EIGENVALUES AND EIGENSACES FOR THE TWISTED DIRAC OPERATOR OVER $SU(N, 1)$ AND $Spin(2N, 1)$

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Abstract. Let $X$ be a symmetric space of noncompact type whose isometry group is either $SU(n, 1)$ or $Spin(2n, 1)$. Then the Dirac operator $D$ is defined on $L^2$-sections of certain homogeneous vector bundles over $X$. Using representation theory we obtain explicitly the eigenvalues of $D$ and describe the eigenspaces in terms of the discrete series.

1. Introduction

Let $G$ be a connected real reductive Lie group. From now on we fix a maximal compact subgroup $K$ of $G$. Let $g_0 = k_0 \oplus p_0$ be the Cartan decomposition of the Lie algebra of $G$, with $k_0$ the Lie algebra of $K$, and let $h_0$ be a Cartan subalgebra of $k_0$. We denote by $g, k, p, h$ the complexifications of $g_0, k_0, p_0, h_0$, and let $\Phi(h, g)$ be the root system of $(g, h)$. Let $\Phi_k$ and $\Phi_n$ be the compact and noncompact rootspaces of $\Phi(h, g)$ respectively; fix $\Phi^+ = \Phi_k^+ \cup \Phi_n^+$, a positive root system; and denote by $\rho$ one-half of the sum of the positive roots of $\Phi(h, g)$.

Let $(\tau, V)$ be a representation of $K$. We denote

$$C^\infty(G/K, V) = \{ f : G \to V, \ C^\infty | \ f(gk) = x(k)^{-1}f(g) \ \forall k \in K \},$$

$$L^2(G/K, V) = \{ f : G \to V \ | \ f(gk) = \tau(k)^{-1}f(g) \ \forall k \in K, \ ||f||_2^2 < \infty \}$$

where $|| ||_2$ is the $L^2$-norm with respect to a fixed Haar measure. Both spaces are representations of $G$ under the left regular action.

Let $V_\sigma$ be an irreducible representation of $K$ with maximal weight $\sigma$ relative to $\Phi_k^+$. The Dirac operator defines a map

$$D : L^2(G/K, V_\sigma \otimes S) \rightarrow L^2(G/K, V_\sigma \otimes S)$$

as in (3.1). $D$ is an elliptic essential selfadjoint $G$-invariant operator.

In this paper the eigenvalues of the Dirac operator are explicitly obtained for $G = SU(n, 1)$ and $Spin(2n, 1)$, and with $\sigma$ far from the walls of the Weyl chambers. In additions, the respective eigenspaces are expressed as a finite...
sum of discrete series using the Harish-Chandra parametrization of the discrete series. To obtain this we derive specific results for these groups which say when a discrete series occurs in $L^2(G/K, V_a \otimes S)$; furthermore, its multiplicity is a power of two. For the case of $G = \text{Sp}(2, \mathbb{R})$, we give examples of discrete series which occur in $L^2(G/K, V_a \otimes S)$ with multiplicity different from a power of two. In general, we show that each discrete series occurring in an eigenspace for a nonzero eigenvalue has even multiplicity. For the kernel the multiplicity is one.

2. Notation

In this section we fix notation and give some known results.

2.1. Let $G$ be a connected real reductive Lie group and, from now on, let $K$ denote a fixed maximal compact subgroup of $G$. Assume that the rank of $G$ is equal to the rank of $K$. Let $g_0 = k_0 \oplus p_0$ be the Cartan decomposition of the Lie algebra of $G$, with $k_0$ the Lie algebra of $K$; and let $h_0$ be a Cartan subalgebra of $k_0$. Because of the rank condition $h_0$ is also a Cartan subalgebra of $G$. The complexification of any Lie algebra is denoted without the subscript. So if $\Phi(h, g)$ is the root system of $g$ (resp. $h$) and $\Phi(h, k)$ that of $k$ (resp. $h$), then $\Phi(h, k) \subset \Phi(h, g)$. $\Phi(h, k) = \Phi_k$ is called the set of compact roots of $\Phi(h, g)$. The complement of $\Phi_k$ is called the set of noncompact roots and is denoted by $\Phi_n$. Let $\Phi^n_+ \subset \Phi_n$ be a fixed positive root system of $\Phi^+_n$. One can choose a subset $\Phi^n_+$ of $\Phi_n$ such that $\Phi^+ = \Phi^n_+ \cup \Phi_+^+$ is a positive root system of $\Phi(h, g)$. The choice of $\Phi^n_+$ is not unique: there are exactly $|W_G|/|W_K|$ choices, where $W_G$ is the Weyl group of $g$ and $W_K$ is that of $k$. When necessary, we will say explicitly which choice will be taken.

Denote by

$$\rho_k = \frac{1}{2} \sum_{\alpha \in \Phi^+_k} \alpha, \quad \rho_n = \frac{1}{2} \sum_{\alpha \in \Phi^+_n} \alpha$$

and by $\rho = \rho_k + \rho_n$. When $\rho$ is not analytically integral in $G$, fix a twofold cover of $G$, which will be also denoted by $G$ without causing confusion, and call $K$ the inverse image of $K$.

2.2. The Killing form is defined at $g_0$ by

$$B(X, Y) = \text{Trace}(\text{ad} X \text{ ad} Y).$$

Its restriction to $h$ is nondegenerate and negative definite, so $-B(\ , \ )$ is an inner product on $h_0$ which gives one on $i h_0$. Let $(i h_0)'$ be the real dual of $i h_0$ and denote by $(\ , \ )$ the inner product at $(i h_0)'$ which comes from the Killing form. Also, $B$ is positive definite in $p_0$ and the $K$-representation on $p_0$ is orthogonal.

Because of the last condition of (2.1), the representation

$$K \to SO(p_0) \simeq SO(\dim p_0)$$

given by the adjoint representation lifts to the universal cover $\text{Spin}(p_0)$ of $SO(p_0)$; that is, the usual spin representation $S$ of $\text{Spin}(p_0)$ gives rise to a $K$-module. Let $(s, S)$ denote this $K$-module.
2.3. Let \((\pi, H)\) be a representation of \(G\) on the Hilbert space \(H\). Without
lost of generality we can suppose that \(\pi(K)\) acts by unitary operators. Hence
\(H\) is an orthogonal sum of irreducible representations of \(K\) as a \(K\)-module
\[
H = \bigoplus_{\tau \in \hat{K}} m(\tau) V_\tau
\]
where \(\hat{K}\) is the set of equivalence classes of irreducible representations of \(K\);
the multiplicity \(m(\tau)\) is a nonnegative integer or \(+\infty\). The subspace \(m_\tau V_\tau\)
is the isotypic \(K\)-submodule of type \(\tau\) of \((\pi, H)\). It is usually denoted by \(H[\tau]\).

We say that \((\pi, H)\) is an admissible representation if \(\pi(K)\) acts by unitary
operators and \(m_\tau\) is finite for all \(\tau \in \hat{K}\).

An admissible representation \((\pi, H)\) is a discrete series if it is irreducible
and all its matrix coefficients \(g \to \langle \pi(g)u, v \rangle\) (with \(u, v \in V_K\) ) are square
integrable.

All discrete series can be parametrized by weights \(\lambda \in (i h_0)'\), the dual of
\(i h_0\), such that \(\lambda\) is nonsingular (i.e., \((\lambda, \alpha) \neq 0 \ \forall \alpha \in \Phi(h, g)\)), and \(\lambda + \rho\) is
integral (i.e., \(\lambda(H) \in 2\pi i \mathbb{Z}, \ \forall H \in i h_0\) such that \(\exp H = 1\) ). The discrete series
\(H_\lambda\) of parameter (or Harish-Chandra parameter) \(\lambda\) has infinitesimal character
\(\chi_\lambda\), and two discrete series are equivalent if and only if their parameters are
conjugate by an element of the Weyl group of \(K\).

