Abstract. Let $T$ be a superstable theory with $< 2^{\aleph_0}$ countable models. We study some special types $p \in S(\emptyset)$ of $U$-rank 2 called skeletal (cf. [Bu4]). We reduce an eventual version of the problem of counting isomorphism types of sets $p(M)$ for countable $M$ to a problem from linear algebra.

0. Introduction

Throughout, $T$ is a superstable theory with $< 2^{\aleph_0}$ countable models and we work with $T^{eq}$. In our notation we usually follow [Ba]. For example, if $A \subseteq B$ and $p \in S(B)$ then we say that $p$ is based on $A$ if $p$ does not fork over $A$ [Ba, IV.1.17 (ii)]. For background on stability theory see [Ba, Sh]. For the definition and basic properties of modularity in the context of stability theory see [Bui, CHL] or [P]. In [Bu4, Ne3], a study of the possible isomorphism types of sets $p(M)$ for a type $p$ of $U$-rank 2 was initiated. This investigation is strongly connected with Vaught's conjecture for superstable theories of finite $\infty$-rank (cf. [Bu4, §0]). Following [Bu4], we call a type $p \in S(\emptyset)$ skeletal if $p$ is stationary, has $U$-rank 2 and for a realizing $p$ there is a $b \in acl(a)$ such that $U(b) = 1$ and $tp(a/b)$ has finite multiplicity. $q = stp(b)$ is called a base type of $p$.

In this paper we often refer to [Bu4]. There are two important contributions from [Bu4, §2] here. Firstly, [Bu4, §2] proves that $I(T, \aleph_0) < 2^{\aleph_0}$ implies $q$ is locally modular (see Lemma 1.9 here). Secondly, it proves a local version of NOTOP (see Theorem 1.16 below).

For a type $p$ over $\emptyset$ let $I(p, \kappa)$ denote the number of isomorphism types of sets $p(M)$, where $M$ is a model of $T$ of power $\kappa$. We consider the following conjecture.

(P) If $T$ is superstable, $p$ is skeletal and $I(T, \aleph_0) < 2^{\aleph_0}$ then $I(p, \aleph_0)$ is countable.

While a proof of (P) may not yield immediately a proof of Vaught’s conjecture for superstable theories of finite rank, it may give strong indications on how to prove Vaught’s conjecture in this case. Moreover, Vaught’s conjecture implies (P). Here we prove some structure theorems connected with (P). We prove that for some finite set $E$, the problem of counting isomorphism types of sets $(p|E)(M)$ for countable $M$ reduces to a problem from linear algebra. The
exchange of information with Steven Buechler helped me greatly to prepare this paper. Also I should thank the referee for some helpful suggestions.

If \( \mathcal{F} \) is a family of nonempty sets, then \( S \subseteq \bigcup \mathcal{F} \) is called a selector from \( \mathcal{F} \) if \( |S \cap X| = 1 \) for every \( X \in \mathcal{F} \). We work within a monster model \( \mathfrak{C} \). We use \( a, b \) and so on to denote tuples of elements of \( \mathfrak{C} \).

For the rest of this paper, \( p \) denotes a skeletal type and \( q \) is a fixed base type of \( p \). Let \( a \) realize \( p \) and \( b \in \text{acl}(a) \) realize \( q \). Let \( p_b \) denote \( \text{tp}(a/b) \). So \( p_b \) has finite multiplicity and necessarily \( U \)-rank 1. For any \( c \) realizing \( q \), let \( p_c \) be the conjugate of \( p_b \) over \( c \). As in [Bu4, §1], \( q \) has only finitely many conjugates over \( \varnothing \). Hence by adding an element of \( \text{acl}(\varnothing) \) to the signature we may assume that \( q \) has only one conjugate over \( \varnothing \), i.e. that \( q \) is just a stationary type over \( \varnothing \).

Our goal is to classify isomorphism types of countable sets of the form \( p(M) \), and to prove that \( I(p, \aleph_0) \) is countable. Some cases are trivial (cf. [Bu4, §1]). Eliminating these cases by [Bu4, §1] we fix the following assumptions from now on.

0.1. **Assumptions.** \( q \) is a complete stationary type over \( \varnothing \). Let \( Q = q(\mathfrak{C}) \). For \( a \in Q \), \( p_a \) is nontrivial properly weakly minimal, nonisolated and of finite multiplicity. Moreover, for \( b \) realizing \( p_a \), \( \text{stp}(b/a) \) is locally modular, nonmodular, nonorthogonal to \( \varnothing \) and almost orthogonal to \( \varnothing \). In particular, \( p_a \) is weakly orthogonal to \( q|a \).

Using these assumptions we prove the following lemma.

0.2. **Lemma.** Let \( b, c \in Q \).

1. Suppose \( a \) realizes \( p_b \) and \( p_c \). Then \( b, c \) are interalgebraic and \( \text{stp}(a/b), \text{stp}(a/c) \) are parallel.

2. All stationarizations of \( p_b \) are nonorthogonal.

*Proof.* (1) We have \( b, c \in \text{acl}(a) \). If \( c \notin \text{acl}(b) \), then we get that \( p_b \) is not almost orthogonal to \( \varnothing \), a contradiction. Hence \( b, c \) are interalgebraic. It follows that \( \text{stp}(a/b), \text{stp}(a/c) \) are parallel.

(2) Let \( r_0, r_1 \) be stationarizations of \( p_b \). Since \( r_i \) is nonorthogonal to \( \varnothing \), for \( d \in Q \setminus \text{acl}(b) \) there is a stationarization \( r'_i \) of \( p_d \) nonorthogonal to \( r_i \). It follows that for a realizing \( p_b \), there are \( d_0, d_1 \in \text{acl}(a) \cap Q \) such that \( a \) realizes \( p_{d_0} \) and \( p_{d_1} \), and \( r'_i = \text{stp}(a/d_i) \) is nonorthogonal to \( r_i \), \( i = 0, 1 \). By (1), \( r'_0, r'_1 \) are parallel, hence \( r_0, r_1 \) are nonorthogonal.

Notice that if \( p_a \) is properly weakly minimal and nonisolated, then by [Ne1], \( I(T, \aleph_0) < 2^{2^{\aleph_0}} \) implies that \( p_a \) has finite multiplicity. An important tool for us will be algebraic dependence ACL, introduced in [Ne2] (and denoted there by \( \text{acl}^* \)).

0.3. **Definition.** Suppose \( A \subseteq \mathfrak{C} \). Let \( P^* \) be a family of properly weakly minimal locally modular pairwise nonorthogonal types over \( \text{acl}(A) \). We define a dependence relation \( \text{ACL}_{\mathcal{A}} \) on \( P^* \). For \( r \in P^* \) and \( R \subseteq P^* \) we define \( r \in \text{ACL}_{\mathcal{A}}(R) \) iff \( r \) is realized in the algebraic closure of \( A \cup \{ r'(\mathfrak{C}) : r' \in R \} \). \( \text{DIM}_{\mathcal{A}} \) denotes the ACL-dimension. When no confusion arises, we omit \( A \) in \( \text{ACL}_{\mathcal{A}}, \text{DIM}_{\mathcal{A}} \). The following lemma was proved in [Ne2, 2.11] (but the proof relies on [H]).

0.4. **Lemma.** For \( R \neq \varnothing \), \( r \in \text{ACL}(R) \) iff \( r \) is modular or whenever \( B \) contains a realization of every \( r' \in R \) then \( r \) is realized in \( \text{acl}(AB) \).
0.5. **Lemma.** Assume \( A \subseteq C \) is countable, \( p^* \) is a stationary properly w.m. nontrivial type over \( \text{acl}(A) \) and \( P^* \) is the family of weakly minimal nonmodular types over \( \text{acl}(A) \) nonorthogonal to \( p^* \).

1. \( \text{ACL} \) satisfies the exchange principle on \( P^* \).
2. \( \text{ACL} \) is modular on \( P^* \).

**Proof.** (1) follows by Lemma 0.4.

(2) is implicit in [Ne2, end of §2] and in [Ne1, 3.3]; the proof appears in [Ne3, 1.2] and essentially also in [Bu4, 1.11].

Assume \( d \) is a modular dependence relation on a set \( X \), for all \( d \)-independent \( x, y \in X \) there is \( z \in d(x, y) \setminus (d(x) \cup d(y)) \), and \( d \)-dimension of \( X \) is \( \geq 4 \). Then after identifying \( d \)-interdependent elements of \( X \), \( X \) with the induced dependence relation becomes a projective space over some division ring \( F \) (we say that \( X \) is projective over \( F \)). Let \( P^* \) be as in Lemma 0.5. Any \( r \in P^* \) is locally modular, hence the pregeometry on \( r(\emptyset) \) gives rise to a division ring \( F_r \), and obviously for \( r \neq r' \in P^* \), \( F_r \) and \( F_{r'} \) are isomorphic. We define \( F_v \) as any \( F_r \), \( r \in P^* \). On the other hand, by Lemma 0.5, \( \text{ACL} \) is modular on \( P^* \). By [Bu2, 2.4] there is a weakly minimal \( \phi \in p^* \) such that every w.m. \( q \) containing \( \phi \) is nonorthogonal to \( p^* \). Hence \( \text{DIM}(P^*) \) is infinite. Also, [Bu1, Theorem 2] yields quickly that \( \text{ACL} \)-dependence on \( P^* \) is nontrivial. Hence \( P^* \) is projective over some division ring \( F_h \). \( h \) and \( v \) in \( F_h, F_v \) stand for "horizontal" and "vertical". In [Ne2, 6.9] we conjectured that \( F_h \) and \( F_v \) are isomorphic. Below we prove this conjecture. The proof, suggested by the referee, is much like the proof of [Ne2, 6.8], but uses 1-basedness of unidimensional theory.

0.6. **Proposition.** \( F_h \) and \( F_v \) are isomorphic.

**Proof.** W.l.o.g. the type \( p^* \) from Lemma 0.5 is modular and we add \( \text{acl}(A) \) to the signature. By [Bu2, 2.4], restricting to some \( \phi \in p^* \), we may assume that \( T \) is weakly minimal and unidimensional, hence in particular 1-based.

We shall find an isomorphism between a projective plane over \( F_h \) in \( P^* \) and a projective plane over \( F_v \) in \( P^*(\emptyset) \). Let \( r_0, r_1, r_2 \in P^* \) be \( \text{ACL} \)-independent. Let \( a, b, c \) realize \( r_0, r_1, r_2 \) respectively. Note first that if \( U(e) = 1 \) and \( e \in \text{acl}(abc) \) then \( \text{stp}(e) \) is nonmodular, hence belongs to \( P^* \). Let \( a'b'c' \) have the same strong type as \( abc \) and be independent from \( abc \) (over \( \emptyset \)). We show

\[
\text{for any } d \in \text{acl}(abca'b'c') \text{ realizing } p^*|a'b'c' \text{ there is an } e \in \text{acl}(abc) \text{ with } \text{stp}(e) \in P^* \text{ and } d \text{ interalgebraic with } e \over a'b'c'.
\]

Indeed, choose \( e \) interalgebraic with \( Cb(a'b'c'd/abc) \). By 1-basedness, \( e \in \text{acl}(abc) \cap \text{acl}(a'b'c'd) \). Since \( abca'b'c' \), we have \( e \perp a'b'c' \). \( d \in \text{acl}(abca'b'c') \) implies \( e \perp d(abca'b'c') \). We get \( U(e) = 1 \), and \((*)\) follows.

Now we define a function \( f: \text{ACL}(r_0r_1r_2) \to P^*(\emptyset) \) as follows. Let \( r \in \text{ACL}(r_0r_1r_2) \). Then there are \( e \in \text{acl}(abc) \) and \( e' \in \text{acl}(a'b'c') \) realizing \( r \). Clearly \( e \) and \( e' \) are independent, hence \( r|e' \), and the more so \( r|a'b'c' \) is modular, and \( e \) realizes \( r|a'b'c' \). Since \( r \) is nonorthogonal to \( p^* \), there is a \( d \in \text{acl}(ea'b'c') \) realizing \( p^*|a'b'c' \). Let \( d = f(r) \).

\((*)\) shows that for every \( d' \in \text{acl}(abca'b'c') \) realizing \( p^*|a'b'c' \) there is a \( d \in \text{acl}(abca'b'c') \) realizing \( p^*|a'b'c' \) which is interalgebraic with \( d' \) over \( a'b'c' \) and \( ii \) in the range of \( f \). It follows that \( f \) induces an isomorphism...
between the projective plane generated by $r_0, r_1, r_2$ in $P^*$ and the projective plane over $a'b'c'$ generated in $p^*(e)$ by $a, b, c$. In particular, $F_v$ and $F_h$ are isomorphic.

1. ACL and Many Model Arguments

Now for the rest of this paper we will fix a specific meaning of $P^*$. Let $P^*$ be the set of weakly minimal, nonmodular types over $acl(Q)$ nonorthogonal to $p_a$ for some (every) $a \in Q$. Let $A \subseteq Q$. We define the following families of types.

$P^*_0 = \{stp(b/a)| acl(A): a \in acl(A) \cap Q \text{ and } b \text{ realizes } p_a\}$,
$P^*_A = \{r| acl(A): r \in P^* \text{ and } r \text{ is based on } A\}$,
$P^*_A = \{r \in P^*_A: \text{for some finite } A' \subseteq A, r \text{ is based on } A' \text{ and has finitely many conjugates over } A'\}$,
$P_A = \{r| acl(A): r \in ACL_Q(P^*_0) \text{ and } r \text{ is based on } A\}$.

We write $P, P^0, P^*$ for $P_Q, P^0_Q, P^*_Q$ and $P^*_Q$. Also we write $P_a$ instead of $P^{(a)}$ (with all possible superscripts). For $A \subseteq B \subseteq Q$ we can naturally regard $P^*_A$ as a subset of $P^*_B$, identifying $r \in P^*_A$ with $r|acl(B)$ (similarly with $P^0_A, P^*_A$ and $P_A$). We say that $R, R' \subseteq P^*$ are isomorphic, if some $f \in Aut(Q)$ induces a bijection from $R$ onto $R'$. In the next lemma we collect basic properties of the families of types defined above. We say that a formula $\phi(x, y)$ is algebraic in $x$ if for every $a, \phi(x, a)$ is algebraic.

1.1. Lemma. Assume $A \subseteq Q$.

(1) $P^*_A \subseteq P_A \subseteq P^*_A \subseteq P^*_A$. $P_A$ is $ACL_A$-closed in $P^*_A$ and $P^*_A$ is $ACL_A$-closed in $P^*_A$.

(2) If $q|A$ is modular or orthogonal to $p_a$, $a \in Q$, then $ACL_A$ on $P^*_A$ and $ACL_Q$ on $P^*_A$ regarded as a subset of $P^*$, agree.

(3) Assume $R \subseteq P^*_A, r \in P^*_A$ and $r \in ACL_Q(R)$. Then for some $r' \in ACL_A(R)$, $r$ and $r'$ are $ACL_Q$-interdependent.

(4) For countable $A, P^*_A$ is countable.

Proof. (1) The only less trivial inclusion is $P_A \subseteq P^*_A$. Let $r \in P_A$ and $r' = r|acl(Q)$. Choose a finite $A' \subseteq A$ such that $r'$ does not fork over $A'$. Since $r' \in ACL(P^*)$, for some finite $B \subseteq Q$ containing $A'$, $r'$ does not fork over $B$ and $\text{Mlt}(r'|B)$ is finite. But $q$ is stationary, hence $\text{Mlt}(B/A')$ is finite, too. It follows that $\text{Mlt}(r'|A')$ is finite.

(2) Suppose $X \subseteq P^*_A, r \in P^*_A$ and $r \in ACL_Q(X)$. Revealing the definition we see that if $q$ is orthogonal to $p_a$ then obviously $r \in ACL_A(X)$. Otherwise, $q|A$ is modular, and by Lemma 0.4, $r \in ACL_Q(X)$ means that $r \in ACL_A(X \cup \{q|A\})$. By Lemma 0.4, $q|A \in ACL_A(X)$, and $r \in ACL_A(X)$.

(3) W.l.o.g. $A$ is finite, and we can assume $r$ is based on $Aa$ for some $a \in Q \setminus acl(A)$. By assumption, there are $r_0, \ldots, r_n \in R \cup B_i$ realizing $r_i$ and $b$ realizing $r' = r|acl(Aa)$ such that $n$ is minimal possible. $b = (b_0, \ldots, b_n)$ is independent over $A$ and

(a) $b \perp a(A) \text{ and } b \in acl(Aab)$. If $T$ is 1-based, then it is easy to prove that if $c$ is a large enough fragment of $Cb(b/Aab)$ then $r' = stp(c/A)$ satisfies our demands (in fact, $T$ is "locally" 1-based, at least close to types $r, r_0, \ldots, r_n$).
This is the idea of the following proof. Let \( \phi(x, y, a') \) be a formula over \( \acl(Aa) \) witnessing \( b \in \acl(Aab) \). So \( D = \phi(c, b, a') \) is finite and w.l.o.g. contained in \( r'(c) \). By the minimality of \( n \) we get

(b) \( \dim_{Aa}(D) = 1 \), that is \( b \) and \( D \) are interalgebraic over \( Aa \). Let \( q' = \stp(a'/A) \) and \( s = \stp(b/A) \). We define an equivalence relation \( E \) on \( s(c) \) by \( b'/Eb'' \) iff for \( a'' \) realizing \( q'\mid Ab'b'' \), \( \phi(c, b', a'') = \phi(c, b'', a'') \). Let \( c = b/E \). Hence \( c \perp a(A) \). Let \( r' = \stp(c/A) \). To finish it suffices to prove that \( b \) and \( c \) are interalgebraic over \( A \). We have

(c) \( b \in \acl(Aac) \). Indeed, suppose \( f \in \Aut(c) \), \( f \) fixes \( Aac \) pointwise, \( f(bb) = b'b' \), and w.l.o.g. \( b'b' \perp bb(Aac) \). Hence in particular

(d) \( b' \perp b(Aac) \)

(a) implies \( b \perp a(Ac) \), hence also \( b' \perp a(Ac) \), and by (d) we get \( b' \perp ba(Ac) \) and \( b' \perp a(ABC) \). Hence \( a \perp b'(A) \) and \( a \perp c(A) \), which gives \( a \perp bb'(A) \), and the more so \( a' \perp bb'(A) \), i.e. \( a' \) satisfies \( q'\mid Ab'b' \). Since \( b/E = c = b/E \), we get \( D = \phi(c, b, a') \), hence \( b' \in D \). This proves (c). Now we show

(e) \( c \in \acl(AaD) \). Suppose \( f \in \Aut(c) \) fixes \( AaD \) pointwise, let \( c'b' = f(cb) \) and w.l.o.g. \( c'b' \perp cb(AaD) \). In particular, \( b' \perp bc(ABC) \) and by (b) and (c), \( D \subseteq \acl(Aac) \), hence \( b' \perp b(Aac) \). This implies as above that \( bb'(A) \perp A'a(A) \), and \( D = \phi(c, b, a') = \phi(c, b', a') \), because \( f \) is elementary. Hence \( bEb' \), and \( c' = c \).

