A BANACH SPACE NOT CONTAINING $c_0$, $l_1$
OR A REFLEXIVE SUBSPACE

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Abstract. An infinite-dimensional Banach space is constructed which does not contain $c_0$, $l_1$ or an infinite-dimensional reflexive subspace. In fact, it does not even contain $l_1$ or an infinite-dimensional subspace with a separable dual.

An old result of James [2] asserts that a Banach space with an unconditional basis is either reflexive or has a subspace isomorphic to one of $c_0$ or $l_1$. This suggests a natural problem, which has been considered by several authors: does every Banach space contain $c_0$, $l_1$ or a reflexive subspace? James's result yields a positive answer for any space containing an unconditional basic sequence, so the problem was thrown into sharper focus by the recent construction [1] of a space without one. In this paper we adapt the construction of [1]. We shall draw attention to the differences and similarities later. We also show that our space has no subspace with a separable dual. Since a theorem of Johnson and Rosenthal [5] states that a subspace of a separable dual space either has a reflexive subspace or a nonseparable dual, this is only a slightly stronger result. However, our proof is direct.

This second statement should be compared with results of James [4] and Lindenstrauss and Stegall [6]. They independently constructed separable spaces not containing $l_1$ but with nonseparable duals, answering in the negative a question of Banach. The space in this paper can therefore be regarded as a hereditary version of those spaces. (This is true, to some extent, not just of the result, but also of the construction.)

The paper is self-contained, but will be easier to read by those familiar with the techniques of [1] and indeed of [8], Schlumprecht's construction of an arbitrarily distortable space, which lies at the heart of the construction here as well as that of [1]. The main difficulty of this result is the proof of Lemma 4 below. This proof is postponed until after the lemma is used to prove the main result. Thus the reader who wishes to understand the main ideas of the construction without wading through pages of technical argument can simply stop reading when the proof of Lemma 4 starts.

We shall begin by giving the definition of our norm, which is fairly complicated.

First, let $f: \mathbb{R} \to \mathbb{R}$ be the function $x \mapsto \sqrt{\log_2(x+1)}$, and note that $f$ has the following properties:

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(i) \( f(1) = 1 \) and \( f(x) < x \) for every \( x > 1 \);
(ii) \( f \) is strictly increasing and tends to infinity;
(iii) \( \lim_{x \to \infty} x^{-q} f(x) = 0 \) for every \( q > 0 \);
(iv) the function \( x f(x)^{-2} \) is concave and nondecreasing;
(v) \( f(xy) \leq f(x) f(y) \) for every \( x, y \geq 1 \).

Let \( c_{00} \) be the vector space of sequences of real numbers all but finitely many
of which are zero, and let the standard basis of \( c_{00} \) be written \( e_1, e_2, \ldots \). An interval \( E \subset \mathbb{N} \) is a subset of the form \( \{a, a + 1, a + 2, \ldots , b\} \) for some
\( a \leq b, a, b \in \mathbb{N} \). Given an interval \( E \), let the letter \( E \) also stand for the
projection from \( c_{00} \) to \( c_{00} \) defined by \( \sum_{i=1}^{\infty} a_i e_i \rightarrow \sum_{i \in E} a_i e_i \). Given two
intervals \( E, F \subset \mathbb{N} \), write \( E < F \) if \( \max E < \min F \). The support of a vector
\( x = \sum_{i=1}^{\infty} x_i e_i \in c_{00} \) is \( \text{supp}(x) = \{i : x_i \neq 0\} \). This is always a finite set.

We define the range of a vector, written \( \text{ran}(x) \), to be the smallest interval
containing \( \text{supp}(x) \). Given \( x, y \in c_{00} \), we shall write \( x < y \) for the statement
\( \text{ran}(x) < \text{ran}(y) \). If \( x_1 < \cdots < x_N \), then we say that \( x_1, \ldots , x_N \) are successive.

Let \( Q \) be the subset of \( c_{00} \) consisting of sequences of rationals in the interval
\([-1, 1]\). Let \( J \subset \mathbb{N} \) be a set such that, if \( m < n \) and \( m, n \in J \), then
\( \log \log \log \log \log n \geq 1000m \). Let us also suppose that \( f(j) \geq 10^{10^3} \) for every
\( j \in J \). Let \( \sigma \) be an injection from the set of finite sequences of successive
elements of \( Q \) to \( J \).

Let \( X = (c_{00}, \| \cdot \|) \) be any normed space such that the standard basis is
bimonotone. For every \( m \in \mathbb{N} \), define \( A_m(X) \) to be the set of linear functionals
on \( X \) of the form \( f(m)^{-1}(x_1^* + \cdots + x_m^*) \), where \( x_1^*, \ldots , x_m^* \) are successive
members of \( c_{00} \) and \( \|x_i^*\| \leq 1 \) for each \( i \). A special sequence of functionals
on \( X \) is defined to be a sequence of the form \( z_1^*, \ldots , z_M^* \), where \( M \in \mathbb{N} \),
the \( z_i^* \) are successive, \( z_1^* \in A_{m_1} \cap Q \) for some \( m \in J \) and, for \( 2 \leq i \leq M \),
we have \( z_i^* \in A_{\sigma(z_1^*, \ldots , z_{i-1}^*)} \cap Q \). A special functional on \( X \) is defined to be
a functional of the form \( E(z_1^* + \cdots + z_M^*) \) such that \( z_1^*, \ldots , z_M^* \) is a special
sequence. To any special sequence \( z_1^*, \ldots , z_M^* \) we can associate a sequence of
integers \( n_1, \ldots , n_M \in J \) such that \( z_i^* \in A_{n_i}^* \) and \( n_i = \sigma(z_1^*, \ldots , z_{i-1}^*) \) for
\( 2 \leq i \leq M \). The first number \( n_1 \) is not necessarily uniquely determined, but
\( n_2, \ldots , n_M \) certainly are. Given a special functional \( z^* = E(z_1^* + \cdots + z_M^*) \),
we say that \( Z \subset J \) is an associated set for \( z^* \) if we can pick such a sequence
\( n_1, \ldots , n_M \) associated to the sequence \( z_1^*, \ldots , z_M^* \) and \( Z \) consists of those
\( n_i \) for which \( E \cap \text{ran}(z_i^*) \neq \emptyset \). A collection of special functionals \( w_1^*, \ldots , w_N^* \)
is called disjoint if we can choose for them disjoint associated sets \( Z_1, \ldots , Z_N \).

We are now ready to define our norm. We shall define it as the limit of a
sequence of norms on \( c_{00} \). First, let \( X_0 \) be defined by \( \|x\|_{X_0} = \|x\|_{\infty} \). For
\( n \geq 1 \), define \( X_n \) by

\[
\|x\|_{X_n} = \|x\|_{X_{n-1}} \vee \sup \left\{ f(N)^{-1} \sum_{i=1}^{N} \|E_i x\|_{X_{n-1}} : N \geq 2, E_1 < \cdots < E_N \right\}
\vee \sup \left( \sum_{i=1}^{M} |x_i^*(x)|^2 \right)^{1/2}
\]

where the second supremum ranges over all sequences \( x_1^*, \ldots , x_M^* \) of disjoint
special functionals on \( X_{n-1} \).