2.4. Let \(f \in C^\infty(G/K, V)\) or \(f \in L^2(G/K, V)\) and consider the action of
\(G\) given by
\[
\pi(g)f(x) = f(g^{-1}x).
\]
We also require the action of the elements of \(g_0\) as left-invariant differential
operators, that is, if \(X \in g_0\)
\[
X f(x) = \frac{d}{dt} \bigg|_{t=0} f(x \exp tX).
\]
Now if \(Z = X + iY \in g\), we define \(Z f = X f + iY f\). Then each \(D \in (\mathbb{Z}(g) \otimes \operatorname{End}(V))^K\) defines a left-invariant differential operator on \(C^\infty(G/K, V)\)
[Wa, Chapter 5]. \(G\) acts on \((\mathbb{Z}(g) \otimes \operatorname{End}(V))^K\) by \(\operatorname{Ad} \otimes (\text{repres. of } K \text{ on }
\operatorname{End}(V))\)

2.5. If \(\{X_i\}\) is an orthonormal base of \(g\) (with respect to the Killing form),
the Casimir element \(\Omega\) is defined by
\[
\Omega = \sum X_i X_i.
\]
It is known that \(\Omega\) belongs to the center of \(\mathbb{Z}(g)\). The Casimir operator acts
on a discrete series \(H_\lambda\) by the constant \(\|\lambda\|^2 - \|\rho\|^2\). An explicit expression for
the Casimir can be computed as follows. Let \(\{H_\alpha\}\) be an orthonormal basis of
\(i h_0\), and for each \(\alpha \in \Phi(h, g)\), let
\[
g_\alpha = \{X \in g / \operatorname{ad}(H) = \alpha(H)X \ \forall H \in h\}.
\]
Choosing appropriately \(X_\alpha \in g_\alpha\), \(\Omega\) is given by
\[
\Omega = \sum_{\alpha \in \Phi^+} H_\alpha^2 + \sum_{\alpha \in \Phi^+} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha) = \sum_{\alpha \in \Phi^+} H_\alpha^2 + \sum_{\alpha \in \Phi^+} (H_\alpha + 2X_{-\alpha} X_\alpha).
\]
3. Eigenvalues of D

If we fix a minimal left ideal in the Clifford algebra of $p_0$, the resulting representation of $so(p_0)$ breaks into two irreducible representations. Composed with the adjoint action of $k_0$ on $p_0$, this lifts to a representation $S$ of $K$, called the spin representation. Let $\{X_i\}_{i=1}^{2n}$ be an orthonormal base of $p_0$, let $c$ be the operation of left Clifford multiplication and let $V_\sigma$ be an irreducible representation of $K$ of maximal weight $\sigma$ ($\Phi_k^+$-dominant). The Dirac operator

$$D : L^2(G/K, V_\sigma \otimes S) \rightarrow L^2(G/K, V_\sigma \otimes S)$$

is defined by

$$D = \sum_{i=1}^{2n} (1 \otimes c(X_i)) X_i$$

where the $X_i$ act as left-invariant differential operators for all $i$. The spin representation $S$ decomposes into a sum of two subrepresentations $S = S^+ \oplus S^-$. If $X \in p_0$, then $c(X)S^\pm = S^\mp$, so

$$D^\pm : L^2(G/K, V_\sigma \otimes S^\pm) \rightarrow L^2(G/K, V_\sigma \otimes S^\mp)$$

are also well defined.

We list some properties of the Dirac operator $D$. $D$ is an elliptic $G$-invariant differential operator, and as the riemannian metric of $G/K$ is complete, $D$ and $D^2$ are essentially selfadjoint in $L^2(G/K, V_\sigma \otimes S)$ [W]; that is, the minimal extension is the unique selfadjoint closed extension starting from the set of smooth compactly supported functions. So, we consider $D$ densely defined by this extension, which coincides with the maximal one [A]. The eigenvalues of $D$ are defined as the eigenvalues of the unique selfadjoint extension.

Let $L^2_d$ be the closure of the sum of all irreducible $G$-invariant closed subspaces of $L^2(G/K, V_\sigma \otimes S)$; Harish-Chandra has proved that $L^2_d$ is the direct sum of a finite number of square integrable $G$-irreducible closed subspaces, that is a finite sum of discrete series

$$L^2 \simeq \bigoplus_{\lambda \in F} n_\lambda H_\lambda$$

with $F$ a finite set and $n_\lambda$ the multiplicity of the discrete series $H_\lambda$ with parameter $\lambda$.

A theorem of Connes and Moscovici [C-M] ensures that if

$$D : L^2(G/K, V_\sigma \otimes S) \rightarrow L^2(G/K, V_\sigma \otimes S)$$

is an elliptic $G$-invariant operator, each eigenspace of $D$ is a finite sum of discrete series and $D$ has a finite number of eigenvalues.

Take $\Phi^+$ such that $\sigma$ is a $\Phi^+$-dominant weight. If $\Omega$ is the Casimir element of the universal enveloping algebra $\mathcal{U}(g)$ of $g$, the Parthasarathy equality for the square of the operator $D$ [A-S] is

$$D^2 = -\Omega + (\sigma - \rho_n, \sigma - \rho_n + 2\rho)I.$$

This equality restricted to an immersion of a discrete series $H_\lambda$ (with infinitesimal character $\chi_\lambda$) in $L^2_d$ is

$$D^2|_{H_\lambda} = \left(-\|\lambda\|^2 + \|\rho\|^2 + (\sigma - \rho_n, \sigma - \rho_n + 2\rho)\right)I$$
because the Casimir acts on $H_\lambda$ by the constant $||\lambda||^2 - ||\rho||^2$ (see (2.5)).

Recall that $n_\lambda$ denotes the multiplicity of the discrete series with parameter $\lambda$ which occur in $L^2(G/K, V_\sigma \otimes S)$, that is

$$n_\lambda = \dim \text{Hom}_G \left( H_\lambda, L^2(G/K, V_\sigma \otimes S) \right) = \dim \text{Hom}_K \left( H_\lambda, V_\sigma \otimes S \right)$$

by Frobenius reciprocity. If the maximal weight $\sigma$ of $V_\sigma$ is sufficiently far from the walls of the Weyl chambers of $K$, or more precisely, if

$$\sigma + \gamma, \alpha > 0 \quad \forall \gamma \in P(S), \forall \alpha \in \Phi^+_K \quad \text{(3.5)}$$

with $P(S)$ the set of weight of $S$, then,

$$V_\sigma \otimes S = \bigoplus_{\gamma \in P(S)} V_{\sigma + \gamma} \quad \text{(3.6)}$$

where $V_{\sigma + \gamma}$ is the irreducible $K$-module with maximal weight $\sigma + \gamma$. This happens because the multiplicity of each weight of $S$ is one, and

$$\chi_{V \otimes S} = \chi_V \cdot \chi_S = \Delta_K^{-1} \sum_{w \in W_K} \det w \ e^{w(\sigma + \rho_k)} \sum_{\gamma \in P(S)} e^\gamma$$

$$= \Delta_K^{-1} \sum_{w \in W_K} \sum_{\gamma \in P(S)} \det w \ e^{w(\sigma + \rho_k) + \gamma} = \Delta_K^{-1} \sum_{w \in W_K} \sum_{\gamma \in P(S)} \det w \ e^{w(\sigma + \rho_k)}$$

$$= \sum_{\gamma \in P(S)} \chi_{\sigma + \gamma} \quad \text{(by (3.5))}$$

where $\chi_w$ denotes the character of the $K$-module $W$. By (3.6), we have that

$$n_\lambda = \sum_{\gamma \in P(S)} \dim \text{Hom}_K \left( H_\lambda, V_{\sigma + \gamma} \right). \quad \text{(3.7)}$$

So, we only have to analyse when the isotypic component $(H_\lambda[\sigma + \gamma])_\gamma$ of the representation $H_\lambda$ restricted to $K$ of maximal weight $\sigma + \gamma$, is not zero. In the cases $G = SU(n, 1)$ and $G = Spin(2n, 1)$ it is known that if $H_\lambda[\sigma + \gamma] \neq 0$, then $H_\lambda[\sigma + \gamma]$ is irreducible because each $K$-type of any principal series has this property; that is,

$$n_\lambda = | \{ \gamma \in P(S) : H_\lambda[\sigma + \gamma] \neq 0 \} |. \quad \text{(3.8)}$$

Denote by $\Eig(D)$ the set of eigenvalues of $D$, and by $W_\alpha(D)$ the eigenspace of the operator $D$ associated to the eigenvalue $\alpha$.

**Proposition 3.1.** Let $D$ be the Dirac operator defined in $L^2(G/K, V_\sigma \otimes S)$. Then,

(i) If $\beta \in \Eig(D^2)$, $\beta \neq 0$, and $\alpha$ is the positive square root of $\beta$,

$$W_{\alpha^+}(D^2) = W_{\alpha}(D) \oplus W_{-\alpha}(D) \quad \text{and} \quad W_0(D^2) = W_0(D).$$

(ii) If $\alpha$ is a nonzero eigenvalue of $D$, $W_\alpha(D)$ is equivalent to $W_{-\alpha}(D)$ as a $G$-module, so that each discrete series which occurs in $W_{\alpha^+}(D^2)$ has even multiplicity.