(4) For a fixed finite \( A' \subseteq A \), there are countably many strong types over \( A' \) with finitely many conjugates over \( A' \), because \( T \) is small. Hence \( P_A^* \) is countable.

Let \( \acl' \) denote the ACL-dependence over \( P^*_\sigma \), that is \( r \in \acl'(R) \) iff \( r \in \acl(RP^*_\sigma) \). We say that \( R \) is essentially ACL-closed in \( R' \) if \( R \subseteq R' \) and for every \( r \in \acl(R) \cap R' \) there is an \( r' \in R \) with \( r \in \acl(r') \). \( R \) is essentially ACL-closed if \( R \) is essentially ACL-closed in \( P^* \). Similarly we define the notion of an essentially ACL'-closed set. Hence Lemma 1.1(3) says that \( P_A^* \) is essentially ACL-closed. Every \( r \in P_A^0 \) is locally modular. Hence we can define \( F_p \) as the division ring corresponding to the pregeometry \( (r(c), acl) \), and the choice of \( a \in Q \) and \( r \in P_A^0 \) does not matter. By Lemma 1.1(2) we can omit \( A \) in \( \acl(A) \) when \( A \neq \varnothing \). By Lemma 0.5 and Proposition 0.6, \((P^*, \acl)\) is projective over \( F_p \). By Lemma 1.1(3), \( P_A^* \) is essentially a projective subspace of \( P^* \), hence is projective over \( F_p \). Similarly, \( P_A \) and \( P_A^* \) are projective over \( F_p \). By [Bu3] or [Ne1, 0.5], \( F_p \) is in fact a locally finite field. For \( X, Y \subseteq P^* \), \( \dim(X/Y) \) denotes the ACL-dimension of \( X \) over \( Y \), and \( \dim(X) \) denotes the ACL-dimension of \( X \). We define a notion of independence on \( P^* \). \( X, Y \subseteq P^* \) are independent over \( Z \subseteq P^* \) (\( X \perp Y(Z) \)) if for all finite \( X' \subseteq X, Y' \subseteq Y \), \( \dim(X'/Z) = \dim(X'/Y'Z) \). Clearly \( X \perp Y(Z) \) implies \( Y \perp X(Z) \) (see [Ne2]).

1.2. Lemma. Assume \( A \subseteq Q, a \in Q \setminus acl(A), r \in P_A^0 \). Then \( r \notin \acl(P_A^*) \).

Proof. Suppose not. W.l.o.g. \( A \neq \varnothing \). By Lemma 1.1, for some \( s \in P_A^*, r \) and \( s \) are ACL,Aa-interdependent. That is for some \( b, c \) realizing \( r|Q, s|Q \) respectively, \( b, c \) are acl-interdependent over \( Aa \). In particular, \( bA \perp a \) and \( cA \perp b(a) \), showing \( r \) is not almost orthogonal to \( \varnothing \). This contradicts Assumptions 0.1.
The next lemma and the following corollary show that ACL has a “local character”, that is for $A, B, C \subseteq Q$ with $C \subseteq A \cap B$ and $A \perp B(C)$, $P_A^* \perp P_B^*(P_C^*)$.

1.3. **Lemma.** Assume $A, B, C \subseteq Q$, $A \perp B(C)$, $r \in P_A^*$, $r' \in P_B^*$, $r \in \text{ACL}(r')$. Then for some $r'' \in P_C^*$, $r \in \text{ACL}(r'')$.

**Proof.** W.l.o.g. $A = \{a\} \cup C$, $B = \{b\} \cup C$, $a, b \in Q \setminus \text{acl}(C)$. For simplicity assume $C = \emptyset$. So we have $a \perp b$, $r_a \in P_a^*$, $r_b \in P_b^*$ and $r_a \in \text{ACL}(r_b)$. We want to find an $r \in P_a^*$ such that $r_a \in \text{ACL}(r)$. Let $a'$ realize $r_a|ab$, $b'$ realize $r_b|ab$ and $b' \in \text{acl}(aa'b)$. Let $c$ be a sufficiently large finite fragment of $Cb(bb'/aa')$ such that $Cb(bb'/aa') \subseteq \text{acl}(c)$. Hence $c \in \text{acl}(aa')$, and since $b \perp aa'$, $c \perp b$. We have $U(c/a) = U(a'/a)$ and $a' \in \text{acl}(ca)$, hence it suffices to prove

(a) $c \perp a$. Then $r = \text{stp}(c/\emptyset)$ will satisfy our demands. To prove (a) it suffices to show $c \subseteq \text{acl}(bb')$. This follows because $T$ is “locally” 1-based. We can prove $c \subseteq \text{acl}(bb')$ directly. Let $b_0b_0' = bb'$, $b_1b_1'$, $b_2b_2'$, ... be a Morley sequence in $\text{stp}(bb'/aa')$. Let $I = \{b_0, b_1, \ldots\}$, $J = \{b_0', b_1', \ldots\}$. $b \perp aa'$ implies $I \perp aa'$, hence $c \perp I$, $b' \in \text{acl}(aa'b)$ implies $J \subseteq \text{acl}(Iaa')$, hence $\text{dim}(J/I) \leq 2$ and $c \subseteq \text{acl}(IJ)$. $\text{dim}(J/I) \leq 1$ would imply $U(c/\emptyset) = 1$, and $c \subseteq \text{acl}(bb')$. So we can assume $\text{dim}(J/I) = 2$. This will lead to a contradiction.

If $q$ is orthogonal to $p_a$, then $c \subseteq \text{acl}(IJ)$ implies $c \perp a$, and we are done. So we can assume $q$ is nonorthogonal to $p_a$, and then $q$ is locally modular. We have $J$ is pairwise $I$-independent, but not $I$-independent. We shall transfer this situation to $r_b(\emptyset)$, to reach a contradiction with modularity.

Let $r_i = \text{stp}(b_i'/b_i)$. We have $b_i' \perp I(b_i)$. By transitivity of ACL, $r_i \in \text{ACL}(r_i)$, hence by Lemma 0.4, there is $b_i''$ realizing $r_i|I$ such that $b_i'' \in \text{acl}(b_i'I)$, and $b_0'' = b'$. Let $J'' = \{b_0'', b_1', \ldots\}$. Hence $\text{dim}(J''/I) = 2$ and for $i \neq j$, $b_i'' \perp b_j''(I)$. Let $b^*$ realize $r_b|IJ$. So $r_b|bb^*$ is modular, and nonorthogonal to $q|b$, which is also modular. Thus we can choose $I^* = \{b_1^*, b_2^*, \ldots\}$, a Morley sequence in $r' = r_b|bb^*$, such that $b_i^*$ is interalgebraic over $bb^*$ with $b_i$. Add $bb^*$ to the signature. So we have $\text{dim}(J''/I^*) = 2$, and for $i \neq j$, $b_i'' \perp b_j''(I^*)$. We can assume that for $i, j > 2$,

(b) $b_i^*b_j'' \equiv b_j^*b_i''(b_i^*b_j''I^*)$. We have $b_i'' \in \text{acl}(b_i''b_j''I^*)$, hence by modularity of $r'$, there is $a'$ realizing $r'$ with $a' \in \text{acl}(b_i^*b_j''b_j'') \cap \text{acl}(b_i'b_j)$. By (b), $a' \in \text{acl}(b_i''b_j''I^*)$ as well. It follows $b_i'' \not \subseteq b_j''(b_j''b_i'')$, hence $b_i'' \not \subseteq b_j''(I^*)$, a contradiction.

1.4 **Corollary.** Assume $A, B \subseteq Q$, $C \subseteq A \cap B$ and $A \perp B(C)$. Then $P_A^* \perp P_B^*(P_C^*)$, $P_A \perp P_B(P_C^*)$ and $P_A \perp P_B(P_C)$.

**Proof.** For example we prove $P_A^* \perp P_B^*(P_C^*)$. Suppose not. Then there are finite $X \subseteq P_A^*$, $Y \subseteq P_B^*$ with $\text{DIM}(X/P_C^*) < \text{DIM}(X/P_A^*)$. By modularity of ACL, there is an $r^* \in \text{ACL}(P_A^*) \cap \text{ACL}(P_B^*) \setminus \text{ACL}(P_C^*)$. By Lemma 1.1(3) there are $r \in P_A^*$ and $r' \in P_B^*$, ACL-interdependent with $r^*$. By Lemma 1.3, $r^* \in \text{ACL}(P_C^*)$, a contradiction.

In the next proposition we see how to reduce $(P)$ to the problem of counting the isomorphism types of ACL-closed countably dimensional subsets of $P^*$.  


1.5. Proposition. Assume $M$ is a countable model of $T$ and $A = q(M)$.

(1) Let $R(M) = \{ r \in P_A : r$ is realized in $M \}$. Then $R(M)$ is ACL-closed in $P_A$.

(2) Conversely, if $R \subseteq P_A^*$ is ACL-closed in $P_A^*$ then there is $N$ with $A = q(N)$ and $R = \{ r \in P_A^* : r$ is realized in $N \}$.

Proof. $R(M)$ is ACL-closed in $P_A$ by Lemma 0.4. If $R \subseteq P_A^*$ is ACL-closed in $P_A^*$, then by Lemma 1.1(4) we can find the desired $N$ by the omitting types theorem.

1.6. Corollary. To compute $I(p, \mathcal{N}_0)$ it suffices to determine the isomorphism types of ACL-closed in $P_A$ subsets of $P_A$, where $A = q(M)$ for a countable $M$.

Proof. We have $p(M) = \bigcup \{ \delta_M(A) : a \in A = q(M) \} \subseteq \bigcup \{ r(M) : r \in R(M) \}$, where $R(M) = \{ r \in P_A : r$ is realized in $M \}$. By local modularity, any two types in $R(M)$ have the same dimension $n_M$ in $M$. Hence the isomorphism type of $p(M)$ is determined by the dimension of $A$, $n_M$ and the isomorphism type of $R(M)$.

Consequently our further study concentrates on ACL-closed subsets of $P^*$. In fact it suffices to investigate only ACL-closed subsets of $P_A$, but it turns out that it is not easier at all. By Proposition 1.6, any two nonisomorphic ACL-closed countably dimensional subsets of $P^*$ correspond to nonisomorphic countable models of $T$. In particular, $I(T, \mathcal{N}_0) < 2^{\mathcal{N}_0}$ implies there are $< 2^{\mathcal{N}_0}$ isomorphism types of ACL-closed countably dimensional subsets of $P^*$. Notice that we need not to bother about subsets of $P_A^*$ with $A$ finite dimensional. The following lemma is a reformulation of [Bu2, 5.2(a)] or [Bu4, 1.14].

1.7. Lemma. If $A \subseteq Q$ has finite dimension then $P_A^*$ has finite ACL-dimension.

1.8. Corollary. If $A \subseteq Q$ is finite then there are only countably many nonisomorphic ACL-closed subsets of $ACL(P_A^*)$. In particular, there are countably many isomorphism types in the set $\{ p(M) : q(M)$ has finite dimension $\}$.

Proof. Follows by Lemmas 1.1, 1.7 and Corollary 1.6.

The following lemma, proved in [Bu4, §2], is fundamental in this paper.

1.9. Lemma. $Q$ is locally modular.

An easy argument from [Bu4, §1] gives also

1.10. Assumption. $Q$ is nontrivial.

By Lemma 1.9 we can define $F_q$ as the division ring corresponding to the pregeometry $(Q, acl)$. Now we have two pregeometries: $(Q, acl)$ and $(P^*, ACL)$ (or $(P, ACL)$ if you like). Counting the isomorphism types of ACL-closed subsets of $P^*$ has mostly a geometrical character, and utilizes the interaction between $(Q, acl)$ and $(P^*, ACL)$. There were two breakthroughs in the understanding of properly weakly minimal sets. The first was the Buechler's discovery that such sets are locally modular [Bu1]. This introduced geometry into the subject. The second was Hrushovski's result, connecting with a modular regular type a connected modular regular group $G$ such that the forking dependence on the generic type of $G$ may be regarded just as a vector space dependence [H]. This introduced more algebra and stable groups into the subject. We do a similar thing in this paper, explaining the structure of $U$-rank 2 types. Now we are at the geometrical stage. Geometrical means are quite restricted, using them we shall be able to prove for example that $F_q$ being finite implies that $F_p$ is
finite. Then in §§2 and 3 we shall translate the ACL'-dependence on \( P^* \) into a linear dependence in some vector space over \( F_p \). This will solve the problem of counting \( I(p|E, \mathfrak{N}_0) \) for some finite set of parameters \( E \). Note however that in the Hrushovski result in \([H]\) mentioned above we also need some parameters.

Later we shall be extending \( T \) by adding new constants. Now we shall discuss this procedure. By Corollary 1.8 we may restrict ourselves to the countable models with \( \dim(q(M)) \) infinite. Hence we can quite harmlessly add any finite independent subset \( A \) of \( Q \) to the signature. Then we replace \( p \) and \( q \) by \( p|A, q|A \), and for \( a \in Q \setminus \acl(A) \), \( p_a \) by \( p_a|Aa \) (by Assumptions 0.1, \( p_a \) is weakly orthogonal to \( tp(A/a) \)). Hence the new \( P^0 \) and \( P \) somewhat change, but \( P^* \) remains essentially the same, and if we classify the countably dimensional ACL'-closed subsets of \( P^* \), then this will give also a classification of the sets \( p(M) \) for the old \( p \). It is more convenient to work with ACL' rather than with ACL. This is because it may happen that for \( A, B \subseteq Q \) with \( A \downarrow B \), \( P^*_A \nsubseteq P^*_B \), while by Corollary 1.4 we have then \( P^*_A \nsubseteq P^*_B(p^0) \). That is, it is really ACL', and not ACL, which has "local character".

In Theorem 2.2 we shall give a reduction of the problem of counting ACL'-closed subsets of \( P^* \) to a problem from linear algebra. Now if there are \( 2^{\aleph_0} \) isomorphism types of countably dimensional ACL'-closed subsets of \( P^* \), then this gives clearly also \( I(T, \mathfrak{N}_0) = 2^{\aleph_0} \).

Suppose there are countably many nonisomorphic countably dimensional ACL'-closed subsets of \( P^* \). We would like to count countably dimensional ACL'-closed subsets of \( P^* \). Here we do not have a general method, and must deal separately with each particular case. In §4 we shall see examples of such a reasoning. From now on we fix the following assumption.

1.11. Assumption. \( Q \) is modular.

The local character of ACL enables us to define the following concept of a basis \( \pi(r) \) for \( r \in P^* \).

1.12. Definition. Let \( r \in P^* \). Define \( \pi(r) \) as the minimal set \( A \) such that \( A = \acl(A) \cap Q \) and \( r \in \ACL(P^*_A) \). Let \( n(r) = \dim(\pi(r)) \). For \( R \subseteq P^* \) let \( \pi(R) = \acl(\bigcup\{\pi(r) : r \in R\}) \cap Q \). For \( r \in P \) let \( n'(r) \) be the minimal \( n \) such that for some \( A \subseteq Q \) of size \( n \), \( r \in \ACL(P^*_A) \).

Notice that by Corollary 1.4 and Assumption 1.11, the definition of \( \pi(r) \) is correct, because if \( r \in \ACL(P^*_A) \cap \ACL(P^*_B) \) and \( C = \acl(A) \cap \acl(B) \cap Q \) then \( A \downarrow B(C) \) and \( r \in \ACL(P^*_C) \). In the next lemma we define important coefficients \( n_a \) and \( n_b \).

1.13. Lemma. (1) \( n_a = \max\{n(r) : r \in P^* \} \) is finite. Hence also \( n_b = \max\{n(r) : r \in P \} \) is finite.

(2) \( n_c = \max\{n'(r) : r \in P \} \) is finite.

Proof. We shall prove only (1) as (2) is similar. First we prove the following triangle inequality.

\[
(\Delta) \quad n'(r') + n'(r'') \leq n'(r''') \leq n(r'') + n(r').
\]

The right-hand side inequality follows by the definition of \( n(r) \). Then the left-hand side inequality follows by the exchange principle.
Now suppose that there are \( r \) with \( n(r) \) arbitrarily large. Then we can choose \( \{ A_i : i < \omega \} \), an independent family of subsets of \( Q \), and \( r_i \in P^*_A \) such that \( n(r_i) = \dim(A_i) \) and if \( k_i = n(r_i) \), then for every \( i > 0 \),
\[
(* ) \quad k_0 + \cdots + k_{i-1} < k_i.
\]
Let \( R = \{ r_i : i < \omega \} \), \( R_{<i} = \{ r_j : j < i \} \), \( R_{>i} = \{ r_j : j > i \} \) and \( R_{\neq i} = \{ r_j : j \neq i \} \).