Now we claim that \( \|x\|_{X_n} \leq \|x\|_1 \) for every \( n \). This is certainly true when
n = 0. If it is true for \( n = k \) then \( \|x^*\|_{X_k^*} \geq \|x^*\|_{\infty} \) for every \( x^* \in c_{00} \).

It follows that \( \|x^*\|_{\infty} \leq f(m)^{-1} \) for every \( x^* \in A_m^*(X_k) \). Given a sequence \( x_1^*, \ldots, x_M^* \) of disjoint special functionals on \( X_k \), we can find disjoint associated sets \( Z_1, \ldots, Z_M \subset J \). We know that \( \|x_i^*\|_{\infty} \leq f(\min Z_i)^{-1} \), so

\[
\left( \sum_{i=1}^{M} |x_i^*(x)|^2 \right)^{1/2} \leq \sum_{i=1}^{M} |x_i^*(x)| \leq \|x\|_1 \sum_{i=1}^{M} \|x_i^*\|_{\infty} \leq \|x\|_1 \sum_{j \in J} f(j)^{-1} \leq \|x\|_1.
\]

It follows easily that \( \| \cdot \|_{X_{k+1}} \) is also dominated by \( \| \cdot \|_1 \).

It is also clear that if \( X \) and \( Y \) are two normed spaces on \( c_{00} \) such that \( \|x\|_X \leq \|x\|_Y \) for every \( x \in c_{00} \), then every sequence of disjoint special functionals on \( X \) is also such a sequence on \( Y \). This implies that \( \| \cdot \|_{X_0}, \| \cdot \|_{X_1}, \ldots \) is an increasing sequence of norms. Since they are bounded above by \( \| \cdot \|_1 \), they tend to a limit, giving a space \( X = (c_{00}, \| \cdot \|_1) \). Strictly speaking, we will be interested in the completion of this space, but it is more convenient for the time being to consider the incomplete space \( X \).

It is easy to check that every \( x \in X \) satisfies the equation

\[
\|x\| = \|x\|_{\infty} \vee \sup \left\{ f(N)^{-1} \sum_{i=1}^{N} \|E_i x\| : N \geq 2, E_1 < \cdots < E_N \right\} \vee \sup \left( \sum_{i=1}^{M} |x_i^*(x)|^2 \right)^{1/2}.
\]

where the second supremum is over all sequences \( x_1^*, \ldots, x_M^* \) of disjoint special functionals on \( X \). Note in particular that the standard basis of \( X \) is bimonotone.

We shall now state and prove some lemmas about \( X \). The first three are very similar to lemmas proved by Schlumprecht and slightly adapted in [1]. First, we say that \( x \in X \) is an \( l_{1+}^n \)-average with constant \( C \) if \( \|x\| = 1 \) and \( x = \sum_{i=1}^{n} x_i \) for some sequence of successive nonzero vectors \( x_1, \ldots, x_n \) such that \( \|x_i\| \leq Cn^{-1} \) for every \( i \). An \( l_{1+}^n \)-vector is simply a nonzero multiple of an \( l_{1+}^n \)-average. That is, it is a vector \( x \) that can be written as \( \sum_{i=1}^{n} x_i \) for some sequence of successive nonzero vectors \( x_1, \ldots, x_n \) such that \( \|x_i\| \leq Cn^{-1} \|x\| \) for each \( i \).

**Lemma 1.** For every \( n \in \mathbb{N} \) and \( C > 1 \) there exists \( N \) such that, for any sequence \( x_1, \ldots, x_N \) of successive nonzero vectors in \( X \), the subspace generated by \( x_1, \ldots, x_N \) contains an \( l_{1+}^n \)-average with constant \( C \).

**Proof.** Suppose the result is false. Without loss of generality the \( x_i \) all have norm one. Let \( k \) be an integer such that \( k \log C > \log f(n^k) \) (such an integer exists because of property (iii) of the function \( f \)), let \( N = n^k \) and let \( x = \sum_{i=1}^{N} x_i \). For every \( 0 \leq i < k \) and every \( 1 \leq j \leq n^{k-i} \), let \( x(i, j) = \sum_{t=(j-1)n^{k-i}+1}^{jn^{k-i}} x_i \). Thus \( x(0, j) = x_j \), \( x(k, 1) = x \) and, for \( 1 \leq i < k \), each \( x(i, j) \) is a sum of \( n \) successive \( x(i-1, j)'s \). By our assumption, no \( x(i, j) \) is an \( l_{1+}^n \)-vector with constant \( C \). It follows easily by induction that
\|x(i, j)\| \leq C^{-n^i}, and, in particular, that \|x\| \leq C^{-k}n^k = C^{-k}N. However, it also follows easily from the definition of the norm on X that \|x\| \geq Nf(N)^{-1}. This is a contradiction, by choice of k. \qed

Note that we have proved the slightly stronger result that if the x_i have norm one then there is an interval E \subset \{1, 2, \ldots, N\} such that \sum_{i \in E} x_i is an l_{l+}^N-vector. The technique used to prove this lemma is essentially due to R. C. James [3].

**Lemma 2.** Let M, N \in \mathbb{N} and C \geq 1, let x be an l_{l+}^N-vector with constant C, and let E_1 < \cdots < E_M be a sequence of intervals. Then

\[
\sum_{j=1}^{M} \|E_jx\| \leq C(1 + 2M/N)\|x\|.
\]

**Proof.** For convenience, let us normalize so that \|x\| = N and x = \sum_{i=1}^{N} x_i, where x_1 < \cdots < x_N and \|x_i\| \leq C for each i. Given j, let A_j be the set of i such that supp(x_i) \subset E_j, and let B_j be the set of i such that \text{E}_j(x_i) \neq 0. By the triangle inequality and the fact that the basis is bimonotone,

\[
\|E_jx\| \leq \left\| \sum_{i \in B_j} x_i \right\| \leq C(|A_j| + 2).
\]

Since \sum_{j=1}^{M} |A_j| \leq N, we get

\[
\sum_{j=1}^{M} \|E_jx\| \leq C(N + 2M)
\]

which gives the result, because of our normalization. \qed

The next definition is extremely important, as it was in [1]. We shall say that a sequence x_1 < \cdots < x_N is a rapidly increasing sequence of l_{l+}-averages, or R.I.S., of length N with constant 1 + \epsilon if, for each k, x_k is an l_{l+}^{n_k}-average with constant 1 + \epsilon, n_1 \geq 2^N and, for k = 2, 3, \ldots, N, we have \epsilon \sqrt{f(n_k)} \geq |\text{ran}(x_1 + \cdots + x_{k-1})|.