(iii) $L^2_d = \bigoplus_{\alpha \in \Eig(D)} W_\alpha(D)$.

(iv) The set of the eigenvalues of $D^2$ is

$$\Eig(D^2) = \{ -||\lambda||^2 + ||\sigma + \rho_k||^2 \mid \lambda \text{ is a } \Phi^+_k \text{-dominant Harish-Chandra parameter and } H_\lambda[\sigma + \gamma] \neq 0 \text{ for some } \gamma \in P(S) \}$$
and the set of the eigenvalues of $D$ is

$$\text{Eig}(D) = \left\{ \alpha : \alpha^2 \in \text{Eig}(D^2) \right\}.$$ 

**Note.** Using the Atiyah-Schmid result, which ensures that the kernel of $D$ is equivalent to $H_{\sigma + p_k}$, this proposition says that the multiplicity of each discrete series which occurs in $L_d^2$ is even except for $H_{\sigma + p_k}$.

**Proof.** Since $\beta = \|Df\|^2/\|f\|^2 > 0$, it makes sense to take the positive square root $\alpha$.

(i) Since $D^2$ is an essentially selfadjoint operator its eigenvalues are real. If $\beta \neq 0$, let $f \in W_\beta(D^2)$, then $f \pm \alpha^{-1} Df \in W_{\pm \alpha}(D)$, with $\alpha$ the positive square root of $\beta$, because

$$D(f \pm \alpha^{-1} Df) = Df \pm \alpha^{-1} D^2f = Df \pm \alpha f = \pm \alpha(\pm \alpha^{-1} Df + f).$$

Then, since

$$f = \frac{1}{2}(f + \alpha^{-1} Df) + \frac{1}{2}(f - \alpha^{-1} Df)$$

we have that $W_{\alpha^2}(D^2) \subset W_\alpha(D) \oplus W_{-\alpha}(D)$.

$D^2$ is essentially selfadjoint, so if $f$ is in the domain of $D^2$, then

$$(D^2f, f) = (Df, Df).$$

If $f$ also is in the kernel of $D^2$, $\|Df\| = 0$, that is $Df = 0$; and as the kernel of $D^2$ is closed, $W_0(D^2) = W_0(D)$.

(ii) If $f \in L^2(G/K, V_\sigma \otimes S) = L^2(G/K, V_\sigma \otimes S^+) \oplus L^2(G/K, V_\sigma \otimes S^-)$, then $f = (f^+, f^-)$ and $Df = (D^-f^-, D^+f^+)$ because of (3.2). The map

$$W_\alpha(D) \rightarrow W_{-\alpha}(D), \quad (f^+, f^-) \rightarrow (f^+, -f^-)$$

is really an isomorphism between $W_\alpha(D)$ and $W_{-\alpha}(D)$. In fact,

$$D(f^+, -f^-) = (D^-f^-, D^+f^+) = (-\alpha f^+, \alpha f^-) = -\alpha(f^+, -f^-).$$

(iii) The equality (3.4) implies that each discrete series in $L_d^2$ is in an eigenspace of $D^2$, the eigenvalue depends on the norm of the parameter $\lambda$. Then $L_d^2$ is the sum of eigenspaces of $D^2$, and by (i), we have

$$L_d^2 \cong \bigoplus_{\beta \in \text{Eig}(D^2)} W_\beta(D^2) \cong \bigoplus_{\alpha \in \text{Eig}(D)} W_\alpha(D).$$

(iv) The equality (3.7) ensures that $n_\lambda \neq 0$ if and only if $H_{\lambda}[\sigma + \gamma] \neq 0$ for some $\gamma \in P(S)$. Then by the equality (3.4) and (iii) if $H_{\lambda}[\sigma + \gamma] \neq 0$ for some $\gamma \in P(S)$, one has that $H_{\lambda} \in \text{Eig}(D^2)$. But

$$\|p\|^2 + (\sigma - \rho_n, \sigma - \rho_n + 2p) = (\rho, \rho) + 2(\sigma - \rho_n, \rho) + (\sigma - \rho_n, \sigma - \rho_n)$$

$$= (\sigma - \rho_n + \rho, \sigma - \rho_n + \rho) = \|\sigma + \rho_k\|^2.$$ 

Thus,

$$\text{Eig}(D^2) = \{ -\|\lambda\|^2 + \|\sigma + \rho_k\|^2 | \lambda \text{ is a } \Phi_k^*\text{-dominant Harish-Chandra parameter, and } H_{\lambda}[\sigma + \gamma] \neq 0 \text{ for any } \gamma \in P(S) \}. \quad \square$$
4. \( G = SU(n, 1) \)

Let \( K \) be the usual immersion of \( S(U(n) \times U(1)) \) in \( G \), so \( K \) is a maximal compact subgroup of \( G \). Let \( T \) be the torus of diagonal matrices of \( K \), so \( T \) is also a compact Cartan subgroup of \( G \). Let \( g_0 \), \( k_0 \), \( h_0 \) be their Lie algebras and \( g \), \( k \), \( h \) the complexifications. Choose an orthonormal base \( \{H_1, \ldots, H_n\} \) of the real Lie algebra \( ih_0 \) with respect to \(-B(\ ,\ )\), where \( B \) is the Killing form of \( g \) (\( B(X, Y) = \frac{1}{n} \text{tr}(XY) \)).

If \( H = \sum ih_j E_{jj} \in ih_0 \), let \( e_j \in (ih_0)' \) be given by
\[
e_j(H) = h_j, \quad j = 1, \ldots, n + 1.
\]
Denote by \( (\ ,\ ) \) the dual symmetric form to the Killing form of \( g \).

The root set of \( (g, h) \) is
\[
\Phi(h, g) = \{e_i - e_j: i \neq j, i, j = 1, \ldots, n + 1\}
\]
and
\[
\Phi_k = \{e_i - e_j: i \neq j, i, j = 1, \ldots, n\}, \quad \Phi_n = \{\pm(e_i - e_{n+1}): i = 1, \ldots, n\}.
\]
Fix\(^{(4.1)}\)
\[
\Phi^+_k = \{e_i - e_j: i < j < n + 1\}.
\]

The number of choices of \( \Phi^+_n \) such that \( \Phi^+_k \cup \Phi^+_n \) is a positive root system of \( \Phi(h, g) \) is \( n + 1 = |W_G|/|W_K| \), because \( W_G \) is the set of permutations of \( n + 1 \) elements and \( W_K \) that of \( n \) elements. The different \( \Phi^+_n \) are\(^{(4.2)}\)
\[
\Psi^r = \{e_i - e_{n+1}: 1 \leq i \leq r - 1\} \cup \{-e_i + e_{n+1}: r \leq i \leq n\}
\]
with \( 1 \leq r \leq n + 1 \).

From now on fix \( r \) such that \( \Phi^+_n = \Psi^r \), then
\[
\rho_k = \frac{1}{2} \sum_{i<j<n+1} (e_i - e_j) = \frac{1}{2} \sum_{i=1}^{n} (n - 2i + 1)e_i,
\]
\[
\rho_n = \frac{1}{2} \left( \sum_{i=1}^{r-1} e_i + \sum_{i=r}^{n} (n - 2r + 2)e_{n+1} \right),
\]
\[
\rho = \frac{1}{2} \left( \sum_{i=1}^{r-1} (n - 2i + 2)e_i + \sum_{i=r}^{n} (n - 2i)e_i + (n - 2r + 2)e_{n+1} \right).
\]

Let \( \lambda \in (ih_0)' \) be an integral weight. Then \( \lambda \) satisfies \( \lambda = \sum_{i=1}^{n+1} \lambda_i e_i \) with \( \sum_{i=1}^{n+1} \lambda_i = 0 \) because the element \( H^\lambda = \sum_{j=1}^{n+1} i\lambda_j E_{jj} \in ih_0 \) such that \( \lambda = -B(\ ,\ H^\lambda) \) has \( \text{Trace}(H^\lambda) = 0 \). Moreover, \( \|e_j - e_{j+1}\| = 2 \) gives
\[
\frac{2(\lambda, e_j - e_{j+1})}{\|e_j - e_{j+1}\|^2} = (\lambda, e_j - e_{j+1}) = \lambda_j - \lambda_{j+1} \in \mathbb{Z} \quad \forall j = 1, \ldots, n.
\]
This implies that for some \( s \in \mathbb{Z}, 0 \leq s < n + 1 \),
\[
\lambda_i = m_i + \frac{s}{n+1}, \quad m_i, s \in \mathbb{Z} \quad \forall i = 1, \ldots, n + 1.
\]
Also note that \( \lambda \) is a \( \Phi^+_k \)-dominant weight if and only if
\[
\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1
\]
and it is $\Psi^r$-dominant if and only if

$$
\lambda_r \leq \lambda_{n+1} \leq \lambda_{r-1}.
$$

Suppose $\lambda$ is a $\Phi^+$-dominant Harish-Chandra parameter. Then as $\lambda + \rho$ and $\rho$ are integral (as $SU(n, 1)$ is simply connected, $\rho$ is integral for any positive root system), $\lambda$ satisfies (4.4), and since $\lambda$ also is nonsingular, at (4.5) and (4.6) the strict inequalities hold.

