GENERALIZATION OF THE WHITNEY-MAHOWALD THEOREM

BANG-HE LI

Abstract. The Whitney-Mahowald theorem gave normal Euler number (mod 4) for embeddings of a closed 2n-manifold in Euclidean 4n-space. We generalize this theorem to embeddings of closed 2n-manifolds in an oriented 4n-manifold with an approach in the framework of unoriented bordism groups of maps.

1. Introduction

Given a map \( f : M \to N \), where \( M \) and \( N \) are smooth connected manifolds with dimensions \( n \) and 2n respectively, and \( M \) is closed, \( N \) is oriented, there is a question:

\( (*) \) What is the set of the normal Euler classes of smooth embeddings in the homotopy class \([f]\)?

An orientation of \( N \) and a map \( f : M \to N \) determine an isomorphism \( H^n(M, \mathbb{Z}) \to \mathbb{Z} \) which sends the normal Euler class of an embedding \( g \) in \([f]\) to an integer \( \chi(g) \) called the normal Euler number \( g \), where \( \mathbb{Z} \) is the local integer coefficients associated to the orientation line bundle of \( M \). For odd \( n \), the normal Euler classes are always zero. Hence Question \( (*) \) is interesting only for even \( n \).

H. Whitney [Wh1] first proved that any \( n \) manifold embeds in \( \mathbb{R}^{2n} \). By using Whitney's technique, J. Milnor proved in [Mi] that if \( N \) is simply connected and \( n > 2 \), then any \( f : M \to N \) is homotopic to embeddings. The following more general result is due to Haefliger [Ha]:

Theorem 1 (Haefliger). If \( n > 2 \) and \( f : M^n \to N^{2n} \) is a map with \( f_* : \pi_1(M) \to \pi_1(N) \) surjective, then \( f \) is homotopic to embeddings.

If \( M \) is orientable, then the normal Euler numbers of embeddings are uniquely determined by their homotopy classes. However, if \( M \) is nonorientable, the situation changes. Whitney [Wh2] in the case \( n = 2 \), and Mahowald [Mah] in the case of \( n \) even, proved that if \( f : M^n \to \mathbb{R}^{2n} \) is an embedding, then

\[
\chi(f) = 2\overline{w}_1(M)\overline{w}_{n-1}(M) \mod 4,
\]

where \( 2\overline{w}_1(M)\overline{w}_{n-1}(M) \) is understood as the image of the dual Stiefel-Whitney number \( \overline{w}_1(M)\overline{w}_{n-1}(M) \) under the natural inclusion \( \mathbb{Z}_2 \to \mathbb{Z}_4 \).

Received by the editors September 2, 1993.
1991 Mathematics Subject Classification. Primary 57N35.
This work is partially supported by the National Funds of Science of P. R. China.
Malyi [Mal] proved that if $n > 2$ is even and $M^n$ is nonorientable, then for any integer $x$ with $x = 2\bar{w}_1(M)\bar{w}_{n-1}(M) \mod 4$, there is an embedding $f : M^n \to \mathbb{R}^{2n}$ with $\chi(f) = x$.

W. S. Massey gave a new proof of Mahowald's theorem in [Mas], by using the following formula proved also by him:

$$P(U_2) = (p_4(X) + e(w_1w_{n-1})) \cdot U,$$

where $U$ is the Thom class of an $n$-dimensional vector bundle $\xi$ over $B$ with $n$ even, $X$ the Euler class of $\xi$, both $U$ and $X$ take local integer coefficients $\tilde{Z}$ determined by $\xi$, $U_2 = U \mod 2$, $w_i$ the $i$th Whitney class of $\xi$, $\tilde{P}$ the Pontryagin square, and

$$\tilde{P} : H^q(B, \tilde{Z}) \to H^q(B, \tilde{Z}_4), \quad \tilde{\theta} : H^q(B, \mathbb{Z}_2) \to H^q(B, \mathbb{Z}_4)$$

the natural homomorphisms.