1.14. **Claim.** (1) If \( r \in \text{ACL}(R_{<i}) \) then \( n(r) \leq k_0 + \cdots + k_{i-1} \).
(2) If \( r \in \text{ACL}(R_{>i}) \) then \( n(r) > k_0 + \cdots + k_i \).
(3) If \( r \in \text{ACL}(R_{\neq i}) \) then \( n(r) \neq k_i \).

**Proof.** (1) follows immediately by \((\Delta)\). In (2), choose a minimal \( R' \subseteq R_{>i} \) such that \( r \in \text{ACL}(R') \). Choose \( r' \in R' \) with maximal \( n(r') \). Then by the exchange principle we get \( n(r') \leq n(r) + \sum \{ n(r'') : r'' \in R' \setminus \{ r' \} \} \). Hence by (*) we are done.

(3) Let \( r \in \text{ACL}(R_{\neq i}) \). Hence there are \( r' \in \text{ACL}(R_{<i}) \) and \( r'' \in \text{ACL}(R_{>i}) \) with \( r \in \text{ACL}(r', r'') \). If \( n(r) = k_i \) then by (1) and \((\Delta)\) we get \( n(r'') \leq k_1 + \cdots + k_i \), contradicting (2) and (*).

For \( X \subseteq \omega \) let \( R_X = \{ r_i : i \in X \} \). By the claim we see that \( i \in X \) iff for some \( r \in \text{ACL}(R_X) \), \( n(r) = k_i \). Hence for distinct \( X \) and \( X' \), \( \text{ACL}(R_X) \) and \( \text{ACL}(R_{X'}) \) are nonisomorphic, contradicting \( I(T, \aleph_0) < 2^{\aleph_0} \).

In the course of proving (P) or Vaught’s conjecture we usually proceed through a series of dichotomies. Each such dichotomy consists in discerning a property \( A \) which implies that \( T \) has \( 2^{\aleph_0} \) countable models (“a many-model argument”). Then theories fall into two categories, depending on whether they have property \( A \) or not. And as theories satisfying \( A \) satisfy also Vaught’s conjecture, we can deal further on only with theories which do not have property \( A \). In our case many model arguments will usually consist in encoding sets of natural numbers into models of \( T \), and a number will be encoded as a dimension of some object within \( M \). These objects will often be subsets of \( P^* \). The many-model arguments in this paper are closely related to ACL. Lemma 1.13 is an example of a many-model argument. Usually such an argument relies on the local character of ACL (Corollary 1.4). As an application of Lemma 1.13 we establish a firm relationship between \( P_A^0 \) and \( P_A \).

1.15. **Proposition.** Assume \( A \subseteq Q \) and \( \dim(A) \geq n_c \). Then \( P_A \subseteq \text{ACL}(P_A^0) \).

**Proof.** W.l.o.g. \( A = \text{acl}(A) \cap Q \). Let \( r \in P_A \). Choose \( B = \text{acl}(B) \cap Q \) with \( \dim(B) \leq n_c \) and \( r' \in \text{ACL}_B(P_A^0) \) such that \( r \in \text{ACL}(r') \). So \( r' \) is based on \( B \). Let \( C = A \cap B \). We have \( A \downarrow B(C) \), hence by Corollary 1.4 there is \( r'' \in P_C \text{-interdependent with both } r \text{ and } r' \). Within \( A \) we can find a copy \( B^* \) of \( B \) over \( C \). Let \( r^* \) be a conjugate of \( r' \) over \( B^* \). Hence \( r^* \in \text{ACL}(r'') \), \( r^* \in \text{ACL}_A(P_A^0) \). Consequently, \( r \in \text{ACL}(r^*) \) and \( r \in \text{ACL}(P_A^0) \subseteq \text{ACL}(P_A^0) \).

We will use the following theorem, which is a local version of NOTOP, proved in [Bu4, §2] (see also [Ne3, 3.1] for a weaker version and [Bu5, 3.3] for a stronger version in the unidimensional case).

1.16. **Theorem.** Assume \( A, B \subseteq Q \). Then \( P^*_{AB} \subseteq \text{ACL}(P^*_A \cup P^*_B) \) and \( P_{AB} \subseteq \text{ACL}(P_A \cup P_B) \).

1.17. **Corollary.** Let \( A \subseteq B \subseteq Q \) and \( \{ b_0, \ldots, b_n \} \) be a basis of \( B \) over \( A \). Then \( P^*_B \subseteq \text{ACL}(P^*_A \cup \bigcup_{i \leq n} P^*_{b_i}) \) and \( P_B \subseteq \text{ACL}(P_A \cup \bigcup_{i \leq n} P_{b_i}) \).
Proof. Apply Theorem 1.16 consecutively.

Now we define some more coefficients.

1.18. Definition. Let \( n_d = \text{DIM}(P^*_d/P^*_a) \) and \( n_e = \text{DIM}(P_d/P^*_a) \) for any \( a \in Q \). By Lemma 1.2, \( 1 \leq n_e \leq n_d \), and by Lemma 1.7, \( n_e, n_d \) are finite. Notice also that since \( P_d \downarrow P^*_a(P_a) \), \( n_e = \text{DIM}(P_d/P^*_a) \). Hence one might try to prove (P) by induction with respect to \( n_e \) or \( n_d \). Any conceivable classification of ACL-closed subsets of \( P^* \) should consist in a decomposition of an ACL-closed subset of \( P^* \) into small pieces. This leads to the notion of a free decomposition introduced below.

1.19. Definition. Assume \( R \subseteq P^* \) is ACL'-closed. We say that \( R' \subseteq P^* \) is a strong subset of \( R \) (\( R' < R \)) if \( R' = \text{ACL}'(R \cap P^*_a(R')) \). We say that \( \{R_i, i \in I\} \) is a free decomposition of \( R \) if

1. (\( R_i \) is ACL'-closed, \( R_i < R \) and \( \text{DIM}(R_i/P^*_a) > 0 \)),
2. \( \{\pi(R_i), i \in I\} \) is independent and
3. \( R = \text{ACL}'(\bigcup_i R_i) \).

We say that \( R \) is decomposable if there is a free decomposition \( \{R_i, i \in I\} \) of \( R \) with \( |I| \geq 2 \). Otherwise we say that \( R \) is indecomposable.

1.20. Remark. (1) Assume \( R \subseteq P^* \) is ACL'-closed and \( \{R_i, i \in I\} \) is a free decomposition of \( R \). Then \( \dim(\pi(R)) = \sum_{i \in I} \dim(\pi(R_i)) \), \( \text{DIM}(R/P^*_a) = \sum_{i \in I} \text{DIM}(R_i/P^*_a) \) and \( \text{DIM}(P^*_a(R)/R) = \sum_{i \in I} \text{DIM}(P^*_a(R_i)/R_i) \).

(2) Assume \( A = \{a_i, i \in I\} \subseteq Q \) is independent. Then \( \{\text{ACL}'(P^*_a), i \in I\} \) is a free decomposition of \( \text{ACL}'(P^*_a) \).

Proof. Follows by Corollary 1.4 and Theorem 1.16.

1.21. Proposition. (1) If \( n_d = 1 \) then every ACL'-closed \( R \subseteq P^* \) is of the form \( \text{ACL}(P^*_a) \), where \( A = \pi(R) \).

(2) If \( n_e = 1 \) then for every \( R \subseteq P \), \( \text{ACL}'(R) \) is of the form \( \text{ACL}'(P_A) \), where \( A = \pi(R) \).

Proof. Trivial.

Let \( \{r_0, \ldots, r_n\} \) be a basis of \( P^*_a \), let \( c_i \) realize \( r_i \), and let \( C = \{c_0, \ldots, c_n\} \). If we add \( C \) to the signature, then every ACL-closed \( R \subseteq P^* \) becomes ACL'-closed, because all types from the old \( P^*_a \) become modular now, hence the new \( P^*_a = \emptyset \). In view of Corollary 1.6 and Proposition 1.21 we get

1.22. Corollary. If \( n_d = 1 \) or \( n_e = 1 \) and \( p' = p|C \) then \( I(p', \aleph_0) = \aleph_0 \).

The main drawback of this corollary is that \( C \) is not embeddable in every model of \( T \). Hence we cannot conclude immediately that \( I(p, \aleph_0) = \aleph_0 \) in this case. We shall deal with this problem in §4. Using the current geometrical set-up, we have a rather limited understanding of the structure of an ACL-closed subset of \( P^* \). Thus we are able to produce a many-model argument only in extreme cases, for example when \( F_p \) and \( F_q \) differ very much. The next theorem will not be used anywhere later, and also in §3 we shall prove it by other means. We include it here as an example of what we can prove using the current geometrical set-up.

1.23. Theorem. If \( n_d > 1 \) and \( F_q \) is finite then \( F_p \) is finite. Supposing \( F_q \) is finite and \( F_p \) is infinite, we will construct \( 2^{\aleph_0} \) nonisomorphic ACL-closed subsets of \( P^* \) with countable dimension.
1.24. **Definition.** We say that \( R \subseteq P^* \) is almost full if \( R \) is ACL'-closed and for \( A = \pi(R) \), \( \dim(P^*_A / R) = 1 \) and there is no \( a \in A \) with \( P^*_a \subseteq R \).

1.25. **Lemma.** Assume there are almost full \( R_i, 0 < i < \omega \), with \( \{\pi(R_i)\}, 0 < i < \omega \) independent and \( \dim(\pi(R_i)) = i \). Then \( I(T, \aleph_0) = 2^{\aleph_0} \).

**Proof.** Let \( A_i = \pi(R_i) \). For \( X \subseteq \omega \setminus \{0\} \) let \( R_X = \text{ACL}(\bigcup_{i \in X} R_i), A_X = \text{acl}(\bigcup_{i \in X} A_i) \cap Q \) (hence \( A_X = \pi(R_X) \)), \( X_{<i} = \{j \in X: j > i \} \), \( X_{\neq i} = \{j \in X: j \neq i \} \). We prove that for \( X \neq Y \), \( R_X \) and \( R_Y \) are nonisomorphic. Fix an \( X \subseteq \omega \). By Corollary 1.4, \( \{P^*_i, i < \omega \} \) is independent (over \( P^*_\omega \)). Hence for \( i \in X \), \( R_i \approx R_X \). We prove

\[ i \notin X \text{ iff for every almost full } R < R_X \text{ maximal in } R_X, \]

\[ \text{(a) if } \dim(\pi(R)) = i \text{ then there are almost full } R^1, \ldots, R^n < R_X \]

\[ \text{ (for some } n) \text{ with } \dim(\pi(R^i)) > i \text{ and } R \subseteq \text{ACL}(R^1 \cup \cdots \cup R^n). \]

\( \rightarrow \). Suppose \( R < R_X \) is almost full and maximal in \( R_X \) with respect to this condition, and \( \dim(\pi(R)) = i \). Let \( A = \pi(R) \) and suppose \( i \notin X \). Hence \( A \subseteq A_X \). By induction on \( j < i \) we prove that \( A \subseteq A_{X_{<j}} \).

\[ \text{Suppose } A \subseteq A_{X_{<j}}. \text{ If } j \notin X, \text{ we are done. Otherwise, by (a), } A \perp A_j(A_{X_{<j}}) \text{(because dim}(A_j) < \text{dim}(A)), \text{ hence } A \subseteq A_{X_{<j}}. \text{ So } A \subseteq A_{X_{<j}} = A_{X_{<j}}. \text{ Hence } i \notin X. \text{ It follows that } R \subseteq R_{X_{<i}}. \]

\( \leftarrow \). Suppose \( i \in X \). We have proved above that for any almost full \( R' < R_X \) with \( \dim(\pi(R')) > i \), \( \pi(R') \subseteq A_{X_{<j}} \). It follows that \( R_i \) is maximal almost full in \( R_X \), and there are no almost full \( R^1, \ldots, R^n < R_X \) with \( \dim(\pi(R^i)) > i \).

**Proof of Theorem 1.23.** By Lemma 1.25 it suffices to find for every \( i < \omega \) an almost full \( R \subseteq P^* \) with \( i = \dim(\pi(R)) \). We find such an \( R \) by induction on \( i \). It is easy to find an almost full \( R \) with \( \dim(\pi(R)) = 1 \). So suppose \( R \) is almost full, \( A = \pi(R) \), \( \dim(A) = i > 0 \) and we must find an almost full \( R' \) with \( \dim(\pi(R')) = i + 1 \). Let \( a \in Q \setminus A \). Choose an ACL'-closed \( R_0 \subseteq \text{ACL}(P^*_a) \) with \( \dim(P^*_a / R_0) = 1 \). Let \( A' = \text{acl}(Aa) \cap Q \). We see that \( \dim(P^*_a / R_0) = 2 \). Since \( F_p \) is infinite, there are \( r_n \in P^*_A \setminus \text{ACL}(RR_0), n < \omega \), such that

\[ (c) r_n \notin \text{ACL}(RR_0r_m) \text{ for } n \neq m < \omega. \]

On the other hand, as \( F_q \) is finite, there are finitely many \( a_0, \ldots, a_k \in A' \) such that for every \( b \in A' \), \( b \in \text{acl}(a_i) \) for some \( i \leq k \). Of course for every \( i \), \( P^*_a \not\subseteq \text{ACL}(R_0) \). By (c) for every \( i \leq k \), there is at most one \( n \) such that \( P^*_a \subseteq \text{ACL}(RR_0r_n) \). Hence for some \( n' \), there is no \( i \leq k \) with \( P^*_a \subseteq \text{ACL}(RR_0r_{n'}). \) Let \( R' = \text{ACL}(RR_0r_{n'}). \) We see that \( R' \) is almost full and \( \pi(R') = A', \dim(A') = i + 1 \).
2. The main result

We shall translate ACL'-dependence on $P^*$ into a linear dependence in some vector space. This is possible if we fix the following assumptions. We can fix them by [H] (also see the discussion in §1).

2.1. Assumptions. $G$ is a type definable over $\emptyset$ weakly minimal connected group in $\mathcal{C}$, $q$ is the generic type of $G$, $Q = q(\emptyset)$ is modular, $F_q$ is the division ring of definable almost over $\emptyset$ pseudo-endomorphisms of $G$.

Let $\mathcal{A} = acl(\emptyset) \cap G$. By [H], $G/\mathcal{A}$ is a vector space over $F_q$ and acl-dependence on $G$ translates just into $F_q$-linear dependence on $G/\mathcal{A}$. We fix the following notation. For $a \in G$ let $a_G$ be $a + \mathcal{A}$, an element of $G/\mathcal{A}$. For $A \subseteq G$, $A_G$ denotes $\{a_G : a \in A\}$. Let $G_G$ be the set $G/\mathcal{A}$ with the structure of right vector space over $F_q$. Suppose $K$ is a division ring, $V$ is a right vector space over $K$ and $L \subseteq M_{n \times n}(K)$ is a division subring of $M_{n \times n}(K)$. Then we can regard $V^n$ as a right vector space over $L$: $L$ acts on $V^n$ by matrix multiplication on the right. For $\nu = (v_1, \ldots, v_n) \in V^n$ let $\pi'(\nu) = K$-span$(v_1, \ldots, v_n) = v_1K + \cdots + v_nK \subseteq V$. For $W \subseteq V^n$, $\pi'(W)$ is the $K$-subspace of $V$ generated by $\pi'(\nu)$, $\nu \in W$.

Assume $W, W'$ are $L$-subspaces of $V^n$. We say that $W, W'$ are isomorphic in $V$ (or just : isomorphic), if there is a $L$-linear isomorphism $f: \pi'(W) \to \pi'(W')$ such that $f$ induces an $L$-linear isomorphism of $W$ and $W'$. Now let $\Gamma$ be a group of automorphisms of the division ring $K$ such that every $\gamma \in \Gamma$ preserves $L$, that is $\gamma[L] = L$. We say that $W, W'$ are $\Gamma$-isomorphic in $V$ if for some $\gamma \in \Gamma$ there is a bijection $f$ of $\pi'(W)$ and $\pi'(W')$ (called a $\gamma$-isomorphism), which is a group isomorphism such that for every $a \in \pi'(W)$ and $\alpha \in K$, $f(\alpha a) = f(a)\gamma(\alpha)$, and $f[W] = W'$. Notice that if $W, W'$ are isomorphic then $W, W'$ are $\Gamma$-isomorphic (with $\gamma = id$).

Let $\Gamma_q$ be the group of automorphisms of $F_q$ induced by $Aut(\mathcal{C})$. We will use $0$ to denote a matrix or tuple of suitable size consisting of zeros only, $I$ will denote the identity matrix of a suitable size. Now we shall formulate the main results of the paper. They will be proved in this and the next section.

2.2. Theorem. There is an embedding $i: F_p \to i[F_p] = F \subseteq M_{n_q \times n_q}(F_q)$ which is a ring monomorphism. Moreover, after adding a finite subset of $acl(\emptyset)$ to the signature, there is a correspondence (that is a binary relation) $\Phi$ between $P^*$ and $G^{n_q}$ such that the following hold.

1. $\Phi$ and $F$ are invariant under automorphisms of $\mathcal{C}$.
2. Dom($\Phi$) = $P^*$ and $G^{n_q \setminus acl(\emptyset)} \subseteq Rng(\Phi) \subseteq G^{n_q}$.
3. For $r^1, \ldots, r^n \in P^*$ and $a^1, \ldots, a^n \in G^{n_q}$ with $\Phi(r^i, a^i)$, $r^1, \ldots, r^n$ are ACL'-independent iff $a^1_G, \ldots, a^n_G$ are linearly independent over $F$.
4. For $r \in P_q^*$, $\Phi(r, (a, 0))$ holds.
5. If $r$ is a stationarization of $p_a$, $a \in Q$, then $\Phi(r, (a, 0))$ holds.