**Lemma 3.** Let 0 < \epsilon < 1/2, let x_1, \ldots, x_N be a R.I.S. of length N with constant 1 + \epsilon, and let x = \sum_{i=1}^{N} x_i. Let M \geq f^{-1}(6N/\epsilon), and let E_1 < \cdots < E_M. Then \[ f(M)^{-1} \sum_{j=1}^{M} \|E_jx\| \leq 1 + 2\epsilon. \]

**Proof.** For each i let n_i be maximal such that x_i is an l_{l+}^{n_i}-average with constant 1 + \epsilon. We obtain three estimates for \sum_{j=1}^{M} \|E_jx_i\|.

First, it follows directly from the definition of the norm that \sum_{j=1}^{M} \|E_jx_i\| \leq f(M). Second, we know that it is at most f(|\text{supp}(x_i)|) and by Lemma 2 it is at most (1 + \epsilon)(1 + 2M/n_i).

Let t be maximal such that n_t \leq M. Then, if i < t, we have n_{i+1} \leq M. Hence, by the definition of a R.I.S., we have

\[
\sum_{i=1}^{t-1} f(|\text{supp}(x_i)|) \leq \epsilon \sqrt{f(M)}
\]
and hence
\[ \sum_{i=1}^{M} \sum_{j=1}^{t-1} \|E_j x_i\| \leq \epsilon \sqrt{f(M)}. \]

Using this and the other two inequalities, we find that
\[ \sum_{j=1}^{N} \sum_{i=1}^{M} \|E_j x_i\| \leq \epsilon \sqrt{f(M) + f(M) + (N-t)(1+\epsilon)(1+2M/n)} \]
\[ \leq f(M) + 3N(1+\epsilon). \]
Since \( f(M) \geq 6N/\epsilon \), the result follows. \( \square \)

We shall now state the main lemma of this paper and deduce from it that \( X \) does not contain \( c_0 \), \( l_1 \) or a reflexive subspace. We will then prove the main lemma.

**Lemma 4.** Let \( M \in J \), and let \( x_1, \ldots, x_M \) be a R.I.S. of length \( M \) with constant \( 3/2 \). Then there exists a choice of signs \( \epsilon_1, \ldots, \epsilon_M \) such that \( \| \sum_{i=1}^{M} \epsilon_i x_i\| < 100M/(M-1) \).

Once we have proved this lemma, the proof of our main theorem is fairly straightforward. First, if the completion of \( X \) contains a subspace that is \( c_0 \), \( l_1 \) or reflexive, then it must contain such a subspace generated by a block basis. Let us call it \( Y \) and the block basis \( y_1, y_2, \ldots \). Since \( \| \sum_{i=1}^{N} y_i\| \geq Nf(N)^{-1} \), we know that \( Y \) cannot be \( c_0 \). Let \( x_1, \ldots, x_M \) be a R.I.S. in \( Y \) with constant \( 3/2 \) and length \( M \in J \). Then by Lemma 4 there is a sequence of signs \( \epsilon_1, \ldots, \epsilon_M \) such that \( \| \sum_{i=1}^{M} \epsilon_i x_i\| < 100Mf(M)^{-1} \). This shows that \( y_1, y_2, \ldots \) is not \( (f(M)/100) \)-equivalent to the unit vector basis of \( l_1 \). Since \( M \) can be arbitrarily large, \( Y \) is not \( l_1 \).

To show that \( Y \) is not reflexive is slightly more complicated, but still easy. Given an integer \( M \in J \), define an \( M \)-pair as follows. Let \( u_1, \ldots, u_M \) be a R.I.S. of length \( M \) and constant \( 3/2 \). By changing signs if necessary, let the norm of \( \sum_{i=1}^{M} u_i \) be at most \( 100Mf(M)^{-1} \). For each \( i \) let \( u_i^* \) be a support functional for \( u_i \), such that \( \text{ran}(u_i^*) \subseteq \text{ran}(u_i) \). Now let \( v \) be the vector \( \sum_{i=1}^{M} u_i / \| \sum_{i=1}^{M} u_i \| \) and let \( v^* \) be the functional \( f(M)^{-1}(u_1^* + \cdots + u_M^*) \). Then \( v^*(v) = Mf(M)^{-1}/\| \sum_{i=1}^{M} u_i \| > 1/100 \) and \( v^* \in A_M \). After a perturbation, we can also get that \( v^* \in Q \), while keeping \( v^* \) in \( A_M \) and keeping the estimate for \( v^*(v) \). Such a pair \( (v, v^*) \) is what we mean by an \( M \)-pair.

We can clearly choose an \( M \)-pair \( (v, v^*) \) for any integer \( M \) in such a way that \( v \in Y \), and we can also make \( \min \text{supp}(v) \) as large as we like. It follows that we can find a sequence of pairs \( (v_i, v_i^*) \) such that \( v_1 < v_2 < \cdots \) are successive elements of \( Y \) and, for each \( i > 1 \), \( (v_i, v_i^*) \) is a \( \sigma(v_i^*, \ldots, v_{i-1}^*) \)-pair. This ensures that \( v_1^*, v_2^*, \ldots \) is a special sequence.

We now claim that the linear functional \( w^* \) defined by
\[ x \mapsto \lim_{N \to \infty} \sum_{n=1}^{N} v_n^*(x) \]
is continuous on the completion of \( X \). This is true because the limit certainly exists for any \( x \in X \) and is bounded on \( B(X) \), since the functionals used in
the limit are all special functionals and hence have norm at most 1. Thus the functional can be extended to the completion. However, \( w^*(v_n) \geq 1/100 \) for every \( n \), so the basic sequence \( v_1, v_2, \ldots \) is not shrinking. (For basic facts about bases and reflexivity see [2] or [7].) This shows that \( Y \) has a nonreflexive subspace and hence is not reflexive.

The same argument shows that \( Y \) does not have a separable dual. Indeed, suppose that \( z^*_1, z^*_2, \ldots \) were a dense subset of \( Y^* \). Then we could pick a block basis \( x_1, x_2, \ldots \) of \( y_1, y_2, \ldots \) such that \( z^*_j(x_j) = 0 \) for every \( j \geq i \) and hence such that every sequence of successive vectors generated by \( x_1, x_2, \ldots \) tended weakly to zero. However, as above, we can find a block basis \( v_1, v_2, \ldots \) of \( x_1, x_2, \ldots \) and a functional \( w^* \) such that \( w^*(v_n) \geq 1/100 \) for every \( n \in N \). This argument is standard and forms part of the proof of a more general result of Johnson and Rosenthal [5] stated in the introduction.