To generalize Mahowald's theorem to embeddings of $M^n$ in an oriented $N^{2n}$, we define first $P : H_n(N, \mathbb{Z}_2) \to \mathbb{Z}_4$ as follows:

For any $x \in H_n(N, \mathbb{Z}_2)$, take a compact submanifold $N^{2n}_x$ of $N$ so that $x = i_\ast y$, where

$$i_\ast : H_n(N_x, \mathbb{Z}_2) \to H_n(N, \mathbb{Z}_2)$$

is the natural homomorphism and $y \in H_n(N_x, \mathbb{Z}_2)$. Let

$$Dy \in H^n(N_x, \partial N_x, \mathbb{Z}_2) \cong H^n(N_x/\partial N_x, \mathbb{Z}_2)$$

be the Lefschetz dual of $y$. Then

$$P(x) = \langle \tilde{P}(Dy), [N_x/\partial N_x] \rangle,$$

where $[N_x/\partial N_x]$ is the fundamental class of $H_{2n}(N_x/\partial N_x, \mathbb{Z}_4) \cong \mathbb{Z}_4$

determined by the orientation of $N_x$ inherited from that of $N$, and $\langle \ , \rangle$ stands for the Kronecker product.

It is easy to see that $P$ is well defined and if $n$ is even, then

$$P(x + y) = P(x) + P(y) + 2x \cdot y,$$

where $x \cdot y$ is the intersection number (the proof depends on a formula for the Pontryagin square; cf. [MT, p. 21]).

Now we are in a position to state

**Theorem 2.** Let $n$ be even.

1. If $f : M^n \to N^{2n}$ is an embedding, then

$$\chi(f) = P(f_\ast [M]) + 2w_1(f)w_{n-1}(f) \mod 4$$

where $[M]$ is the generator of $H_n(M, \mathbb{Z}_2) \cong \mathbb{Z}_2$, and $w_1(f)$ the $i$th normal Whitney class of $f$.

2. If $n > 2$, $M$ is nonorientable, and $f$ is a map with $f_\ast : \tilde{\pi}_1(M) \to \pi_1(N)$ surjective, where $\tilde{\pi}_1(M)$ is the subgroup of $\pi_1(M)$ consisting of orientation-preserving elements, then normal Euler numbers of embeddings in $[f]$ fill the mod 4 residue class $P(f_\ast [M]) + 2w_1(f)w_{n-1}(f)$.

**Corollary 1.** If $n$ is even, and $f, g : M^n \to N^{2n}$ are homotopic embeddings, then $\chi(f) = \chi(g) \mod 4$.  

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Corollary 2 (The generalized Whitney congruence of Rohlin [Ro]). Let \( f : M^2 \to N^4 \) be an embedding, where \( N \) is oriented closed and \( f_*[M] \) is characteristic, then
\[
\sigma(N) = \chi(f) + 2\chi(M) \mod 4,
\]
where \( \sigma(N) \) is the signature of \( N \), and \( \chi(M) \) stands for the Euler characteristic number of \( M \).

Theorem 3. Let \( \mathcal{N}_n(N) \) be the bordism group of maps from closed (possibly non-orientable) \( n \)-manifolds into the oriented \( 2n \)-manifold \( N \). Then \( P(f_*[M]) + 2w_1(f)w_{n-1}(f) \) gives a map \( q : \mathcal{N}_n(N) \to \mathbb{Z}_4 \) with the following properties:

1. Any element \( x \in \mathcal{N}_n(N) \) includes embeddings with normal Euler numbers \mod 4 equal to \( q(x) \).
2. A self-transversal immersion with only double points in \( x \) has \mod 4 normal Euler number equal to \( q(x) \) (or \( q(x) + 2 \)) if and only if its number of self-intersection points is even (or odd).
3. Let \( x, y \in \mathcal{N}_n(N) \) be represented by \( f : M_1 \to N \) and \( g : M_2 \to N \), and define \( x \cdot y \) as \( f_*[M_1] \cdot g_*[M_2] \in \mathbb{Z}_2 \). Then \( x \cdot y \) gives a bilinear form on \( \mathcal{N}_n(N) \) and
\[
q(x + y) = q(x) + q(y) + 2x \cdot y.
\]

In general, it is difficult to calculate Pontryagin squares. We shall give another formula for \( q(x) \) in a special case that \( N \) is the total space of the orientation line bundle of a manifold \( K^{2n-1} \), e.g. \( N = R^{2n-1} \times R = R^{2n} \).