2.3. Corollary. Let $i, \Phi$ be as in Theorem 2.2. $\Phi$ induces a bijection $\Psi$ between ACL'-closed subsets of $P^*$ and $F$-subspaces of $G^{n_q}_G$. Assume $R, R' \subseteq P^*$ are ACL'-closed. Then $R, R'$ are isomorphic iff $\Psi(R), \Psi(R')$ are $\Gamma_q$-isomorphic in $G_G$.
Proof. For ACL'-closed $R \subseteq P^*$ let $\Psi(R)$ be the set $\{a_{\text{of}} \in G_{\text{of}}^{n_q} : a_{\text{of}} = 0$ or for some $r \in R$, $\Phi(r, a)$ holds$. By Theorem 2.2, $\Psi(R)$ is an $F$-subspace of $G_{\text{of}}^{n_q}$. By Theorem 2.2(1), (3), for $r \in P^*$ and $a \in G_{\text{of}}^{n_q}$ with $\Phi(r, a)$, $a$ and $\pi(r)$ are interalgebraic, hence if $A(R) = \pi(R) \cup \{0\}$, then $\pi'(\Psi(R)) = A(R)_{\text{of}}$. It is clear that $\Psi$ is a bijection between ACL'-closed subsets of $P^*$ and $F$-subspaces of $G_{\text{of}}^{n_q}$. Also by Theorem 2.2, $\gamma[F] = F$ for every $\gamma \in \Gamma_q$. Now suppose $R, R'$ are ACL'-closed and isomorphic subsets of $P^*$. That is, for some $f \in \text{Aut}(\mathcal{C})$, $f[R] = R'$. By Theorem 2.2(1), $f$ induces a $\gamma$-isomorphism of $\Psi(R)$ and $\Psi(R')$ (with $\gamma \in \Gamma_q$ induced by $f$).

Conversely, suppose $W = \Psi(R), W' = \Psi(R') \subseteq G_{\text{of}}^{n_q}$ are $\gamma$-isomorphic for some $\gamma \in \Gamma_q$. Hence there is a $\gamma$-isomorphism $f_0$ of $\pi'(W')$ and $\pi'(W')$ with $f_0[W] = W'$. By compactness it is easy to see that there is an $f \in \text{Aut}(\mathcal{C})$ with $f[F_q] = \gamma$ such that $f_0(a_{\text{of}}) = f(a)_{\text{of}}$ for $a \in \pi(R)$. By Theorem 2.2, $f[R] = R'$.

If we want to consider only $P$ (which is sufficient to determine the value of $I(p, n_0)$), we get the following versions of Theorem 2.2 and Corollary 2.3.

2.2'. Theorem. There is an embedding $i': F_p \to i'[F_p] = F' \subseteq M_{n_{\text{of}} \times n_0}[F_q]$ which is a ring monomorphism. Moreover, after adding a finite subset of $\text{acl}(\varnothing)$ to the signature, there is a correspondence $\Phi'$ between $P$ and $G_{\text{of}}^{n_h}$ such that conditions (1)-(5) from Theorem 2.2 hold with $\Phi$ replaced by $\Phi'$, $P^*$ by $P$ and $n_a$ by $n_h$.

2.3'. Corollary. Let $i', \Phi'$ be as in Theorem 2.2'. $\Phi'$ induces a bijection $\Psi'$ between ACL'-closed subsets of ACL'(P) and $F'$-subspaces of $G_{\text{of}}^{n_h}$. If $R, R' \subseteq \text{ACL}'(P)$ are ACL'-closed then $R, R'$ are isomorphic iff $\Psi'(R)$ and $\Psi'(R')$ are $\Gamma_q$-isomorphic in $G_{\text{of}}$.

This and the next section is devoted to the proof of Theorem 2.2. After the proof of this theorem we shall indicated how to prove Theorem 2.2'. Then we shall discuss some further implications of Theorems 2.2 and 2.2'. The key to the proof of Theorem 2.2 is an analysis of ACL' on $P^*$. Notice that if $X \subseteq P^*$ is essentially ACL'-closed then ACL' is modular on $X$ and $X$ with ACL'-dependence is projective over $F_p$. It would be nice if just $P^0$ were essentially ACL'-closed, then we would not have to bother about "imaginary" types in $P^*$. Our current goal is to choose a nice essentially ACL'-closed $X \subseteq P^*$ with ACL'(X) = $P^*$. The point is that for every $r \in P^*$, out of the many types $r' \in P^*$ ACL'-interdependent with $r$, we need to include into $X$ only one of a specific "sort". This will make the ACL'-dependence on $X$ easier to understand. We shall eventually find finitely many types $r_1, \ldots, r_k \in P^*$ (for some $k$) such that the set $X$ of copies of $r_1, \ldots, r_k$ will be essentially ACL'-closed and ACL'(X) = $P^*$.

We cannot expect that for every $r \in P^*$ with $A = \pi(r)$, and $b$ realizing $r$, $\text{tp}(b/A)$ is stationary. To overcome this difficulty in the process of proving Theorem 2.2 we will add to the signature a finite subset of $\text{acl}(\varnothing)$. We will use the following trivial fact.

2.4. Fact. Assume $\text{tp}(ab)$ is stationary. Then for every $A \perp b$, $\text{tp}(a/b)$ has a unique nonforking extension over $Ab$.

The next definition formalizes our idea of "sorts of types", mentioned above.
Recall that \( r' \) is a copy of \( r \) iff \( r' = f(r) \) for some \( f \in \text{Aut}(\mathfrak{C}) \). In Definition 2.5 we define an equivalence relation \( \simeq \) on \( P^* \).

2.5. **Definition.** (1) Let \( r, r' \in P^* \). Then \( r \simeq r' \) iff \( r \) is ACL'-interdependent with a copy of \( r' \).

(2) For \( n \leq n_a \) let \( S_n = \{ r | \simeq : r \in P^* \) and \( n(r) = n \} \) and let \( S = \bigcup_{0 < n \leq n_a} S_n \). We call the \( \simeq \)-class of \( r \) the sort of \( r \).

As we mentioned above, we want to find a nice essentially ACL'-closed \( X \subseteq P^* \) with \( ACL'(X) = P^* \). In Definition 2.5 we define \( \simeq \) so that if \( r \simeq r' \) then in order to have an \( r'' \in X \) with \( r \in ACL'(r'') \) we may include into \( X \) a copy of \( r' \). In fact in the proof of Theorem 2.2 we could do well without sorts from \( S_n, n < n_a \). But I think they are interesting in themselves, and have properties parallel to the sorts from \( S_{n_a} \). In the next lemma we collect basic properties of \( \simeq \).

2.6. **Lemma.** (1) Suppose \( a, b, E \subseteq Q \), \( a \perp E \), \( a \perp E \), \( a, b \) are interalgebraic over \( E \) and \( r \in P_b^* \). Then for some copy \( r' \in P_b^* \) of \( r \), \( r \) and \( r' \) are ACL-interdependent over \( P_b^* \).

(2) (an alternative definition of \( \simeq \)). For \( r, r' \in P^* \), \( r \simeq r' \) iff \( n(r) = n(r') \) and for some \( E \subseteq Q \) independent from \( \pi(r) \), for some copy \( r'' \) of \( r' \), \( r \) and \( r'' \) are ACL-interdependent over \( P_b^* \).

(3) \( S_0 = ACL(P_a^*)/\simeq \) and all types in \( ACL(P_a^*) \) are \( \simeq \)-equivalent.

(4) \( S_1, \ldots, S_{n_a} \) are nonempty and disjoint.

(5) Assume \( A \subseteq Q \) is independent, \( |A| \geq n \). Then \( S_n = \{ r | \simeq : r \in P_A^* \) and \( n(r) = n \} \).

**Proof.** (1) W.l.o.g. both \( a \) and \( b \) are independent, \( n = n(r) = |b| > 0 \). Choose an independent \( n \)-tuple \( d \subseteq Q \) with \( d \perp abE \), and let \( c = d + b \). By Theorem 1.16, \( P_b^* \subseteq ACL(P_c^* P_d^*) \), hence by modularity of ACL, there are \( r_c \in P_c^* \), \( r_d \in P_d^* \) with \( r \in ACL(r_c, r_d) \), and \( r, r_c, r_d \) are pairwise ACL-independent. Since \( b \subseteq acl(aE) \), by [H] there are \( a' \subseteq acl(a) \cap G, c \subseteq acl(E) \cap G \) with \( b = a' + c \). Since \( a, b \) are interalgebraic over \( E \), \( a', c \subseteq Q \) and \( a, a' \) are interalgebraic over \( E \), hence \( a' \) is independent. Choose \( f \in \text{Aut}(\mathfrak{C}) \) fixing \( d \) and \( r_d \), and sending \( b \) to \( a' \). Let \( r' = f(r), r' = f(c) \) and \( r'_c = f(r_c) \). Hence \( r'_c = d + a', r_d \in ACL(r'_c, r_c) \). We have \( c \perp abE \) and \( c - c' = b - a' = c \subseteq acl(E) \), hence \( c_c' \perp ab(E) \). We have \( r \in ACL(r_c r_d) \subseteq ACL(r_c r'_c) \). \( c_c' \perp ab(E) \) implies \( P_{ab}^* \perp P_{c c'}^*(P_E^* \), hence since \( r' \in P_d^* \), we get \( r \) and \( r' \) are ACL-interdependent over \( P_E^* \).

(2) \( \rightarrow \) is clear. \( \leftarrow \). Let \( A = \pi(r), A'' = \pi(r'') \). Since \( r, r'' \) are ACL-interdependent over \( P_E^* \), \( A \) and \( A'' \) are interalgebraic over \( E \), hence \( \dim(A/E) = \dim(A''/E) \). Also, \( \dim(A) = \dim(A'') = n(r) = n(r') \), hence \( A \perp E \) gives \( A'' \perp E \). By (1) there is a copy \( r^* \in P_A^* \) of \( r'' \) such that \( r^*, r'' \) are ACL-interdependent over \( P_E^* \), hence also \( r \) and \( r^* \) are ACL-interdependent over \( P_E^* \). Now \( E \perp A \) implies \( r, r^* \) are ACL'-interdependent.

(3), (4), (5) are trivial.

We shall use the next lemma in a many-model argument to show that \( S \) is finite.

2.7. **Lemma.** Assume \( 0 < n \leq n_a \), \( R = \{ r_i | i \in I \} \subseteq P^* \), \( n(r_i) = n \), \( \{ \pi(r_i), i \in I \} \) is independent, \( r \in P^* \) is ACL'-interdependent with \( r_i \) over \( R' = R \setminus \{ r_i \} \) for some \( t \in I \), and \( n(r) \leq n \). Then \( n(r) = n \) and \( r \simeq r_i \).
Proof. Let \( A_i = \pi(r_i), \ B = \pi(r), \ A' = \bigcup\{A_i: i \in I\setminus\{t\}\} \). We have \( B \) and \( A_i \) are interalgebraic over \( A' \) and \( A_i \downarrow A' \), hence also \( B \downarrow A' \) and \( n(r) = n(r_i) = n \). By Lemma 2.6(2) with \( E := A' \), \( r \simeq r_i \), so we are done.

2.8. Theorem. \( S \) is finite. Moreover, \( |S_{n_a}| = 1 \).

Proof. Let \( n > 0 \). First we prove that \( S_n \) is finite. Suppose not. Choose \( R = \{r_i, i < \omega\} \subseteq P^* \) with \( n(r_i) = n \) and \( \{r_i, i < \omega\} \) pairwise \( \simeq \)-nonequivalent. We can assume \( \{\pi(r_i), i < \omega\} \) is independent. For \( X \subseteq \omega \) we define \( R_X \) as \( \{r_i, i \in X\} \). By Lemma 2.7, if \( r \in ACL'(R_X) \setminus ACL'(0) \) and \( n(r) \leq n \), then for some \( i \in X \), \( r \simeq r_i \). It follows that for \( X \neq Y \subseteq \omega \), \( ACL'(R_X) \) and \( ACL'(R_Y) \) are nonisomorphic. This shows \( I(T, \aleph_0) = 2^{\aleph_0} \), a contradiction.

Now suppose \( r, r' \in P^* \), \( n(r) = n(r') = n_a \). W.l.o.g. \( \pi(r), \pi(r') \) are independent. Choose \( r'' \in ACL'(r'^r) \) such that \( r, r', r'' \) are pairwise \( ACL' \)-independent. By Lemma 2.7 we see that \( r' \simeq r \) and \( r'' \simeq r' \). This shows \( r \simeq r' \), hence \( |S_{n_a}| = 1 \).

Let \( \sigma \) be the only sort in \( S_{n_a} \). Now we are going to choose for any independent \( B \subseteq Q \) with size \( n \leq n_w \) and for \( s \in S_n \), a representative of \( s \) in \( P^*_B \), uniformly in \( B \). That is, for \( B' \subseteq Q \) with \( B'' \equiv B \), the representatives of \( s \) in \( P^*_{B'} \) should be independent. But the notion of conjugate is somewhat ambiguous. If \( r \in S(B) \) and \( f: B \rightarrow B' \) is any bijection, then \( f \) is elementary, and we might consider \( f(r) \) a conjugate of \( r \) over \( B' \). The picture is clarified if we fix enumerations of \( B \) and \( B' \) and insist that \( f \) preserves these enumerations. Then \( f \) is unique, hence also \( f(r) \) is unique. This is the reason for the notation we introduce now.

For \( n \leq n_a \) let \( Q(n) \) be the set of independent \( n \)-tuples of elements of \( Q \), and we stipulate \( Q = Q^{(1)} \). If \( \bar{a}, \bar{a}' \in Q(n) \), \( r \) is a strong type over \( \bar{a}, r' \) is a strong type over \( \bar{a}' \), then we say that \( r \) and \( r' \) are conjugate if \( b, b' \) realizing \( r, r' \) respectively, we have \( \bar{a}b \equiv \bar{a}'b' \). Let \( 0 < n \leq n_a \) and \( s \in S_n \). For every \( \bar{a} \in Q(n) \) we choose \( s_{\bar{a}} \in P^*_{\bar{a}} \) so that \( s_{\bar{a}} \simeq s \) and for \( \bar{a}, \bar{a}' \in Q(n) \), \( s_{\bar{a}} \) and \( s_{\bar{a}'} \) are conjugate. Moreover, let \( S \subseteq S \) be the \( \simeq \)-class of some \( \sigma \in P^0_a \), \( a \in Q \). We choose \( s^*_a \) as a stationarization of \( P^a_\sigma \). Let \( b \) realize \( s_\sigma \). Hence \( \text{Mlt}(b/a) \) is finite and \( \text{Mlt}(a) = 1 \), and so \( \text{Mlt}(ab) \) is finite. It follows that for some \( E_s \in \text{FE}(\sigma) \), if \( c = ab/E \) then \( c \in acl(\sigma) \) and \( \text{tp}(ab/c) \) is stationary. We add names for the elements of the finite set \( C \) of all \( E_s \)-classes, \( s \in S \), to the signature. The next lemma shows that this does not really affect \( \simeq \).

2.9. Lemma. If we define, according to Definition 2.5, \( \simeq_C \) and \( S_C \) in the new signature, then for \( 0 < n \leq n_a \) and \( a \in Q(n) \), for every \( s_c \in S^C_n \) there is an \( s \in S_n \) and a conjugate \( r \) of \( s_{\bar{a}} \) over \( \bar{a} \) such that \( r \simeq_C s_c \).

Proof. By Lemma 2.6(5), in \( P^*_{\bar{a}} \) there are representatives of all sorts in \( S^C_n \). Suppose \( r \in P^*_{\bar{a}} \), \( r \simeq_C s_c \). For some \( s \in S_n \), \( r \simeq s_{\bar{a}} \), hence for some copy \( r' \) of \( s_{\bar{a}} \), \( r \) and \( r' \) are \( ACL' \)-interdependent. It follows that \( r \simeq_C r' \). Obviously, for some \( \bar{a}' \in Q(n) \) interalgebraic with \( a \), \( r' \) is a conjugate over \( \bar{a}' \) of \( s_{\bar{a}'} \). Choose \( f \in \text{Aut}_C(\sigma) \) sending \( \bar{a}' \) to \( a \). Let \( r'' = f(r') \). Clearly, \( r \simeq_C r'' \), and \( r'' \) is a conjugate over \( \bar{a} \) of \( s_{\bar{a}} \), as required.

We can assume now that the original \( \simeq \) and \( S \) have been defined after adding
C to the signature, and by Lemma 2.10 we stipulate the following assumption.

2.10. Assumption. If $0 < n \leq n_a$, $s \in S_n$, $a \in Q(n)$ and $b$ realizes $s_a$ then $\text{tp}(ab)$ is stationary.

The next lemma shows that all conjugates over $a$ of $s_a$ are $\text{ACL}'$-interdependent. The original author's proof of this lemma consisted in constructing arbitrarily large almost full sets. Here we give a simplified version of this proof, suggested by S. Buechler. The proof is similar to that of Lemma 2.6(1).

2.11. Lemma. Suppose $0 < n \leq n_a$, $a, a' \in Q(n)$, $s \in S_n$, $r$ is a conjugate over $a$ of $s_a$ and $r'$ is a conjugate over $a'$ of $s_{a'}$. Assume $E \subseteq Q$, $a \perp E$, $a' \perp E$ and $a - a' \subseteq \text{acl}(E)$ (the subtraction is pointwise, in $G$). Then $r$ and $r'$ are $\text{ACL}'$-interdependent over $P_E$.

