KINEMATIC FORMULAS FOR MEAN CURVATURE POWERS OF HYPERSURFACES AND HADWIGER'S THEOREM IN $\mathbb{R}^{2n}$

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Abstract. We first discuss the theory of hypersurfaces and submanifolds in the $m$-dimensional Euclidean space leading up to high dimensional analogues of the classical Euler's and Meusnier's theorems. Then we deduce the kinematic formulas for powers of mean curvature of the $(m-2)$-dimensional intersection submanifold $S_0 \cap S_1$ of two $C^2$-smooth hypersurfaces $S_0$, $S_1$, i.e.,

$$\int_G \left( \int_{S_0 \cap S_1} H^{2k} \, d\sigma \right) \, dg.$$ 

Many well-known results, for example, the C-S. Chen kinematic formula and Crofton type formulas are easy consequences of our kinematic formulas. As direct applications of our formulas, we obtain analogues of Hadwiger's theorem in $\mathbb{R}^{2n}$, i.e., sufficient conditions for one domain $K_B$ to contain, or to be contained in, another domain $K_A$.

0. Introduction

Let $M^p$, $N^q$ be submanifolds of dimensions $p$, $q$, respectively, in a homogeneous space $G/H$ and let $I$ be an integral invariant (e.g., volume, surface area, etc.) of the submanifold $M^p \cap gN^q$. Then many works in integral geometry have been concerned with computing integrals of the following type

$$\int_{\{g \in G : M^p \cap gN^q \neq \emptyset\}} I(M^p \cap gN^q) \, dg,$$

where $dg$ is the normalized kinematic density of $G$. For example in the case that $G$ is the group of motions in an $n$-dimensional Euclidean space $\mathbb{R}^n$, $M^p$ and $N^q$ are submanifolds of $\mathbb{R}^n$ and

$$I(M^p \cap gN^q) = \text{Vol}(M^p \cap gN^q)$$

evaluation of (0.1) leads to the formulas of Poincaré, Blaschke, Santaló and others (see [1]). R. Howard [10] obtained a kinematic formula for $I(M^p \cap gN^q) = \text{Vol}(M^p \cap gN^q)$ in a homogeneous space. If $I(M \cap gN) = \chi(M \cap gN)$, where $\chi(\cdot)$ is the Euler-Poincaré characteristic of the intersection $M \cap gN$ of domains $M$ and $N$ in $\mathbb{R}^n$ with smooth boundaries, then (0.1) leads to S. S. Chern's kinematic fundamental formula [2]. Next, assume that $I(M^p \cap gN^q) = \mu(M^p \cap gN^q)$ is one of the integral invariants from the Weyl tube formula. Then (0.1) leads to the Chern-Federer kinematic formula [22].

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for submanifolds of $\mathbb{R}^n$. This integral also leads to the C-S. Chen kinematic formula [15] if we take $I(M \cap gN) = \int_{M \cap gN} \kappa^2(M \cap gN) \, d\sigma$, the total square of curvature of the intersection curve $M \cap gN$ of two compact closed surfaces $M$, $N$ in $\mathbb{R}^3$. T. Shifrin [24] also obtained a kinematic formula of type (0.1) by letting $I(M^p \cap gN^q)$ be the integral of a Chern class. R. Howard [14] achieved more general kinematic formula in the case that $I(M^p \cap gN^q)$ is a homogeneous polynomial of the second fundamental form of $M^p \cap gN^q$ in a homogeneous space.

As is evident by the above examples, many known kinematic formulas are intrinsic. Only a few of them deal with the extrinsic invariants, for example, C-S. Chen’s formula. In fact, Howard [14] gives some extrinsic kinematic formulas. In this paper, we will give kinematic formulas for powers of mean curvature in the case that $M$, $N$ are hypersurfaces of Euclidean space $\mathbb{R}^n$. One of our results is the following.

**Main Theorem.** Let $S_i$ ($i = 0, 1$) be compact smooth hypersurfaces of class $C^2$ in $\mathbb{R}^n$. Denote by $H$ the mean curvature of the $(n-2)$-dimensional intersection submanifolds $S_0 \cap gS_1$ and by $\tilde{\kappa}_n^i(S_i)$ (defined in equation (4.15) below) the $i$th integral of the principal curvatures of $S_i$, respectively, where $g \in G$, the rigid motions in $\mathbb{R}^n$. Then for any integer $k$ with $0 \leq 2k \leq n - 1$ we have the kinematic formula

\[
\int_G \left( \int_{S_0 \cap gS_1} H^{2k} \, d\sigma \right) \, dg = \sum_{\substack{i, j, l, \ell \text{ (even)}}} c_{ijkl} \tilde{\kappa}_n^{l+2j}(S_0) \tilde{\kappa}_n^{l+2j}(S_1),
\]

where $dg$ is the kinematic density for $\mathbb{R}^n$, $d\sigma$ is the volume element and the $c$'s are constants depending on the indices.

In the first part of this paper we seek formulas for submanifolds of $\mathbb{R}^n$ in which the integrand in (0.1) can be expressed as a homogeneous polynomial in the components of the curvature tensor of the submanifolds. If $(M, N)$ is a pair of hypersurfaces in $\mathbb{R}^n$, we obtain kinematic formulas in which the right-hand side of (0.1) is expressed explicitly in terms of the total principal curvatures of the given hypersurfaces. Many well-known results (for example, the C-S. Chen formula [15] and Crofton type formulas) are easy consequences of our theorems. It is possible to use the ideas of this paper to obtain the general kinematic formula for $p$-dimensional submanifolds $M^p$ and $q$-dimensional submanifolds $N^q$, i.e., for integrals of type $\int_G \left( \int_{M^p \cap gN^q} H^k \, d\sigma \right) \, dg$, $(0 \leq k \leq p+q-n)$, in which the right-hand side of (0.1) can be expressed as a polynomial of the curvatures of the submanifolds.

In section 1 we give some basic concepts of differential geometry of hypersurfaces in $\mathbb{R}^n$ which are generalizations of the results of classical differential geometry (for example, Meusnier’s and Euler’s theorem). Some preliminaries of integral geometry are introduced in section 2. The basic concepts are in Santaló’s book [1]. We obtain our kinematic formula (Theorem 1) and its consequences (for example, C-S. Chen’s formula, Crofton type formulas) in section 3 and the proof of Main Theorem is given in detail in section 4. The methods used in this part are mostly based on those of Chern [26] and Howard [14].
In the second part of this paper we derive the analogues of Hadwiger's containment theorem. The general principle underlying this investigation can be briefly described as follows:

**Containment problem.** Let $K_\alpha$, $K_\beta$ be two suitable domains in the Euclidean space $\mathbb{R}^n$, for example, two convex bodies with interior points. Let $G$ be the group of rigid motions of $\mathbb{R}^n$ and let $m$ be its (suitably normalized) kinematic measure. Then

\[
m\{g \in G : gK_\beta \subset K_\alpha \text{ or } gK_\beta \supset K_\alpha\} = m\{g \in G : K_\alpha \cap gK_\beta \neq \emptyset\} - m\{g \in G : \partial K_\alpha \cap g\partial K_\beta \neq \emptyset\}.
\]

(If $K_\alpha$, $K_\beta$ are not convex, one assumes that their boundaries are connected.)

By integral geometric methods it is possible to estimate the measure $m\{g \in G : K_\alpha \cap gK_\beta \neq \emptyset\}$ from below and the measure $m\{g \in G : \partial K_\alpha \cap g\partial K_\beta \neq \emptyset\}$ from above in terms of geometric invariants of $K_\alpha$ and $K_\beta$. This will result in an inequality of the form

\[
m\{g \in G : gK_\beta \subset K_\alpha \text{ or } gK_\beta \supset K_\alpha\} \geq f(A^1_\alpha, \ldots, A^k_\alpha; A^1_\beta, \ldots, A^k_\beta),
\]

where each $A^l_\gamma$ is an integral geometric invariant of $K_\gamma$ ($\gamma = \alpha, \beta$). One can then state the following conclusion: If $f(A^1_\alpha, \ldots, A^k_\alpha; A^1_\beta, \ldots, A^k_\beta) > 0$, then there is a rigid motion $g$ such that either $gK_\beta$ is contained in $K_\alpha$ or $gK_\beta$ contains $K_\alpha$.