Let \( f : M^n \to K^{2n-1} \) be a map, \( K' \) a compact \((2n - 1)\)-dimensional submanifold of \( K \) containing \( f(M) \), and \( D : H_n(K', Z_2) \to H^{n-1}(K', \partial K'; Z_2) \) the Lefschetz dual. Then
\[
u(f) = f^*Df_*[M] \in H^{n-1}(M, Z_2)
\]
is well defined. We have

Theorem 4. If \( N \) is the total space of the orientation line bundle of \( K^{2n-1} \) with \( n \) even, and \( f : M \to N \) is an embedding, then
\[
\chi(f) = 2\bar{\omega}_1(f)(u(f) + \bar{\omega}_{n-1}(f)) \mod 4,
\]
where \( \bar{\omega}_i(f) \) is the \( i \)th stable normal Whitney class of \( f \) regarded as a map \( M \to K \), and \( u(f) \) is understood similarly.

Remark 1. The proof of Theorem 4 is geometric, hence different from those of Mahowald and of Massey for \( N = R^{2n} \).

Remark 2. By Theorem 3 and the proof of Theorem 4, we have
\[
q(x) = 2\bar{\omega}_1(f)(u(f) + \bar{\omega}_{n-1}(f)),$
\]
where \( f \in x \in \mathcal{N}_n(K) = \mathcal{N}_n(N) \), and a formula expressing \( P(f_*[M]) \) in terms of \( u(f), \omega_1(K), \) and \( \omega_i(f), i \leq n - 1 \), can be gotten. From this we see that if \( K^{2n-1} \) is orientable with \( n \) even and \( N = K \times R \), \( y \in H_n(N, Z_2) \) with \( P(y) \neq 0 \), then there is no map \( f : M^n \to N \) with \( M \) orientable and \( f_*[M] = y \).
Example. Let $g : M = RP^n \# RP^n \to RP^n$ be the map collapsing the second copy to a point, and $f$ be the composition

$$M \xrightarrow{g} RP^n \subset RP^{2n-1} \subset RP^{2n-1} \times R.$$ 

Then

$$f_* : \tilde{\pi}_1(M) \to \pi_1(RP^{2n-1} \times R)$$

is surjective, and it follows from (2) of Theorem 2 that if $n > 2$ is even, then the normal Euler numbers of the embeddings homotopic to $f$ fill a mod 4 residue class. Now Theorem 4 tells us that this class is $4\mathbb{Z}$. And we see that for the generator $x$ of $H_n(RP^{2n-1} \times R)$, $P(x) = 2$, and hence $x$ is not represented by maps from orientable $n$-manifolds.

Remark 3. The earliest preprint of this paper was typed in 1989, under the title “Embedding $n$-manifolds in $2n$-manifolds” and was used and quoted in [Li3]. The main content of the paper was given in a talk in a conference held at Tokyo University, September, 1990.

Remark 4. Recently, Yamada [Ya] found the formula in part (1) of Theorem 2 for the case $n = 2$ with $H_1(M, \mathbb{Z}) = 0$ independently, using a geometric method.

2. Proof of Theorem 2

Let $f : M \to N$ be an embedding, $N_f$ a compact tubular neighbourhood of $f(M)$. Regard $N_f$ as the disk bundle of the normal bundle of $f$; we have by Massey’s formula

$$\tilde{P}(U_2) = (\tilde{\rho}_4(X(f)) + \theta(w_1(f)w_{n-1}(f))) \cdot U$$

where $X(f)$ is the normal Euler class.

Let $[N_f/\partial N_f]$ be the fundamental class of

$$H^{2n}(N_f/\partial N_f, \mathbb{Z}_4) \cong \mathbb{Z}_4$$

corresponding to the orientation of $N_f$ inherited from that of $N$. Since $U_2 \in H^n(N_f, \partial N_f; \mathbb{Z}_2)$ is the Lefschetz dual of $f_*[M] \in H_n(N_f, \mathbb{Z}_2)$, we have

$$\tilde{P}(U_2) = P(f_*[M])[N_f/\partial N_f].$$

Let $\chi(f)$ be the normal Euler number determined by the orientation of $N$. Then the Thom isomorphism

$$H^n(M, \tilde{\mathbb{Z}}_4) \cong H^{2n}(N_f/\partial N_f, \mathbb{Z}_4)$$

given by $x \to x \cdot U$ sends

$$(\tilde{\rho}_4(X(f)) + \theta(w_1(f)w_{n-1}(f))) \cdot U$$

to

$$(\chi(f) + 2w_1(f)w_{n-1}(f))[N_f/\partial N_f];$$

hence

$$\chi(f) = P(f_*[M]) + 2w_1(f)w_{n-1}(f) \mod 4$$

and (1) is proved.