This concludes the less technical part of the paper. The rest of the paper is devoted to proving Lemma 4. We will need some preliminary lemmas before we begin the proof in earnest. The first is similar to, but more complicated than, Lemma 3. We need a few more definitions. A special combination is a functional of the form \( \sum_{i=1}^{N} a_i x^*_i \) where \( \sum_{i=1}^{N} |a_i|^2 = 1 \) and \( x^*_1, \ldots, x^*_N \) is a sequence of disjoint special functionals. A particularly simple sort of special combination can be defined as follows. Pick distinct integers \( j_1, \ldots, j_N \in J \), pick \( x^*_i \in A_{j_i}^* \) (recall that the sets \( A_m^* \) were defined at the same time as special sequences, etc.), let \( E_1, \ldots, E_N \) be any sequence of intervals, let \( \sum_{i=1}^{N} |a_i|^2 = 1 \), and let \( x^* = \sum_{i=1}^{N} a_i E_i(x^*_i) \). We shall call these basic special combinations. Thus, a basic special combination is one where the special sequences used to build it have length at most 1.

For the proof of the next lemma and of the main lemma later it will be convenient to make one other definition. Let \( x_1 < \cdots < x_M \) be a R.I.S. with constant \( 1 + \epsilon \), for some \( \epsilon > 0 \). For each \( i \), let \( n_i \) be maximal such that \( x_i \) is an \( l_{1+\epsilon}^n \)-average with constant \( 1 + \epsilon \) and let us write it out as \( x_i = x_{i1} + \cdots + x_{in_i} \), where \( \|x_{ij}\| \leq (1 + \epsilon)n_{ij}^{-1} \) for each \( j \). Given an interval \( E \subset N \), let \( i = i_E \) and \( j = j_E \) be respectively minimal and maximal such that \( E x_i \) and \( E x_j \) are nonzero, and let \( r = r_E \) and \( s = s_E \) be respectively minimal and maximal such that \( E x_{ir} \) and \( E x_{js} \) are nonzero. Define the length \( \lambda(E) \) of the interval \( E \) to be \( j_E - i_E + (s_E/n_{js}) - (r_E/n_{ir}) \). Thus the length of \( E \) is the number of \( x_i \)'s contained in \( E \), allowing for fractional parts. Obviously this definition depends on the R.I.S. in question, but it will always be clear from the context which one is being considered.

It is easy to check that if \( E_1 < \cdots < E_M \) and \( E = \bigcup E_i \) then \( \sum \lambda(E_i) \leq \lambda(E) \). It follows from the triangle inequality and the lower bound for \( n_1 \) in the definition of a R.I.S. that if \( x = x_1 + \cdots + x_M \) with \( x_1, \ldots, x_M \) a R.I.S. with constant \( 1 + \epsilon \), then \( \|E x\| \leq (1 + \epsilon)(\lambda(E) + 2^{2^{-M}}) \).

**Lemma 5.** Let \( l \in J \), and let \( x_1, \ldots, x_M \) be a R.I.S. of length \( M \) with constant \( 3/2 \) such that \( \exp \sqrt{\log l} \leq M \leq l \). Suppose also that \( x = x_1 + \cdots + x_M \) is an \( l_{1+\epsilon}^M \)-vector with constant 2 for some \( M' \geq \log M \). Let \( x^* = \sum_{i=1}^{N} a_i x^*_i \) be a basic special combination. Then \( \|x^*(x)\| \leq 10^{-100}\|x\| + 2M f(M)^{-1} \).

**Proof.** There exist distinct integers \( k_1, \ldots, k_N \in J \) such that, for each \( i \), \( x_i \) is an interval projection of some functional in \( A_{k_i}^* \). Suppose that there is some
i such that $k_i = l$. Then $x_i^* = f(l)^{-1} E(y_i^* + \cdots + y_i^*)$ for some sequence $y_i^*, \ldots, y_i^*$ of successive norm-one functionals and some interval $E$. Let $E_r = \text{ran}(y_i^*)$ for each $r$. Then certainly

$$
|x_i^*(x)| \leq f(l)^{-1} \sum_{r=1}^{l} \|E_r x\| \leq f(l)^{-1} \left(2^{2-M} l + (3/2) \sum_{r=1}^{l} \lambda(E_r)\right) \\
\leq f(l)^{-1} (1 + 3M/2) \leq 2M f(M)^{-1}.
$$

Let us now suppose that $\sum_{i=1}^{N} a_i x_i^*$ is a basic combination and no $k_i$ is equal to $l$. It follows that, for each $i$, either $k_i \leq \log \log \log l$ or $k_i \geq \exp \exp \exp l$.

For each $j = 1, 2, \ldots, M$, let $n_j$ be the greatest integer such that $x_j$ is an $l_1^M$-average with constant $3/2$, and, for each $i = 1, 2, \ldots, N$, let $t_i$ be the greatest value of $j$ such that $k_i \geq n_j$ (or zero if there is no such $j$). We shall now examine the effect of the functional $x_i^*$ on $x$.

We have

$$
x_i^* \left(\sum_{j=1}^{M} x_j\right) = x_i^* \left(\sum_{j=1}^{t_i-1} x_j\right) + x_i^* (x_{t_i}) + x_i^* \left(\sum_{j=t_i+1}^{M} x_j\right).
$$

When $j < t_i$, we know that $n_{j+1} \leq k_j$ which implies, by the definition of a R.I.S., that $|\text{supp}(x_j) + \cdots + x_{t_j-1})| \leq \sqrt{f(k_j)}$. The first part of the right-hand side is therefore at most $\sqrt{f(k_j)} \|x_i^*\|_\infty \leq f(k_i)^{-1/2}$ in modulus. As for the third part, if we temporarily set $y_j = E_{t_i+1} x_j$, we find that it is at most

$$
sup \left\{ f(k_i)^{-1} \sum_{r=1}^{t_i} \|E_r y_j\| : E_1 < \cdots < E_{k_i} \right\}
$$

in magnitude. Given an optimal sequence $E_1, \ldots, E_{k_i}$, at most $M$ of the intervals contains part of more than one $x_j$. We now consider two cases.

If $k_i \geq M$ then $k_i \geq \exp \exp \exp M$, so the best splitting is no greater than the greatest possible value of $f(k_i)^{-1} \sum_{j=t_i+1}^{M} \sum_{r=1}^{s_j} \|E_r x_j\|$ such that $\sum_{j=t_i+1}^{M} s_j \leq k_i + M$. However, when $j \geq t_i + 1$ in this case, we certainly know that $n_j \geq k_i + M$, so the greatest possible value is, by Lemma 2, at most

$$
\sum_{j=t_i+1}^{M} f(k_i)^{-1} (1 + 2n_j^{-1}(k_i + M)) \leq 3(M - t_i) f(k_i)^{-1} \leq f(k_i)^{-1/2}.
$$

On the other hand, if $k_i \leq M$, then $k_i \leq \log \log \log M$. Using Lemma 2 again and the fact that $x$ is an $l_1^M$-average, we have

$$
\sum_{r=1}^{N_i} \|E_r y_j\| \leq \sum_{r=1}^{N_i} \|E_r x\| \leq 2(1 + 2k_i/M') \|x\|,
$$

giving $|x^*(y_j)| \leq 6 f(k_i)^{-1} \|x\|$.