Proof. Choose $b \in Q(n)$ independent from $aE$, and let $c = a + b$ (the addition is pointwise). Hence $c \in Q(n)$ and $a \perp c(E)$. By Theorem 1.16 and modularity of $\text{ACL}$, there are $r_b \in P_b^*$ and $r_c \in P_c^*$ such that $r \in \text{ACL}(r_br_c)$ and $r, r_b, r_c$ are pairwise $\text{ACL}$-independent. By Fact 2.4 and Assumption 2.10 there is an $f \in \text{Aut}(c)$ fixing $E$ pointwise, with $f(r_b) = r_b$, $f(a) = a'$ and $f(r) = r'$. Let $c' = f(c)$ and $r_{c'} = f(r_c)$. So we have $c' = a + b$, $a - a' \subseteq \text{acl}(E)$ implies $c - c' \subseteq \text{acl}(E)$, hence $a \perp c(E)$ gives $aa' \perp cc'(E)$. We have also $r' \in \text{ACL}(r_br_c)$. Since $r_b \in \text{ACL}(rr_c)$, $r' \in \text{ACL}(rr_c'c')$. Thus $aa' \perp cc'(E)$ gives $rr' \perp r_r'r_c(P_E)$. It follows that $r$ and $r'$ are $\text{ACL}$-interdependent over $P_E$.

In particular, applying Lemma 2.11 to the case $a = a'$, $E = \varnothing$, we see that all conjugates of $s_a$ over $a$ are $\text{ACL}'$-interdependent. The next remark shows that we have achieved our goal of choosing a nice essentially $\text{ACL}'$-closed subset $X$ of $P^*$ with $\text{ACL}'(X) = P^*$.

2.12. Remark. Let $X = \{s_a : s \in S_n, n > 0, a \in Q(n)\}$. Then $X$ is essentially $\text{ACL}$-closed and $\text{ACL}'(X) = P^*$.

Proof. Let $r \in P^*$. It suffices to prove that for some $r' \in X$, $r \in \text{ACL}'(r')$. So we may assume $n(r) > 0$. By the definition of $s_a$, for some $s \in S_a$ and $a \in Q(n)$, $r \simeq s_a$. That is $r$ is $\text{ACL}'$-interdependent with a copy of $s_a$ over some $a' \in Q(n)$. By Lemma 2.11, any copy of $s_a$ over $a'$ is $\text{ACL}'$-interdependent with $s_{a'}$. Hence $r \in \text{ACL}'(s_{a'})$.

Now we shall investigate $\text{ACL}'$-dependence on $X$ from Remark 2.12. The next theorem is a generalization of [Bu4, 3.14]. Theorem 3.14 in [Bu4] deals with a single $s \in S$ (namely with $s^*$, corresponding to $p_a$), while we deal here with all $s \in S$ at once. Also we give a different proof.

2.13. Theorem. Let $s \in S_n, n > 0$. Then there is a division subring $F_s$ of the ring of matrices $M_{n \times n}(F_q)$ such that if $A \subseteq Q(n)$ and $a_{\alpha} \in F_{s^*}\cdot\text{span}(A_{\alpha})$ then $s_a \in \text{ACL}'(\{s_b: b \in A\})$. Moreover, if $A$ is independent, then the converse is true, that is for $a \in Q(n)$, $s_a \in \text{ACL}'(\{s_b: b \in A\})$ iff $a_{\alpha} \in F_{s^*}\cdot\text{span}(A_{\alpha})$.

Proof. It is easy to define $F_s$: let $F_s^* = \{\alpha \in M_{n \times n}(F_q) : a, b \in Q(n)$ with $b_{\alpha} = a_{\alpha}a, s_b \in \text{ACL}'(s_a), \}$, and let $F_s = F_s^* \cup \{0\}$. By this definition $F_s^*$ is a multiplicative subgroup of $M_{n \times n}(F_q)$. So if we prove that $F_s$ is also an additive subgroup of $M_{n \times n}(F_q)$, then the theorem will be proved for $|A| = 1$. 

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The key to the proof will be however the case \( |A| = 2 \). Choose \( a, b \in Q^n \) with \( a \perp b \). Let \( H = \{ (\alpha, \beta) : \alpha, \beta \in M^{*}_{n \times n}(F_q) \cup \{ 0 \}, \ (\alpha, \beta) \neq (0, 0) \} \) and for \( c \in Q^n \) with \( c_{a,b} = a_{a,b} + b_{a,b} \), \( s_c \in \text{ACL}'(s_{a,b}) \cup \{ (0, 0) \} \).

2.14. Claim. (1) \((\alpha, \beta) \in H \rightarrow (\beta, \alpha) \in H \).
(2) \((\alpha, \beta) \in H , \ \gamma \in F_\mathcal{S} \rightarrow (\gamma \alpha, \beta) \in H \).
(3) \gamma \in F_\mathcal{S} \rightarrow (0, \gamma) \in H \).
(4) \((\alpha, \beta) \in H \rightarrow \beta \in F_\mathcal{S} \).
(5) \( H = F_\mathcal{S} \times F_\mathcal{S} \).

Proof. (1)-(3) are obvious. (4). W.l.o.g. \( \beta \neq 0 \). We have \( s_c \in \text{ACL}'(s_{a,b}) \), where \( c_{a,b} = a_{a,b} + b_{a,b} \). Since \( \beta \) is invertible, \( c \perp a \), hence by Lemma 2.11 (with \( E = a \), if \( \beta' \in Q^n \) and \( b_{a,b}' = b_{a,b} \beta \) then \( s_c \) and \( s_{b'} \) are ACL-interdependent over \( P_\mathcal{S}^\ast \). Hence \( s_{b'} \in \text{ACL}(s_{a}, P_\mathcal{S}^\ast) \). But \( a \perp b' \), hence \( s_{b'} \in \text{ACL}(s_{a}, P_\mathcal{S}^\ast) \), giving \( \beta \in F_\mathcal{S} \).

(5) By (1)-(4) it suffices to prove that for some \( \alpha, \beta \in M^{*}_{n \times n}(F_q) \), \((\alpha, \beta) \in H \). Let \( c = a + b \). By Theorem 1.16 there are \( r \in P_\mathcal{S}^\ast \) and \( r' \in P_\mathcal{S}^\ast \) such that \( s_c \in \text{ACL}(r, r') \). By Lemma 2.6(2), \( r \sim r' \sim s_c \), hence there are \( \alpha, \beta \in M^{*}_{n \times n}(F_q) \) such that \( a_{a,b} = a_{a,b} \alpha, \ b_{a,b} = b_{a,b} \beta \) then \( r \in \text{ACL}'(s_{a,b}) \). Thus \( \alpha, \beta \in H \).

In particular, if \( c_{a,b} = a_{a,b} + b_{a,b} \) then \( s_c \in \text{ACL}(s_{a,b}) \). By the exchange principle, \( s_b \in \text{ACL}'(s_{a,b}) \) and \( b_{a,b} = c_{a,b} - a_{a,b} \), hence \( (1, -1) \in H \), and \(-1 \in F_\mathcal{S} \). So to prove that \( F_\mathcal{S} \) is a division subring of \( M^{*}_{n \times n}(F_q) \) it suffices to prove that \( F_\mathcal{S} \) is closed under \( + \). Let \( \alpha, \beta \in F_\mathcal{S} \). \( c_{a,b} = a_{a,b} \alpha + b_{a,b} \) and \( c_{a,b} = a_{a,b} \beta + \delta_{a,b} \). \((\alpha, 1), (\beta, 1) \in H \), hence \( s_{c_{a,b}} \in \text{ACL}'(s_{a,b}s_{b}^\ast) \subseteq \text{ACL}'(s_{a,b}s_{b}^\ast) \). But \( c_{a,b} = a_{a,b} \alpha + b_{a,b} \), i.e. \((\alpha + \beta, 1) \in H \). Hence \( \alpha + \beta \in F_\mathcal{S} \). Now we prove that for \( A \subseteq Q^n \) and \( a \in Q^n \),

(6) if \( a_{a,b} \in F_\mathcal{S} - \text{span}(A_{a,b}) \) then \( s_a \in \text{ACL}'(s_{b} : b \in A) \).

W.l.o.g. \( A \) is finite. First suppose \( A \) is independent. Then (6) follows by an easy induction, by Claim 2.14(5). Now we approach the general case. Let \( A = \{ a^1, \ldots, a^k \} \) for some \( k \), and choose \( c^1, \ldots, c^k \) \( \in Q^n \) such that \( \{ A, c^1, \ldots, c^k \} \) is independent. Let \( C = \{ c^1, \cdots, c^k \} \). Suppose \( a \in Q^n \) and \( a_{a,b} = \sum a_{a,b}^i \alpha^i \), \( \alpha^i \in F_\mathcal{S}^\ast \). Let \( b^i = a^i + c^i \), and choose \( b \in Q^n \) with \( b_{a,b} = \sum b_{a,b}^i \alpha^i \). Notice that \( B = \{ b^1, \ldots, b^k \} \) is independent, hence we get \( s_b \in \text{ACL}'(s_{b}^1 \cdots s_{b}^k) \). Also, \( a^i, b^i, c^i \) are pairwise independent for each \( i \), hence \( s_{b}^i \in \text{ACL}'(s_{a}^i, s_{c}^i) \), and \( s_{a}^i \), \( s_{b}^i \) are ACL-interdependent over \( P_C \). Similarly, \( s_a \) and \( s_b \) are ACL-interdependent over \( P_{C}^\ast \). Hence we get \( s_{a} \in \text{ACL}(s_{a}^1, \ldots, s_{a}^k, P_{C}^\ast) \). Since \( A \perp C \), we get \( s_{a} \in \text{ACL}(s_{a}^1, \ldots, s_{a}^k) \).

To finish the proof, suppose \( A \subseteq Q^n \) is independent, \( a \in Q^n \) and \( s_{a} \in \text{ACL}'(s_{b} : b \in A) \). We want to prove \( a_{a,b} \in F_\mathcal{S} - \text{span}(A_{a,b}) \). We may assume \( A \) is finite, and proceed by induction on \(|A| \). If \(|A| = 1 \) we are done by the definition of \( F_\mathcal{S} \). Suppose \( b \in A \), \( A' = A \setminus \{ b \} \), and \( s_a, s_b \) are ACL-interdependent over \( \{ s_c : c \in A' \} \). It follows that \( a \perp A' \) and \( a, b \) are interalgebraic over \( A' \). Hence there are \( a' \subseteq \text{acl}(b), \ a'' \subseteq \text{acl}(A') \) with \( a = a' + a'' \). \( a \perp A' \) implies \( a' \in Q^n \), hence for some \( \gamma \in M^{*}_{n \times n}(F_q) \), \( a_{a,b} = b_{a,b} \gamma \). By Lemma 2.11, \( s_a \) and \( s_{a,b} \) are ACL-interdependent over \( P_{A}^\ast \), hence \( s_{a,b} \in \text{ACL}(s_a P_{A}^\ast) \), and since
Let \( b \perp A' \), \( s_g' \in \text{ACL}'(s_b) \) and \( y \in F_s \). We may suppose that for every \( b' \in A \), \( s_g' \) and \( s_g'' \) are \( \text{ACL}' \)-interdependent over \( \{ s_z' : z \in A \} \{ b' \} \). Let \( b' \in A' \). By the same argument as above, \( a \perp A \{ b' \} \), hence \( a \perp b \). It follows that \( a'' \in Q(n) \).

We have \( a'' = a - a' \), hence \( s_g'' \in \text{ACL}'(s_g a) \subseteq \text{ACL}'(\{ s_z : z \in A \}) \), and since \( a'' \subseteq \text{acl}(A') \), \( s_g'' \in \text{ACL}'(\{ s_z' : z \in A' \}) \). By the inductive hypothesis, \( a_g' \in F_s - \text{span}(A_g') \). Together we get \( a_g' = b_g' y + a_g'' \in F_s - \text{span}(A_g') \). Theorem 2.13 describes to some extent \( \text{ACL}' \)-dependence on the set \( X_s = \{ s_g : a \in Q(n) \} \) (for \( s \in S_n \)). If we could waive in the “moreover” part the assumption that \( A \) is independent, then the description would be full, \( \text{ACL}' \) would be modular on \( X_s \), and \( X_s \) with \( \text{ACL}' \)-dependence would be projective over \( F_s \). Notice however that even this would not imply \( F_s \cong F_p \), because \( X_s \) is not necessarily essentially \( \text{ACL}' \)-closed. When \( s \in S_1 \) then Theorem 2.13 gives \( F_s \), a division subring of \( F_q \), and if \( A \subseteq Q \), \( a_g' \in F_s - \text{span}(A_g') \) and \( a \in Q \) then \( s_g \in \text{ACL}'(\{ s_g : b \in A \}) \). But if \( A \) is dependent over \( \varnothing \), there may be other \( a \in \text{acl}(A) \cap Q \) with \( a_g' \notin F_s - \text{span}(A_g') \) and yet \( s_g \in \text{ACL}'(\{ s_g : b \in A \}) \). When \( s \in S_n \), \( n > 1 \), we encounter an additional difficulty. Namely, for \( A \subseteq Q(n) \) there may be an \( n \)-tuple \( a \in G^n \setminus Q(n) \) with \( a_g' \in F_s - \text{span}(A_g') \). Then we cannot conclude that \( s_g \in \text{ACL}'(\{ s_g, b \in A \}) \) just because \( s_g \) is not defined! Defining suitably \( s_g \) by “projecting types” will be the main remaining trick in the proof of Theorem 2.2. This will be done in the next section. Notice yet that \( F_s \) is preserved by \( \text{Aut}(\mathcal{C}) \).

### 3. Projecting types

In this section we conclude the proof of Theorem 2.2. First we show that for \( s \in S \), \( F_s \) may be regarded as a division subring of \( F_p \) (hence \( F_s \) is a field), and in fact \( F_s \) is isomorphic to \( F_p \), which gives an embedding \( i : F_p \to M_{n_s \times n_s}(F_q) \) such that \( i(F_p) \) is a division subring of \( M_{n_s \times n_s}(F_q) \). Then, projecting types, we extend the definition of \( s_g \) to all \( a \in G^n \setminus \text{acl}(\varnothing) \) so that a counterpart of Theorem 2.13 holds. This gives a correspondence between \( P^* \) and \( G^n \) satisfying all the conditions of Theorem 2.2, except possibly for (5). We fulfill this last condition by a suitable change of coordinates. At the end of this section we give a hint on how to prove Theorem 2.2'. We begin with a closer look at some special \( R \subseteq P^* \).

#### 3.1. Lemma

Assume \( s \in S_n \), \( n > 0 \), \( A \subseteq Q(n) \) is independent. Let \( R = \text{ACL}'(\{ s_g, a \in A \}) \).

1. Assume \( r \in R \) and \( 0 < n(r) \leq n \). Then \( n(r) = n \) and for some \( b \in Q(n) \), \( b_g' \in F_s - \text{span}(A_g') \) and \( r, s_b \) are \( \text{ACL}' \)-interdependent.

2. Suppose \( a^0, \ldots, a^k \in Q(n), a_g' \in F_s - \text{span}(A_g') \) for \( i \leq k \). Then the following conditions are equivalent.
   a. \( a^0, \ldots, a^k \) are independent.
   b. \( a^0_g, \ldots, a^k_g \) are linearly independent over \( F_s \).
   c. \( s_g^0, \ldots, s_g^k \) are \( \text{ACL}' \)-independent.

**Proof.**

1. By Lemmas 2.7 and 2.11, \( r \) and \( s_b \) are \( \text{ACL}' \)-interdependent for some \( b \in Q(n) \). By Theorem 2.13, \( b_g' \in F_s - \text{span}(A_g') \).

2. (b) \( \rightarrow \) (a) is an easy exercise from linear algebra. W.l.o.g. \( A \) is finite. Suppose \( a^0_g, \ldots, a^k_g \) are \( F_s \)-linearly independent. By induction on \( i \leq k + 1 \)
we find sets \( A_i, i \leq k + 1 \), with \( A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_{k+1} \) such that for every \( i \leq k \), \( A_i \cup \{a^0, \ldots, a^i\} \) is independent and \( F_s\text{-span}(A') = F_s\text{-span}(A_{i+1}) \). Suppose we have \( A_i \) and want to find \( A_{i+1} \).

By inductive hypothesis, \( a_{ir}^i \in F_s\text{-span}(A'_{ir}) \). Since \( A_i \) is independent, \( a_{ir}^i \) can be presented in a unique way as an \( F_s \)-linear combination of elements of \( A'_{ir} \). As \( a_0^i, \ldots, a_{ir}^i \) are \( F_s \)-linearly independent, there must be an \( a \in A_i \) such that \( a_{ir}^i \) and \( a_{ir}^j \) are \( F_s \)-dependent over \( A'_{ir} \setminus \{a_{ir}^i\} \). Let \( A_{i+1} = A_i \setminus \{a\} \). We see that \( A_{i+1} \) satisfies our requirements. For \( i = k + 1 \) we get that \( \{a^0, \ldots, a^k\} \subseteq A_{k+1} \) is independent.

(a) \( \rightarrow \) (c) follows by Corollary 1.4, (c) \( \rightarrow \) (b) follows from Theorem 2.13.

Using lemma 3.1 we get an embedding of \( F_s \) into \( F_p \).

3.2. Lemma. For every \( s \in S \) there is a ring monomorphism \( i_s : F_s \rightarrow F_p \). \( i_s \) is an isomorphism.

Proof. Let \( s \in S_n, n > 0 \). Choose independent \( a, b, c \in Q(n) \), let \( R = \text{ACL}'(s_a, s_b, s_c) \) and \( X_s = \{ r \in R : r = s_d \text{ for some } d \in Q(n) \} \). Hence after identifying \( \text{ACL}' \)-interdependent types, \( R \) becomes a projective plane over \( F_p \), and by Lemma 3.1, \( X_s \) with \( \text{ACL}' \)-dependence becomes a projective plane over \( F_s \). Consider the line \( \mathcal{L} \) in \( R \) generated by \( s_a, s_b \). If we call \( s_a, s_b \) 0 and \( s_{a+b} \) 1, then the well-known algorithm endows \( \mathcal{L} \setminus \{\infty\} \) with a structure of a division ring isomorphic to \( F_p \). The same algorithm, restricted to \( \mathcal{L}_X = \mathcal{L} \cap X_s \), endows \( \mathcal{L}_X \setminus \{\infty\} \) with a structure of a division ring isomorphic to \( F_s \). So \( \mathcal{L}_X \setminus \{\infty\} \) becomes a division subring of \( \mathcal{L} \setminus \{\infty\} \), which gives the required \( i_s : F_s \rightarrow F_p \). In case when \( s = s \), by Lemma 3.1(1), \( X_s \) is essentially \( \text{ACL}' \)-closed and \( \text{ACL}'(X_s) = R \), hence \( \mathcal{L} = \mathcal{L} \cap X_s \). This means that \( i_s \) is an isomorphism.