To determine when a $K$-type occurs at a discrete series of $G$, fix $\Phi^+ = \Phi^+_k \cup \Psi^r$. Denote by $m_\lambda(\tau)$ the multiplicity of the irreducible representation of highest weight $\tau$ in $H_\lambda$.

**Proposition 4.1.** Let $\lambda = \sum_{i=1}^{n+1} \lambda_ie_i$ be a Harish-Chandra parameter of a discrete series of the group $SU(n, 1)$ which is $(\Phi^+_k \cup \Psi^r)$-dominant, and let $\tau = \sum_{i=1}^{n+1} \tau_ie_i$ be a $\Phi^+_k$-dominant weight. If $\mu = \lambda + \rho_n - \rho_k = \sum_{i=1}^{n+1} \mu_ie_i$, then

$$
m_\lambda(\tau) = 1 \Leftrightarrow \left\{ \begin{array}{l}
\tau_n \leq \mu_n \leq \tau_{n-1} \leq \cdots \leq \tau_r \leq \mu_r < \mu_{r-1} \leq \tau_{r-1} \leq \cdots \leq \mu_1 \leq \tau_1, \\
\tau_i - \mu_i \in \mathbb{Z} \quad \forall i = 1, \ldots, n.
\end{array} \right.
$$

**Proof.** If $\tau' = \tau + \rho_k$ and $\mu' = \mu + \rho_k$, then the inequality of the proposition is equivalent to

$$
\tau_n' \leq \mu_n' < \tau_{n-1}' \leq \cdots < \tau_r' \leq \mu_r' < \mu_{r-1}' \leq \tau_{r-1}' \leq \cdots < \mu_1' \leq \tau_1'
$$

because $(\rho_k)_i = (\rho_k)_i + 1$ for each $i$.

The Blattner formula is

$$
m_\lambda(\tau) = \sum \det s Q(s^{-1}\tau' - \mu')
$$

where $Q(\sigma)$ is the number of expressions of the weight $\sigma$ as a sum of positive noncompact roots.

Suppose $m_\lambda(\tau) \neq 0$, so $Q_s = Q(s^{-1}\tau' - \mu') \neq 0$ for some $s \in W_K$. Since $\Phi^+ = \Phi^+_k \cup \Psi^r$, from (4.2) we get $(s^{-1}\tau' - \mu', e_i) \in \mathbb{Z}$ and

$$
(s^{-1}\tau' - \mu', e_i) \left\{ \begin{array}{l}
\geq 0, \\
\leq 0,
\end{array} \right. \quad 1 \leq i \leq r - 1,
$$

$$
(s^{-1}\tau' - \mu', e_i) \leq 0, \quad r \leq i \leq n,
$$

because $s^{-1}\tau' - \mu' = \sum_{i=1}^{n} n_i(e_i - e_{n+1})$ with $n_i \geq 0$ for $i < r$ and $n_i \leq 0$ for $r \leq i < n + 1$. Now $W_K$ is the permutation set of the elements $\{e_1, \ldots, e_n\}$, so if $\pi$ is a permutation of $n$ elements, then

$$
(s^{-1}\tau' - \mu')_{(i)} = \left\{ \begin{array}{l}
\tau'_{\pi(i)} - \mu'_i \geq 0, \\
\tau'_{\pi(i)} - \mu'_i \leq 0,
\end{array} \right. \quad 1 \leq i \leq r - 1,
$$

$$
(s^{-1}\tau' - \mu')_{(i)} \leq 0, \quad r \leq i \leq n.
$$

Since $\mu_n' < \mu_{n-1}' < \cdots < \mu_1'$, (4.8) ensures that $\pi$ leaves invariant the sets $\{1, \ldots, r - 1\}$ and $\{r, \ldots, n\}$, because if $1 \leq i < r$ and $r \leq j \leq n$ (because $\tau$ is dominant), then $\tau'_{\pi(i)} \leq \mu'_j < \mu'_i \leq \tau'_{\pi(i)}$, implies $\pi(j) > \pi(i)$ $\forall i, j$ in the given intervals.

Let $H$ be the permutation set that permute the $\tau'_i$'s in each interval $[\mu'_i, \mu'_{i-1})$ with $1 \leq i < r$ ($\mu'_0 = \infty$). For $s_1 \in H$, since $Q_s = Q_{ss_1}$,

$$
m_\lambda(\tau) = \sum \det s Q_s = \sum \det s Q_{ss_1} = \sum \det s(s_1)^{-1} Q_s = \det(s_1)^{-1} m_\lambda(\tau).
$$
$H$ always contains a transposition unless $H = 1$, and the sign of a transposition (its determinant) is $-1$, so $H = 1$. Then, because of the decreasing order of $\tau_j^r$'s ($j \neq n + 1$) and (4.8)
$$
\mu_{r-1}^t \leq \tau_1^r < \mu_{r-2}^t \leq \cdots \leq \mu_1^t \leq \tau_1^t.
$$
The same argument for the intervals $(\mu_{r+1}^t, \mu_r^t)$ with $r \leq i < n + 1$ ($\mu_{n+1}^t = -\infty$) yields
$$
\tau_n^r \leq \mu_r^t < \tau_{n-1}^r \leq \cdots \leq \tau_1^r.
$$
Thus, the unique $s$ such that $Q_s \neq 0$ is $s = 1$, so $m_\chi(\tau) = \det 1 Q_1 = 1$. □

The proposition will be used for $\tau = \sigma + \gamma$ with $\sigma$ a $\Phi^+_k$-dominant weight and $\gamma$ a weight of $S$. In this case

$$
P(S) = \left\{ \frac{1}{2}(\pm \alpha_1 \pm \alpha_2 \pm \cdots \pm \alpha_n) : \alpha_i \in \Psi'' \right\}
$$

$$
= \left\{ \frac{1}{2}(\pm e_1 \pm \cdots \pm e_n + m_{n+1}) : m = \text{number of } (-) - \text{number of } (+) \right\}
$$

$$
\sigma = \sum_{i=1}^{n+1} \sigma_i e_i, \quad \sigma_i = m_i + s \quad \frac{n+1}{n+1}, \quad s, m_i \in \mathbb{Z}, \quad 0 \leq s < n + 1,
$$

$$
\sigma + \gamma = \sum_{i=1}^{n+1} (\sigma_i + \varepsilon_i) e_i, \quad \varepsilon_i = (\gamma, e_i) = \begin{cases} 
\pm \frac{1}{2}, & i \neq n + 1, \\
-\sum_{i=1}^{n} \varepsilon_i, & i = n + 1.
\end{cases}
$$

We retain the notation of §3.

**Proposition 4.2.** Let $\lambda = \sum_{i=1}^{n+1} \lambda_i e_i$ be a $\Psi''$-dominant Harish-Chandra parameter, and let $L^2_d$ be the discrete part of $L^2(G/K, V_\sigma \otimes S)$ as in (3.3) and $\sigma$ be as in §3. Then

(i) \( n_\lambda \neq 0 \iff \exists \quad \begin{cases} 
(\sigma + \rho_k - \lambda_i) \in \mathbb{Z}, & i = 1, \ldots, n, \\
\sigma_i \in [\sigma_{i+1} + \frac{1}{2}(n - 2i - 1), \sigma_i + \frac{1}{2}(n - 2i + 1)], & 1 \leq i < r - 1, \\
\sigma_{r-1} \in [\sigma_r + \frac{1}{2}(n - 2r + 1), \sigma_{r-1} + \frac{1}{2}(n - 2r + 3)], \\
\sigma_r \in [\sigma + \frac{1}{2}(n - 2r + 1), \sigma_{r-1}], \\
\sigma_{i+1} \in [\sigma_i + \frac{1}{2}(n - 2i + 1), \sigma_{i+1} + \frac{1}{2}(n - 2i + 3)], & r < i \leq n.
\end{cases} \)

(ii) \( n_\lambda \neq 0 \Rightarrow n_\lambda = 2^m, \quad 0 \leq m \leq n. \)

(iii) \( n_\lambda = 1 \iff \lambda = \sigma + \rho_k. \)

**Remark.** If $\sigma + \rho_k$ is a Harish-Chandra parameter, then $W_0(D^2) = W_0(D) \supset H_{\sigma + \rho_k}$ by (iii) of the last proposition and (iv) of Proposition 3.1. Actually, the equality is true by the irreducibility of $W_0(D)$ [A-S].