Since $k_1, \ldots, k_N$ are distinct, we get

$$
\sum_{i=1}^{N} \left|\sum_{j \neq t_i} x_i^*(x_j)\right| \leq 2 \sum_{s \in J} f(s)^{-1/2} + 6 \sum_{s \in J} f(s)^{-1} \|x\| \leq 10^{-101} \|x\|.
$$
On the other hand, \( \sum_{i=1}^{N} a_i x_i^*(x_i) \) is at most \( (\sum_{i=1}^{N} |x_i^*(x_i)|^2)^{1/2} \) which equals \( (\sum_{j=1}^{M} \sum_{i=1}^{j} |x_i^*(x_j)|^2)^{1/2} \). But \( \sum_{i=1}^{j} |x_i^*(x_j)|^2 \leq 1 \) for every \( j \), so this is at most \( \sqrt{M} \) which is less than \( 10^{-101} M f(M)^{-1} \leq 10^{-101} \|x\| \).

It follows in this case that \( |x^*(x)| \leq 10^{-100} \|x\| \). It follows from our two calculations that in general \( |x^*(x)| \leq 10^{-100} \|x\| + 2M f(M)^{-1} \), as stated. \( \Box \)

Some later arguments will make use of the following easy Chebyshev-type lemma.

**Lemma 6.** Let \( \epsilon > 0 \) and \( \delta = \sqrt{2\epsilon} \). Let \( \sum_{i=1}^{N} a_i^2 \leq 1 \), let \( \sum_{i=1}^{N} b_i^2 \leq 1 \), and let \( \sum_{i=1}^{N} a_i b_i \geq 1 - \epsilon \). Then there exists a subset \( A \subset \{1, 2, \ldots, N\} \) such that \( \sum_{i \in A} a_i^2 \geq 1 - \delta \) and, for every \( i \in A \), we have \( 1 - \sqrt{\delta} \leq b_i/a_i \leq 1 + \sqrt{\delta} \).

**Proof.** First, we have \( \sum_{i=1}^{N} (a_i - b_i)^2 \leq 2\epsilon \). Suppose that there exists \( A \subset \{1, 2, \ldots, N\} \) such that \( \sum_{i \in A} a_i^2 \geq 1 - \delta \) and \( (a_i - b_i)^2 \geq \delta a_i^2 \) for every \( i \in A \). Then

\[
\sum_{i=1}^{N} (a_i - b_i)^2 \geq \sum_{i \in A} (a_i - b_i)^2 > \delta \sum_{i \in A} a_i^2 \geq \delta^2 = 2\epsilon
\]

contradicting the first estimate. It follows that there exists a subset \( A \subset \{1, 2, \ldots, N\} \) such that \( \sum_{i \in A} a_i^2 \geq 1 - \delta \) and \( (a_i - b_i)^2 \leq \delta a_i^2 \) for every \( i \in A \). This implies that \( |a_i - b_i| \leq \sqrt{\delta} a_i \) for each \( i \in A \) which implies the lemma. \( \Box \)

The next lemma is very well known and has been used extensively in the local theory of Banach spaces.

**Lemma 7.** Let \( f: \{-1, 1\}^n \rightarrow \mathbb{R} \) be a function that is 1-Lipschitz with respect to the Hamming distance on \( \{-1, 1\}^n \), let \( P \) be the uniform distribution on \( \{-1, 1\}^n \), and let \( M \) be the median of \( f \). Then

\[
P[|f(\epsilon) - M| \geq \delta n] \leq 2 \exp(-\delta^2 n/2).
\]

We are now ready to prove the main lemma.

**Proof of Lemma 4.** We shall prove the following stronger statement. There is a choice of signs \( \epsilon_1, \ldots, \epsilon_M \) such that, for every interval \( E \),

\[
\left\| E \left( \sum_{i=1}^{M} \epsilon_i x_i \right) \right\| < 100 \lambda(E) f(M) f(\lambda(E))^{-2}.
\]

Indeed, suppose this statement is false. We shall derive a contradiction in several stages. First, define a seminorm \( \| \cdot \| \) (actually it is easy to see that it is a norm) on \( X \) by

\[
\|x\| = \sup\{|x^*(x)| : x^* \text{ is a special combination}\}.
\]

Also, for the rest of the paper, let \( \epsilon = 10^{-50} \).

**Step 1.** There exists an interval \( A \subset \{1, 2, \ldots, M\} \) of cardinality

\[
N \geq 20 \exp \sqrt{\log M}
\]

such that, with probability at least \( M^{-2} \) (over \( \{-1, 1\}^4 \)) the following statements are true:
(i) \( \| \sum_{i \in E} \epsilon_i x_i \| \geq (1 - \epsilon) \| \sum_{i \in A} \epsilon_i x_i \| \vee (1 - \epsilon) 100Nf(M)f(N)^{-2} \); (ii) for every subinterval \( B \subseteq A \) we have
\[
\left\| \sum_{i \in B} \epsilon_i x_i \right\| < 100|B|f(M)f(|B|)^{-2}.
\]

Proof. For every \( \epsilon \in \{-1, 1\}^M \) there must be a minimal interval \( E \) such that
\( \| E\epsilon(x)\| \geq 100\lambda(E)f(M)f(\lambda(E))^{-2} \), where \( x(\epsilon) \) stands for the vector \( \sum_{i=1}^M \epsilon_i x_i \). (Recall also that \( \lambda(E) \) is the length of the interval \( E \) defined before Lemma 5.) Now \( \| E\epsilon(x)\| \leq (3/2)\lambda(E) + 2 \), so this tells us that \( (3/2)\lambda(E) + 2 > 100\lambda(E)f(M)f(\lambda(E))^{-2} \) which implies that \( \lambda(E) \geq 20\exp \sqrt{\log M} \).

First, we shall show that, for such an \( E \), we have \( \| E\epsilon(x)\| = \| E\epsilon(x)\| \), i.e., \( x(\epsilon) \) is normed by a special combination. Indeed, if this is not the case, then we can find a sequence of intervals \( F_1 < \cdots < F_k \) with \( \bigcup_{i=1}^k F_i = E \) such that, writing \( x = x(\epsilon) \),
\[
\left\| \sum_{i=1}^k F_i x_i \right\| = f(k)^{-1} \sum_{i=1}^k ||F_i x_i||.
\]
Setting \( \lambda_i = \lambda(F_i) \) and \( \lambda = \lambda(E) \), we know that \( \sum_{i=1}^k \lambda_i \leq \lambda \). By Lemma 3, we also know that \( k \leq f^{-1}(6M/\epsilon) \). We also know that, for each \( i \), \( \| F_i x_i \| \leq (3/2)\lambda_i + 2^{-M} \). It follows that \( \sum_{i=1}^k ||F_i x_i|| \leq (3/2)\lambda + 2^{-M/2} \). Since \( \| E\epsilon(x)\| > 100\lambda f(M)f(\lambda)^{-2} \) this tells us that
\[
((3/2)\lambda + 2^{-M/2})f(k)^{-1} > 100\lambda f(M)f(\lambda)^{-2} > 100\lambda f(\lambda)^{-1}.
\]
If \( k \geq \lambda^{1/100} \) then this is a contradiction.