Suppose $n > 2$ and $f_* : \tilde{\pi}_1(M) \to \pi_1(N)$ is surjective. Then by Theorem 1, $f$ is homotopic to embeddings and we may assume $f$ is an embedding.
For any \( m \in \mathbb{Z} \), take a self-transversal immersion \( h : S^n \to S^{2n} \) with \( 2|m| \) self-intersection points and \( \chi(h) = 4m \). Making a suitable connected sum of \( f \) and \( h \), we get a self-transversal immersion \( g = f \# h : M \to N \) homotopic to \( f \) with \( 2|m| \) self-intersection points and \( \chi(g) = \chi(f) + 4m \). Assume \( a_1, b_1, a_2, b_2, \ldots, a_{2|m|}, b_{2|m|} \) are distinct points in \( M \) such that \( g(a_i) = g(b_i) \).

Choose simple curves \( I_i, i = 1, 2 \), connecting \( a_i \) and \( b_i \) such that \( I_1 \cap I_2 = \emptyset \) and \( I_1 \cap \{a_3, b_3, \ldots, a_{2|m|}, b_{2|m|}\} = \emptyset \). Then \( g(I_1) \) and \( g(I_2) \) form a simple closed curve \( \gamma \) in \( N \). Given orientations of \( T(M) \) on \( I_1 \) and \( I_2 \), the signs of self-intersections of \( g \) at \( g(a_1) \) and \( g(a_2) \) are determined. The surjectivity of \( f_* : \pi_1(M) \to \pi_1(N) \) together with its implicit that there exist elements in \( \pi_1(M) \setminus \pi_1(N) \) which are in the kernel of \( f_* \) allows us to choose \( I_2 \) so that \( \gamma \) is nullhomotopic and the signs at \( g(a_1) \) and \( g(a_2) \) are opposite. Thus, by Whitney’s technique, we can get an immersion regularly homotopic to \( g \) with \( 2|m| - 2 \) self-intersection points. Continuing in this way, we will get at last an embedding regularly homotopic to \( g \). This proves (2).

**Proof of Corollary 2.** By a formula of Wu (see [Wu] or [Th]),

\[
P(f_*[M]) = \tilde{P}(w_2(N)) = p_1(N) \mod 4 + 2w_4(N),
\]

where \( p_1 \) is the first Pontryagin class of \( N \), \( w_4(N) + \sigma(N) \mod 2 \) is an \( \mathcal{N}_4^{so} \)-invariant, and \( w_4(CP^2) + \sigma(CP^2) \mod 2 = 0 \); hence \( w_4(N) + \sigma(N) \mod 2 = 0 \) for any \( N \). This fact together with \( p_1(N) = 3\sigma(N) \) implies \( P(f_*[M]) = \sigma(N) \mod 4 \). Now, Corollary 2 follows from (1) of Theorem 2.

### 3. Proof of Theorem 3

First, since \( f_*([M]) \) and \( w_1(f)w_{n-1}(f) \) are bordism invariants, \( q \) is well defined.

Now, let \( f : M \to N \) be a self-transversal immersion, and \( b \in N \) be a self-intersection point of \( f \) such that \( b = f(a_1) = f(a_2) \), but \( a_1 \neq a_2 \). Take a neighbourhood of \( b \) which corresponds to \( R^{2n} \) diffeomorphically such that the image of \( f \) in this neighbourhood corresponds to \( R^n \times 0 \cup 0 \times R^n \). For \( x \in R^n \) with \( |x| < 2 \), let \( v(x, 0) = (0, x) \), \( v(0, x) = (x, 0) \). Then \( v \) extends to a normal vector field of \( f \) denoted by \( \nu \) also. We may assume \( v \) is transversal to the zero section since it is already so at \( a_1 \) and \( a_2 \). The normal Euler number of \( f \) is the algebraic sum of the zeros of \( v \), and the total contribution of \( a_1 \) and \( a_2 \) is \( \pm 2 \).