The first classical result is due to Hadwiger (see [1, 5, 6], 1941) who was the first to use the method of integral geometry to obtain some sufficient conditions for containment problem in Euclidean plane $\mathbb{R}^2$ and later in projective plane $\mathbb{R}P^2$ and hyperbolic plane $H^2$. Since then many mathematicians have been interested in getting a version of the containment problem in space $\mathbb{R}^n$, that is, getting sufficient conditions to insure that a given domain $K_\beta$ of surface area $F_\beta$, bounded by a piecewise smooth boundary $\partial D_\beta$, of volume $V_\beta$ may be moved 'inside' another domain $K_\alpha$ of surface area $F_\alpha$, bounded by a piecewise smooth boundary $\partial K_\alpha$, of volume $V_\alpha$. Grinberg, Ren and Zhou (see [4, 28]) obtained a variation of Hadwiger's theorem in the plane of constant curvature $e$ and reinterpreted those sufficient conditions by the isoperimetric deficits of domains involved. But there was no general result or analogue of Hadwiger's theorem in space $\mathbb{R}^n$ ($n \geq 3$) until works [7, 9, 10, 11, 12, 17] appeared, even if some very strong restrictions are put onto the domains involved (for example, the domains are supposed to be convex bodies and some topological conditions are put onto their boundaries and intersection). The situation of $n$-dimensional space $\mathbb{R}^n$ ($n \geq 3$) is much more complex and difficult than that of 2-dimensional plane $\mathbb{R}^2$. None of the formulas and methods in $\mathbb{R}^2$ can be parallely carried out in higher dimensions. One also hopes that analogous results may be achieved for hyperbolic and projective spaces of higher dimension. We have some results in a later paper [29].

By restricting the domains $K_\gamma$ to the convex category, the author [9, 10, 11] obtained some sufficient conditions in $\mathbb{R}^n$ ($n \geq 3$) which are the generalizations of Hadwiger's theorem to $\mathbb{R}^n$. Zhang [7] derived a sufficient condition in $\mathbb{R}^3$ for the domains belonging to the convex category. Goodey [13] obtained some related results by putting topological restrictions on the convex domains.
involved and their intersection $\partial K_\alpha \cap g^\circ K_\beta$. Later we removed the convexity restrictions and obtained analogues of Hadwiger's theorem in $\mathbb{R}^3$ [12, 17].

In section 5 of the second part of this paper we will give direct applications of our kinematic formulas and obtain the $\mathbb{R}^{2n}$ analogues (Theorem 4 and Theorem 5) of Hadwiger's theorem. It is surely possible to obtain analogues of Hadwiger's theorem in $\mathbb{R}^{2n+1}$. But for this purpose we must seek new kinematic formulas. In fact, we can only obtain kinematic "inequalities".

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1. Differential geometry of the hypersurface in $\mathbb{R}^m$

Let $M$ be an $n$-dimensional submanifold of $\mathbb{R}^m$. The second fundamental form of $M$ at $x \in M$ is a symmetric bilinear mapping

$$h_x : M_x \times M_x \rightarrow M_x^\perp,$$

where $M_x$ is the tangent bundle of $M$ and $M_x^\perp$ is the normal bundle of $M$ at $x$. If $e_1, \cdots, e_m$ is a orthonormal basis of $\mathbb{R}^m$ such that $e_1, \cdots, e_n$ is a basis of $M_x$ and $e_{n+1}, \cdots, e_m$ is a basis of $M_x^\perp$, then the components of $h_x$ in this basis are the numbers $(h_x^a)_{ij} = \langle h_x(e_i, e_j), e_a \rangle$, $1 \leq i, j \leq n$, $n + 1 \leq \alpha \leq m$, where $\langle , \rangle$ is the usual inner product in $\mathbb{R}^n$. Associated to the bundle of orthonormal frames $\{e_A\}$ over $M$ we have the coframes, which consist of $m$ linearly independent linear differential forms $\{\omega_A\}$, such that the metric in $M$ is

$$ds^2 = \sum_i \omega_i^2.$$

Restricted to $M$, we have $\omega_\alpha = 0$ and $0 = d\omega_\alpha = \sum_i \omega_\alpha \wedge \omega_i$. By Cartan's lemma, we get

$$\omega_{i\alpha} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha,$$

the second fundamental form of $M$ with respect to frames $\{e_\alpha\}$ and coframes $\{\omega_i\}$ is

$$II = \sum_{i,j,\alpha} (h_{ij}^\alpha \omega_i \omega_j) e_\alpha.$$

The mean curvature vector is defined by

$$\overline{H} = \sum_\alpha \left( \frac{1}{n} \sum_i h_{ii}^\alpha \right) e_\alpha.$$
The square of mean curvature of $M$ is

$$H^2 = \frac{1}{n^2} \sum_{\alpha} \left( \sum_i h_{ii}^\alpha \right)^2. \tag{1.6}$$

Let $X^n \subseteq Y^m \subseteq \mathbb{R}^{n+p}$ be two submanifolds. If we choose the frame

$$(x; e_1, \ldots, e_n, e_{n+1}, \ldots, e_m, e_{m+1}, \ldots, e_{n+p}) \tag{1.7}$$

such that $e_1, \ldots, e_n \in T(X^n)$ and $e_1, \ldots, e_m \in T(Y^m)$, then we have the mean curvature vector $\vec{H}_X$ of $X^n$, the mean curvature vector $\vec{H}_Y$ of $Y^m$, respectively, are

$$\vec{H}_X = \frac{1}{n} \sum_{\alpha=n+1}^{m} \left( \sum_i h_{ii}^\alpha \right) e_{\alpha} + \frac{1}{n} \sum_{\beta=m+1}^{n+p} \left( \sum_i h_{ii}^\beta \right) e_{\beta} = \vec{G}_{Y^m} + \vec{N}_{Y^m}, \tag{1.8}$$

$$\vec{H}_Y = \frac{1}{m} \sum_{\beta=m+1}^{n+p} \left( \sum_i h_{ii}^\beta \right) e_{\beta} = \frac{n}{m} \vec{N}_{Y^m}, \tag{1.9}$$

where $\vec{G}_{Y^m}$ is defined as the mean geodesic curvature of $Y^m$ at $x \in X^n$ and $\vec{N}_{Y^m}$ is the mean normal curvature of $Y^m$ at $x \in X^n$. Their lengths, i.e., $|\vec{G}_{Y^m}| = \kappa_g(Y^m)$, $|\vec{N}_{Y^m}| = \kappa_n(Y^m)$, are called, respectively, the geodesic curvature, normal curvature of $Y^m$ (at $x \in Y^m$ in the direction $X^n$). Of course they are invariants. It is obvious that two submanifolds $X$ and $X'$ of same dimension which are tangent at $x$ have the same normal mean curvatures. This actually is the classical Meusnier's theorem when $X$ is a smooth curve in a surface $Y^2$ of 3-dimensional Euclidean space $\mathbb{R}^3$.

Let $\Sigma$ be a hypersurface in an $m$-dimensional Euclidean space $\mathbb{R}^m$. For each point $x \in \Sigma$, choose the frames $\{x; e_1, e_2, \ldots, e_{m-1}, e_m\}$ such that $e_1, \ldots, e_{m-1} \in T_x(\Sigma)$, and $e_m \in N_x(\Sigma)$, the normal vector at $x \in \Sigma$. Then we have the fundamental equations

$$dx = \sum_{i=1}^{m-1} \omega_i e_i, \tag{1.10}$$

$$de_i = \sum_{j=1}^{m-1} \omega_{ij} e_j + \omega_{im} e_m, \quad \omega_{ij} = \omega_{ji},$$

$$de_m = -\sum_{i=1}^{m-1} \omega_{im} e_i,$$

and the integrability conditions

$$\sum_{i=1}^{m-1} \omega_i \wedge \omega_{im} = 0, \quad d\omega_i = \sum_{j=1}^{m-1} \omega_j \wedge \omega_{ij}, \tag{1.11}$$

$$d\omega_{ij} = \sum_{k=1}^{m-1} \omega_{ik} \wedge \omega_{kj} + \omega_{im} \wedge \omega_{mj},$$

$$d\omega_{im} = \sum_{j=1}^{m-1} \omega_{ij} \wedge \omega_{jm}.$$
From the above equations, there are functions \( h_{ij} \) such that

\[
\omega_{im} = \sum_{j=1}^{m-1} h_{ij} \omega_j, \quad h_{ij} = h_{ji}. \tag{1.12}
\]

Now we get

\[
I = \langle dx, dx \rangle = \sum_{i=1}^{m-1} (\omega_i)^2, \tag{1.13}
\]

\[
\Pi = -\langle dx, de_m \rangle = \sum_{j=1}^{m-1} \omega_j \omega_{jm} = \sum_{i,j=1}^{m-1} h_{ij} \omega_i \omega_j, \tag{1.14}
\]

which are called, respectively, the first and second fundamental form of \( \Sigma \) (\( \Pi \) is only defined up to a sign). The principal directions are the directions which diagonalize \( (h_{ij}) \), i.e., the eigenvalue directions of \( (h_{ij}) \). It is known that at each point of a hypersurface \( \Sigma \) in \( \mathbb{R}^m \) there are \( m-1 \) principal directions and \( m-1 \) principal curvatures \( \kappa_1, \ldots, \kappa_{m-1} \).