**Proof.** (i) Suppose that $n_\lambda \neq 0$, then $m_\chi(\sigma + \gamma) \neq 0$ for some $\gamma \in P(S)$, so by Proposition 4.1 and (4.3)

$$
\sigma_i + \varepsilon_i + (\rho_k)_i - \mu_i = \sigma_i + \varepsilon_i + (\rho_k)_i - (\lambda_i \pm \frac{1}{2}) \in \mathbb{Z} \quad \forall i
$$

if and only if

\begin{align*}
\lambda_i \in [\sigma_{i+1} + \sigma_{i+1} + \frac{1}{2}(n - 2i)], & \sigma_i + \varepsilon_i + \frac{1}{2}(n - 2i)], & 1 \leq i < r - 1, \\
\lambda_{r-1} \in (\sigma_r + \sigma_{r-1} + \frac{1}{2}(n - 2(r - 1)), & \sigma_{r-1} + \sigma_{r-1} + \frac{1}{2}(n - 2(r - 1))], \\
\lambda_r \in [\sigma_r + \sigma_r + \frac{1}{2}(n - 2r - 1), \lambda_{r-1}], \\
\lambda_{i+1} \in [\sigma_i + \sigma_i + \frac{1}{2}(n - 2i - 1)], & \sigma_{i+1} + \sigma_{i+1} + \frac{1}{2}(n - 2(i - 1))], & r < i \leq n.
\end{align*}
As $e = \pm \frac{1}{2}$, the components of $\lambda$ are in the given intervals.

Conversely, we want to know when there exist $\gamma \in P(S)$ such that $m_i(\sigma + \gamma) \neq 0$. Denote

for $i \leq r - 1$

\[
N_i = [\sigma_{i+1} + \frac{1}{2}(n - 2i - 1), \sigma_{i+1} + \frac{1}{2}(n - 2i + 1)],
\]

\[
B_i = [\sigma_{i+1} + \frac{1}{2}(n - 2i + 1), \sigma_{i+1} + \frac{1}{2}(n - 2i - 1)],
\]

\[
M_i = (\sigma_{i+1} + \frac{1}{2}(n - 2i - 1), \sigma_{i+1} + \frac{1}{2}(n - 2i + 1)];
\]

for $i = r - 1$

\[
N_{r-1} = (\sigma_r + \frac{1}{2}(n - 2(r - 1) - 1), \sigma_r + \frac{1}{2}(n - 2(r - 1) + 1)),
\]

\[
B_{r-1} = [\sigma_r + \frac{1}{2}(n - 2(r - 1) + 1), \sigma_{r-1} + \frac{1}{2}(n - 2(r - 1) - 1)],
\]

\[
M_{r-1} = (\sigma_{r-1} + \frac{1}{2}(n - 2(r - 1) - 1), \sigma_{r-1} + \frac{1}{2}(n - 2(r - 1) + 1)];
\]

for $i = r$

\[
N_r = [\sigma_r + \frac{1}{2}(n - 2(r - 1) - 1), \sigma_r + \frac{1}{2}(n - 2(r - 1) + 1)),
\]

\[
B_r = [\sigma_r + \frac{1}{2}(n - 2(r - 1) + 1), \lambda_{r-1}],
\]

\[
M_r = \emptyset;
\]

for $r < i \leq n$

\[
N_i = [\sigma_i + \frac{1}{2}(n - 2(i - 1) - 1), \sigma_i + \frac{1}{2}(n - 2(i - 1) + 1)),
\]

\[
B_i = [\sigma_i + \frac{1}{2}(n - 2(i - 1) + 1), \sigma_{i-1} + \frac{1}{2}(n - 2(i - 1) - 1)],
\]

\[
M_i = (\sigma_{i-1} + \frac{1}{2}(n - 2(i - 1) - 1), \sigma_{i-1} + \frac{1}{2}(n - 2(i - 1) + 1)).
\]

Observe that the intervals $N_i$ and $M_i$ have length one, except when they are empty. Suppose $H_i[\sigma + \gamma] \neq 0$. When $\lambda_i \in N_i$, for $i < r$, set $e_{i+1}(\gamma) = -\frac{1}{2}$ and for $i \geq r$, set $e_i(\gamma) = -\frac{1}{2}$. Similarly, for $\lambda_i \in M_i$, put $e_i(\gamma) = \frac{1}{2}$, when $i < r$ and $e_{i+1}(\gamma) = \frac{1}{2}$ when $i > r$. If $\lambda$ is a Harish-Chandra parameter whose components satisfy the conditions on the right-hand side of (i), then two consecutive components $\lambda_i$ and $\lambda_{i+1}$ of $\lambda$ cannot be at $N_i$ and $M_{i+1}$ respectively. So, either case determines the value of the corresponding component of $\gamma$. If $\lambda \in B_i$, $e_i(\gamma)$ can take either value. So, there exist a $\gamma$ such that $H_i[\sigma + \gamma] \neq 0$.

(ii) Suppose that $\lambda_{ij} \notin B_{ij}$, $j = 1, \ldots, m$, and $\lambda_k \in B_k$ for $k \neq i_j$. Then $\lambda_{ij} \in N_{ij} \cup M_{ij}$, so this determines exactly $m$ components values of the $\gamma$'s such that $m_i(\sigma + \gamma) \neq 0$. Thus there exist $2^n-m$ weight $\gamma$ such that $m_i(\sigma + \gamma) \neq 0$.

(iii) $n_1 = 1$ is equivalent to the existence of a unique $\gamma \in P(S)$ such that $m_i(\sigma + \gamma) \neq 0$, so the components of $\lambda$ determine every components of $\gamma$, or equivalently $\lambda_i \in N_i \cup M_i \ \forall i = 1, \ldots, n$. Note that $M_r = \emptyset$, so $\lambda_r \in N_r$. This implies that $\lambda_i \in N_i \ \forall i > r$. The component $\lambda_{r-1} \in M_{r-1}$, because

\[
\lambda_{r-1} \geq \lambda_r + 1 \geq \sigma_r + \frac{1}{2}(n - 2(r - 1) - 1) + 1 = \text{right extreme of the open set } N_{r-1}.
\]
So \( \lambda_i \in M_i \) for \( i < r \). Again, as the lengths of \( N_i \) and \( M_i \) are one,
\[
(\sigma + \rho_k - \lambda)_i \in \mathbb{Z} \quad \forall i = 1, \ldots, n, \\
(\sigma + \rho_k)_i \in M_i, \quad i < r, \\
(\sigma + \rho_k)_i \in N_i, \quad i \geq r,
\]
so the conclusion is \( \lambda = \sigma + \rho_k \).