Let \( \gamma \) be a curve in the plane as shown in Figure 1.
We assume \( \gamma(t) = (\gamma_1(t), \gamma_2(t)), 0 \leq t \leq 1, \) such that \( \gamma(0) = (0, 1), \gamma(1) = (1, 0), \) and \( \gamma \) contacts with the \( x \)-axis and \( y \)-axis smoothly.

Let
\[
D^n = \{ x \in \mathbb{R}^n | |x| < 1 \},
S^{n-1} = \{ x \in \mathbb{R}^n | |x| = 1 \}.
\]

Take off the neighbourhoods of \( a_1 \) and \( a_2 \) in \( M \) which correspond to \( D^n \times 0 \) and \( 0 \times D^n \), and then add \( S^{2n-1} \times I \) naturally; we get a new manifold \( M' \).

Mapping \( S^{n-1} \times I \) to \( \mathbb{R}^{2n} \) by
\[
(\omega, t) \rightarrow (\gamma_1(t)\omega, \gamma_2(t)\omega),
\]
we have an immersion \( f' \) of \( M' \) in \( N \) and a normal vector field \( v' \) of \( f' \) which are identical with \( f \) and \( v \) respectively outside the neighbourhoods of \( a_1 \) and \( a_2 \). On \( S^{n-1} \times I \), we may assume \( v' \) has no zeros, as can be seen from Figure 2.

Now it is clear that the numbers of self-intersection points of \( f \) and \( f' \) differ by 1 and their normal Euler numbers differ by 2.

Since \( f \) and \( f' \) are obviously bordant, and any bordism class in \( \mathcal{N}_n(N) \) is represented by a self-transversal immersion, we prove property (1) by repeating the process from \( f \) to \( f' \) and Theorem 2, (1).

If \( f \) has even (odd) number of self-intersection points, then \( f \) is bordant to an embedding \( g \) with \( \chi(f) - \chi(g) = 2 \times \) even number (2\( \times \) odd number). This proves (2).

Property (3) is straightforward since
\[
P(x + y) = P(x) + P(y) + 2x \cdot y
\]
and
\[
w_1(x + y)w_{n-1}(x + y) = w_1(x)w_{n-1}(x) + w_1(y)w_{n-1}(y).
\]

The proof is complete.

4. Proof of Theorem 4

We divide the proof into three steps.

**Step 1.** First, we notice that the homotopy class \([f] \in [M, N]\) is represented by a self-transversal immersion \( g : M \to K \) (cf. [LP]). Regard \( g \) as an immersion of \( M \) in \( N \); then its normal bundle includes a line bundle, hence has an orientation-reversing automorphism. Thus \( 2\chi(g) = 0 \) (cf. [Li1]), and \( \chi(g) = 0 \). We will prove in Steps 2 and 3 that \( g \) is regularly homotopic to a self-transversal immersion \( f_1 \) of \( M \) in \( N \) with mod 2 number of
self-intersection points \( w_1(g)(u(g) + w_{n-1}(g)) \). Since an immersion regularly homotopic to embeddings must have an even number of self-intersections, and \( \chi(g) = 0 \), Theorem 3 follows immediately from the classification theorem of immersions of \( M \) in \( N \) (cf. \([\text{Li2}]\)).

The aim of Step 2 (for \( n > 2 \)) and Step 3 (for \( n = 2 \)) is to construct an immersion \( f_1 : M \to N \) which is self-transversal and regularly homotopic to \( g \), and calculate the number of self-intersection points of \( f_1 \).

**Step 2.** Suppose \( n > 2 \). Then the multiple points of \( g \) consist only of double points whose set \( X \subset M \) is the disjoint union of some circles \( S_1, \ldots, S_k \) such that \( g(S_{2i-1}) = g(S_{2i}) \), \( i = 1, \ldots, j \), and \( S_i \to g(S_i) \) is a nontrivial 2-sheet covering if \( i > 2j \). By formula (1) and the introduction of \([\text{He}]\), we see that the homology class \([X]\) in \( H_1(M, \mathbb{Z}_2) \) represented by \( X \) is the Poincaré dual of \( u(g) + w_{n-1}(g) \).