Let \( i = i_s^{-1} \). Hence \( i : F_p \rightarrow M_{n_s \times n_s}(F_q) \), as required in Theorem 2.2.

We shall see that we can regard \( P^* \) as a vector space over \( F \), so that \( \text{ACL}' \)-dependence becomes just linear dependence over \( F \). We shall define \( \sigma_a \in P^*_a \) for any \( a \in G_n \setminus \text{acl}(\varnothing) \). We would like to have always \( \sigma_a \in P^*_a \). For that reason, for \( a \in G_n \) define \( P^*_a \) as \( P_A \), where \( A = \text{acl}(a) \cap Q \). This generalizes naturally the convention introduced in §1.

3.3. Definition. Let \( a \in G_n \setminus \text{acl}(\varnothing) \). Choose \( d \in Q(n) \) with \( d \perp a \). Let \( c = a + d \). So \( c \in Q(n) \). We define \( \sigma_a \) as any \( r \in P^*_c \) which is \( \text{ACL}' \)-interdependent with \( \sigma_c \) over \( P^*_c \). If \( a \in G(n) \) and \( c \in \mathcal{A} \), we can define \( \sigma_a \) as any type in \( P^*_c \), provided \( P^*_c \) is nonempty (this case is really not important).

Let me explain the idea underlying Definition 3.3. If \( a \in Q(n) \), we will see below that the new and the old \( \sigma_a \) are \( \text{ACL}' \)-interdependent, so that Definition 3.3 extends the previous notation. For such an \( a \) we may assume the new \( \sigma_a \) equals the old one. If \( a \) is not independent, we may think of \( \sigma_a \) as a degenerated version of a "real" \( \sigma_c \), a projection of \( \sigma_c \). Suppose \( c \in Q(n) \), \( c = (c_1, \ldots, c_{n_a}) \), and we add \( c_1 \) to the signature. Then \( \sigma_c \), in the new signature, is a degenerated version of \( \sigma_c \) in the old signature, because \( c \) is no longer in \( Q(n) \). Still it bears some resemblance to the old \( \sigma_c \). This is expressed in Definition 3.3. In the next lemma we prove some basic properties of \( \sigma_a, a \in G_n \).
3.4. Lemma. Let \( a \in G^{n_a} \), \( a_{\sigma} \neq 0 \).

(1) \( \sigma_a \) exists, moreover, up to \( \text{ACL}' \)-interdependence, the choice of \( \sigma_a \) is unique, hence Definition 3.3 is correct.

(2) If \( a \in Q^{(n_a)} \) then the new \( \sigma_a \) as defined in Definition 3.3 is \( \text{ACL}' \)-interdependent with the old \( \sigma_a \), defined after Theorem 2.8, hence we may assume that the new \( \sigma_a \) equals the old one in this case.

(3) \( \pi(\sigma_a) = \text{acl}(a) \cap Q, \ n(\sigma_a) = \dim(a) \).

(4) For every \( r \in P^* \) there is \( b \in G^{n_a} \) such that \( r \) and \( \sigma_b \) are \( \text{ACL}' \)-interdependent.

(5) Suppose \( d \in Q^{(n_a)} \) is independent from \( a \), \( c = a + d \). Then \( \sigma_a, \sigma_c, \sigma_d \) are pairwise \( \text{ACL}' \)-independent, and \( \sigma_a \in \text{ACL}'(\sigma_c, \sigma_d) \).

Proof. (1) First we prove that \( \sigma_a \) exists. Choose \( d \in Q^{(n_a)} \) with \( d \perp a \) and let \( c = a + d \). So \( c \in Q^{(n_a)} \) and \( c \perp a \). By Theorem 1.16, \( P^*_c \subseteq \text{ACL}(P^*_d P^*_a) \), hence there is \( r \in P^*_a \text{ACL}' \)-interdependent with \( \sigma_a \) over \( P^*_c \). Hence \( \sigma_a \) exists. If \( r'' \in P^*_a \) is another type \( \text{ACL}' \)-interdependent with \( \sigma_a \) over \( P^*_c \), then \( r \in \text{ACL}(r'' P^*_a) \), hence \( d \perp r \) implies \( r \in \text{ACL}(r'') \). To prove that the choice of \( \sigma_a \) is unique up to \( \text{ACL}' \)-interdependence, choose \( d' \in Q^{(n_a)} \) independent from \( ad \), let \( c' = a + d' \), and suppose \( r' \in P^*_a \) is \( \text{ACL}' \)-interdependent with \( \sigma_a \) over \( P^*_d \). By Assumption 2.10, there is an \( f \in \text{Aut}(c) \) fixing \( a \) and \( r' \), with \( f(c') = c \) and \( f(\sigma_a) = \sigma_a \). Let \( d'' = f(d') \). We see that \( d'' = f(c') - a = c - a = d \), and we have that \( f(r') = r' \) is \( \text{ACL}' \)-interdependent with \( \sigma_a \) over \( P^*_c \). By the previous paragraph, \( r' \) is \( \text{ACL}' \)-interdependent with \( \sigma_a \).

(2) If \( a \in Q^{(n_a)} \) then by Theorem 2.13, the old \( \sigma_a \in \text{ACL}'(\sigma_a, \sigma_d) \) because \( a_{\sigma} \in F^*_r \text{-span}(\sigma_c, \sigma_d, \sigma_a) \), hence by (1) we are done.

(3) Let \( c, d \) be as in Definition 3.3. Hence \( \dim(c/d) = \dim(a) \). Since \( \sigma_c \) and \( \sigma_d \) are \( \text{ACL}' \)-interdependent over \( P^*_d \),

\[
\dim(\pi(\sigma_c)/d) = \dim(\pi(\sigma_d)/d) = \dim(\pi(\sigma_a)).
\]

Since \( \pi(\sigma_c) = \text{acl}(c) \cap Q \), \( \dim(c/d) = \dim(\pi(\sigma_c)/d) \). Hence \( \dim(\pi(\sigma_c)) = \dim(\pi(\sigma_a)) \).

(4) W.l.o.g. \( r \notin P^*_a \). Let \( A = \pi(r) \). Choose \( c \in Q^{(n_a)} \) independent from \( A \), let \( r' = \sigma_a \). Since \( P^* \) is projective over \( F^*_a \), there is \( r'' \in \text{ACL}(r'r') \) such that \( r, r', r'' \) are pairwise \( \text{ACL}' \)-independent. Choose \( D \subseteq \text{acl}(A) \) with \( D = \pi(r'') \). \( r' \) and \( r'' \) are \( \text{ACL}' \)-interdependent over \( P^*_a \), hence \( \dim(D/A) = \dim(D/A) \).

It follows that \( \dim(D/A) = n_a \). Also, since \( n(r'') \leq n_a \), \( \dim(D) \leq n_a \). We conclude \( D \perp A \) and \( \dim(D) = n_a \). As \( c \subseteq \text{acl}(AD) \), there are \( b \subseteq \text{acl}(A) \cap G \), \( d \subseteq \text{acl}(D) \cap G \) with \( c = b + d \). Since \( c \perp A \), \( d \in Q^{(n_a)} \) and \( d \perp b \). Also, \( r \) is \( \text{ACL}' \)-interdependent with \( r' = \sigma_a \) over \( P^*_a \). Hence \( r \) is \( \text{ACL}' \)-interdependent with \( \sigma_b \) over \( P^*_d \). But \( b \subseteq \text{acl}(A) \) and \( A \perp d \), hence \( r \) is \( \text{ACL}' \)-interdependent with \( \sigma_b \).

(5) By Lemma 2.11 (with \( E := a \)), \( \sigma_c \) and \( \sigma_d \) are \( \text{ACL}' \)-interdependent over \( P^*_c \), so there is an \( r \in P^*_a \) such that \( r \in \text{ACL}(\sigma_a \sigma_d) \). But \( \sigma_a \) and \( \sigma_c \) are \( \text{ACL}' \)-interdependent over \( P^*_d \), hence \( r \in \text{ACL}(\sigma_a P^*_a) \). Now \( P^*_a \perp P^*_a (P^*_a) \) implies \( r \) and \( \sigma_a \) are \( \text{ACL}' \)-interdependent.

3.5. Corollary. The set \( X = \{ \sigma_a ; a \in G^{n_a} \} \) is essentially \( \text{ACL}' \)-closed and \( \text{ACL}'(X) = P^* \).
The next theorem shows that $ACL'$-dependence on the set $X$ from Corollary 3.5 is essentially a linear dependence over $F_r$.

3.6. Theorem. Assume $A \subseteq G^{n_2}$ and $a \in G^{n_2}$. Then $\varphi_a \in ACL'\{\tau_b, b \in A\}$ iff $a_{\varphi_a} \in F_r \cdot \text{span}(A_{\varphi_a})$.

Proof. Since $G^{n_2}$ is a vector space over $F_r$, it suffices to prove the theorem for $|A| \leq 2$.

$\leftarrow$. First we consider the case when $A = \{b\}$. W.l.o.g. $a_{\varphi_a} \neq 0 \neq b_{\varphi_a}$. Suppose $\alpha \in F_r$ and $a_{\varphi_a} = b_{\varphi_a} \alpha$. Thus $\alpha \neq 0$ and $\alpha$ is invertible. We want to prove that $\varphi_a \in ACL'(\tau_b)$. Choose $d, d' \in Q^{(n_2)}$ independent from $b$ so that $d_{\varphi_a} = d_{\varphi_a} \alpha$. Let $c = b + d$, $c' = a + d'$. By Corollary 1.4 it suffices to prove $\varphi_a \in ACL'(\tau_{bc'})$. By Definition 3.3, $\varphi_a \in ACL'(\tau_{bc'})$ and $\varphi_a \in ACL'(\tau_{bc'}).$ Since $c_{\varphi_a} = c_{\varphi_a} \alpha$ and $c \in Q^{(n_2)}$, by Theorem 2.13, $\varphi_c \in ACL'(\tau_c).$ Hence we get $\varphi_a \in ACL'(\tau_{bc'})$, and $\varphi_a \in ACL'(\tau_{bc}).$ The case $|A| = 2$ is similar, we leave it to the reader.

$\rightarrow$. For a change, let us consider now the case $A = \{b, c\}$, with $b, c \in G^{n_2}$ and $b_{\varphi_a}, c_{\varphi_a}$, linearly independent over $F_r$, the case $|A| = 1$ being similar. Suppose $\varphi_a \in ACL'(\tau_{bc})$ and $\varphi_b, \varphi_c$ are pairwise $ACL'$-independent. We want to find $\alpha, \beta \in F_r$ such that $a_{\varphi_a} = b_{\varphi_a} \alpha + c_{\varphi_a} \beta.$ Obviously, $\varphi_a \in \text{acl}(bc)$. Choose $d, d', d'' \in Q^{(n_2)}$ with $\{d, d', d'', bc\}$ independent. Let $e = c + d$, $e' = a + d'$. Consider $X = \{\tau_b, \tau_c, \tau_d, \tau_d', \tau_d''\}$. $\text{DIM}(X/P^*) = 5$, and by Lemma 3.4(5), since $\varphi_a \in ACL'(\tau_{bc})$, we have $\varphi_e, \varphi_{e'}, \varphi_{e''} \in ACL'(X)$. Each of the sets $\{d, d', d''\}$ and $\{e, e', e''\}$ is independent. By modularity of $ACL$, there is $r \in ACL'\{\tau_{d_{\varphi_a}}, \tau_{d_{\varphi_a}}\} \cap ACL'\{\tau_{e_{\varphi_a}}, \tau_{e_{\varphi_a}}, \tau_{e_{\varphi_a}}\} \setminus ACL'(\emptyset)$. By Lemma 3.1, $r$ is $ACL'$-interdependent with $\varphi_g$ for some $g \in Q^{(n_2)}$. By Theorem 2.13 we find $\alpha_i, \beta_i \in F_r$ for $i < 3$ such that

$$g_{\varphi_a} = d_{\varphi_a} \alpha_0 + d'_{\varphi_a} \alpha_1 + d''_{\varphi_a} \alpha_2 = e_{\varphi_a} \beta_0 + e'_{\varphi_a} \beta_1 + e''_{\varphi_a} \beta_2 = d_{\varphi_a} \beta_0 + d'_{\varphi_a} \beta_1 + d''_{\varphi_a} \beta_2 + c_{\varphi_a} \beta_0 + a_{\varphi_a} \beta_1 + b_{\varphi_a} \beta_2,$$

Comparing the second and the last step in this sequence of equalities, we see that $\alpha_i = \beta_i$, some $\alpha_i$ is $\neq 0$ and $c_{\varphi_a} \beta_0 + a_{\varphi_a} \beta_1 + b_{\varphi_a} \beta_2 = 0$. But $b_{\varphi_a}, c_{\varphi_a}$ are linearly independent over $F_r$, hence $\beta_1 = 0$ would imply $\beta_0 = \beta_2 = 0$, a contradiction. Thus $\beta_i \neq 0$, i.e. $a_{\varphi_a} = -c_{\varphi_a} \beta_0 \beta_1 - b_{\varphi_a} \beta_0 \beta_2$. Hence $\alpha = -\beta_0 \beta_1^{-1}, \beta = -\beta_2 \beta_1^{-1}$ satisfy our demands. Now we shall relate types $s_a$, $s \in S_n, a \in Q^{(n_2)}$, and division rings $F_s$ with types $\tau_b, b \in G^{n_2}$ and field $F_r$. For $s \in S_n, n > 0$, and for an $a^* \in Q^{(n)}$ we choose $\alpha_s \in M_{n \times n_2}(F_q)$ such that if $b^* \in G^{n_2}$ and $a^*_{\varphi_a} = a^*_{\varphi_a} \alpha_s$, then $\varphi_{b^*}$ and $s_{a^*}$ are $ACL'$-interdependent. Clearly, for $s = s$ we can take $\alpha_s = I$.

3.7. Proposition. Assume $s \in S_n, n > 0$.

(1) $\alpha_s$ has rank $n$, hence for some $\beta_s \in M_{n \times n_2}(F_q), \alpha_s \beta_s = I$.

(2) For every $a \in Q^{(n)}$, if $b \in G^{(n_2)}$ and $b_{\varphi_a} = a_{\varphi_a} \alpha_s$ then $\varphi_b$ and $s_a$ are $ACL'$-interdependent.

(3) $\beta \in F_r$ iff for some $\gamma \in F_r, \alpha_s \gamma = \beta \alpha_s$.

(4) For every $\gamma \in F_r$ there is at most one $\beta \in M_{n \times n_2}(F_q)$ with $\alpha_s \gamma = \beta \alpha_s$.

(5) By (4) we define a partial function $f_s$ with $\text{Dom}(f_s) \subseteq F_r, \text{Rng}(f_s) \subseteq M_{n \times n_2}(F_q)$ so that $f_s(\gamma) = \beta$ iff $\alpha_s \gamma = \beta \alpha_s$. Then $F_s' = \text{Dom}(f_s)$ is a division subring of $F_s$ and $f_s: F_s' \to F_r$ is an isomorphism.
Proof. (1) We know that \( s_y^* \) and \( s_x^* \) are ACL'-interdependent. By Lemma 3.5(3) and the definition of \( s_y^* \), \( \dim(b^*) = n(s_y^*) = n(s_x^*) = \dim(a^*) = n \).

Since \( b^*_{\alpha} = a^*_{\gamma} \alpha_s \), rank of \( \alpha_s \) equals \( n \).

(2) Let \( f \) be an automorphism of \( \epsilon \) fixing \( \alpha_s \), sending \( a \) to \( a^* \) and \( b \) to \( b' \). Let \( r = f(s_y) \), \( r' = f(s_x) \). We have \( b' - b^* \subseteq \alpha_s \), hence by Theorem 3.6 and Lemma 2.11, \( r \in \text{ACL}'(s_y^*) \) and \( r' \in \text{ACL}'(s_x^*) \). It follows that \( r \) and \( r' \) are ACL'-interdependent, hence also \( s_y \) and \( s_x \) are ACL'-interdependent.

(3) We neglect the case \( \beta = 0 = \gamma \). By Theorems 2.13 and 3.6, we have \( \beta \in F^*_x \) iff for \( a, b \in Q(n) \) with \( b_{\alpha} = a_{\alpha} \beta \), \( s_y \in \text{ACL}'(s_y^*) \) iff for \( a', b' \in G^n \) with \( a'_{\alpha} = a_{\alpha} \alpha_s \), \( b'_{\alpha} = b_{\alpha} \alpha_s \), \( s_y' \in \text{ACL}'(s_y^*) \) iff for some \( \gamma \in F^*_x \), \( b'_{\alpha} = a_{\alpha} \alpha_s \gamma \) iff for some \( \gamma \in F^*_x \), \( s_{\alpha} \beta \alpha_s = a_{\alpha} \alpha_s \gamma \) iff for some \( \gamma \in F^*_x \), \( \beta \alpha_s = \alpha_s \gamma \).

(4) If \( \beta \alpha_s = \beta' \alpha_s \) for some \( \beta, \beta' \in M_{n \times n}(F_q) \), then multiplying this equality on the right by \( \beta' \), we get \( \beta = \beta' \).