The converse is true because each component of \( \lambda \) is in \( N_i \cup M_i \) and this determine exactly \( \gamma = \rho_n^\tau \) by a similar argument to that used before. This \( \gamma \) satisfies \( H_2[\sigma + \gamma] \neq 0 \), that is \( n_\lambda = 1 \). \( \square \)

5. \( G = \text{Spin}(2n, 1) \)

In this case the maximal compact subgroup \( K \) is \( \text{Spin}(2n) \). Fix \( T \) a maximal torus in \( \mathfrak{k} \) with Cartan subalgebra \( \mathfrak{h}_0 \), and an ordered orthonormal base \( \{H_1, \ldots, H_n\} \) of the real Lie algebra \( i\mathfrak{h}_0 \). Let \( \{e_1, \ldots, e_n\} \) be the dual base to \( \{H_1, \ldots, H_n\} \), so
\[
e_j(H_j) = \delta_{ij}.
\]

The root system \( \Phi(h, g) \) lies in \( (i\mathfrak{h}_0)' \), the real dual of \( i\mathfrak{h}_0 \). It is known that
\[
\Phi_k = \{e_i \pm e_j : i \neq j, i, j = 1, \ldots, n\}, \quad \Phi_n = \{\pm e_i : i = 1, \ldots, n\}.
\]

Fix
\[
(5.1) \quad \Phi^+_n = \{e_i \pm e_j : i < j\}.
\]

Now we have two choices of \( \Phi^+_n \) such that \( \Phi^+ = \Phi^+_k \cup \Phi^+_n \) is a positive root system, these are
\[
(5.2) \quad \Psi^1 = \{e_1, \ldots, e_n\}, \quad \Psi^2 = \{e_1, \ldots, e_{n-1}, -e_n\}.
\]

With (5.1) in mind
\[
(5.3) \quad 0 \leq |\rho_n| \leq 1 \quad \rho_n = \frac{1}{2} \sum_{i=1}^n e_i, \quad \rho^2_n = \frac{1}{2} \left( \sum_{i=1}^{n-1} e_i - e_n \right)
\]
where \( \rho^2_n \) correspond to choice of \( \Psi^1 \) as positive noncompact root system. Let \( \lambda \in (i\mathfrak{h}_0)' \) be an integral weight, so \( \lambda = \sum \lambda_i e_i \) with \( \lambda_i \in \mathbb{Z} \) \( \forall i = 1, \ldots, n \) or \( \lambda_i = \frac{1}{2}(2k_i + 1) \) with \( k_i \in \mathbb{Z} \) \( \forall i = 1, \ldots, n \). Note that \( \lambda \) is \( \Phi^+_k \)-dominant, is equivalent to
\[
(5.5) \quad 0 \leq |\lambda_n| \leq \lambda_{n-1} \leq \cdots \leq \lambda_1.
\]

The next proposition gives a necessary and sufficient condition for when a \( K \)-type occurs in a discrete series of \( \text{Spin}(2n, 1) \) of parameter \( \lambda \). Denote by \( m_\lambda(\tau) \) the multiplicity of the irreducible component of maximal weight \( \tau \) in this discrete series.
Proposition 5.1. Let \( \lambda = \sum_{i=1}^{n} \lambda_i e_i \) be a \( \Phi^+ \)-dominant Harish-Chandra parameter (for either of the two choices of \( \Phi_+ \)). Let \( \tau = \sum_{i=1}^{n} \tau_i e_i \) be a \( \Phi^+_k \)-dominant weight and set \( \mu = \lambda + \rho_n - \rho_k = \sum_{i=1}^{n} \mu_i e_i \). Then,
\[
m_{\lambda}(\tau) = 1 \iff \left\{ \begin{array}{l}
|\lambda_i| + \frac{1}{2} \leq |\tau_i| \leq \mu_{n-1} \leq \tau_{n-1} \leq \cdots \leq \mu_1 \leq \tau_1,
\sgn \lambda_n = \sgn \tau_n.
\end{array} \right.
\]

Proof. Fix \( \Phi^+_k = \Phi^+ \), and let \( \lambda \) be \( \Psi^+ \)-dominant, or equivalently \( \lambda_n > 0 \). Let \( \tau' = \tau + \rho_k \) and \( \mu' = \mu + \rho_k = \lambda + \rho_n \), then we have to prove
\[
m_{\lambda}(\tau) = 1 \text{ if and only if } \mu_j' \leq \tau_j' < \mu_{j-1}', \quad j = 1, \ldots, n \quad (\mu_0 = \infty).
\]

In this case the Weyl group \( W_K \) of \( K \) is the set of maps
\[
s: (e_1, \ldots, e_n) \rightarrow (\pm e_{\pi(1)}, \ldots, \pm e_{\pi(n)})
\]
with an even number of minus signs where \( \pi \) is a permutation of a set of \( n \) elements; the determinant of \( s \) is the sign of \( \pi \). The Blattner formula says that
\[
m_{\lambda}(\tau) = \sum_{s \in W_K} \det s Q(s^{-1} \tau' - \mu')
\]
where \( Q(\sigma) \) is the number of expressions of \( \sigma \) as a sum of positive noncompact roots. If \( s \in W_K \), one has that \( Q_s = Q(s^{-1} \tau' - \mu') \neq 0 \) if and only if \( \pm \tau_n(s) - \mu_k' \) is a nonnegative integer for all \( k \). Since the number of minus sign is even, and \( \mu_n' \), \( \tau_j' \geq 0 \), except for \( \tau_n' \), then \( s \) cannot change signs, so \( \tau_n' \geq 0 \). Besides, since \( \mu_n' \leq \mu_j' \forall j \), it follows that \( \tau_j' \geq \mu_n' \forall j \) (otherwise \( Q_s = 0 \forall s \)). Suppose that \( m_{\lambda}(\tau) \neq 0 \), so \( Q_s \neq 0 \) for some \( s \). Let \( H \) be the permutation subgroup which changes the elements \( \tau_j' \) which are in the interval \( [\mu_k', \mu_{k-1}'] \). Since the order of \( \tau_j' \) in the interval is irrelevant, if \( \pi \in H \) and \( s \pi \) corresponds to \( n \), then \( Q_s = Q_s \).
\[
m_{\lambda}(\tau) = \sum \det s Q_s = \sum \det s Q_{s\pi} = \sum \det s(s_1)^{-1} Q_s = \det(s_1)^{-1} m_{\lambda}(\tau).
\]
But \( H \) always has a transposition, except when \( H = \{1\} \), in which case there is only one \( \tau_j' \) in each interval \( [\mu_k', \mu_{k-1}'] \). This holds for \( k = 1, \ldots, n \) where \( \mu_0 = \infty \). Since \( \tau_n' \geq \mu_n' \) and the coefficients \( \tau_j' \) are ordered, \( m_{\lambda}(\tau) \neq 0 \) only if the condition of the proposition holds.

Conversely if the condition of the proposition holds, \( \tau_n' - \mu_k' \geq 0 \) if and only if \( \pi = 1 \), so \( Q_1 = 1 \) and \( Q_s = 0 \) if \( s \neq 1 \), that is \( m_{\lambda}(\tau) = \det 1 Q_1 = 1 \) (we know that in the case of \( Spin(2n, 1) \) that \( m_{\lambda}(\tau) \) is at the most 1).

Now consider \( \lambda_n < 0 \), or equivalently \( \lambda \) is \( \Psi^+ \)-dominant. If we change the positive noncompact root set \( \Psi^+ \) to \( \Psi^2 \), then \( \lambda = \sum_{i=1}^{n} \lambda_i e_i + (-\lambda_n)(-e_n) \) with \( -\lambda_n > 0 \), so the conditions are the same as in the first part of the proof. In this situation we must have
\[
-\tau_n \geq |\lambda_n| + \frac{1}{2} > 0 \Rightarrow \tau_n < 0 \Rightarrow \sgn \lambda_n = \sgn \tau_n
\]
and the proof is complete. \( \square \)

We will use the last proposition in the case \( \tau = \sigma + \gamma \) with \( \sigma \) a \( \Phi^+_k \)-dominant weight and \( \gamma \) a weight of \( S \), because that is what we need to obtain the set of elements of \( \text{Eig}(D^2) \) (see Proposition 3.1(iv)). In this case
\[
P(S) = \{ \frac{1}{2}(\pm e_1 \pm \cdots \pm e_n) \}.
\]
Let 

$$\sigma = \sum \sigma_i e_i, \quad \sigma_i \in \mathbb{Z} \quad \forall i, \quad \text{or} \quad 2\sigma_i \text{ is odd} \quad \forall i.$$ 

Thus,

$$\sigma + \gamma = \sum (\sigma_i + e_i)e_i, \quad e_i = (\gamma, e_i) = \pm \frac{1}{2}.$$ 

**Proposition 5.2.** Let \( \lambda = \sum_{i=1}^{n} \lambda_i e_i \) be a \( \Phi^+_k \)-dominant Harish-Chandra parameter, and let \( L^2_d \) be the discrete part of \( L^2 (G/K, V_\sigma \otimes S) \) as in (3.3), and \( \sigma \) as in (3.5). Then,

(i) \( n_\lambda \neq 0 \Longleftrightarrow \begin{cases} 
\sigma_i - \lambda_i \in \mathbb{Z} \quad \forall i, \\
\lambda_i \in [\sigma_{i+1} + n - i - 1, \sigma_i + n - i], \quad i < n, \\
|\lambda_n| \in (0, |\sigma_n|], \\
\lambda \text{ and } \sigma \text{ are in the same Weyl chamber for } \Phi^+. 
\end{cases} \)

(ii) \( n_\lambda \neq 0 \Rightarrow n_\lambda = 2^m, \quad 0 \leq m \leq n. \)