On the other hand, by minimality of \( E \), we also know that
\[
\| F_i x_i \| \leq 100\lambda_i f(M)f(\lambda_i)^{-2}.
\]
This tells us that
\[
f(k)^{-1} \sum_{i=1}^k 100\lambda_i f(M)f(\lambda_i)^{-2} > 100\lambda f(M)f(\lambda)^{-2}.
\]
Since \( x/f(x)^2 \) is concave and \( \lambda_1 + \cdots + \lambda_k = \lambda \), Jensen's inequality gives us that
\[
f(k)^{-1}k.(\lambda/k).f(M)f(\lambda/k)^{-2} > \lambda f(M)f(\lambda)^{-2}
\]
which implies that \( f(k)f(\lambda/k)^2 < f(\lambda)^2 \). But if \( 2 \leq k < \lambda^{1/100} \) this gives us that \( f(2)f(\lambda^{29/100})^2 < f(\lambda)^2 \) which is clearly false, by the definition of the function \( f \).

This establishes that \( \| E\epsilon(x)\| = \| E\epsilon(x)\| \). Now let \( A = \{i: \text{ran}(x_i) \subseteq E\} \). We shall abuse notation in the following way. When it is understood that an interval \( A \) is a subset of \( \{1, 2, \ldots, M\} \), we shall use the letter \( A \) to refer to the projection \( \sum_{i=1}^M \epsilon_i x_i \mapsto \sum_{i \in A} \epsilon_i x_i \). Then, setting \( N = |A| \), we have
\[
\| A\epsilon(x)\| \geq \| E\epsilon(x)\| - 2 \geq (1 - \epsilon).100f(M)f(N)^{-2}.
\]
Since \( \| E\epsilon(x)\| = \| E\epsilon(x)\| \) and the basis of \( X \) is bimonotone, it is also clear that \( \| A\epsilon(x)\| \geq \| A\epsilon(x)\| - 2 \geq (1 - \epsilon)\| A\epsilon(x)\| \). If \( B \subseteq A \) is any subinterval, then, by the minimality of \( E \), we have \( \| B\epsilon(x)\| \leq 100|B|f(M)f(|B|)^{-2} \).
Now, there are at most $M^2$ such intervals $A \subset \{1, 2, \ldots, M\}$ and there is one for each collection of signs $(\epsilon_i)_{i=1}^M$. Hence, some interval $A$ is used at least $M^{-2}2^M$ times. If $|A| = N$, then for at least $M^{-2}2N$ choices of $\epsilon \in \{-1, 1\}^A$ parts (i) and (ii) of the claim hold.

Before stating the next step, we shall introduce some notation. Let $K = \lceil \log N \rceil$ and let $B_1 < \cdots < B_{5K}$ be subintervals of $A$, each of cardinality between $(1 - \epsilon)N/5K$ and $(1 + \epsilon)N/5K$. Let $v_i = \sum_{j \in B_i} \epsilon_j x_j$ and for $r = 1, 2, 3, 4, 5$ let $u_r = \sum_{i=(r-1)K+1}^{rK} v_i$. Thus the $u_r$ and the $v_i$ are variables depending on the $(\epsilon_j : j \in A)$.

**Step 2.** There exists a choice of signs $(\epsilon_i : i \in A)$ such that

$$\left\| \sum_{j=1}^5 \eta_j u_j \right\| > (1 - \epsilon)200Nf(M)f(N)^{-2}$$

for every choice of signs $\eta_1, \ldots, \eta_5$, and also such that for each $i$

$$\|v_i\| \leq (1 + 3\epsilon)20Nf(M)/Kf(N)^2.$$

**Proof.** For a fixed choice of $\eta_1, \ldots, \eta_5$ we know that $\| \sum_{j=1}^5 \eta_j u_j \|$ is a 2-Lipschitz function on $\{-1, 1\}^A$. Hence, by Lemma 7,

$$\mathbb{P}\left[ \left\| \sum_{j=1}^5 \eta_j u_j \right\| - M \left\| \sum_{j=1}^5 \eta_j u_j \right\| > \frac{\epsilon}{20}200Nf(M)f(N)^{-2} \right] \leq \exp\left(-\frac{1}{2} \left( \frac{5\epsilon}{2f(N)} \right)^2 N \right).$$

Since $M^{-2}$ is greater than this, Step 1 implies that

$$M \left\| \sum_{j=1}^5 \eta_j u_j \right\| \geq \left(1 - \epsilon - \frac{\epsilon}{20}\right)200Nf(M)f(N)^{-2},$$

and hence, by Lemma 7 again,

$$\mathbb{P}\left[ \left\| \sum_{j=1}^5 \eta_j u_j \right\| < (1 - 2\epsilon)200Nf(M)f(N)^{-2} \right] \leq \exp\left(-\left(\frac{\epsilon}{40f(N)}\right)^2 N \right).$$

Similarly,

$$\mathbb{P}[\|v_i\| > (1+\epsilon)200(1+\epsilon)(N/5K)f(M)f(N/5K)^{-2}] \leq \exp\left(-\left(\frac{\epsilon}{40f(N)}\right)^2 \frac{N}{5K} \right).$$

But $(1 + \epsilon)^2 f(N/5K)^{-2} \leq (1 + 3\epsilon)f(N)^{-2}$, so

$$\mathbb{P}[\|v_i\| > (1 + 3\epsilon)200(N/5K)f(M)f(N)^{-2}] \leq \exp\left(-\left(\frac{\epsilon}{40f(N)}\right)^2 \frac{N}{5K} \right).$$

We have $5K + 32$ events that we want to occur simultaneously, and the probability of each individual event failing is at most $\exp(-\epsilon/40f(N))^2(N/5K))$ which is less than $(5K + 32)^{-1}$. This proves the second step.
Let us now fix a choice of signs \( \varepsilon \) satisfying the conditions of the previous step, so that \( u_1, \ldots, u_5 \) and \( v_1, \ldots, v_5 \) are now fixed vectors. Note that \( u_j \) is the sum of a R.I.S. of length approximately \( N/5 \) (certainly at least \( \exp \sqrt{\log M} \)) and constant \( 3/2 \). Also, since
\[
\|u_j\| \geq (1 - \varepsilon).20Nf(M)f(N)^{-2}
\]
and \( u_j = \sum_{i=(j-1)K+1}^{jK} v_i \) with \( \|v_i\| \leq (1 + 3\varepsilon)(20N/K)f(M)f(N)^{-2} \), we have that \( u_j \) is an \( l_{\infty}^K \)-average with constant \( (1 + 5\varepsilon) \). This remark will be useful later when we shall apply Lemma 5 to the vector \( u_4 \).