(5) It is easy to check that \( f_s \) is a homomorphism, and by (3), \( \text{Rng}(f_s) = F_s \).

Multiplying the equation \( \beta \alpha_s = \alpha_s \gamma \) on both sides by \( \beta^{-1} \) on the left and \( \gamma^{-1} \) on the right, we get \( \beta^{-1} \alpha_s = \alpha_s^{-1} \). This shows that \( f_s(\gamma^{-1}) = \beta^{-1} \). Hence \( f_s : F_s' \rightarrow F_s \) is an epimorphism. Since \( F_s' \), \( F_s \) are division rings, \( f_s \) is an isomorphism of \( F_s' \) and \( F_s \).

One obstacle prevents us from concluding the proof of Theorem 2.2. Namely it may happen that for \( s = s^* \) (= the sort of \( p_a \)), \( \alpha_s \neq (1, 0) \). To deal with this difficulty we shall uniformly change coordinates. Recall that we have chosen \( s_a \) for \( a \in Q(n) \) in a rather arbitrary way, as a uniform representative in \( P_a^* \) of the sort \( s \). We could well choose at the beginning another \( s_a \) to serve the same purpose. So now let \( \alpha \in M_{n \times n}(F_q) \) be invertible. For \( a \in Q(n) \) define \( s'_a \) as \( s_a \), where \( a_{\alpha} = a_{\alpha} \alpha \) (by Lemma 2.11 this definition is correct, up to ACL'-interdependence). We shall see below how the change of coordinates affects \( F_s \) and \( \alpha_s \), \( s \in S \).

Let \( F_s = \{ \beta \in M_{n \times n} \} : for a, c \in \text{ACL}'(s_{\alpha}) \} \cup \{ 0 \} \).

3.8. Lemma. (1) \( \beta \in F_s \) iff \( \alpha^{-1} \beta \alpha \in F_s \), hence \( F_s \) is the \( \alpha^{-1} \)-conjugate of \( F_s \).

(2) For \( s \in S_n \), \( n > 0 \), if \( s'_a \alpha_s = \alpha_s^{-1} \alpha_s \), \( a \in Q(n) \), \( b \in G^n \) and \( b_{\alpha} = a_{\alpha} \alpha_s \), then \( s'_{b_{\alpha}} \) and \( s_{a_{\alpha}} \) are ACL'-interdependent.

Proof. (1) We have \( \beta \in F_s \) iff for \( a, b \in Q(n) \) with \( b_{\alpha} = a_{\alpha} \beta \), \( s'_{b_{\alpha}} \in \text{ACL}'(s'_{a_{\alpha}}) \) iff for \( a', b' \in Q(n) \) with \( a'_{\alpha} = a_{\alpha} \alpha \), \( b'_{\alpha} = b_{\alpha} \alpha \), \( s'_{b'_{\alpha}} \in \text{ACL}'(s'_{a_{\alpha}}) \) iff for some \( \gamma \in F^*_s \), \( b'_{\alpha} = a_{\alpha} \alpha \gamma \) iff for some \( \gamma \in F^*_s \), \( \beta \alpha \gamma = \alpha \gamma \beta \alpha \).

(2) Let \( s \in S_n \), \( a \in Q(n) \), \( b \in G^n \) and \( b_{\alpha} = a_{\alpha} \alpha_s \). Then \( s_{b_{\alpha}} \) and \( s_{a_{\alpha}} \) are ACL'-interdependent. Choose \( d \in Q \) independent from \( a, b \), let \( c = b + d \). By Definition 3.3, \( s_{b_{\alpha}} \) and \( s_{c_{\alpha}} \) are ACL'-interdependent over \( P^*_d \). Choose \( c' \), \( d' \in Q(n) \) with \( c'_{\alpha} = c_{\alpha} \), \( d'_{\alpha} \alpha = d_{\alpha} \). Let \( b' = c' - d' \). Then also \( b'_{\alpha} = b_{\alpha} \).

We see that \( s'_{b'_{\alpha}} \) and \( s_{c_{\alpha}} \) are ACL'-interdependent, hence \( s'_{b'_{\alpha}} \) and \( s_{b_{\alpha}} \) are ACL'-interdependent over \( P^*_d \), and so \( s_{b_{\alpha}} \) and \( s'_{b_{\alpha}} \) are ACL'-interdependent. We have \( b'_{\alpha} = a_{\alpha} \alpha^{-1} = a_{\alpha} \alpha_{\alpha} = a_{\alpha} \alpha_{\alpha} \), so we are done.

We can choose \( \alpha \) so that \( \alpha \alpha = (1, 0) \). Hence we can assume that the choice of the original \( s_a \) is such that the following holds.
3.9. Assumption. For \( a \in Q \), \( s^*_a \) is \( \text{ACL}'\)-interdependent with \( \omega_\alpha \), where \( a = (a, 0) \).

Now we can conclude the proof of Theorem 2.2. We define \( \Phi \) as follows. For \( r \in P^* \) and \( a \in G^{n_a} \), \( \Phi(r, a) \) holds iff

\[
(*) \quad r \text{ and } \omega_\alpha \text{ are } \text{ACL'}-\text{interdependent}.
\]

\( \Phi \) is invariant under automorphisms of \( \mathcal{E} \), because (*) is. \( \text{Dom}(\Phi) = P^* \) and \( G^{n_a} \setminus \{0\} \subseteq \text{Rng}(\Phi) \) by Lemma 3.4 and Definition 3.3. Condition (3) in Theorem 2.2 follows from Theorem 3.6. Condition (4) is obvious, (5) follows from Assumption 3.9.

Theorem 2.2' may be proved as follows. One way is to repeat all the proof of Theorem 2.2, but with \( P^* \) replaced by \( P \). Another way consists in noticing that in \( S_{n_b} \) there is a unique sort of types from \( P \). Call this sort \( \omega^0 \). Since \( P \) is essentially \( \text{ACL}'\)-closed in \( P^* \), \( P \) with \( \text{ACL}'\)-dependence is projective over \( F_p \), as well, hence \( i_\rho \) from Lemma 3.2 is an isomorphism of \( F_\rho \) and \( F_p \). This gives an embedding \( i': F_p \rightarrow M_{n_b \times n_b}(F_q) \), as required in Theorem 2.2'. Then define types \( \omega^0_\alpha \) for \( a \in G^{n_a} \), as projections of types \( \omega^0_\alpha \), \( \zeta \in Q^{(n_a)} \), similarly as in Definition 3.3. The rest is much the same as the proof of Theorem 2.2.

Now we shall draw some more corollaries from Theorems 2.2, 2.2' and their proofs. \( F \) and \( F' \) are as in Theorems 2.2 and 2.2'. So \( F \) and \( F' \) are isomorphic copies of \( F_p \). First, using Lemma 2.11 we improve Proposition 1.15 a little.

3.10. Proposition. For every \( A \subseteq Q \), \( P_A \subseteq \text{ACL}(P^0_{a\phi}) \subseteq \text{ACL}'(P_A^0) \).

Proof. Notice that \( P \nsubseteq P^*_{a\phi} \). We will tacitly use this in the proof. By Theorem 1.16 we may assume \( A = \{a\} \). Choose an infinite \( E \subseteq Q \) with \( E \nsubseteq a \). Clearly, \( P_E \subseteq \text{ACL}(P^0_E) \) and \( P_Ea \subseteq \text{ACL}(P^0_{Ea}) \). Suppose \( r \in P_a \).

We can assume that we have added to the language a sufficiently large finite subset of acl(\( \alpha \)). So there are \( a_1, \ldots, a_k \in \text{acl}(Ea) \cap Q \setminus \text{acl}(E) \) (for some \( k \)), such that \( r \in \text{ACL}(s^*_{a_1} \cdots s^*_{a_k} P_E) \). \( a_1 \in \text{acl}(Ea) \setminus \text{acl}(E) \) implies there are \( b_i \in \text{acl}(a) \cap Q \) and \( e_i \in \text{acl}(E) \cap G \) such that \( a_1 = b_i + e_i \). By Lemma 2.11, \( s^*_{a_i} \) and \( s^*_{b_i} \) are ACL-interdependent over \( P_E \), hence over \( P_E \). Thus also \( r \in \text{ACL}(s^*_{b_1} \cdots s^*_{b_k} P_E) \). Now \( E \nsubseteq a \) implies \( P_E \nsubseteq P_a(P_\phi) \), hence \( r \in \text{ACL}(s^*_{b_1} \cdots s^*_{b_k} P_\phi) \), which finishes the proof.

In the next corollary we show various connections between \( F_p \) and \( F_q \), following from Theorems 2.2, 2.2' (or their proofs). In Theorem 1.23, using rough geometrical means, we proved that if \( n_d > 1 \) and \( F_q \) is finite, then \( F_p \) is finite as well. Here we get more: \( F_p \) is finite iff \( F_q \) is finite. Corollary 3.11(4) below was proved also in [Bu4, §4].

3.11. Corollary. (1) The dimension of \( (F_q)^{n_a} \) as a right vector space over \( F \) equals \( n_d \).

(2) The dimension of \( (F_q)^{n_a} \) over \( F' \) equals \( n_e \).

(3) \( F_p \) is finite iff \( F_q \) is finite. In this case \( |F_p|^{n_d} = |F_q|^{n_a} \), \( |F_p|^{n_e} = |F_q|^{n_b} \).

(4) \( F_p \), \( F_q \) have the same characteristic.

(5) \( F_q \)-span of the set \( \{(1, 0)\beta: \beta \in F'\} \) is the whole of \( (F_q)^{n_b} \), regarded as the left vector space over \( F_q \).
(6) \([F_p^* : F_{s^*}] \geq n_b\).

(7) If \(F_p, F_q\) are finite then
\[
\frac{|F_p|^n - 1}{|F_p| - 1} = \sum_{s \in S_1} \frac{|F_q|^n - 1}{|F_s| - 1}.
\]

Proof. (1) Let \(a \in Q\) and let \(V_a = \{a_s \beta : \beta \in (F_q)^{n_a}\}\), \(V_a\) is an \(F\)-subspace of \(G_{o^w}^q\). By Theorem 2.2, \(\text{DIM}(P_a/P_{o^w})\) equals the \(F\)-dimension of \(V_a\). The action of \(F\) on \(V_a\) is isomorphic to the action of \(F\) on \((F_q)^{n_a}\).

(2) The same proof.

(3) By Theorem 2.2, \(F_p\) is embeddable into \(M_{n_a \times n_{q^w}}(F_q)\), hence if \(F_q\) is finite then \(F_p\) is finite, too. Conversely, suppose \(F_p\) is finite. By (1), (2), \(|F_q|^{n_a} = |F_p|^{n_d}\) and \(|F_q|^n = |F_p|^n\), hence \(F_q\) is finite.

(4) \(I\) is the identity of \(F\).

(5) Let \(a \in Q\). By Proposition 3.10, \(\text{ACL}'(\{s_b^* : b_{o^w} = a_{o^w} \alpha, \alpha \in F^*_q\}) = \text{ACL}'(P_a)\). By Theorem 2.2' this means that the set \(\{\alpha(1, 0) \beta : \alpha \in F_q, \beta \in F'\}\) is the whole of \((F_q)^{n_{a^w}}\).

(6) This is an application of the proof of Theorem 2.2'. As in Proposition 3.7 we find there \(F_{s^*}^* \subseteq F'\) and an isomorphism \(f_{s^*} : F_{s^*}^* \rightarrow F_{s^*}, \alpha_{s^*} = (1, 0)\) by Assumption 3.9. By (5), \(\{\alpha_{s^*} \beta, \beta \in F'\}\) are \(F_q\)-linear generators of \((F_q)^{n_{a^w}}\).

\[\text{dim}_{F_{s^*}}(F_q)^{n_{a^w}} = n_b.\]

Now if for \(\beta, \beta' \in F'^{n_{a^w}}, \beta' \beta^{-1} \in F_{s^*}^*\), then by Proposition 3.7(5), for some \(\alpha \in F_{s^*}^* \subseteq F_q\), \(\alpha_{s^*} = \alpha_{s^*} \beta' \beta^{-1}\), which means \(\alpha \alpha_{s^*} \beta = \alpha_{s^*} \beta'\), hence \(\alpha_{s^*} \beta \) and \(\alpha_{s^*} \beta'\) are \(F_{s^*}^*\)-dependent. Thus \(\{F'^{n_{a^w}} : F_{s^*}^*\} \geq n_b\). But \(F_p, F'\) and \(F_{s^*}, F_{s^*}^*\) respectively are isomorphic and locally finite, hence we are done.

(7) Let \(a \in Q\). After identification of \(\text{ACL}'\)-interdependent types, \(P_{s^*}^*\) becomes a projective space over \(F_p\) of dimension \(n_d\), hence has \((|F_p|^{n_d} - 1)/(|F_p| - 1)\) points. For each \(r \in P_{a^w}\) there is a unique \(s \in S_1\) such that for some \(b \in \text{acl}(a) \cap Q, r \in \text{ACL}'(s_b)\). For each \(s \in S_1\), the number of pairwise \(\text{ACL}'\)-independent types \(s_b, b \in \text{acl}(a)\), equals \((|F_q| - 1)/(|F_s| - 1)\), so we are done.

In the next section we shall discuss in greater detail the algebraic problem of counting nonisomorphic \(F\)-subspaces of \(G_{o^w}^q\), to which in Theorem 2.2 we reduced the problem of counting nonisomorphic \(\text{ACL}'\)-closed subsets of \(P^*\).

In fact, S. Buechler has proved in [Bu4] that both \(F_p\) and \(F_q\) are locally finite fields. Hence we shall consider the case when \(K \) and \(L\) are fields with \(L \subseteq M_{n \times n}(K)\) for some \(n\). We shall discuss also the relevance of this problem to the original problem (P).

4. A PROBLEM FROM LINEAR ALGEBRA AND A PROBLEM OF PARAMETERS

Assume \(K\) is a countable field (or even a locally finite field), \(n^* < \omega\), \(L \subseteq M_{n^* \times n^*}(K)\) is a division subring of \(M_{n^* \times n^*}(K)\), and is a field in its own right. Assume \(V\) is a very large right vector space over \(K\) (that is \(V\) is a “monster” \(K\)-space). Before Theorem 2.2 defined the notion of isomorphism between \(L\)-subspaces of \(V^{n^*}\). Let \(I(K, L)\) be the number of isomorphism types of countable \(L\)-subspaces of \(V^{n^*}\) and \(I'(K, L)\) be the number of isomorphism types of countable \(L\)-subspaces \(W\) of \(V^{n^*}\) such that \(W = L\)-span(\(\{(a, 0) \in W : a \in V\}\)). Assume \(\Gamma\) is a group of automorphisms of \(K\) such that each \(\gamma \in \Gamma\) preserves \(L\) and the \(\Gamma\)-orbit of every element of \(K\) is finite. Then \(I(K, L, \Gamma)\) denotes the number of \(\Gamma\)-isomorphism types...
of countable $L$-subspaces of $V^{n*}$. Similarly define $I'(K, L, \Gamma)$. Notice that $I(K, L, \Gamma) \leq I(K, L)$ and $I'(K, L, \Gamma) \leq I'(K, L)$.

In [BN] we show that $I(K, L) < 2^{n0}$ implies $I(K, L) = \aleph_0$. It is not clear in general whether $I(K, L, \Gamma) < 2^{n0}$ implies $I(K, L, \Gamma) = \aleph_0$. However, in many cases it turns out that $I(K, L) = 2^{n0}$ usually implies the existence of $2^{n0}$ nonisomorphic $L$-subspaces $W_\alpha$, $\alpha < 2^{n0}$, of $V^{n*}$ constructed using finitely many distinguished elements $D$ of $K$. If we name these elements, then $W_\alpha$, $\alpha < 2^{n0}$, become non-$\Gamma$-isomorphic, where $\Gamma' = \{\gamma \in \Gamma : \gamma(\alpha) = \alpha \text{ for every } \alpha \in D\}$. Since $[\Gamma : \Gamma']$ is finite, every $\Gamma$-isomorphism class splits into finitely many $\Gamma'$-isomorphism classes, which gives $I(K, L, \Gamma) = 2^{n0}$.

On the other hand $I(K, L) = \aleph_0$ implies of course $I(K, L, \Gamma) = \aleph_0$. Hence $I(K, L, \Gamma)$ and $I(K, L)$ are often equal. We shall discuss these problems in [BN] in greater detail, relying on [DR]. Now notice that certainly $I(K, L, \Gamma) = I(K, L)$ for finite $K$. This is because for finite $K$, also $\Gamma$ is finite, hence every $\Gamma$-isomorphism class splits into finitely many isomorphism classes. Corollaries 2.3 and 2.3' yield the following remark. $F$ and $F'$ are as in Theorems 2.2, 2.2'.

4.1. Remark. (1) $I(F_q, F, \Gamma_q)$ equals the number of isomorphism types of $ACL'$-closed countably dimensional subsets of $P^*$.

(2) $I'(F_q, F, \Gamma_q) = I'(F_q, F', \Gamma_q)$ equals the number of isomorphism types of sets $ACL'({r \in P^0 : r \text{ is realized in } M})$, where $M$ is a countable model of $T$.

This remark shows the relevance of $I(F_q, F, \Gamma_q)$ to the original problem (P). Also, by Remark 4.1, $I(T, \aleph_0) < 2^{n0}$ implies $I(F_q, F, \Gamma_q)$, $I'(F_q, F, \Gamma_q)$, $I(F_q, F', \Gamma_q)$, $I'(F_q, F', \Gamma_q)$ are all $< 2^{n0}$. Even if calculating $I(F_q, F, \Gamma_q)$ does not yet immediately solve (P), it determines $I(p|E, \aleph_0)$ for some finite $E$. We say that $E$ is a basis of $P^*_\varphi$ if $E$ is a selector from $\{r(\xi) : r \in R\}$, where $R$ is an $ACL$-basis of $P^*_\varphi$. Notice that by Lemma 1.7 $R$ is necessarily finite, hence $E$ is finite, too.