(iii) \( n_\lambda = 1 \Longleftrightarrow \lambda = \sigma + \rho_k. \)

(iv) \( \|\lambda\|^2 \leq \|\sigma + \rho_k\| \quad \text{and} \quad \|\lambda\|^2 = \|\sigma + \rho_k\| \iff \lambda = \sigma + \rho_k. \)

**Remark.** Using the notation of the Proposition 3.1, the equality \( W_0(D^2) = W_0(D) = H_{\sigma + \rho_k} \) holds.

**Proof.** (i) Suppose that \( n_\lambda \neq 0 \), so \( m_\lambda (\sigma + \gamma) \neq 0 \) for some \( \gamma \in P(S) \), so

$$\sigma_i + e_i - \mu_i = \sigma_i + e_i - (\lambda_i + 1) \in \mathbb{Z} \quad \forall i \iff \sigma_i - \lambda_i \in \mathbb{Z} \quad \forall i, \\
\lambda_i \in [\sigma_{i+1} + e_{i+1} + n - i - 1/2, \sigma_i + e_i + n - i - 1/2] \quad \text{for } i < n, \\
|\lambda_n| \in (0, |\sigma_n + e_n| - 1/2], \\
\text{sgn } \lambda_n = \text{sgn } (\sigma_n + e_n) = \text{sgn } \sigma_n$$

by the last proposition and (5.4). Note that \( |\lambda_n| + 1/2 \leq |\sigma_n + e_n|, \lambda \text{ integral and nonsingular, ensures that } \text{sgn } (\sigma_n + e_n) = \text{sgn } \sigma_n \).

Conversely, we want to find \( \gamma \in P(S) \) such that \( m_\lambda (\sigma + \gamma) \neq 0 \). Denote

for \( i < n \)

\[ N_i = [\sigma_{i+1} + n - i - 1, \sigma_i + n - i), \]
\[ B_i = [\sigma_{i+1} + n - i, \sigma_i + n - i - 1], \]
\[ M_i = (\sigma_i + n - i - 1, \sigma_i + n - i]; \]

for \( i = n \)

\[ N_n = \emptyset, \]
\[ B_n = (0, |\sigma_n| - 1], \]
\[ M_n = (|\sigma_n| - 1, |\sigma_n|]. \]

This is the situation graphically:

\[
\begin{array}{ccc}
N_i & B_i & M_i \\
N_{i+1} & B_{i+1} & M_{i+1} \\
& \ldots & \\
& \ldots & \\
B_n & M_n \\
\end{array}
\]
If \( \lambda_i \in N_i \), this fixes the value of \( e_{i+1}(\gamma) = -\frac{1}{2} \) for \( \gamma \)'s such that \( H_\lambda[\sigma + \gamma] \neq 0 \). Similarly, \( \lambda_{i+1} \in M_{i+1} \) ensures \( H_\lambda[\sigma + \gamma] = 0 \) for \( e_{i+1}(\gamma) = \frac{1}{2} \). But both cannot occur simultaneously, because \( N_i \) and \( M_{i+1} \) have both length one and equal extremes, and \( \lambda_{i+1} - \lambda_i \in \mathbb{Z} \), that is that only one of the cases determines the value of \( e_{i+1}(\gamma) \). So there is a \( \gamma \) such that \( m_\lambda(\sigma + \gamma) \neq 0 \).

(ii) Suppose that \( \lambda_i \notin B_j \), \( j = 1, \ldots, m \), and \( \lambda_k \in B_k \) for \( k \neq i \). Then \( \lambda_{ij} \in N_{ij} \cup M_{ij} \), this determines exactly \( m \) component values of the \( \gamma \)'s for which \( m_\lambda(\sigma + \gamma) \neq 0 \). So there exist \( 2^{n-m} \) weights \( \gamma \) such that \( m_\lambda(\sigma + \gamma) \neq 0 \).

(iii) \( n_1 = 1 \) is equivalent to the existence of a unique \( \gamma \in P(S) \) such that \( m_\lambda(\sigma + \gamma) \neq 0 \), so that the components of \( \lambda \) determine every component of \( \gamma \), or equivalently \( \lambda_i \in N_i \cup M_i \) \( \forall i \). Now note that \( N_n = \emptyset \) and this ensures that \( \lambda_n \in M_n \). But two consecutive components of \( \lambda \) cannot be in the same interval ( \( M_i \) and \( N_{i-1} \) have the same extremes), so \( \lambda_{n-1} \in M_{n-1} \). Repeating the same argument we obtain that \( \lambda_i \in M_i \) \( \forall i \). Then, as \( \lambda_i - \sigma_i \in \mathbb{Z} \), \( \lambda = \sigma + \rho_k \).

(iv) By (i) \( |\lambda_i| \leq |(\sigma + \rho_k)_i| \) \( \forall i \), so
\[
\|\lambda\|^2 = \sum \lambda_i^2 \leq \sum (\sigma + \rho_k)_i^2 = \|\sigma + \rho_k\|^2
\]
and the equality holds if and only if \( \lambda = \sigma + \rho_k \). \( \square \)

6. \( G = Sp(2, \mathbb{R}) \)

In the cases \( G = SU(n, 1) \) and \( G = Spin(2n, 1) \) we proved that the multiplicity \( n_\lambda \) of the discrete series \( H_\lambda \) of parameter \( \lambda \) which occurs in \( L^2(G/K, V_\sigma \otimes S) \) is a power of 2 with exponent less than or equal \( n \). For the \( G = Sp(2, \mathbb{R}) \) we will show that there exist parameters \( \lambda \)'s such that \( n_\lambda \) is nonzero and is not a power of 2. By (3.7) we know that
\[
n_\lambda = \sum_{\gamma \in P(S)} \dim \text{Hom}_K(H_\lambda, V_{\sigma + \gamma}).
\]
We will give some examples where the number of elements \( \gamma \in P(S) \) such that \( H_\lambda[\sigma + \gamma] \neq 0 \) is not a power of 2.

Let \( G = Sp(2, \mathbb{R}) \). The Lie algebra is
\[
g_0 = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1 \end{pmatrix} : X_1, X_2, X_3 \in \mathbb{R}^{2\times2}, X_2, X_3 \text{ symmetric} \right\}.
\]
Let \( g_0 = k_0 + p_0 \) be the Cartan decomposition of \( g_0 \), where
\[
k_0 = \left\{ \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} : X_1 = -^tX_1, X_2 = ^tX_2 \right\},
\]
\[
p_0 = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} : X_1 = ^tX_1, X_2 = ^tX_2 \right\}.
\]

There is an algebra isomorphism \( k_0 = g_0 \cap u(4) \cong u(2) \) given by
\[
k_0 \rightarrow u(2), \quad \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \rightarrow X_1 + iX_2.
\]
A Cartan subalgebra of $k_0$ and $g_0$ is

$$h_0 = \mathbb{R} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

where the first summand is the center $z_0$ of $k_0$. Let $g$, $k$, $p$, $h$, $z$ be the complexifications of $g_0$, $k_0$, $p_0$, $h_0$, $z_0$ respectively. The root system of $(g, h)$ is

$$\Phi(h, g) = \{\pm e_1 \pm e_2\} \cup \{\pm 2e_1, \pm 2e_2\}$$

where

$$e_j = \begin{pmatrix} 0 & 0 & ih_1 & 0 \\ 0 & 0 & 0 & ih_2 \\ -ih_1 & 0 & 0 & 0 \\ 0 & -ih_2 & 0 & 0 \end{pmatrix} = h_j, \quad j = 1, 2.$$ 

Let

$$\Phi_k = \{\pm (e_1 - e_2)\}, \quad \Phi_n = \{\pm (e_1 + e_2), \pm 2e_1, \pm 2e_2\}$$

and fix

$$\Phi_+ = \{e_1 - e_2\}, \quad \Phi_+ = \{e_1 + e_2, 2e_1, 2e_2\}, \quad \Phi_+ = \Phi_k^+ \cup \Phi_n^+.$$

Let $E_\alpha$ be the root vectors such that $B(E_\alpha, E_{-\alpha}) = 2||\alpha||^2$, where $B$ is the Killing form. Define $H_\alpha = [E_\alpha, E_{-\alpha}]$, so $H_\alpha$ satisfies $\alpha(H_\alpha) = 2$. Thus

$$h = z + \mathbb{C}H_{e_1-e_2} = \mathbb{C}H_{e_1+e_2} \oplus \mathbb{C}H_{e_1-e_2}.$$

Let $(ih_0)'$ be the dual space of $ih_0$; if $\mu \in (ih_0)'$, then

$$\mu = \mu_1(e_1 + e_2) + \mu_2(e_1 - e_2).$$

Denote

$$p^+ = \sum_{\alpha \in \Phi_+^*} g_\alpha, \quad p^- = \sum_{\alpha \in \Phi_-^*} g_{-\alpha}.$$ 

It is known that if $\lambda$ is $\Phi^+$-dominant with $\Phi^+$ as in (6.2), $H_\lambda$ is a holomorphic discrete series of $Sp(2, \mathbb{R})$. Then (see [S]) the restriction of the representation to $K$ of the $K$-finite elements of $H_\lambda$ is equivalent to the representation $S(p^+) \otimes V_\lambda$, where $S(p^+)$ is the symmetric algebra of $p^+$ and $\Lambda = \lambda + \rho_+ - \rho_-$.