Assume that \( e_1, \ldots, e_{m-1} \) are \( m-1 \) principal directions, for each unit vector \( v \in T_x(\Sigma) \), \( v = \sum v_i e_i \). Then we have the normal mean curvature at \( x \in \Sigma \) along the direction \( v \),

\[
\kappa_{\Sigma}(v) = \frac{\Pi(v)}{I(v)} = \frac{\sum \kappa_i \omega_i^2(v)}{\sum \omega_i^2(v)} = \sum_{i=1}^{m-1} \kappa_i v_i^2. \tag{1.15}
\]

This is Euler's formula for higher dimensional hypersurface. In particular, the normal curvatures along the directions of frames \( e_j \) are

\[
\kappa_{\Sigma}(e_j) = \frac{\Pi(e_j)}{I(e_j)} = h_{jj} = \kappa_j, \quad j = 1, \ldots, m-1. \tag{1.16}
\]

If \( v \in T_x(\Sigma) \), a unit vector, then \( v \) can be expressed by the spherical coordinates

\[
v = (\cos \phi_1 \cos \phi_2 \cdots \cos \phi_{m-2} \cos \phi_{m-3}, \cos \phi_1 \cos \phi_2 \cdots \cos \phi_{m-3} \sin \phi_{m-2}, \cos \phi_1 \cos \phi_2 \cdots \cos \phi_{m-4} \sin \phi_{m-3}, \ldots, \cos \phi_1 \sin \phi_2, \sin \phi_1), \tag{1.17}
\]

with

\[
0 \leq \phi_1 < 2\pi; \quad 0 \leq \phi_2, \ldots, \phi_{m-2} \leq \pi, \tag{1.18}
\]

where \( \phi_i \) are the angles between \( v \) and the principal directions \( e_i \). Then we have

\[
\kappa_{\Sigma}(v) = \kappa_1 \cos^2 \phi_1 \cos^2 \phi_2 \cdots \cos^2 \phi_{m-3} \cos^2 \phi_{m-2}
+ \kappa_2 \cos^2 \phi_1 \cos^2 \phi_2 \cdots \cos^2 \phi_{m-3} \sin^2 \phi_{m-2}
+ \kappa_3 \cos^2 \phi_1 \cos^2 \phi_2 \cdots \cos^2 \phi_{m-4} \sin^2 \phi_{m-3} + \cdots
+ \kappa_{m-2} \cos^2 \phi_1 \sin^2 \phi_2 + \kappa_{m-1} \sin^2 \phi_1. \tag{1.19}
\]
This is Euler's formula in classical differential geometry when \( m = 3 \). The mean curvature of \( \Sigma \) is

\[
H_\Sigma = \frac{1}{m-1} \text{Tr}(\Pi) = \frac{1}{m-1} \sum_{i=1}^{m-1} h_{ii} = \frac{1}{m-1} \sum_{i=1}^{m-1} \kappa_i.
\]

Now for the frame \( e_1, \ldots, e_n \) and \( e'_1, \ldots, e'_n \) attached to \( S_0 \) and \( S_1 \) the mean curvature vectors of \( S_0, S_1 \), respectively, are

\[
\overline{H}_0 = \frac{1}{n-1} \sum_{i=1}^{n-1} h_{ii} e_n, \tag{1.21}
\]

\[
\overline{H}_1 = \frac{1}{n-1} \sum_{i=1}^{n-1} h'_i e'_n. \tag{1.22}
\]

The mean curvature vector of \( S_0 \cap gS_1 \) is

\[
\overline{H} = \frac{1}{n-2} \left[ \sum_{i=1}^{n-2} h_{ii} e_{n-1} + \sum_{i=1}^{n-2} h_{ii} e_n \right]. \tag{1.23}
\]

Let \( U, V \) be an orthonormal basis of the normal space of \( S_0 \cap gS_1 \) and \( H_0, H_1 \) be the mean curvatures of \( S_0, S_1 \), respectively. We choose \( U \) in the direction of \( \overline{H} \), the direction of the mean curvature of \( S_0 \cap gS_1 \). Then

\[
\text{Tr}(U, d^2x) = (n-2)H_0, \quad \text{Tr}(V, d^2x) = 0. \tag{1.24}
\]

Let

\[
e_n = \cos \theta U + \sin \theta V,
\]

then

\[
(e_n, d^2x) = \cos \theta (U, d^2x) + \sin \theta (V, d^2x). \tag{1.26}
\]

Hence

\[
\text{Tr}(e_n, d^2x) = h_{11} + \cdots + h_{n-2,n-2}
= \kappa(e_1) + \cdots + \kappa(e_{n-2}) = (n-1)H_0 - \kappa(e_{n-1}). \tag{1.27}
\]

Thus

\[
\langle \overline{H}, e_n \rangle = (\overline{H} \cdot e_n) = \frac{1}{n-2} [(n-1)H_0 - \kappa(e_{n-1})]
\]

is an invariant, where \( \kappa(e_{n-1}) \) is the principal curvature of \( S_0 \) in the direction \( e_{n-1} \). Similarly,

\[
\langle \overline{H}, e'_n \rangle = (\overline{H} \cdot e'_n) = \frac{1}{n-2} [(n-1)H_1 - \kappa'(e'_{n-1})]
\]

is also an invariant and where \( \kappa'(e'_{n-1}) \) is the principal curvature of \( S_1 \) in the direction \( e'_{n-1} \).
By the following formula

\[(a_1 + \cdots + a_{m-1})^P = \sum_{i_1+\cdots+i_{m-1}=P} \frac{P!}{i_1! \cdots i_{m-1}!} a_{i_1} \cdots a_{i_{m-2}} a_{i_{m-1}} \]

we have

\[\kappa^P_S(v) = \sum_{i_1, \ldots, i_{m-1}} c_{i_1 \cdots i_{m-1}} v_1^{2i_1} \cdots v_{m-1}^{2i_{m-1}} \kappa_1^{i_1} \kappa_2^{i_2} \cdots \kappa_{m-1}^{i_{m-1}},\]

where

\[c_{i_1 \cdots i_{m-1}} = \frac{P!}{i_1! \cdots i_{m-1}!}, \quad i_1 + \cdots + i_{m-1} = P.\]

If \(d\sigma\) denotes the area element of \(\Sigma\), we define the \(r\)th homogeneous mean curvature as

\[\tilde{M}_r^{i_1 \cdots i_{m-1}}(\Sigma) = \left(\begin{array}{c} \frac{m-1}{r} \\ \end{array} \right)^{-1} \int_{\Sigma} \left(\kappa_1^{i_1} \cdots \kappa_{m-1}^{i_{m-1}}\right) d\sigma,\]

where \((\kappa_1^{i_1} \cdots \kappa_{m-1}^{i_{m-1}})\) denotes the \(r\)th homogeneous polynomial of the principal curvatures. In particular, \(\tilde{M}^0\) is the area and \(\tilde{M}^1\) is the total mean curvature of \(\Sigma\).

When \(m = 3\), all the results here agree with those in classical differential geometry.