Let \( \delta_i = \pm 1 \) (\( \delta = \pm 1 \)) so that \( \epsilon_i = 1 \) (\( \delta_i = 1 \)) iff \( S_i \) (\( g(S_i) \)) is orientation-preserving in \( M \) (in \( K \)). Then it is easy to see that

\[
\delta_{2i-1} = \delta_{2i} = \epsilon_{2i-1} \epsilon_{2i} \quad \text{if} \ 1 < i \leq j, \\
\delta_i = -\epsilon_i \quad \text{if} \ 2j < i \leq k.
\]

Denoting by \( \xi \) the orientation bundle of \( K \), and \( \xi_0 \) the bundle of nonzero vectors of \( \xi \), we define \( f_1 \) on \( X \) as follows:

1. If \( 1 \leq i \leq j \), then \( f_1 = g \) on \( S_{2i-1} \) and \( f_1 = u \circ g \) on \( S_{2i} \), where \( u \) is a smooth section of \( g(S_{2i}) \) transversal to the zero section.
2. If \( i > 2j \) and \( \delta_i = -1 \), then there is a smooth map \( f_1 : S_i \to \xi_0 \) such that \( p \circ f_1 = g \), where \( p \) is the projection of \( \xi \).
3. If \( i > 2j \) and \( \delta_i = 1 \), then there is a smooth map \( f_1 : S_i \to \xi \) with \( p \circ f_1 = g \) such that \( f_1 \) has only one transversal self-intersection point.

Extend \( f_1 \) to an immersion of \( M \) in \( N \) so that \( f_1 = g \) outside a tubular neighbourhood of \( X \), and \( p \circ f_1 = g \) on this neighbourhood. Then \( f_1 \) is regularly homotopic to \( g \) in \( N \) and has only transversal self-intersection points in \( f_1(X) \).

Now we calculate

\[
\langle w_1(g), [X] \rangle = \sum_{i=1}^{k} \langle w_1(g), [S_i] \rangle.
\]

Letting \( s(1) = 0 \) and \( s(-1) = 1 \), we have

\[
\langle w_1(M), [S_i] \rangle = s(\epsilon_i), \\
\langle g^*w_1(K), [S_i] \rangle = \begin{cases} 
  s(\delta_i), & 1 \leq i \leq 2j, \\
  0, & 2j < i \leq k.
\end{cases}
\]

It follows then from

\[
w_1(M) + w_1(g) = g^*w_1(K)
\]

that

1. if \( 1 \leq i \leq j \),

\[
\langle w_1(g), [S_{2i-1}] + [S_{2i}] \rangle = \begin{cases} 
  0, & \text{if} \ \delta_{2i} = 1, \\
  1, & \text{if} \ \delta_{2i} = -1.
\end{cases}
\]

2. if \( 2j < i \leq k \) and \( \delta_i = -1 \),

\[
\langle w_1(g), [S_i] \rangle = 0;
\]

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
(3) if $2j < i \leq k$ and $\delta_i = 1$,

$$\langle w_1(g), [S_i] \rangle = 1.$$  

This shows that $\langle w_1(g), [X] \rangle$ is equal to mod 2 number of self-intersection points of $f_i$. Since

$$\langle w_1(g), [X] \rangle = w_1(g)(u(g) + w_{n-1}(g)),$$  

the proof for the case $n > 2$ is complete.

**Step 3.** Suppose $n = 2$. Now, the multiple points of $g$ consist of double points and triple points whose set $X \subset M$ is the image of a self-transversal immersion $h$ of the disjoint union of some $l$ copies of the circle $S$. Denote by $h_i$ the restriction of $h$ on the $i$th copy of $S$.