To obtain the irreducible representations of $K$ that occur at $S(p^+)$ we will need the fact that $S(p^+)$ is the dual of $S(p^-)$ and the result of [S]. Select the maximal ordered subset $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ of $p^-$ selected such that $\alpha_1$ is the small root of $p^-$, and if $\alpha_1, \ldots, \alpha_s$ has been chosen, $\alpha_{s+1}$ is the small root of $p^-$ strongly orthogonal to $\alpha_1, \ldots, \alpha_s$ ($\alpha_{s+1} \pm \alpha_i \notin \Phi$, $i = 1, \ldots, s$). Then, the results of [S] says any irreducible representation of $K$ which occurs in $S(p^+)$ has multiplicity one and its maximal weight is $k_1 \gamma_1 + \cdots + k_r \gamma_r$; $k_i \in \mathbb{Z}_{\geq 0}$; $\gamma_i = -\alpha_1 - \cdots - \alpha_i$. Moreover, this representation occurs in polynomials of degree at most $k_1 + 2k_2 + \cdots + rk_r$. In our case $\Delta = \{-2e_1, -2e_2\}$, so

$$\gamma_1 = 2e_1, \quad \gamma_2 = 2e_1 + 2e_2$$

and the highest weight of the irreducible representations of $S(p^+)$ is

$$\mu = k_1 2e_1 + k_2(2e_1 + 2e_2) = (k_1 + 2k_2)(e_1 + e_2) + k_1(e_1 - e_2), \quad k_i \in \mathbb{Z}_{\geq 0}.$$
Therefore,
\[ S(p^+) = \bigoplus_{k_1, k_2 \geq 0} C_{(k_1+2k_2)(e_1+e_2)} \otimes V'_{k_1(e_1-e_2)} \]

where \( V'_{k_1(e_1-e_2)} \) is an \( SU(2) \)-module of maximal weight \( k_1(e_1 - e_2) \), and \( C_{(k_1+2k_2)(e_1+e_2)} \) is the one-dimensional representation of the center of \( U(2) \) given by \( \det()^{k_1+2k_2} \). The \( U(2) \)-module \( V_{\Lambda} \) is equivalent to \( C_{a(e_1+e_2)} \otimes V_{b(e_1-e_2)} \) if \( \Lambda = a(e_1 + e_2) + b(e_1 - e_2) \), so using the Clebsh-Gordon formula for the tensor product of two \( SU(2) \)-modules,
\[ S(p^+) \otimes V_{\Lambda} = \bigoplus_{k_1, k_2 \geq 0} C_{(k_1+2k_2)(e_1+e_2)} \otimes C_{a(e_1+e_2)} V'_{b(e_1-e_2)} \]
\[ = \bigoplus_{k_1, k_2 \geq 0} C_{(k_1+2k_2+a)(e_1+e_2)} \left( V'_{k_1(e_1-e_2)} \otimes V''_{b(e_1-e_2)} \right) \]
\[ = \bigoplus_{k_1, k_2 \geq 0} \left( \min(2k_1, 2b) \bigoplus_{t=0} C_{(k_1+2k_2+a)(e_1+e_2)} V'_{(k_1+b-t)(e_1-e_2)} \right) \]

If the discrete series \( H_{\lambda} \) occurs in \( L^2(G/K, V_{\sigma} \otimes S) \) where \( V_{\sigma} \) is the irreducible representation of \( K \) of maximal weight \( \sigma = \sigma_1 e_1 + \sigma_2 e_2 \), where \( \sigma \) is sufficiently far from the walls as in (3.5); then the \( K \)-type \( H_{\lambda}[\sigma + \gamma] \) is nonzero for some \( \gamma \in P(S) \).

Denote the noncompact roots by
\[
\alpha_1 = 2e_1 = (e_1 + e_2) + (e_1 - e_2), \\
\alpha_2 = 2e_2 = (e_1 + e_2) - (e_1 - e_2), \\
\alpha_3 = e_1 + e_2.
\]

Then \( P(S) = \{ \rho_n - \sum m_i \alpha_i: m_i = 0 , 1 \} \).

We will give one example of a parameter \( \lambda \) such that \( n_{\lambda} \) is not a power of 2. In the cases of \( Spin(2n, 1) \) and \( SU(2n, 1) \) it happens that
\[ n_{\lambda} = | \{ \gamma \in P(S): H_{\lambda}[\sigma + \gamma] \neq 0 \} | \]

but for \( Sp(2, \mathbb{R}) \) this is not true.

Take \( \lambda = \sigma + \rho_k - \alpha_1 - \alpha_2 \) with \( \sigma \) chosen so that \( \lambda \) is \( \Phi^+ \)-dominant.

The highest weight of the minimal \( K \)-type of \( H_{\lambda} \) is
\[ \Lambda = \lambda + \rho_n - \rho_k = \sigma + \rho_n - \alpha_1 - \alpha_2. \]

Since \( \rho_n - \alpha_1 - \alpha_2 \in P(S) \), \( H_{\lambda} \) occurs in \( L^2(G/K, V_{\sigma} \otimes S) \). The multiplicity of each \( K \)-type is equal to the number of expressions of its maximal weight in the form
\[ (k_1 + 2k_2 + a)(e_1 + e_2) + (k_1 + b - t)(e_1 - e_2) \]
with \( k_i \geq 0 \) and \( 0 \leq t \leq \min(2k_1, 2b) \). Since \( \sigma \) is nonsingular and \( \Phi^+ \)-dominant, \( b = \sigma_1 - \sigma_2 > 0 \). To obtain \( n_{\lambda} \) we need the multiplicity of each \( K \)-type \( \sigma + \gamma \) in \( H_{\lambda} \) with \( \gamma \in P(S) \).

\[
\sigma + \rho_n - \alpha_1 - \alpha_2 = a(e_1 + e_2) + b(e_1 - e_2), \\
k_1 = 0, \\
k_2 = 0, \\
t = 0, \\
multiplicity = 1,
\]
\[
\begin{align*}
\sigma + \rho_n &= (2 + a)(e_1 + e_2) + b(e_1 - e_2), \\
k_1 &= 0, \quad k_2 = 1, \quad t = 0, \\
k_1 &= 2, \quad k_2 = 0, \quad t = 2, \\
multiplicity &= 2, \\
\sigma + \rho_n - \alpha_1 &= (1 + a)(e_1 + e_2) + (-1 + b)(e_1 - e_2), \\
k_1 &= 1, \quad k_2 = 0, \quad t = 2, \\
multiplicity &= 1, \\
\sigma + \rho_n - \alpha_2 &= (1 + a)(e_1 + e_2) + (1 + b)(e_1 - e_2), \\
k_1 &= 1, \quad k_2 = 0, \quad t = 0, \\
multiplicity &= 1, \\
\sigma + \rho_n - \alpha_3 &= (1 + a)(e_1 + e_2) + b(e_1 - e_2), \\
k_1 &= 1, \quad k_2 = 0, \quad t = 1, \\
multiplicity &= 1, \\
\sigma + \rho_n - \alpha_2 - \alpha_3 &= a(e_1 + e_2) + (1 + b)(e_1 - e_2), \\
multiplicity &= 0, \\
\sigma + \rho_n - \alpha_1 - \alpha_3 &= a(e_1 + e_2) + (-1 + b)(e_1 - e_2), \\
multiplicity &= 0, \\
\sigma + \rho_n - 2\rho_n &= (-1 + a)(e_1 + e_2) + b(e_1 - e_2), \\
multiplicity &= 0,
\end{align*}
\]

Then \( n_4 = 6 \neq 2^m \) and \( |\{\gamma \in P(S) : H_4[\gamma] \neq 0\}| = 5 \neq 2^m \).

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References