Before moving on to the next step, we shall need some more notation. First, we know that there must exist special combinations \( x^* = \sum_{i=1}^n a_i x_i^* \) and \( y^* = \sum_{j=1}^m b_j y_j^* \) such that
\[
5\alpha < |x^*(u_1 + u_2 + u_3 + u_4 + u_5)| \leq 5\alpha(1 + 3\varepsilon)(1 - \varepsilon)^{-1}
\]
and
\[
5\alpha < |y^*(u_1 + u_2 - u_3 + u_4 + u_5)| \leq 5\alpha(1 + 3\varepsilon)(1 - \varepsilon)^{-1}
\]
while
\[
\|u_j\| \leq \alpha(1 + 3\varepsilon)(1 - \varepsilon)^{-1} \quad \text{for each } j.
\]
Here \( \alpha \) stands for \( (1 - \varepsilon).20Nf(M)f(N)^{-2} \).

There are four cases for the signs of \( x^*(u_1 + u_2 + u_3 + u_4 + u_5) \) and \( y^*(u_1 + u_2 - u_3 + u_4 + u_5) \). We shall only look at the case when both are positive. The other cases are similar. Let us define probability measures \( \mu \) and \( \nu \) on \( \{1, 2, \ldots, n\} \) and \( \{1, 2, \ldots, m\} \) by \( \mu(A) = \sum_{i \in A} |a_i|^2 \) and \( \nu(B) = \sum_{j \in B} |b_j|^2 \). Let us also define a sequence of signs \( \eta_1, \ldots, \eta_5 \) by \( \eta_3 = -1 \) and otherwise \( \eta_r = 1 \).

Step 3. Let \( \delta = \sqrt{100\varepsilon} \). Then there exist \( C \) and \( D \) with \( \mu(C) \geq 1 - 5\delta \) and \( \nu(D) \geq 1 - 5\delta \) such that, for every \( i \in C \), \( j \in D \) and \( 1 \leq r \leq 5 \), we have
\[
(1 - \sqrt{\delta})(1 + 5\varepsilon)a_i \alpha \leq x_i^*(u_r) \leq (1 + \sqrt{\delta})(1 + 5\varepsilon)a_i \alpha
\]
and
\[
(1 - \sqrt{\delta})(1 + 5\varepsilon)b_j \alpha \leq \eta_r y_j^*(u_r) \leq (1 + \sqrt{\delta})(1 + 5\varepsilon)b_j \alpha.
\]

Proof. We know that \( x^*(u_r) \leq \alpha(1 + 3\varepsilon)(1 - \varepsilon)^{-1} \leq \alpha(1 + 5\varepsilon) \) for each \( r \), so \( x^*(u_r) \geq \alpha(1 - 20\varepsilon) \) for each \( r \) (since we know \( x^*(u_r) \leq \alpha(1 + 5\varepsilon) \) and \( |x^*(u_1 + \cdots + u_5)| \geq 5\alpha \)). In other words, \( \sum_{i=1}^n a_i x_i^*(u_r) \geq \alpha(1 - 20\varepsilon) \) while \( \sum_{i=1}^m b_j x_j^*(u_r) \leq \alpha^2(1 + 5\varepsilon)^2 \). The existence of \( C \) now follows from Lemma 6 applied once for each \( u_4 \). Similarly, we get the set \( D \).

Let us take a closer look at the \( x_i^* \) and \( y_j^* \). Each \( x_i^* \) is of the form \( E_i(x_i^*_{r_1} + \cdots + x_i^*_{r_p}) \) for some special sequence \( x_i^*_{r_1}, \ldots, x_i^*_{r_p} \). Let \( k_i \) be minimal such that \( \text{ran}(x_i^*_{r_k}) \cap \text{ran}(u_5) \neq \emptyset \). Then it may be the case that \( \text{ran}(x_i^*_{r_k}) \cap \text{ran}(u_3) = \emptyset \) or it may not. Let us define \( C_1 \) to be \( \{ i : \text{ran}(x_i^*_{r_k}) \cap \text{ran}(u_3) \neq \emptyset \} \). Similarly, let \( D_1 = \{ j : \text{ran}(y_j^*_{l_j}) \cap \text{ran}(u_3) \neq \emptyset \} \), where \( l_j \) is minimal such that \( \text{ran}(y_j^*_{l_j}) \cap \text{ran}(u_3) \neq \emptyset \).

Step 4. \( \max\{\mu(C_1), \mu(D_1)\} \leq 1/50 \).
Proof. Let \( C_2 = \{1, 2, \ldots, n\} \setminus C_1 \) and \( D_2 = \{1, 2, \ldots, n\} \setminus D_1 \). Then

\[
\sum_{i=1}^{n} a_i x_i^*(u_4) = \sum_{i \in C_1} a_i x_i^*(u_4) + \sum_{i \in C_2} a_i x_i^*(u_4).
\]

Writing \( U_r \) for \( \text{ran}(u_r) \), we have that \( U_4(\sum_{i \in C_1} a_i x_i^*) \) is \( \mu(C_1)^{1/2} \) multiplied by a basic special combination. We also remarked earlier that \( u_4 \) satisfied the conditions required in Lemma 5 (with \( N/5 \) replacing \( M \) and \( M \) replacing \( l \)). Hence, we may apply that lemma and deduce that

\[
\sum_{i \in C_1} a_i x_i^*(u_4) \leq \mu(C_1)^{1/2}(10^{-100}\|u_4\| + (2N/5).f(N/5)^{-1}).
\]

Now

\[
\|u_4\| \geq \|u_4\| \geq \alpha(1 - 20\varepsilon) \geq (1 - 25\varepsilon).20f(M).f(N)^{-2},
\]

so this is at most \( \mu(C_1)^{1/2}(10^{-100} + 1/45)\|u_4\| \).

Meanwhile, \( \sum_{i \in C_2} a_i x_i^*(u_4) \geq \alpha(1 - 20\varepsilon) \geq (1 - 25\varepsilon)\|u_4\| \), we find that \( (1/20)\mu(C_1)^{1/2} + \mu(C_2)^{1/2} \geq 1 - 25\varepsilon \). We also know that \( \mu(C_1) + \mu(C_2) = 1 \). If \( \mu(C_1) \geq 1/50 \) then \( \mu(C_2) \leq 49/100 \) which implies that \( \mu(C_1)^{1/2} \leq 99/100 \). This tells us that \( (99/100) + (1/20\sqrt{5\alpha}) \geq 1 - 25\varepsilon \), which is false. The argument for \( D_1 \) is similar.

