lambda\}\mathbf{1}$, so $|\mathbf{L}'| \leq \{\lambda\}\mathbf{1}$. Since each of the nonzero entries in $\{\lambda\}$ is less than 1 we have $\{\lambda\}\mathbf{1} < |E|$, so $|\mathbf{L}'| \leq |E| - 1$. Thus $|\mathbf{L}| \leq |\Lambda| + |\mathbf{L}'| \leq 2|E| - 1$. Incidentally, this argument shows that (6.1) is true as claimed above.

As $\max\{p'(e) \mid e \in E(G)\} \leq |\mathbf{L}'| < |E|$, Step 3 is strongly polynomial in the running time of oracle \mathbf{O} . It remains to show that Step 1 of CirCov2 can be done in time bounded by a polynomial in $|E(G)| \log(r)$ despite the exponential number of variables λ_C . We give an indirect method which involves the dual linear program.

$$(6.6) \quad \begin{array}{l} \min px \\ Mx \geq \mathbf{1} \end{array}$$

The *separation problem* for (6.6) is the following:

Given a rational weight vector x on $E(G)$ either determine that x satisfies $Mx \geq \mathbf{1}$, or display a violated inequality (that is, a circuit in G having total weight less than 1).

A deep theorem of Grötschel, Lovász and Schrijver (See Corollary 14.1g(v) of [Sch]) implies that a basic optimal solution to (6.5) can be found via the ellipsoid method in time polynomially bounded by $|E|$ and the input length of w provided that

(i) the polyhedron $P := \{x \mid Mx \geq \mathbf{1}\}$ is *full dimensional* and *pointed* (see 8.3(6) in [Sch]), and

(ii) the separation problem for (6.6) can be solved in time polynomially bounded by $|E|$ and the input length of x .

That P is full dimensional follows from the fact that any edge $e = st$ in a 3-edge-connected graph is a $\{0, \pm \frac{1}{2}\}$ -linear combination of three cycles (consider two edge-disjoint (s, t) -paths in $G - e$). To prove pointedness, suppose that weight vectors x and x' are such that $x + \alpha x' \in P$ for all rational scalars α . Then we must have $Mx' = \mathbf{0}$. As P is full dimensional, the columns of M are linearly independent so $x' = \mathbf{0}$, and thus P is pointed. To solve (ii) it suffices to check for each $e \in E(G)$ that $(G, x - \chi^{e_0})$ has no negative-weight circuits or display one if one exists. This can be done using $|E(G)|$ calls to a shortest-path algorithm for undirected weighted graphs with no negative-weight circuits (e.g., Chapter 6.2 in [Law]). This completes the proof. \square

It is possible that a direct algorithm for solving (6.5) can be obtained using the proof of Seymour's "sums of circuits" result [(2.5) in Sey1], though we do not investigate this here. We do not know whether there exists a polynomial-time algorithm for (6.3), even when input is restricted to cubic graphs.

ACKNOWLEDGMENT

We thank A. Sebő for stimulating discussion regarding the complexity of circuit covers.

REFERENCES

- [Alo] N. Alon and M. Tarsi, *Covering multigraphs by simple circuits*, SIAM J. Algebraic Discrete Methods **6** (1985), 345–350.
- [Als] B. R. Alspach and C-Q. Zhang, *Cycle coverings of cubic graphs*, Discrete Math. **111** (1993), 11–17.
- [Arc] D. Archdeacon, *Face colorings of embedded graphs*, J. Graph Theory **8** (1984), 387–398.
- [Ber] J. C. Bermond, B. Jackson, and F. Jaeger, *Shortest coverings of graphs with cycles*, J. Combin. Theory Ser. B **35** (1983), 297–308.
- [Bon] J. A. Bondy, *Small cycle double covers of graphs*, Cycles and Rays (G. Hahn, G. Sabidussi, and R. Woodrow, eds.), Nato ASI Series C, vol. 301, Kluwer, Dordrecht and Boston, 1990, pp. 21–40.
- [Cel] U. A. Celmins, *On cubic graphs that do not have an edge 3-coloring*, Ph.D. thesis, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada, 1984.
- [Cat] P. Catlin, *Double cycle covers and the Petersen graph*, J. Graph Theory **13** (1989), 465–483.
- [Coo] W. Cook, J. Fonloupt, and A. Schrijver, *An integer analogue of Carathéodory's theorem*, J. Combin. Theory Ser. B **40** (1986), 63–70.
- [Edm] J. Edmonds and E. L. Johnson, *Matching, Euler-tours, and the Chinese postman*, Math. Programming **5** (1973), 88–124.
- [Ell] M. N. Ellingham, *Petersen subdivisions in some regular graphs*, Congr. Numer. **44** (1984), 33–40.
- [Fan1] G. Fan, *Covering weighed graphs by even subgraphs*, J. Combin. Theory Ser. B **49** (1990), 137–141.
- [Fan2] ———, *Integer flows and cycle covers*, J. Combin. Theory Ser. B **54** (1992), 113–122.
- [Fle1] H. Fleischner, *Eulersche Linien und Kreisüberdeckungen die vorgegebene Durchgänge in den Kanten vermeiden*, J. Combin. Theory Ser. B **2** (1980), 145–167.
- [Fle2] H. Fleischner and A. Frank, *On circuit decompositions of planar Eulerian graphs*, J. Combin. Theory Ser. B **50** (1990), 245–253.
- [For] L. R. Ford and D. R. Fulkerson, *Flows in networks*, Princeton Univ. Press, Princeton, NJ, 1962.
- [Fu] X. Fu and L. A. Goddyn, *Matroids with the circuit cover property* (in preparation).
- [Ful] D. R. Fulkerson, *Blocking and antiblocking pairs of polyhedra*, Mat. Programming **1** (1971), 168–194.
- [Gar] M. R. Gary and D. S. Jonsonson, *Computers and intractability: a guide to the theory of NP-completeness*, Freeman, San Francisco, 1979.
- [God1] L. A. Goddyn, *Cycle double covers of graphs with Hamilton paths*, J. Combin. Theory Ser. B **46** (1989), 253–254.
- [God2] ———, *Cycle covers of graphs*, Ph.D. thesis, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada, 1988.
- [Gua] M. Guan and H. Fleischner, *On the minimum weight cycle covering problem for planar graphs*, Ars Combin. **20** (1985), 61–68.
- [Hag] G. Haggard, *Loops in duals*, Amer. Math. Monthly **87** (1980), 654–656.
- [Ita] A. Itai and M. Rodeh, *Covering a graph by circuits*, Automata, Languages, and Programming, Lecture Notes in Computer Science, no. 62, Springer, Berlin, 1978, pp. 289–299.
- [Jac] B. Jackson, *Shortest circuit covers and postman tours in graphs with a nowhere zero 4-flow*, SIAM J. Comput. **19** (1990), 659–665.
- [Jae1] F. Jaeger, *Flows and generalized coloring theorems in graphs*, J. Combin. Theory Ser. B **26** (1979), 205–216.
- [Jae2] ———, *A survey of the cycle double cover conjecture*, Cycles in Graphs (B. R. Alspach and C. D. Godsil, eds.), Ann. Discrete Math., vol. 27, North-Holland, Amsterdam, 1985, pp. 1–12.