2. Preliminaries on integral geometry

Let \(S_0\) and \(S_1\) be two piecewise smooth hypersurfaces in \(\mathbb{R}^n\) of class \(C^2\). We assume \(S_0\) fixed and \(S_1\) moving under the rigid motion \(g\) of \(\mathbb{R}^n\) with the kinematic density \(dg\). Consider generic positions \(gS_1\) so that the intersection \(S_0 \cap gS_1\) is an \((n - 2)\)-dimensional manifold. Let \(x \in S_0 \cap gS_1\) and consider the orthonormal frames \((x; e_1, \cdots, e_n)\) such that \(\text{span}\{e_1, \cdots, e_{n-2}\} = T_x(S_0 \cap gS_1)\), the tangent \((n - 2)\)-space to \(S_0 \cap gS_1\) at \(x\). Let \(e'_{n-1}, e'_n\) be the unit vectors that are, respectively, tangent and normal to \(S_1\) at \(x\), so that \((x; e_1, \cdots, e_{n-2}, e'_{n-1}, e'_n)\) is a second orthonormal frames of origin \(x\). We denote the kinematic density \(dT_0\) on \(S_0\) (i.e., the density for sets of frames \((x; e_1, \cdots, e_n)\) such that \(e_n\) is the unit normal to \(S_0\)) by

\[dT_0 = \bigwedge_i (dx \cdot e_i) \bigwedge_{h<j} (e_h \cdot de_j) \wedge (dx \cdot e'_{n-1}) \wedge (de_1 \cdot e'_1) \wedge \cdots \wedge (de_{n-2} \cdot e'_{n-2}) \wedge (de_{n-1} \cdot e'_{n-1}).\]

Similarly, the kinematic density on \(S_1\) (density for frames \((x; e_1, \cdots, e_{n-2}, e'_{n-1}, e'_n)\) such that \(e'_n\) is the unit normal to \(S_1\)) is denoted by

\[dT_1 = \bigwedge_i (dx \cdot e_i) \bigwedge_{h<j} (e_h \cdot de_j), \quad i, h, j = 1, 2, \cdots, n - 1,\]

so that the kinematic density for \(\mathbb{R}^n\) is

\[dg = dT_1 \wedge (dx \cdot e_n) \wedge (de_1 \cdot e_n) \wedge \cdots \wedge (de_{n-1} \cdot e_n).\]
The kinematic density on \( S_0 \cap gS_1 \) (i.e., density for frames \((x; e_1, \cdots, e_{n-2})\) attached to \( S_0 \cap gS_1 \)) is defined by

\[
\begin{align*}
\mathcal{D}_{0i} &= \bigwedge_i (dx \cdot e_i) \bigwedge_{h<j} (e_h \cdot de_j), \quad i, h, j = 1, 2, \cdots, n - 2.
\end{align*}
\]

Then we have the basic formula (see [1, pp. 262, (15.35)])

\[
\begin{align*}
\mathcal{D}_{01} \land dg &= \sin^{n-1} \phi \, d\phi \land \mathcal{D}_{00} \land \mathcal{D}_{11},
\end{align*}
\]

where \( \phi \) is the angle between \( S_0 \) and \( S_1 \) (i.e., between \( e_n \) and \( e'_n \)).

If \( L_1 \) is a hyperplane, assume that \( dL_{n-1|[x]} \) is the density of the \((n-1)\)-plane about \( x \) and that \( dL_{n-1} \) is the density for sets of \((n-1)\)-plane in \( \mathbb{R}^n \).

Then we have (see [1, pp. 244, (14.64)])

\[
\begin{align*}
\int \sigma(x) \land dL_{n-1} &= \sin^{n-1} \phi \, d\phi \land \int \sigma^{(0)}(x) \land dL_{n-1},
\end{align*}
\]

where \( \sigma(x) \) is the volume element of the intersection submanifolds \( S_0 \cap gL_1 \) and \( \sigma^{(0)}(x) \) is the volume element of \( S_0 \).

Denote by

\[
\begin{align*}
\kappa_n(S_0) &= \langle \vec{H}, e_n \rangle = \frac{1}{n-2}[(n-1)H_0 - \kappa(e_{n-1})],
\kappa_n(S_1) &= \langle \vec{H}, e'_n \rangle = \frac{1}{n-2}[(n-1)H_1 - \kappa'(e'_{n-1})],
\end{align*}
\]

the normal mean curvatures of \( S_0, S_1 \), respectively. Then by the laws of sines and cosines we have

**Lemma 1.** Let \( H \) be the mean curvature of the intersection manifold \( S_0 \cap gS_1 \) of two compact hypersurfaces \( S_0 \) and \( S_1 \), i.e., \( \cos \phi = \langle e_n, e'_n \rangle \). Then we have

\[
\begin{align*}
H^2 \sin^2 \phi &= (\vec{H} \cdot e_n)^2 + (\vec{H} \cdot e'_n)^2 - 2\langle \vec{H} \cdot e_n \rangle \langle \vec{H} \cdot e'_n \rangle \cos \phi
= \kappa_n^2(S_0) + \kappa_n^2(S_1) - 2\kappa_n(S_0) \kappa_n(S_1) \cos \phi,
\end{align*}
\]

or

\[
H \sin \phi = (\kappa_n^2(S_0) + \kappa_n^2(S_1) - 2\kappa_n(S_0) \kappa_n(S_1) \cos \phi)^{1/2}; \quad 0 \leq \phi < \pi.
\]

When \( n = 3 \), (2.8) becomes a formula of classical differential geometry in \( \mathbb{R}^3 \).

**Lemma 2.** Let \( \kappa_n^{(i)} \) \((i = 0, 1)\) be the normal curvatures of surfaces \( S_i \) in 3-dimensional Euclidean space \( \mathbb{R}^3 \). Denoted by \( \kappa \) the curvature of the intersection curve \( S_0 \cap gS_1 \) of two compact surfaces \( S_0 \) and \( S_1 \) and by \( \phi \) the angle between \( S_0 \) and \( S_1 \), i.e., \( \cos \phi = \langle e_3, e'_3 \rangle \). Then we have

\[
\kappa \sin \phi = \left[(\kappa_n^{(0)})^2 + (\kappa_n^{(1)})^2 - 2\kappa_n^{(0)} \kappa_n^{(1)} \cos \phi\right]^{1/2}.
\]

From the formula

\[
(a_1 + a_2 + a_3)^k = \sum_{j_1 + j_2 + j_3 = k} \frac{k!}{j_1! j_2! j_3!} a_1^{j_1} a_2^{j_2} a_3^{j_3},
\]
for $0 \leq 2k \leq n - 1$ we have

$$H^{2k} \sin^{2k} \phi = (\kappa_n^{2}(S_0) + \kappa_n^{2}(S_1) - 2\kappa_n(S_0) \kappa_n(S_1) \cos \phi)^k$$

(2.12) 

$$= \sum_{i,j,l, i+j+l=k} \frac{(-2)^i k!}{i!j!l!} \kappa_n^{i+2j}(S_0) \kappa_n^{l+2j}(S_1) \cos^l \phi.$$

From (2.5) and (2.12), we come to

$$H^{2k} dT_0 \wedge dg$$

(2.13) 

$$= \sum_{i,j,l, i+j+l=k} \frac{(-2)^i k!}{i!j!l!} \kappa_n^{i+2j}(S_0) \kappa_n^{l+2j}(S_1) \cos^l \phi \sin^{n-2k-1} \phi d\phi \wedge dT_0 \wedge dT_1,$$

with

(2.14) 

$$0 \leq \phi < \pi,$$

the angle between $S_0$ and $S_1$ and $dg$ the density for $\mathbb{R}^n$. Also

(2.15) 

$$\int_0^\pi \cos^l \phi \sin^{n-2k-1} \phi d\phi = \begin{cases} 
\frac{(l-1)!!(n-2k-2)!!}{(n-k-i-j-1)!!} \cdot \pi, & l \text{ even}, n \text{ odd}; \\
\frac{(l-1)!!(n-2k-2)!!}{(n-k-i-j-1)!!} \cdot 2, & l \text{ even}, n \text{ even}; \\
0, & l \text{ odd}.
\end{cases}$$