Step 5. There exist \( C_3 \subset C \cap C_2 \) and \( D_3 \subset D \cap D_2 \) such that \( \mu(C_3) \) and \( \nu(D_3) \) both exceed \( 19/20 \) and a bijection \( \phi : C_3 \rightarrow D_3 \) such that, for every \( i \in C_3 \), the special functionals \( U_5 x_i^* \) and \( U_5 y_{\phi(i)}^* \) are not disjoint.

Proof. Without loss of generality all \( a_i \) and \( b_j \) are nonzero. If \( i \in C \) this implies that \( x_i^*(u_r) \neq 0 \) for every \( r \), and in particular that \( \text{ran}(x_i^*) \cap \text{ran}(u_1) \neq \emptyset \). This remark will be useful later. Similarly, if \( j \in B \), then \( \text{ran}(y_j^*) \cap \text{ran}(u_1) \neq \emptyset \).

If \( i \in C \cap C_2 \), then \( \text{ran}(x_i^*,k_i) \cap \text{ran}(u_3) = \emptyset \), but \( \text{ran}(x_i^*) \cap \text{ran}(u_3) \neq \emptyset \). It follows that \( E_i(x_i^*,k_i-1) \neq 0 \), so the associated set of \( U_5 x_i^* \) is uniquely determined. Let

\[
C_4 = \{i \in C \cap C_2 : U_5 x_i^* \text{ and } U_5 y_j^* \text{ are disjoint for every } j \in D \cap D_2\}.
\]

By the definition of \( C_4 \), we know that

\[
\sum_{i \in C_4} |x_i^*(u_5)|^2 + \sum_{j \in D \cap D_2} |y_j^*(u_5)|^2 \leq \|u_5\|^2.
\]

From the properties of \( C \) and \( D \) we can deduce that

\[
(1 - \sqrt{\delta})^2(1 + 5\varepsilon)^2\alpha^2 \mu(C_4) + (1 - \sqrt{\delta})^2(1 + 5\varepsilon)^2\alpha^2 \nu(D \cap D_2) \leq (1 + 5\varepsilon)^2\alpha^2.
\]

Hence,

\[
(1 - \sqrt{\delta})^2\mu(C_4) + (1 - 5\delta - (1/50))(1 - \sqrt{\delta})^2 \leq 1,
\]

which implies that \( \mu(C_4) \leq 1/40 \). Let \( C_3 = (C \cap C_2) \setminus C_4 \). Then \( \mu(C_3) \leq (39/40) - (1/50) - 5\delta \geq 19/20 \) and, for every \( i \in C_3 \), there exists some \( j \in D \cap D_2 \) (which must be unique) such that the associated sets of \( U_5 x_i^* \) and \( U_5 y_j^* \) are not disjoint. By the same argument for the \( y_j^* \) we can define a set \( D_3 \). It is easy to see that \( C_3 \) and \( D_3 \) have the properties claimed above.
Step 6. Let \( V = \text{ran}(u_2 + u_3) \). If \( i \in C_3 \), then \( Vx_i^* = Vy_{\phi(i)}^* \).

Proof. Setting \( j = \phi(i) \), we know that \( E_ix_i^*, l_{k_i-1} \neq 0 \) and similarly \( F_jy_j^*, l_{l_j-1} \neq 0 \) (where \( y_j^* = F_j(y_j^* + \cdots + y_j^*_{j_{q_j}}) \)). We also know that \( \sigma(x_i^*, \ldots, x_i^*, l_{k_i-1}) = \sigma(y_j^*, \ldots, y_j^*, l_{l_j-1}) \). It follows from the definition of a special sequence and the fact that \( \sigma \) is an injection that \( k_i = l_j \) and \( x_i^* = y_j^* \) for every \( t < k_i \). Finally, we know that \( U_1x_i^* \) and \( U_1y_j^* \) are both nonzero and that \( U_3 \cap \text{ran}(x_i^*, l_{k_i}) = U_3 \cap \text{ran}(y_j^*, l_{l_j}) = \emptyset \). Putting all these facts together tells us that \( Vx_i^* = Vy_j^* \) as claimed.

The contradiction. The set \( C_3 \) is certainly not empty. Let \( i \in C_3 \). Then since \( C_3 \subset C \) it follows from Step 3 that \( \alpha/2 \leq x_i^*(u_2)/a_i \leq 2\alpha \). Since \( x_i^*(u_2) = y_{\phi(i)}^*(u_2) \) and \( D_3 \subset D \), we also get \( \alpha/2 \leq x_i^*(u_2)/b_{\phi(i)} \leq 2\alpha \). This implies that \( b_{\phi(i)}/a_i \geq 1/4 \). By the same argument we have \( \alpha/2 \leq x_i^*(u_3)/a_i \leq 2\alpha \) and \( \alpha/2 \leq -x_i^*(u_3)/b_{\phi(i)} \leq 2\alpha \). This implies that \( b_{\phi(i)}/a_i \leq -1/4 \). (Recall that we restricted our attention to nonzero \( a_i \) and \( b_j \).) We have arrived at the contradiction we promised. \( \square \)

Remarks. 1. It is not hard to see that the space constructed in this paper has a predual, and that this predual does not contain \( c_0 \) or a boundedly complete basic sequence. Since the predual certainly does not contain \( l_1 \), it also gives a space not containing \( c_0, l_i \) or a reflexive subspace.

2. An overview of the proof of Lemma 4 might be helpful. The aim of the first two steps is to obtain an almost isometric copy of \( l_1^n \) for a suitable \( n \), in which every vector is normed by a special combination. The remaining steps are designed to show that this cannot happen. There is a technical problem about basic special combinations, which is dealt with in Step 4. The main point of Steps 3 to 6 is to show that the special combinations norming \( u_1 + \cdots + u_4 \) and \( u_1 + u_2 - u_3 + u_4 + u_5 \) are roughly equal on \( u_2, u_3 \) and \( u_4 \), which clearly cannot happen. Of key importance is that any term in a special sequence determines the whole of the sequence up to that point, but this fact is harder to apply than it was in [1], where one did not deal with combinations of special sequences.

3. The most important difference between the problem solved in this paper and that solved in [1] is that reflexivity is an infinite-dimensional phenomenon, whereas the property of not containing an unconditional basic sequence is equivalent to saying that every infinite-dimensional subspace contains arbitrarily large finite-dimensional (block) subspaces of a certain kind. The resulting need to consider infinite special sequences is a serious difficulty, dealt with by the \( l_2 \)-sum in the definition of the norm and the nonexplicitness of Lemma 4.

4. There are possible extensions of the main result. For example, James constructed a nonreflexive space with nontrivial type and cotype. It seems likely that the construction of this paper could be adapted to give a space with nontrivial type and cotype and no reflexive subspace (answering a question of Casazza). Another hereditary version of a James space would be the following—a uniformly nonoctahedral space with no reflexive subspace. Again, such a space probably exists, but a proof is unlikely to be pleasant.
References

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