4.2. Theorem. Assume $I(F_q, F, \Gamma_q) = \aleph_0$. There is a finite set $E = E' \cup E''$, where $E' \subseteq acl(\varnothing)$ and $E''$ is a basis of $P^*_\varphi$, such that $I(p|E, \aleph_0)$ is countable.

Proof. Let $E'$ be a finite subset of $acl(\varnothing)$ required by Theorem 2.2, and $E''$ be a basis of $P^*_\varphi$. W.l.o.g. add $E''$ to the signature. Suppose $I(p|E, \aleph_0)$ is uncountable. Then there are $ACL'$-closed countably dimensional sets $R_\alpha \subseteq P^*$, $\alpha < \omega_1$, which are pairwise nonisomorphic over $E$, that is for $\alpha \neq \beta$ there is no $f \in Aut_E(\xi)$ with $f[R_\alpha] = R_\beta$. For $f \in Aut(\xi)$, $f[R_\alpha]$ is determined by $f' = f|acl(Q)$. By Remark 4.1 w.l.o.g. $R_\alpha$, $\alpha < \omega_1$ are isomorphic. Choose for $\alpha > 0$ $f_\alpha \in Aut(\xi)$ such that $f_\alpha[R_\alpha] = R_0$. Let $e_\alpha = f_\alpha(\xi)$. We get an $\alpha \neq \beta < \omega_1$ with $e_\alpha \equiv e_\beta(acl(Q))$. Choose $f \in Aut(\xi)$ with $f|acl(Q) = id$, $f(e_\alpha) = e_\beta$. Then $f' = f_\beta^{-1}f_\alpha \in Aut_E(\xi)$ and $f'[R_\alpha] = R_\beta$, a contradiction.

The problem of determining $I(K, L)$ is similar to the problem of counting $ACL'$-closed subsets of $P^*$. It may be instructive to note that many ideas from the proof of Theorem 2.2 have a counterpart here. It is so with sorts of types $s$ and fields $F_s$. Let us review shortly how these notions appear in the purely algebraic setting. We assume now that $I(K, L) < 2^{n0}$. For a matrix $\alpha$, $\text{rk}(\alpha)$ denotes the rank of $\alpha$. 

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Let $0 < n \leq n^*$. We denote $\simeq$ on \( \{\alpha \in M_{n \times n}(K) : \text{rk}(\alpha) = n\} \) by $\alpha \simeq \beta$ iff for some $\gamma \in L^{*}$ and $\delta \in M_{n \times n}(K)$, $\alpha \gamma = \delta \beta$. We define $S_n(K, L) = \{\alpha \simeq : \text{rk}(\alpha) = n\}$, $S(K, L) = \bigcup_{0 < n \leq n^*} S_n(K, L)$. This corresponds to Definition 2.5. For $s \in S_n(K, L)$ choose $\beta_s \in M_{n \times n^*}(K)$ with $\beta_s/\simeq = s$ so that if $s^* = (1, 0)/\simeq$ then $\beta_{s^*} = (1, 0)$ and for $s = I/\simeq$ (here $I \in M_{n \times n^*}(K)$), $\beta_s = I$.

Now it is trivial to see that $S_n^*(K, L) = \{\delta\}$. Let $V^{(n)}$ denote the set of $K$-independent $n$-tuples from $V$. For $s \in S_n(K, L)$ and $a \in V^{(n)}$ define $s_a$ as $a \beta_s \in V^{n^*}$. Notice that for every $a \in V^{n^*}$ with $\text{dim}_K(a) = n$ there is a unique $s \in S_n(K, L)$ such that for some $b \in \pi'(a)(n)$, $a$ and $s_b$ are $L$-interdependent. Recall that for $\nu = (v_1, \ldots, v_n) \in V^n$, $\pi'(\nu) = K$-span$(v_1, \ldots, v_n)$. Now there is no more trouble with projecting types: for $a \in V^{n^*}$ we define $s_a$ as $a \beta_s = a$.

However the counterparts of Lemma 2.7, Theorems 2.8 and 2.13, Lemma 3.8 and Corollary 3.11 seem more interesting. The proofs are always parallel to the original ones, but easier, so we omit them.

4.3. **Proposition.** Suppose $s^i \in S_n(K, L)$, $i \in I$, $\{b^i \in V^{(n)}, i \in I\}$ is $K$-independent (that is $\pi'(b^i) \cap \pi'(\{b^j : j \neq i\}) = \{0\}$),

$$a \in V^{n^*} \cap L$-$\text{span} \{s^i_b, i \in I\}$$

and $\text{dim}(a) \leq n$. Then $\text{dim}(a) = n$ and if $a$ and $s^i_b$ are $L$-interdependent over $\{s^j_b, j \neq i\}$, then for some $b \in \pi'(a)(n)$, $a$ and $s^i_b$ are $L$-interdependent.

4.4. **Corollary.** $S(K, L)$ is finite. For $s \in S_n(K, L)$, $F_s = \{\beta \in M_{n \times n}(K) : \text{for some } \gamma \in L, \beta s \gamma = \beta \beta_s \} \cup \{0\}$, $F'_s = \{\gamma \in L : \text{for some } \beta \in F_s, \beta s \gamma = \beta \beta_s \}$. 

4.5. **Theorem.** Let $s \in S_n(K, L)$. For $A \subseteq V^{(n)}$ and $a \in V^{(n)}$, $a \in F_s$-$\text{span}(A)$ implies $s_a \in L$-$\text{span}\{s^i_b, b \in A\}$. Moreover, if $A$ is $K$-independent, then the converse is true, that is for $a \in V^{(n)}$, $s_a \in L$-$\text{span}\{s^i_b, b \in A\}$ iff $a \in F_s$-$\text{span}(A)$.

4.6. **Proposition.** $F_s$ and $F'_s$ are isomorphic, the function $\{(\beta, \gamma) \in F_s \times F'_s : \beta s \gamma = \beta \beta_s\}$ is an isomorphism of $F_s$ and $F'_s$.

4.7. **Remark.** $n' = \text{dim}_L(K^{n^*})$ is finite. If $K$ is finite then

$$\frac{|L|^{n' - 1}}{|L| - 1} = \sum_{s \in S_1(K, L)} \frac{|K| - 1}{|F_s| - 1}.$$ 

In Definition 1.19 we defined the notion of free decomposition of an ACL'-closed subset of $P^*$. Correspondence $\Phi$ from Theorem 2.2 translates this notion into the notion of free decomposition of an $L$-subspace of $V^{n^*}$.

4.8. **Definition.** Assume $W$ is an $L$-subspace of $V^{n^*}$. \( \{W_i, i \in I\} \) is a free decomposition of $W$ (in $V$) if

1. $W_i$ is an $L$-subspace of $V^{n^*}$, $W_i \neq \{0\}$,
2. $\pi'(W) = \bigoplus \{\pi'(W_i), i \in I\}$, in particular $\{\pi'(W_i), i \in I\}$ is $K$-independent,
3. $W_i = \pi'(W_i)^{n^*} \cap W$ and
(4) \( W = L \)-span(\( \bigcup_i W_i \)).

We say that \( W \) is decomposable if there is a free decomposition \( \{W_i, i \in I\} \) of \( W \) with \( |I| \geq 2 \). Otherwise we say that \( W \) is indecomposable.

4.9. Remark. Assume that there are finitely many isomorphism types of indecomposable \( L \)-subspaces of \( V^n \), and every \( L \)-subspace \( W \) of \( V^n \) has a free decomposition into indecomposables. Then \( I(K, L) \) is countable.

The following proposition corresponds to Remark 1.20 and Proposition 1.21.

4.10. Proposition. Assume \( n^* = 1 \) and \( n' = \dim_L(K^{n^*}) = 1 \). Then every indecomposable \( L \)-subspace of \( V^{n'} = V \) is isomorphic to \( aK, a \in V \setminus \{0\} \), and every \( L \)-subspace of \( V \) has a free decomposition into indecomposables. Every \( L \)-subspace of \( V \) is a \( K \)-subspace of \( V \).

In [Bu4] S. Buechler proved that if \( n^* = 1, n' = 2 \) (that is \( [K : L] = 2 \)), then \( I(K, L) \) is countable. He proves there that in this case every \( L \)-subspace \( W \) of \( V \) has a free decomposition \( \{W_i, i \in I\} \cup \{W_j, j \in J\} \) into indecomposables, such that \( \dim_K(\pi'(W_i)) = \dim_K(\pi'(W_j)) = 1 \) for every \( i, j \), every \( W_i \) is of the form \( aK \) and \( W_j \) of the form \( aL \) for some \( a \in V \).

Clearly the isomorphism type of \( W \) is determined by the powers of \( I \) and \( J \), hence \( I(K, L) = \aleph_0 \) here. Notice also, that \( W \) uniquely determines \( K \)-span(\( \bigcup_i W_i \)). Indeed, \( a \in K \)-span(\( \bigcup_i W_i \)) iff \( aK \subseteq W \). Call \( K \)-span(\( \bigcup_i W_i \)) the full part of \( W \).

It is important to know that we can get the remaining part of the decomposition of \( W \) in a rather arbitrary way. Namely, for any choice of \( \{W_j, j \in J\} \) such that \( W_j = a_j L \subseteq W \), if \( \{a_j, j \in J\} \) is \( K \)-independent over the full part of \( W \) and

\[ \pi'(W) = K \text{-span}(\{\pi'(W_i), i \in I\} \cup \{a_j, j \in J\}) \]

then \( \{W_i, i \in I\} \cup \{W_j, j \in J\} \) is a free decomposition of \( W \).

In the case \( n^* = 1 \) and \( [K : L] = n' \geq 4 \) [Bu4], it is proved that \( I(K, L) = 2^{\aleph_0} \) and \( I(K, L, \Gamma) = 2^{\aleph_0} \). A proof is essentially contained also in [DR] (which uses different language). In [BN] we analyze more deeply the problem of determining \( I(K, L) \) for various \( K, L \). Now let us check the impact on \( (P) \) of the few cases we considered above. This is summarized in the following proposition. \( n_a \) and \( n_b \) play the role of \( n^* \) here, and \( n_d, n_e \) the role of \( n' \).

4.11. Proposition. (1) Assume \( n_a = 1 \). Then \( n_d \leq 4 \). Moreover, if \( n_d = 1 \) or 2, then \( I(p, \aleph_0) \) is countable.

(2) Assume \( n_b = 1 \). Then \( n_e \leq 4 \). Moreover, if \( n_e = 1 \) or 2 then \( I(p, \aleph_0) \) is countable.

The rest of this paper is devoted to the proof of this proposition. Since parts (1) and (2) are similar, we shall concentrate on (1). We assume that we have added to the signature a sufficiently large finite subset of \( \text{acl}(\emptyset) \), as required in Theorem 2.2. Assume \( n_a = 1 \). If \( n_d \geq 4 \) then as mentioned above, \( I(F_q, F, \Gamma_q) = 2^{\aleph_0} \). By Corollary 2.3, \( I(T, \aleph_0) = 2^{\aleph_0} \), a contradiction. Hence \( n_d < 4 \). Now suppose \( n_d = 1 \) or 2. Theorem 4.2 gives a finite set \( E \) such that \( I(p|E, \aleph_0) \) is countable. However, since \( E \) is not embeddable into every model of \( T \), we cannot conclude immediately that \( I(p, \aleph_0) \) is countable (this is the problem of parameters from the title of this section). To do this we have to prove in each case that there are countably many isomorphism types of
ACL-closed countably dimensional subsets of $P^*$. But we know by Corollary 2.3 that there are countably many isomorphism types of $ACL'$-closed countably dimensional subsets of $P^*$.

We shall consider only the case $n_d = 2$, as the case $n_d = 1$ is similar and easier. Let us enumerate $acl(\emptyset)$ as $\{b_n, n < \omega\}$, and let $\{p_i, i < \omega\}$ be an enumeration of all complete stationary types over finite subsets of $acl(\emptyset)$, with $p_i$ being over $\{b_n, n < i\}$. Suppose $r = (r_1, \ldots, r_n)$ is an $ACL'$-independent tuple of types from $P^*$, $\zeta \subseteq Q$ is a basis of $\pi(r)$. We say that $(r, \zeta)$ is $k$-determined if whenever $a_i$ realizes $r_i$ and $a = (a_1, \ldots, a_n)$ then $tp(ac/\{b_i: i < k\})$ is stationary and parallel to $p_i$ (for some $i < k$). Then we say that $(r, a)$ corresponds to this $p_i$.

4.12. Lemma. Assume $R \subseteq P^*$ is $ACL$-closed, $A = \pi(R)$ and either $P^*_a \subseteq ACL'(R)$ or for no $a \in A$, $P^*_a \subseteq ACL'(R)$. Then there is $k < \omega$ such that $R \subseteq ACL'(R_k)$, where $R_k = \{r: \text{for some } a \in A, r \subseteq P^*_a \cap R \text{ is an } ACL'$-basis of $P^*_a \cap R, P^*_a \cap R \not\subseteq ACL'(\emptyset)$ and $(r, a)$ is $k$-determined\}.

Proof. First let us consider the case when $P^*_a \subseteq ACL'(R)$. Suppose the lemma is false. Then we can choose $n_1$, $i < \omega$, so that $n_0$ is the minimal $k$ such that $R_k \not\subseteq ACL'(\emptyset)$ and $n_{i+1}$ is the minimal $k$ such that $R_k \not\subseteq ACL'(R_{n_i})$. Of course, $R \subseteq \bigcup_{n \in \omega} R_n$.

Choose $r_i \in R_{n_i} \setminus ACL'(\bigcup_{k < n_i} R_k)$ and $a_i \in A$ such that $(r_i, a_i)$ is $n_i$-determined. Let $R^0 = R \cap P^*_a$. Hence $R \downarrow P^*_a(R^0)$. For $X \subseteq \omega$ let $R(X) = ACL(R^0 \cup \{r_i: i \in X\})$. Notice that $R(X) \subseteq R$. We can recover the set $\{n_i: i \in X\}$ from $R(X)$ as follows.

Let $R(X)_k$ be defined just as $R_k$ but with $R$ replaced everywhere by $R(X)$. Define by induction a sequence $m_i$, $i < \omega$, so that $m_0$ is the minimal $k$ such that $R(X)_k \not\subseteq ACL'(\emptyset)$ and $m_{i+1}$ is the minimal $k$ such that $R(X)_k \not\subseteq ACL'(R(X)_{m_i})$. We see that $m_i$, $i < \omega$, is an increasing enumeration of $\{n_i: i \in X\}$.

This shows that if $X \neq Y \subseteq \omega$ then $R(X), R(Y)$ are nonisomorphic. This gives $I(T, \aleph_0) = 2^{2^{\aleph_0}}$.

The case when for no $a \in A$, $P^*_a \subseteq ACL'(R)$ is handled similarly, modulo the following claim, which follows from the discussion of the case $n^* = 1$, $n^* = 2$ before Proposition 4.11.

4.13. Claim. Suppose $A' = \{a_i, i \in J\} \subseteq A$ is independent, $r_i \subseteq P^*_a$ is an $ACL'$-basis of $P^*_a \cap R$ and $P^*_a \cap R \not\subseteq ACL'(\emptyset)$. Then

$$P^*_a \cap R \subseteq ACL'(\{r_i, i \in J\}).$$

Now we can finish the proof of Proposition 4.11(1) in case $n_d = 1$, $n_d = 2$. Let $R$ be an $ACL$-closed subset of $P^*$ with countable dimension, let $R^0 = R \cap P^*_a$, $A = \pi(R)$, $R' = ACL'(R)$ and let $W = \Psi(R')$, $\Psi$ being defined in Corollary 2.3. By the discussion before Proposition 4.11, there is a free decomposition $\{W_i, i \in I\} \cup \{W_j, j \in J\}$ of $W$, as described there. We can assume that for every $j \in J$, $\Psi^{-1}(W_j) \cap R \not\subseteq ACL'(\emptyset)$. Let $W_l = F-span(\cup_{i \in I} W_i)$, $W_j = F-span(\cup_{j \in J} W_j)$, $R_l = \Psi^{-1}(W_l) \cap R$, $R_j = \Psi^{-1}(W_j) \cap R$. We have $R_l \downarrow R_j(R^0)$, $R = ACL(R_lR_jR^0)$. Let $r^0$ be an $ACL$-basis of $R^0$. By Lemma 4.12 choose $k < \omega$ such that $R_l, R_j$ and $(r^0, \emptyset)$ are $k$-determined. Then we can find $k$-determined $(r_i, a_i)$, $i \in I$, $(r_i, a_j)$, $j \in J$ such that

(1) for $i \in I, a_i \in \pi(R_l)$, $r_i \subseteq P^*_a \cap R$ is an $ACL'$-basis of $P^*_a \cap R$,
(2) for $j \in J$, $a_j \in \pi(R_j)$, $P_{a_j}^* \cap R \not\subseteq \text{ACL}'(\emptyset)$ and $r_j \subseteq P_{a_j}^* \cap R$ is an $\text{ACL}'$-basis of $P_{a_j}^* \cap R$.

(3) $\{a_i, i \in I\} \cup \{a_j, j \in J\}$ is independent and $A \subseteq \text{acl}(\{a_i, i \in I\} \cup \{a_j, j \in J\})$. For $t < k$ let $I_t = \{i \in I: (r_i, a_i) \text{ corresponds to } p_t\}$, $J_t = \{j \in J: (r_j, a_j) \text{ corresponds to } p_t\}$ and $i_0$ is the minimal $i$ such that $(r_0, \emptyset)$ corresponds to $p_i$. W.l.o.g. add $\{b_t, t < k\}$ to the signature. We see that the sequences $(|I_t|: t < k)$, $(|J_t|: t < k)$ and $i_0$ determine the isomorphism type of $R$.

References


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