Let

(2.16) 

$$C_{ijkln} = \begin{cases} 
\frac{2^l k!(l-1)!!(n-2k-2)!!}{i!j!l!!(n-k-i-j-1)!!} \cdot \pi, & n \text{ odd}; \\
\frac{2^{l+1} k!(l-1)!!(n-2k-2)!!}{i!j!l!!(n-k-i-j-1)!!}, & n \text{ even}.
\end{cases}$$

Then we have

Lemma 3. Let $S_i$ ($i = 0, 1)$ be smooth compact hypersurfaces of class $C^2$ in an $n$-dimensional Euclidean space $\mathbb{R}^n$. Then for any $k$ with $0 \leq 2k \leq n - 1$ we have

$$\int_{\{g : S_0 \cap g_{S_1} \neq \emptyset\}} \left( \int H^{2k} dT_0 \right) dg$$

(2.17) 

$$= \sum_{i,j,l, i+j+l=k, i \text{ even}} C_{ijkln} \int \kappa_n^{l+2i}(S_0) dT_0 \int \kappa_n^{l+2j}(S_1) dT_1.$$ 

3. Kinematic formulas for hypersurfaces

Let $S$ be a smooth hypersurface of class $C^2$ in an $n$-dimensional Euclidean space $\mathbb{R}^n$ and $\kappa_n(S)$ be the normal mean curvature of $S$. Denoted by $\tilde{\kappa}_n(S)$ the $r$th total normal mean curvature of $S$, i.e.,

(3.1) 

$$\tilde{\kappa}_n(S) = \int_S \kappa_n(S) d\sigma,$$
where $d\sigma$ is the volume element. Let
\[ C(n) = O_{n-2}^2 O_{n-3} \cdots O_1, \]

(3.2)
\[
J_{i_1 \cdots i_{n-1}}^{(1)} = \binom{n-1}{l+2i} c_{i_1 \cdots i_{n-1}} I(i_1, \cdots, i_{n-1}),
\]
\[
J_{j_1 \cdots j_{n-1}}^{(0)} = \binom{n-1}{l+2j} c_{j_1 \cdots j_{n-1}} I(j_1, \cdots, j_{n-1}),
\]

(3.3)
\[
i_1 + \cdots + i_{n-1} = l + 2i, \quad j_1 + \cdots + j_{n-1} = l + 2j,
\]

(3.4)
\[
I(i_1, \cdots, i_{n-1}) = O_{n-2} \frac{(2i_1 - 1)!! \cdots (2i_{n-1} - 1)!!}{(n-1)(n+1) \cdots (n+2p-3)},
\]

where $p = \sum_{k=1}^{n-1} i_k$. $O_m$ is the surface area of the $m$-dimensional unit sphere and its value is

(3.5)
\[
O_m = \frac{2\pi^{(m+1)/2}}{\Gamma((m+1)/2)},
\]

where $\Gamma$ denotes the gamma function. Then we have the following kinematic formula (3.6) which can be explicitly expressed as integrals of the principal curvatures of $S_0$ and $S_1$, i.e.,

**Theorem 1.** Let $S_i (i = 0, 1)$ be two compact smooth hypersurfaces of class $C^2$ in an $n$ dimensional Euclidean space $\mathbb{R}^n$. Then for any integer $k$ with $0 < 2k < n - 1$ we have the following kinematic formula

\[
\int_G \left( \int_{S_0 \cap (\nu S_0)} H^{2k} d\sigma \right) \, dg
= C(n) \sum_{i,j,l} C_{ijkl} \sum_{i_1, \cdots, i_{n-1}} J_{i_1 \cdots i_{n-1}}^{(1)} M_{i_1 \cdots i_{n-1}}^{l+2i} (S_1)
\times \sum_{j_1, \cdots, j_{n-1}} J_{j_1 \cdots j_{n-1}}^{(0)} M_{j_1 \cdots j_{n-1}}^{l+2j} (S_0).
\]

(3.6)

Let $\Sigma$ be a $C^2$-smooth hypersurface in $\mathbb{R}^n$, and let $R_\Sigma$, $H_\Sigma$ be, respectively, the scalar curvature, the mean curvature of $\Sigma$. Denote by $\bar{R}_\Sigma$ and $\bar{H}_\Sigma^{(2)}$ the total scalar curvature and the total square of mean curvature of $\Sigma$, i.e.,

(3.7)
\[
\bar{R}_\Sigma = \int_\Sigma R_\Sigma \, d\sigma, \quad \bar{H}_\Sigma^{(2)} = \int_\Sigma H_\Sigma^2 \, d\sigma.
\]

If $k = 1$ then we have

**Corollary 1.** Let $S_i (i = 0, 1)$ be $C^2$-smooth closed hypersurfaces in $\mathbb{R}^n$. Denote by $F_i$ the surface area of $S_i$. Then we have

(3.8)
\[
\int_G \left( \int_{S_0 \cap (\nu S_0)} H^2 \, d\sigma \right) \, dg = \frac{\pi O_{n-2}^2 O_{n-3} \cdots O_1}{(n+1)(n-1)}
\times \left\{ 3(n-1)^2 (F_0 \bar{H}_1^{(2)} + F_1 \bar{H}_0^{(2)}) - 4(F_0 \bar{R}_1 + F_1 \bar{R}_0) \right\}.
\]

When $k = 1$ and $n = 3$, the scalar curvature is Gaussian curvature. Then C-S. Chen's kinematic formula [15] is an easy consequence of our Corollary 1.
Corollary 2. Let $S_i (i = 0, 1)$ be compact $C^2$-smooth surfaces in $\mathbb{R}^3$. Denote by $H_i$, $R_i$, $\tilde{H}_i^{(2)}$ and $\tilde{R}_i$, respectively, the mean curvature, Gaussian curvature, the total square of mean curvature and the total Gaussian curvature of $S_i$. Then we have

$$\int_G \left( \int_{S_0 \cap gS_1} \kappa^2 \, ds \right) \, dg = 2\pi^3 (3 \tilde{H}_0^{(2)} - \tilde{R}_0) F_1 + 2\pi^3 (3 \tilde{H}_1^{(2)} - \tilde{R}_1) F_0,$$

where $\kappa$ is the curvature of the intersection curves $S_0 \cap gS_1$, $ds$ is the arc element of $S_0 \cap gS_1$ and $F_i$ are the surface areas of $S_i$.

Let

$$C_{nk} = \int_0^\pi \sin^{n-2k-1} \phi \, d\phi$$

$$= \begin{cases} \frac{n-2k-2}{n-2k-1} \cdot \frac{n-2k-4}{n-2k-3} \cdot \ldots \cdot \frac{3}{2} \cdot \frac{2}{1}, & n \text{ even,} \\ \frac{n-2k-2}{n-2k-1} \cdot \frac{n-2k-4}{n-2k-3} \cdot \ldots \cdot \frac{1}{2} \cdot \frac{1}{1}, & n \text{ odd,} \end{cases}$$

we also have

$$c'_{i_1 \ldots i_{n-1}} = \binom{n-1}{2k} \frac{(2k)!}{i_1! \ldots i_{n-1}!} \frac{(2i_1-1)!! \ldots (2i_{n-1}-1)!!}{(n-1)(n+1) \ldots (n+4k-3)},$$

we also have

Theorem 2. Let $L_1$ be a hyperplane and let $S_0$ be a closed hypersurface in $\mathbb{R}^n$. Denote by $\tilde{M}_i^{(r)}$ the $r$th homogeneous mean curvature of $S_0$ and $H$ the mean curvature of submanifold of $S_0 \cap gL_1$. Then for any $k \ (0 < 2k < n-1)$, we have

$$\int_G \left( \int_{S_0 \cap gL_1} H^{2k} \, d\sigma \right) \, dg$$

$$= O_{n-2}^3 O_{n-3} \ldots O_1 C_{nk} \sum_{i} \sum_{i_1, \ldots, i_{n-1}} c'_{i_1 \ldots i_{n-1}} \tilde{M}_i^{(r)} (S_0).$$

Theorem 2 can be restated as follows:

Corollary 3. Let $AG(n, n-1)$ be the Grassmannian manifold of all affine hyperplanes in $\mathbb{R}^n$. Then for any compact hypersurface $S$ in $\mathbb{R}^n$ and for $k \ (0 \leq 2k \leq n-1)$, we have the following Crofton type formula

$$\int_{AG(n, n-1)} \left( \int_{S \cap L} H^{2k} \, d\sigma \right) \, dL$$

$$= C_{nk} O_{n-2}^3 O_{n-3} \ldots O_1 \sum_{i} \sum_{i_1, \ldots, i_{n-1}} c'_{i_1 \ldots i_{n-1}} \tilde{M}_i^{(r)} (S).$$

---

1 The formula was first proved by C-S. Chen in 1972 (see [15]) and reproved by this author in 1992 (see [12]) by a different method.
4. Proofs of Theorems

Proof of Theorem 1. Equation (2.2) may be written as
\[
\begin{align*}
    dT_i &= \bigwedge_{i < j} (dx \cdot e_i) \bigwedge_{h < j} (e_h \cdot de_j) = d\sigma^{(1)} \wedge dg_i, \\
    (4.1)
\end{align*}
\]
where \( d\sigma^{(1)} \) is the volume element of \( S_1 \) at \( x \) and \( dg_i \) is the kinematic density for \( S_1 \), i.e., for \( SO(n-1) \). Then for any positive integer \( p \) we have
\[
\begin{align*}
    \int \kappa^p_n(S_1(v)) \, dT_1 &= \int \kappa^p_n(S_1(v)) \, d\sigma^{(1)} \wedge dg_i \\
    &= \sum_{i_1+\cdots+i_n-1=p} c_{i_1\cdots i_n-1} \int_{S_1} \left( \kappa^{i_1}_{i_1} \cdots \kappa^{i_n-1}_{i_n-1} \right) d\sigma^{(1)} \int_{SO(n-1)} v^{2i_1}_{i_1} \cdots v^{2i_n-1}_{i_n-1} \, dg_i \\
    &= \operatorname{Vol}(SO(n-2)) \sum_{i_1, \cdots, i_n-1} \binom{n-1}{p} c_{i_1\cdots i_n-1} \tilde{M}^p_{i_1\cdots i_n-1}(S_1) I(i_1, \cdots, i_n-1),
\end{align*}
\]
where
\[
\tilde{M}^p_{i_1\cdots i_n-1}(S_1) = \left( \frac{n-1}{p} \right)^{-1} \int_{S_1} \left( \kappa^{i_1}_{i_1} \cdots \kappa^{i_{n-1}}_{i_{n-1}} \right) d\sigma^{(1)}
\]
are the \( p \)th homogeneous mean curvature of \( S_1 \),
\[
(4.3) \quad c_{i_1\cdots i_n-1} = \frac{p!}{i_1! \cdots i_{n-1}!}, \quad i_1 + \cdots + i_{n-1} = p;
\]
\[
\begin{align*}
    \int_{SO(n-1)} v^{2i_1}_{i_1} \cdots v^{2i_{n-1}}_{i_{n-1}} \, dg_i &= \operatorname{Vol}(SO(n-2)) \int_{S^{n-2}} v^{2i_1}_{i_1} \cdots v^{2i_{n-1}}_{i_{n-1}} \, d\sigma \\
    &= \operatorname{Vol}(SO(n-2)) I(i_1, \cdots, i_{n-1}).
\end{align*}
\]
Equality (4.4) follows from the fibering of \( SO(n-1) \) over \( S^{n-2} \) with the fiber \( SO(n-2) \), and the values of the integrals are known (see, for example, H. Weyl [18]),
\[
I(i_1, \cdots, i_{n-1}) = \int_{S^{n-2}} v^{2i_1}_{i_1} \cdots v^{2i_{n-1}}_{i_{n-1}} \, d\sigma \\
= \frac{(2i_1-1)!! \cdots (2i_{n-1}-1)!!}{(n-1)(n+1) \cdots (n+2p-3)},
\]
where \( p = \sum_{k=1}^{n-1} i_k \).
So we have
\[
\begin{align*}
    \int \kappa^p_n(S_1(v)) \, dT_1 &= \operatorname{Vol}(SO(n-2)) \sum_{i_1, \cdots, i_{n-1}} J^{(1)}_{i_1\cdots i_{n-1}} \tilde{M}^p_{i_1\cdots i_{n-1}}(S_1),
\end{align*}
\]
where
\[
(4.7) \quad J^{(1)}_{i_1\cdots i_{n-1}} = \binom{n-1}{p} c_{i_1\cdots i_{n-1}} I(i_1, \cdots, i_{n-1}).
\]
Similarly, for any \( q \) we have
\[
\begin{align*}
    \int \kappa^q_n(S_0(v)) \, dT_0 &= \operatorname{Vol}(SO(n-2)) \sum_{j_1, \cdots, j_{n-1}} J^{(0)}_{j_1\cdots j_{n-1}} \tilde{M}^q_{j_1\cdots j_{n-1}}(S_0),
\end{align*}
\]

\[
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\]
where

\( J_{j_1 \cdots j_{n-1}}^{(0)} = \binom{n-1}{q} c_{j_1 \cdots j_{n-1}} I(j_1, \cdots, j_{n-1}) \) ;

\( c_{j_1 \cdots j_{n-1}} = \frac{q!}{j_1! \cdots j_{n-1}!} \), \( j_1 + \cdots + j_{n-1} = q \);

and the \( I(j_1, \cdots, j_{n-1}) \) are computed as in (4.5). Also we have

\[ \text{Vol}(SO(n-2)) = O_{n-2} \cdots O_1. \]

Putting formula (4.6), (4.8) and (4.11) into (2.17) we obtain

\[
O_{n-3} \cdots O_1 \int_G \left( \int_{S_0 \cap S_1} H^{2k} d\sigma \right) dg
= (O_{n-2} \cdots O_1)^2 \sum_{i, j, l} C_{ijklm} \sum_{i_1, \cdots, i_{n-1}} J_{i_1 \cdots i_{n-1}}^{(1)} \tilde{M}^{l+2i}_{i_1 \cdots i_{n-1}}(S_1)
\times \sum_{j_1, \cdots, j_{n-1}} J_{j_1 \cdots j_{n-1}}^{(0)} \tilde{M}^{l+2j}_{j_1 \cdots j_{n-1}}(S_0),
\]

i.e.,

\[
\int_G \left( \int_{S_0 \cap S_1} H^{2k} d\sigma \right) dg
= O_{n-2}^2 \cdot O_{n-3} \cdots O_1 \sum_{i, j, l} C_{ijklm} \sum_{i_1, \cdots, i_{n-1}} J_{i_1 \cdots i_{n-1}}^{(1)} \tilde{M}^{l+2i}_{i_1 \cdots i_{n-1}}(S_1)
\times \sum_{j_1, \cdots, j_{n-1}} J_{j_1 \cdots j_{n-1}}^{(0)} \tilde{M}^{l+2j}_{j_1 \cdots j_{n-1}}(S_0),
\]

with

\[ i_1 + \cdots + i_{n-1} = l + 2i, \quad j_1 + \cdots + j_{n-1} = l + 2j. \]

We complete the proof of Theorem 1.

Let

\[
c_{ijklm} = C(n) C_{ijklm},
\]

\[
\tilde{\kappa}^{l+2i}_n(S_1) = \sum_{i_1, \cdots, i_{n-1}} J_{i_1 \cdots i_{n-1}}^{(1)} \tilde{M}^{l+2i}_{i_1 \cdots i_{n-1}}(S_1),
\]

\[
\tilde{\kappa}^{l+2j}_n(S_0) = \sum_{j_1, \cdots, j_{n-1}} J_{j_1 \cdots j_{n-1}}^{(0)} \tilde{M}^{l+2j}_{j_1 \cdots j_{n-1}}(S_0),
\]

and we obtain our Main Theorem.

Proof of Theorem 2. If \( S_1 = L \) is a hyperplane, then all principal curvatures of \( L \) vanish. Then (2.9) becomes

\[ H^2 \sin^2 \phi = \kappa_n^2(S), \]

where \( H \) is the mean curvature of submanifold \( S \cap L \) and \( \phi \) is the angle between \( S \) and \( L \). From (2.6) we have

\[ H^{2k} d\sigma \wedge dL_{n-1} = \kappa_n^{2k}(S) \sin^{n-2k-1} \phi d\phi \wedge d\sigma^{(0)} \wedge dL_{n-1}. \]
and thus
\[
\int_G \left( \int_{S \cap gL} H^{2k} d\sigma \right) dg = C_{nk} O_{n-2}^3 O_{n-3} \cdots O_1 \sum_{i_1, \ldots, i_{k-1}} c'_{i_1 \cdots i_{k-1}} M'_{i_1 \cdots i_{k-1}}(S).
\]

This proves Theorem 2.