Let

$$X_1 = \bigcup_{i=1}^{k} h_i(S), \quad X_2 = \bigcup_{i=k+1}^{l} h_i(S)$$

such that each $h_i(S)$ in $X_1$ includes triple points, while $X_2$ does not. Then $X_1 \cap X_2 = \emptyset$ and $h_\alpha(S) \cap h_\beta(S) = \emptyset$, if $k < \alpha < \beta \leq l$, and $X = X_1 \cup X_2$. Moreover, we have

$$g(X_1) \cap g(X_2) = \emptyset.$$  

We are able to cope with $X_2$ exactly as in Step 2. To cope with $X_1$, we may assume first that

$$g(h_{2i-1}(S)) = g(h_{2i}(S)), \quad \text{if } 1 \leq i \leq j,$$

$$g(h_i(S)) \neq g(h_\alpha(S)), \quad \text{if } 2j < i \leq k, \alpha \neq i, 1 \leq \alpha \leq k.$$  

We have also

$$\delta_{2i-1} = \delta_{2i} = \varepsilon_{2i-1} \varepsilon_{2i}, \quad \text{if } 1 \leq i \leq j,$$

$$\delta_i = -\varepsilon_i, \quad \text{if } 2j < i \leq k,$$

where $\varepsilon_j = \pm 1$, $\delta_i = \pm 1$, and $\varepsilon_i = 1$ ($\delta_i = 1$) iff $h_i : S \to M$ ($g \circ h_i : S \to K$) is an orientation-preserving loop.

Let $X_3 = \{d_1, d_2, \ldots, d_{3s}\} \subset X_1$ be the set of triple points of $g$ which is the set of self-intersection points on $h$ such that

$$g(d_{3i-2}) = g(d_{3i-1}) = g(d_{3i}), \quad 1 \leq i \leq s,$$

and let $u$ be a nonzero section of $\xi$ over $g(X_3)$. Let $t : X_3 \to R$ be given by

$$t(d_{3i-2}) = -1, \quad t(d_{3i-1}) = 0, \quad t(d_{3i}) = 1.$$  

**Step 3(a).** Suppose $j \geq 1$ and

$$h_1^{-1}(X_3) = \{a_1, a_2, \ldots, a_\alpha\}.$$  

Then

$$h_2^{-1}(X_3) = \{b_1, b_2, \ldots, b_\alpha\}$$

has the following properties.

1. $g(h_1(a_i)) = g(h_2(b_i))$,
2. $h_1(a_i) \neq h_2(b_i).$
Regarding $S$ as $[0, 1]/\{0, 1\}$, we may assume that
\[
a_1 = 0 < a_2 < \cdots < a_\alpha < 1 = a_{\alpha+1},
\]
\[
a_i = b_i, \quad i = 1, 2, \ldots, \alpha + 1,
\]
\[
g(h_1(q)) = g(h_2(q)), \quad \text{for } q \in [0, 1].
\]
Define $f_1$ on $X_3$ by
\[
f_1(x) = t(x)u(g(x));
\]
then extend $f_1$ to a smooth map on $h_1(S)$ so that
\[
f_1(h_1(q)) = t_i(h_1(q))u_i(g(h_1(q))), \quad \text{if } q \in [a_i, a_{i+1}],
\]
where $u_i$ is a nonzero smooth section of $\xi$ over $g(h_1([a_i, a_{i+1}]))$ with
\[
(u_i \circ g \circ h_1)(q) = (u \circ g \circ h_1)(q) \quad \text{for } q = a_i, a_{i+1},
\]
and $t_i \circ h_1$ is a smooth function on $[a_i, a_{i+1}]$ such that
\[
t_i(h_1(q))u_i(g(h_1(q))) = t(h_1(q))u_i(g(h_1(q))), \quad \text{for } q = a_i, a_{i+1},
\]
and
\[
\frac{d}{dq} t_i(h_1(q))\left\{\begin{array}{l}
0 \quad \text{on } [a_i, a_i + \varepsilon] \cup [a_{i+1} - \varepsilon, a_{i+1}],
\end{array}\right.
\]
is either identically zero or nonzero on $(a_i + \varepsilon, a_{i+1} - \varepsilon)$, where $0 < \varepsilon < \frac{1}{2}(a_{i+1} - a_i)$ for $i = 1, 2, \ldots, \alpha$.