- [Jam1] U. Jamshy and M. Tarsi, *Cycle covering of binary matroids*, J. Combin. Theory Ser. B **46** (1989), 154–161.
- [Jam2] U. Jamshy, A. Raspaud, and M. Tarsi, *Short circuit covers for regular matroids with a nowhere zero 5-flow*, J. Combin. Theory Ser. B **42** (1987), 354–357.
- [Jam3] U. Jamshy and M. Tarsi, *Short cycle covers and the cycle double cover conjecture*, J. Combin. Theory Ser. B (submitted).
- [Law] E. L. Lawler, *Combinatorial optimization: networks and matroids*, Holt, Rinehart and Winston, New York, 1976.
- [Lit] C. Little and R. Ringeisen, *On the strong graph embedding conjecture*, Proc. Ninth Southeastern Conf. on Combinatorics, Graph Theory and Computing, Utilitas Math., Winnipeg, 1978, pp. 479–487.
- [Mat] K. R. Matthews, *On the Eulericity of a graph*, J. Combin. Theory Ser. B **2** (1978), 143–148.
- [Sch] A. Schrijver, *Theory of linear and integer programming*, Wiley, New York, 1986, pp. 310–312.
- [Sey1] P. D. Seymour, *Sums of circuits*, Graph Theory and Related Topics (J. A. Bondy and U. S. R. Murty, eds.), Academic Press, New York, 1979, pp. 341–355.
- [Sey2] ———, *Matroids and multicommodity flows*, Europ. J. Combin. **2** (1981), 257–290.
- [Sey3] ———, *Even circuits in planar graphs*, J. Combin. Theory Ser. B **31** (1981), 327–338.
- [Sey4] P. D. Seymour and N. Roberson, *Graph Minors. XIII: The disjoint paths problem* (submitted).
- [Sze] G. Szekeres, *Polyhedral decompositions of cubic graphs*, J. Austral. Math. Soc. **8** (1973), 367–387.
- [Tar1] M. Tarsi, *Nowhere zero flows and circuit covering in regular matroids*, J. Combin. Theory Ser. B **39** (1985), 346–352.
- [Tar2] ———, *Semi-duality and the cycle double cover conjecture*, J. Combin. Theory Ser. B **41** (1986), 332–340.
- [Tut] W. T. Tutte, *On the imbedding of linear graphs in surfaces*, Proc. London Math. Soc. (2) **51** (1950), 474–483.
- [Wel] D. Welsh, *Matroid theory*, Academic Press, San Francisco, 1976.
- [You] D. H. Younger, *Integer flows*, J. Graph Theory **7** (1983), 349–357.
- [Zel] B. Zelinka, *On a problem of P. Vestergaard concerning circuits in graphs*, Czechoslovak Math. J. **37** (1987), 318–319.
- [Zha1] Cun-Quan Zhang, *Minimal cycle coverings and integer flows*, J. Graph Theory **14** (1990), 537–546.
- [Zha2] ———, *On compatible cycle decompositions of eulerian graphs*, preprint.
- [Zha3] ———, *On even cycle decompositions of eulerian graphs*, preprint.

(Brian Alspach and Luis Goddyn) DEPARTMENT OF MATHEMATICS AND STATISTICS, SIMON FRASER UNIVERSITY, BURNABY, BRITISH COLUMBIA, CANADA V5A 156
E-mail address: alspach@cs.sfu.ca
goddyn@math.sfu.ca

(Cun-Quan Zhang) DEPARTMENT OF MATHEMATICS, WEST VIRGINIA UNIVERSITY, MORGANTOWN, WEST VIRGINIA 26506
E-mail address: cqzhang@wvnm.wvnet.edu