In particular, we have the following kinematic inequality.

**Theorem 3.** Let \( S_i \ (i = 0, 1) \), \( F_i \), \( R_i \), \( H_i \), \( \tilde{R}_i \), and \( \tilde{H}_i^{(2)} \) be as in Corollary 1. Then we have
\[
\int_G \left( \int_{S_0 \cap gS_1} H \, d\sigma \right) dg \leq \pi A_n \left\{ B_n F_0 F_1 \left[ \left( 3(n-1)^2 \tilde{H}_0^{(2)} - 4 \tilde{R}_0 \right) F_1 \right. \right.
\]
\[
+ \left( 3(n-1)^2 \tilde{H}_1^{(2)} - 4 \tilde{R}_1 \right) F_0 \right\}^{\frac{1}{2}}
\]
\[
= \pi A_n \left\{ B_n F_0 F_1 \left[ 3(n-1)^2 \left( \tilde{H}_0^{(2)} F_1 + \tilde{H}_1^{(2)} F_0 \right) - 4 \left( \tilde{R}_0 F_1 + \tilde{R}_1 F_0 \right) \right] \right\}^{\frac{1}{2}},
\]

where \( A_n = O_{n-2} O_{n-3} \cdots O_1 \) and \( B_n = O_n O_{n-2} O_{n-1}^{-1} / (n^2 - 1) \).

**Proof of Theorem 3.** By Hölder’s inequality we have
\[
\int_{S_0 \cap gS_1} H \, d\sigma \leq \left( \int_{S_0 \cap gS_1} 1^2 \, d\sigma \right)^{\frac{1}{2}} \cdot \left( \int_{S_0 \cap gS_1} H^2 \, d\sigma \right)^{\frac{1}{2}}
\]
\[
= \left( \text{Vol}(S_0 \cap gS_1) \right)^{\frac{1}{2}} \cdot \left( \int_{S_0 \cap gS_1} H^2 \, d\sigma \right)^{\frac{1}{2}}.
\]

The Santaló kinematic formula [1] for the volume of intersection \( S_0 \cap gS_1 \) reads
\[
\int G \text{Vol}(S_0 \cap gS_1) \, dg = \frac{O_n \cdots O_1 O_{n-2}}{O_{n-1}^2} F_0 F_1.
\]

Integrating (4.21) with respect to kinematic density \( dg \) and using Hölder’s inequality, (3.8) and (4.22) we come to
\[
\int_G \left( \int_{S_0 \cap gS_1} H \, d\sigma \right) dg
\]
\[
\leq \left( \int G \text{Vol}(S_0 \cap gS_1) \, dg \right)^{\frac{1}{2}} \cdot \left( \int_{S_0 \cap gS_1} H^2 \, d\sigma \right)^{\frac{1}{2}}
\]
\[
\leq \left( \int G \text{Vol}(S_0 \cap gS_1) \, dg \right)^{\frac{1}{2}} \cdot \left( \int_G \left( \int_{S_0 \cap gS_1} H^2 \, d\sigma \right) \, dg \right)^{\frac{1}{2}}
\]
\[
= \pi A_n \left\{ B_n F_0 F_1 \left[ 3(n-1)^2 \left( \tilde{H}_0^{(2)} F_1 + \tilde{H}_1^{(2)} F_0 \right) - 4 \left( \tilde{R}_0 F_1 + \tilde{R}_1 F_0 \right) \right] \right\}^{\frac{1}{2}}.
\]

This proves Theorem 3.

When \( n = 3 \), we have the following
Corollary 4. Let $S_i$ $(i = 0, 1)$ be two simply closed surfaces of class $C^2$ in $\mathbb{R}^3$. Assume that $F_i$, $R_i$, $H_i$, $\tilde{R}_i$ and $\tilde{H}_i^{(2)}$ are as in Corollary 1. Then we have
\[
\int_G \left( \int_{S_0 \cap g S_1} \kappa \, ds \right) \, dg 
\leq 2\pi^3 \left\{ F_0 F_1 \left[ (3\tilde{H}_0^{(2)} - \tilde{R}_0)F_1 + (3\tilde{H}_1^{(2)} - \tilde{R}_1)F_0 \right] \right\}^{\frac{1}{4}}.
\]

5. Analogues of Hadwiger's theorem—sufficient conditions for one domain to contain another in $\mathbb{R}^{2n}$

It is known that at each point of a hypersurface $\Sigma$ in $\mathbb{R}^n$ there are $n - 1$ principal directions and $n - 1$ principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$. Let $d\sigma$ denote the volume element of $\Sigma$, then the $r$th integral of mean curvature
\[
M_r(\Sigma) = \left( \frac{n-1}{r} \right)^{-1} \int_{\Sigma} \{\kappa_i, \ldots, \kappa_i\} \, d\sigma,
\]
where $\{\kappa_i, \ldots, \kappa_i\}$ denotes the $r$th elementary symmetric function of the principal curvatures. In particular, $M_0$ is the area, $M_{n-1}$ is a numerical multiple of the degree of mapping of $\Sigma$ into the unit hypersphere defined by the normal field and $M_1$ is the total mean curvature of $\Sigma$.

In this section, we will use indices $\alpha$ and $\beta$ for the two domains and their integral geometric invariants. We assume that $K_\alpha$ and $K_\beta$ are bounded by the simple hypersurfaces $\partial K_\alpha$ and $\partial K_\beta$, respectively, which we assume to be of class $C^2$. One restriction we put on domains is that for all $g \in G$, the group of rigid motions in $\mathbb{R}^{2n}$, the Euler-Poincaré characteristic $\chi(K_\alpha \cap g K_\beta)$ of the intersection $K_\alpha \cap g K_\beta$ is at most $n_0$, a finite integer, i.e., $\chi(K_\alpha \cap g K_\beta) \leq n_0$. Denote by $M_\alpha^i$, $M_\beta^i$ the $i$th integrals of mean curvatures of $\partial K_\alpha$, $\partial K_\beta$, respectively. Then Chern's kinematic fundamental formula [1, 2] reads
\[
\int_G \chi(K_\alpha \cap g K_\beta) \, dg
= O_{n-2} \cdots O_1 \left[ O_{n-1} (V_\alpha \chi(K_\beta) + V_\beta \chi(K_\alpha)) + \frac{1}{n} \sum_{h=0}^{n-2} \left( \binom{n}{h+1} M_h^\alpha M_n^\beta \right) \right],
\]
where $\chi(\cdot)$ is the Euler-Poincaré characteristic and $V_\gamma$ $(\gamma = \alpha, \beta)$ are the volumes of $K_\gamma$.

Let
\[
\hat{C}_{ijkl}^{(n)} = \frac{2^{l+k}l!((l-1)!!(2n-2k-2)!!)}{i!j!((2n-k-i-j-1)!!)},
\]
\[
f(i, j, k, l, n, \tilde{M}_\beta^{l+2i}, \tilde{M}_\alpha^{l+2j}) = \sum_{i, j, l} \hat{C}_{ijkl}^{(n)} \sum_{i_1, \ldots, i_{2n-1}} J_{i_1 \cdots i_{2n-1}}^{(\beta)} \tilde{M}_{i_1 \cdots i_{2n-1}}^{l+2i} (\partial K_\beta)
\times \sum_{j_1, \ldots, j_{2n-1}} J_{j_1 \cdots j_{2n-1}}^{(\alpha)} \tilde{M}_{j_1 \cdots j_{2n-1}}^{l+2j} (\partial K_\alpha).
\]