Let $v(q)$ be a vector in the fiber of $\xi$ over $g(h_1(q))$ such that
\[
v(a_i) = (u_i \circ g \circ h_1)(a_i),
\]
\[
v(q) = \pm(u_i \circ g \circ h_1)(q), \quad \text{if } q \in [a_i, a_{i+1}],
\]
and $v$ as a map $[a_1, a_{\alpha+1}] = [0, 1] \rightarrow N$ is continuous. Then $v$ is smooth,
\[
v(a_{\alpha+1}) = s_1v(a_1)
\]
and
\[
f_1(h_1(a)) = s_1(q)v(q),
\]
where $s_1$ is a smooth real-valued function on $[0, 1]$ such that
\[
\frac{ds_1}{dq}\left\{\begin{array}{l}
0 \quad \text{if } |q - a_i| \leq \varepsilon \text{ for some } i,
\end{array}\right.
\]
is either identically zero or nonzero on $(a_i + \varepsilon, a_{i+1} - \varepsilon)$.

Similarly, we can define
\[
f_1(h_1(q)) = s_2(q)v(q), \quad q \in [0, 1],
\]
with the same $v$, and $s_2$ sharing the same properties of $s_1$ stated above. Since
\[
f_1(h_1(q)) = g(h_1(q)), \quad \text{for } q = 0, 1 \text{ and } i = 1, 2,
\]
we have
\[
s_i(0) = \delta_is_i(1), \quad i = 1, 2.
\]
Moreover, $s_1(a_i)$ and $s_2(a_i)$ belong to the set $\{0, \pm 1\}$ and
\[
s_1(a_i) \neq s_2(a_i), \quad i = 1, 2, \ldots, \alpha + 1,
\]
because $f_1(h_1(a_i)) \neq f_1(h_2(a_i))$. Therefore, the graphs of $s_1$ and $s_2$ intersect transversally, and the number of their intersections is even iff $\delta_1 = 1$. Using the same method, we define $f_1$ on $h_{2i-1}(S) \cup h_{2i}(S)$ for any $i \in [2, j]$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Step 3(b). Assume \( k > 2j \) and

\[ h_k^{-1}(X_3) = \{ a_1, \ldots, a_\alpha, b_1, \ldots, b_\alpha \} \]

so that \( g(h_k(a_i)) = g(h_k(b_i)) \), and \( a_2, \ldots, a_\alpha \) are located in a half-circle bounded by \( a_1 \) and \( b_1 \). Let \( p_1 \) and \( p_2 \) be diffeomorphisms of \([0, 1]\) onto the half-circles containing \( a_2 \) and \( b_2 \) respectively such that

\[ g \circ h_k \circ p_1 = g \circ h_k \circ p_2 \]

and \( f \circ h_k \circ p_i \) is smooth as a map defined on \([0, 1]/\{0, 1\}\). Then there are a map \( v : [0, 1] \to \xi \) and real-valued functions \( s_1 \) and \( s_2 \) on \([0, 1]\) as in step 3(a), and

\[
\begin{align*}
  v(1) &= \delta_k v(0), \\
  (f_1 \circ h_k \circ p_i)(q) &= s_i(q)v(q), \quad \text{for } q \in [0, 1] \text{ and } i = 1, 2, \\
  (f_1 \circ h_k \circ p_i)(q) &\neq (f_1 \circ h_k \circ p_2)(q), \quad q \in \{0, 1\}, \\
  (f_1 \circ h_k \circ p_i)(0) &\neq (f_1 \circ h_k \circ p_i)(1), \quad i = 1, 2, \\
  (f_1 \circ h_k \circ p_1)(0) &= (f_1 \circ h_k \circ p_2)(1), \\
  (f_1 \circ h_k \circ p_1)(1) &= (f_1 \circ h_k \circ p_2)(0).
\end{align*}
\]

Therefore, the graphs of \( s_1 \) and \( s_2 \) are transversal and the number of their intersections is even iff \( \delta_k = -1 \). Do the same for \( h_i(S) \) with \( 2j < i \leq k \).

Combining Steps 2, 3(a), and 3(b), we have defined \( f_1 \) on \( X = X_1 \cup X_2 \). \( f_1 \) can be extended to an immersion of \( M \) in \( N \) regularly homotopic to \( g \) with transversal self-intersection points whose mod 2 number contributed by \( X_1 \) is

\[
\sum_{i=1}^j s(\delta_{2i}) + \sum_{i=2j+1}^k s(-\delta_i) = \langle w_1(g), [X_1] \rangle.
\]

This together with Steps 1 and 2 proves Theorem 4.

**References**