Denoted by $\tilde{\kappa}_r(\partial K_\gamma)$ the $r$th integral of homogeneous mean curvature of $\partial K_\gamma$. As a direct application of our kinematic formula (0.2) or (3.6) we have an analogue of Hadwiger's theorem as follows
Theorem 4. Let $K_\alpha, K_\beta$ be domains in $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ bounded by connected hypersurfaces $\partial K_\alpha, \partial K_\beta$, respectively, which we suppose to be $C^2$-smooth. We assume that for all $g \in G$, the group of rigid motions in $\mathbb{R}^{2n}$, the Euler-Poincaré characteristic satisfy $\chi(K_\alpha \cap gK_\beta) \leq n_0$. Then either of following inequalities gives a sufficient condition for $K_\alpha$ to enclose $K_\beta$ or for $K_\beta$ to enclose $K_\alpha$:

\begin{align*}
0_{2n-1}(V_\alpha \chi(K_\beta) + V_\beta \chi(K_\alpha)) + \frac{1}{2n} \sum_{h=0}^{2n-2} \left( \frac{2n}{h+1} \right) M_\alpha^h M_\beta^{2n-2-h} & - n_0 \sum_{i,j,l} \tilde{C}_{ijkl} \kappa_i^{l+2i}(\partial K_\alpha) \kappa_j^{l+2j}(\partial K_\beta) > 0; \\
0_{2n-1}(V_\alpha \chi(K_\beta) + V_\beta \chi(K_\alpha)) + \frac{1}{2n} \sum_{h=0}^{2n-2} \left( \frac{2n}{h+1} \right) M_\alpha^h M_\beta^{2n-2-h} & - n_0 \sum_{i,j,k,l,n} \tilde{M}_\alpha^{l+2i} \tilde{M}_\beta^{l+2j} > 0.
\end{align*}

Moreover, if $V_\alpha \geq V_\beta$, then $K_\alpha$ can enclose $K_\beta$.

Proof of Theorem 4. Let $M$ be an $n$-dimensional closed submanifold in Euclidean space $\mathbb{R}^m$ and $H$ be the mean curvature of $M$. Then B-Y. Chen's formula (i.e., the generalized Fenchel's theorem) \[8\] says

\[
\int_M |H|^n \, d\sigma \geq O_n,
\]

the equality of (5.6) holds if and only if $M$ is imbedded as an $n$-sphere of $\mathbb{R}^m$.

Now, we estimate integral

\[
\int_{\{g : \partial K_\alpha \cap g \partial K_\beta \neq \emptyset\}} \, dg.
\]

For almost every rigid motion $g \in G$ in $\mathbb{R}^{2n}$, the intersection submanifold $\partial K_\alpha \cap g \partial K_\beta$ may be composed of several components, i.e., $\partial K_\alpha \cap g \partial K_\beta = \bigcup_{i=1}^{N_g} \Sigma_i$, where $\Sigma_i$ is a connected imbedded closed $(2n-2)$-dimensional submanifold and $N_g$ is always finite and only depends on $g$. By using B-Y. Chen's formula (5.6) to the generic $(2n-2)$-dimensional submanifolds $\partial K_\alpha \cap g \partial K_\beta$ we have

\[
\int_{\partial K_\alpha \cap g \partial K_\beta} H^{2n-2} \, d\sigma \geq O_{2n-2}.
\]

The above equality holds if and only if $\partial K_\alpha \cap g \partial K_\beta$ is composed of only one component and is a $(2n-2)$-sphere in $\mathbb{R}^{2n}$. If the equality in (5.8) holds for almost all rigid motions $g \in G$, that is, the intersections $\partial K_\alpha \cap g \partial K_\beta$ of the boundaries of two domains $K_\alpha$ and $K_\beta$ are always balls (or empty). Then the two domains $K_\alpha$ and $K_\beta$ must be balls (a consequence of a result of P. Goodey \[13, 25\]). Therefore

\[
O_{2n-2} \int_{\{g : \partial K_\alpha \cap g \partial K_\beta \neq \emptyset\}} \, dg \leq \int_G \left( \int_{\partial K_\alpha \cap g \partial K_\beta} H^{2n-2} \, d\sigma \right) \, dg,
\]
and thus by (0.2) we have

\[ (5.10) \quad \int_{\{g : \partial K_\alpha \cap g \partial K_\beta \neq \emptyset\}} dg \leq C_n \sum_{i, j, k} \tilde{C}_{ijkln} \tilde{k}_n^{l+2i}(\partial K_\alpha) \tilde{k}_n^{l+2j}(\partial K_\beta), \]

where \( C_n = O_{2n-2} \cdots O_1 \). From

\[ (5.11) \quad \int_{\{g : K_\alpha \cap g K_\beta \neq \emptyset\}} dg \leq n_0 \int_G \chi(K_\alpha \cap g K_\beta) \, dg, \]

(5.2), (5.10) and (5.11) we have the kinematic measure of one domain moving into another under the group \( G \) of the rigid motions in \( \mathbb{R}^{2n} \), i.e.,

\[ (5.12) \quad m\{g \in G : g K_\beta \subset K_\alpha \text{ or } g K_\beta \supset K_\alpha\} \]

\[ = \int_{\{g : K_\alpha \cap g K_\beta \neq \emptyset\}} dg - \int_{\{g : \partial K_\alpha \cap g \partial K_\beta \neq \emptyset\}} dg \]

\[ \geq \frac{1}{n_0} C_n \left[ O_{2n-1}(V_\alpha \chi(K_\beta) + V_\beta \chi(K_\alpha) + \frac{1}{2n} \sum_{h=0}^{2n-2} \left( \frac{2n}{h+1} \right) M_h^\alpha M_h^\beta_{2n-2-h} \right] \]

\[ - C_n \sum_{i, j, l} \tilde{C}_{ijkln} \tilde{k}_n^{l+2i}(\partial K_\alpha) \tilde{k}_n^{l+2j}(\partial K_\beta). \]

Equality holds above if and only if the two domains are balls. Hence a sufficient condition for \( K_\alpha \) to contained, or to be contain in, \( K_\beta \) is

\[ (5.13) \quad m\{g \in G : g K_\beta \subset K_\alpha \text{ or } g K_\beta \supset K_\alpha\} \geq 0. \]

Therefore (5.2) or (5.3) is a sufficient condition for \( K_\alpha \) to enclose, or to be enclosed in \( K_\beta \). We have proved Theorem 4.

If \( K_\alpha \) and \( K_\beta \) are convex bodies in \( \mathbb{R}^{2n} \), then we have \( \chi(K_\alpha) = \chi(K_\beta) = \chi(K_\alpha \cap g K_\beta) = n_0 = 1 \) for almost all \( g \in G \). So we have

**Theorem 5.** Let \( K_\alpha, K_\beta \) be two \( C^2 \)-smooth convex bodies in \( 2n \)-dimensional Euclidean space \( \mathbb{R}^{2n} \). Then either of the following inequalities is a sufficient condition for \( K_\alpha \) to contain, or to be contained in, \( K_\beta \):

\[ (5.14) \quad O_{2n-1}(V_\alpha + V_\beta) + \frac{1}{2n} \sum_{h=0}^{2n-2} \left( \frac{2n}{h+1} \right) M_h^\alpha M_h^\beta_{2n-2-h} \]

\[ - \sum_{i, j, l} \tilde{C}_{ijkln} \tilde{k}_n^{l+2i}(\partial K_\alpha) \tilde{k}_n^{l+2j}(\partial K_\beta) > 0; \]

\[ (5.15) \quad O_{2n-1}(V_\alpha + V_\beta) + \frac{1}{2n} \sum_{h=0}^{2n-2} \left( \frac{2n}{h+1} \right) M_h^\alpha M_h^\beta_{2n-2-h} \]

\[ - f(i, j, k, l, n, \tilde{M}_\alpha^{l+2i}, \tilde{M}_\beta^{l+2j}) > 0. \]

Moreover, if \( V_\alpha \geq V_\beta \), then \( K_\alpha \) can contain \( K_\beta \).

**Remark.** Of course, these conditions are not necessary. It would be very interesting to remove the 'smoothness' restriction to the convex bodies involved in this paper. All the notations except \( \tilde{k}_n^r(\partial K_\gamma) \) (or \( \tilde{M}_r^\gamma \)) are well-defined for nonsmooth convex bodies. If we could find substitutes for \( \tilde{k}_n^r(\partial K_\gamma) \) (or \( \tilde{M}_r^\gamma \)), the conditions in this paper can be interpreted for arbitrary convex bodies. This is definitely worth investigating.
REFERENCES


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